Localization for divergence operators with long range random perturbations

Hatem NAJAR*

Abstract

The object of this paper is to study Anderson and strong dynamical localizations on the internal band edges of the spectrum for random perturbations of periodic divergence operators of the form $A_\omega = -\nabla \cdot a_\omega \cdot \nabla$, where a_ω is a long range perturbation of a periodic matrix function. Our results rely on the study of Lifshitz tails from which we get the initial estimate necessary for the multiscale analysis.

1 Introduction

This paper is devoted to study one of the most important properties of random operators: localization. By localization we mean a pure point spectrum associated with exponential decaying eigenfunctions. In this case we often say that we have Anderson localization. Localization is said to be a strong dynamical localization to an order p, on an interval I of the spectrum, when for P_I, the spectral projector onto that interval, and compact set K, we have

$$\mathbb{E}\left\{\sup_{t>0}\left\||x|^p\, e^{itA_\omega} P_I(A_\omega)\chi_K\right\|\right\} < +\infty.$$

Those two types of localization have been intensively studied during last years for their important physical consequences. We quote localization of classical (for instance, electromagnetic or acoustic) waves in a periodic medium perturbed by random impurities. See [8, 9, 10, 11].

*Laboratoire Analyse, Géometrie et Application. Institut Galillée, Université Paris 13. 93430 Villetaneuse (France). Email: najar@math.univ-paris13.fr

Localization at the edge of the spectrum was conjectured for the first time by the physicists at the end of the fifties [2] for Schrödinger operators. It is widely accepted that random models should exhibit localization near fluctuation boundaries. The latter are those parts of the spectrum which are determined by rather rare events. Such phenomenon occurs in various cases (large disorder, extreme energies, Lifshitz tails, etc.). In this context the multiscale analysis is a powerful technique, initially developed for the purpose of proving Anderson localization [7, 8]. It was later shown to also yield dynamical localization (non spreading of the wave packets) [14] and more recently strong dynamical localization (dynamical localization not only with probability one, but in expectation) up to some order [6]. Strong dynamical localization up to any order and in the Hilbert-Schmidt norm is proved in [13].

In literature, operators considered have, in general, either the Schrödinger form [3, 4, 6, 19, 20, 36]:

$$A_\omega = -\Delta + W_{per} + V_\omega,$$

or the divergence form [8, 34]

$$A_\omega = -\nabla \cdot a_\omega \cdot \nabla.$$

In these references, localization near the so called fluctuation boundaries is proved. In what follows, we consider the self-adjoint operator on $L^2(\mathbb{R}^d)$ formally defined by

$$A_\omega = A(a_\omega) = -\nabla \cdot a_\omega \cdot \nabla = - \sum_{1 \le i,j \le d} \partial_{x_i} a_{i,j,\omega} \partial_{x_j}, \tag{1.1}$$

where $a_\omega = (a_{i,j}(\omega))_{1 \le i,j \le d}$ is assumed to be a random bounded and uniformly elliptic matrix; i.e. there exists constants $a_+ > a_- > 0$ such that for any $\xi \in \mathbb{R}^d$ we have

$$a_- \|\xi\|^2 \le \langle a_\omega \xi, \xi \rangle \le a_+ \|\xi\|^2 . \tag{1.2}$$

Let us notice that when $a_\omega = \frac{1}{\rho_\omega} \cdot I_d$ (where I_d is the identity matrix) A_ω described by (1.1) will be the acoustic operator treated in [8, 29, 30, 34].

The main difference with [8, 34] is that our approach to prove localization near the band edges is based on the fact that near those edges the integrated

density of states of the divergence operator exhibits Lifshitz tails. This, enables us to use a weaker assumption on the probability distribution. Such technique was used for the Schrödinger case in [27, 36]. By this our result extends in sense the results of [8, 34].
Briefly, the paper is orgaized as follows:
In section 1, we define the model to be studied and specify various assumptions. We introduce a periodic reference operator A_0. We state the main results (Theorem 1.1 and 1.2). Section 2 contains the proof of Lifshits tails (Theorem 1.2). It is composed of three subsections: introduction to the periodic approximations and statements of some useful lemmas, lower bound and upper bound. In section 3 we prove Theorem 1.1. From [6] and [13] this theorem is proved using a powerful technique called multiscale analysis. (See section 3.1.) So the proof is based on verifying the two main ingredients necessary for this analysis called *initial estimate* and *uniform Wegner estimate*, see [17, 20]. In section 3.2, Theorem 1.2 is used to get the initial estimate. We end this work by an appendix where we prove some technical lemmas used previously.

1.1 The model

We start this section by giving the expression of the coefficients of the matrix $a_\omega = (a_{i,j,\omega}(x))_{1\leq i,j\leq d}$. We assume that a_ω is a matrix which satisfies the following
(A.0)

$$a_{i,j,\omega}(x) = a_{i,j,0}(x) + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma^{i,j} u_{i,j}(x-\gamma);$$

where
(i) $(a_{i,j,0})_{1\leq i,j\leq d}$ is a family of measurable, real valued and $\mathbb{Z}^d$-periodic functions i.e.

$$a_{i,j,0}(x+\gamma) = a_{i,j,0}(x),\ \forall x \in \mathbb{R}^d, \gamma \in \mathbb{Z}^d.$$

(ii) There exists constants $a_{0,+} > a_{0,-} > 0$ such that for any $\xi \in \mathbb{R}^d$ and almost all $x \in \mathbb{R}^d$, we have

$$a_{0,-}\|\xi\|^2 \leq a_{i,j,0}(x)\xi_i\xi_j \leq a_{0,+}\|\xi\|^2. \tag{1.3}$$

(iii) $(u_{i,j})_{1\leq i,j\leq d}$ is a family of real-valued functions, such that there exists $\nu \in (d, d+2]$ and $0 < g_- \leq g_+$ two non vanishing functions, such that for any $\gamma \in \mathbb{Z}^d$ and almost every x in the unit box, C_0 one has

$$g_- \cdot \|\xi\|^2 \leq \sum_{1\leq i,j\leq d} u_{i,j}(x+\gamma)\cdot(1+|\gamma|)^\nu \xi_i\xi_j \leq g_+ \cdot \|\xi\|^2 ; \qquad (1.4)$$

and for all $1 \leq s \leq d$,

$$g_- \leq |(\partial_{x_s} u_{i,j})(x+\gamma)| \cdot (1+|\gamma|)^\nu \leq g_+. \qquad (1.5)$$

(iv) $(\omega_\gamma^{i,j})_{\substack{\gamma\in\mathbb{Z}^d\\ 1\leq i,j\leq d}}$ is a family of independents identically distributed random variables taking values in the interval $[0,1]$, whose common probability distribution $\mathbb{P}_0$ has a bounded density g a.e in $[0,1]$. More precisely if we note the probability space by $(\Omega, \mathcal{F}, \mathbb{P})$ then, our probability space has a product structure i.e. $\Omega = [0,1]^{d^2\mathbb{Z}^d}$ and $\mathbb{P} = \otimes_{\substack{\gamma\in\mathbb{Z}^d\\ 1\leq i,j\leq d}} \mathbb{P}_0$.

Let $\mathcal{A}(a_\omega)$ the quadratic form defined as follow: for $u \in H^1(\mathbb{R}^d) = \mathcal{D}(\mathcal{A}(\varrho_\omega))$

$$\begin{aligned}\mathcal{A}(a_\omega)[u,u] &= \int_{\mathbb{R}^d} a_\omega(x)\cdot\nabla u(x)\overline{\nabla u(x)}dx \\ &= \sum_{1\leq i,j\leq d}\int_{\mathbb{R}^d} a_{i,j,\omega}(x)\cdot\partial_{x_i}u(x)\overline{\partial_{x_j}u(x)}dx.\end{aligned}$$

$\mathcal{A}(a_\omega)$ is a symmetric positive and closed quadratic form. A_ω given by (1.1) is defined as the self adjoint operator associated to $\mathcal{A}(a_\omega)$ [33].
Assumption **(A.0)** ensures that A_ω is a measurable family of self adjoint and ergodic operators [32]. Indeed, if τ_γ refers to the translation by γ, then $(\tau_\gamma)_{\gamma\in\mathbb{Z}^d}$ is a group of unitary operators on $L^2(\mathbb{R}^d)$ and for $\gamma \in \mathbb{Z}^d$ we have

$$\tau_\gamma A_\omega \tau_{-\gamma} = A_{\tau_\gamma\omega}.$$

According to [32] we know that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and Σ_{sc} closed and non random sets of $\mathbb{R}$ such that Σ is the spectrum of A_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) notice the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of A_ω, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one.

1.2 Reference operator

Notice that A_ω can be written as a perturbation of some background operator A_0. More precisely we write:

$$A_\omega = A_0 + \Delta A_\omega,$$

with

$$A_0 = A(a_0)$$

and

$$\Delta A_\omega = A(a_\omega - a_0).$$

1.2.1 Some Floquet Theory

Now we review some standard facts from the Floquet theory for periodic operators. Basic references of this material are [33, 35].
As $(a_{0,i,j})_{1\leq i,j\leq d}$ is a family of $\mathbb{Z}^d$-periodic functions, for any $\gamma \in \mathbb{Z}^d$, we have

$$\tau_\gamma A_0 \tau_\gamma^* = \tau_\gamma A_0 \tau_{-\gamma} = A_0.$$

Let $\mathbb{T}^* = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. We define $\mathcal{H}$ by

$$\mathcal{H} = \{u(x,\theta) \in L^2_{loc}(\mathbb{R}^d)\otimes L^2(\mathbb{T}^*); \forall (x,\theta,\gamma) \in \mathbb{R}^d\times\mathbb{T}^*\times\mathbb{Z}^d;\ u(x+\gamma,\theta) = e^{i\gamma\theta}u(x,\theta)\}.$$

There exists U a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$ such that A_0 admits the Floquet decomposition [35]

$$UA_0U^* = \int_{\mathbb{T}^*}^{\oplus} A_0(\theta)d\theta.$$

Here $A_0(\theta)$ is the operator A_0 acting on $\mathcal{H}_\theta$, defined by

$$\mathcal{H}_\theta = \{u \in L^2_{loc}(\mathbb{R}^d); \forall \gamma \in \mathbb{Z}^d, u(x+\gamma) = e^{i\gamma\theta}u(x)\}.$$

As A_0 is elliptic, we know that, $A_0(\theta)$ has a compact resolvent; hence its spectrum is discrete [33]. We denote its eigenvalues, called Floquet eigenvalues of A_0, by

$$E_0(\theta) \leq E_1(\theta) \leq \cdots \leq E_n(\theta) \leq \cdots.$$

The corresponding eigenfunctions are denoted by $(w(x,\cdot)_j)_{j\in\mathbb{N}}$. The functions $(\theta \to E_n(\theta))_{n\in\mathbb{N}}$ are Lipshitz-continuous, and we have

$$E_n(\theta) \to +\infty \ \text{ as } n \to +\infty \ \text{ uniformly in } \ \theta.$$

The spectrum $\sigma(A_0)$ of A_0 has a band structure. (i.e $\sigma(A_0) = \cup_{n\in\mathbb{N}} E_n(\mathbb{T}^*)$.)

1.2.2 The main assumptions

We assume that A_0 has a spectral gap below E_+. More precisely we assume that:
(**A.1**)
There exists E_+ and $\delta > 0$ such that $\sigma(A_0) \cap [E_+, E_+ + \delta) = [E_+, E_+ + \delta)$ and $\sigma(A_0) \cap (E_+ - \delta, E_+] = \emptyset$.

As, $\Delta A_\omega \geq 0$, Σ contains an interval of the form $[E_+, E_+ + a]$ $(a > 0)$ [18]. As we are interested in the behavior of the IDS in the neighborhood of E_+, we require that E_+ remains always the edge of a gap for Σ, when the perturbation is turned on. More precisely, if for all $t \in [0, 1]$, we define $A_{\omega,t} = A_0 + t\Delta A_\omega$ and Σ_t is the almost sure spectrum of $A_{\omega,t}$, then one requires that the following assumption holds.
(**A.2**)
There exists $\delta' > 0$ such that for all $t \in [0, 1]$, $\Sigma_t \cap [E_+ - \delta', E_+) = \emptyset$.
We assume also the following:
(**A.3**)

$$\limsup_{\varepsilon \to 0^+} \frac{\log|\log \mathbb{P}_0([0, \varepsilon])|}{|\log \varepsilon|} = 0 \tag{1.6}$$

.

1.3 The main results

Let us consider the wave equation:

$$\frac{\partial^2 v}{\partial t^2} = -A_\omega v, \;\; v(0) = v_0, \;\; \frac{\partial v}{\partial t}\Big|_{t=0} = v_1. \tag{1.7}$$

The solution of the equation (1.7) is given by:

$$v(t) = \cos\left(t\sqrt{A_\omega}\right)v_0 + \sin\left(t\sqrt{A_\omega}\right)w_1,$$

where $v_1 = \sqrt{A_\omega}w_1$ and v_0, w_1 are in the domain of A_ω.
The result which we present can be summarized as follow:
• the spectrum of A_ω is pure point in a neighborhood of the edge of $\Sigma\backslash\{0\}$.
• the eigenfunctions corresponding to eigenvalues in this neighborhood decay exponentially.

• for a good choice of v_0 and w_1, $v(t)$ does not disperse in the space.
The first and second assertion are often named **Anderson localization**. The last one is called **strong dynamical localization**.
The main theorem is:

Theorem 1.1 *Let A_ω defined by (1.1). We assume that* **(A.1)**, **(A.2)** *and* **(A.3)** *hold. There exists $\varepsilon_0 > 0$ such that*
(i) $\Sigma \cap [E_+, E_+ + \varepsilon_0] = \Sigma_{pp} \cap [E_+, E_+ + \varepsilon]$.
(ii) an eigenfunction corresponding to an eigenvalue in $[E_+, E_+ + \varepsilon_0]$ decays exponentially .
(iii) there exist $p > 0$ such that

$$\mathbb{E}\Big\{ \sup_{t>0} \Big|\Big| \Big|X\Big|^p \cos\left(t\sqrt{A_\omega}\right) P_{[E_+, E_+ + \varepsilon_0]}(A_\omega)\chi_K \Big|\Big| \Big\} < +\infty;$$

and

$$\mathbb{E}\Big\{ \sup_{t>0} \Big|\Big| \Big|X\Big|^p \sin\left(t\sqrt{A_\omega}\right) P_{[E_+, E_+ + \varepsilon_0]}(A_\omega)\chi_K \Big|\Big| \Big\} < +\infty.$$

Here $P_I(A_\omega)$ is the spectral projection on the interval I, χ_K is the characteristic function of K, K is a compact of $\mathbb{R}^d$ and X is the position operator.

The result of Theorem 1.1 can be related to the behavior of the integrated density of states in the neighborhood of the so-called fluctuation boundary E_+ [25, 27, 31, 32]. For this, we recall that the integrated density of states is defined as follows: we note by $A_{\omega,\Lambda}$ the restriction of A_ω to Λ with self adjoint boundary conditions. As A_ω is elliptic, the resolvent of $A_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $A_{\omega,\Lambda}$ is discrete and is made of isolated eigenvalues of finite multiplicity [33]. We define

$$N_\Lambda(E) = \frac{1}{\mathrm{vol}(\Lambda)} \cdot \#\{\mathrm{eigenvalues\ of\ A_{\omega,\Lambda} \leq E}\}. \tag{1.8}$$

Here $\mathrm{vol}(\Lambda)$ is the volume of Λ in the Lebesgue sense and $\#E$ is the cardinal of E.
It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists almost surely and is independent of the boundary conditions. It is called the **integrated density of states** of A_ω (IDS for the short form). See [16].
For what concern the IDS, we prove that an operator defined by (1.1) exhibits Lifshitz tails in the neighborhood E_+, this was done for various families of random operators (See [18, 23, 24, 28, 29, 30, 32]). For acoustic operators this is proved in [29] under the assumption of non-degeneracy of the IDS of the back-ground operator A_0 which is relaxed here.

Theorem 1.2 *Let A_ω the operator defined by (1.1). We assume that* **(A.1)**, **(A.2)** *and* **(A.3)** *hold. Then*

$$\lim_{\varepsilon\to 0,\varepsilon>0} \frac{\log|\log\big(N(E_+ + \varepsilon) - N(E_+)\big)|}{\log\varepsilon} = -\frac{d}{\nu - d}. \tag{1.9}$$

We end this section by a remark.

Remark 1.3 *Theorems 1.1 and 1.2 are stated for upper bands edges of spectral gap. Under adequate assumption their results are also true for lower bands edges.*

2 Lifshitz tails: the proof of Theorem 1.2

In this section we will prove that the IDS of the operator defined by (1.1) exhibits Lifshitz tails. Such result is proved for several random models. For relevant results on Lifshitz tails we refer to [23, 24, 29, 30]. The proof that we give use the technique of periodic approximations [23, 29] and it consists of two essential parts; lower and upper bounds.

2.1 The periodic approximations

Let us consider the following periodic operator

$$A_{\omega,k} = -\nabla \cdot a_{\omega,k} \cdot \nabla,$$

where the function $a_{\omega,k}$ is the matrix with the coefficient

$$a_{i,j}(\omega,k) = a_{0,i,j} + \sum_{\gamma\in C_k\cap\mathbb{Z}^d} \omega_\gamma \sum_{\beta\in(2k+1)\mathbb{Z}^d} u_{i,j}(\cdot - \gamma - \beta)$$

C_k is the cube

$$C_k = \{x \in \mathbb{R}^d; \forall 1 \le j \le d,\ -\frac{2k+1}{2} < x_j \le \frac{2k+1}{2}\}.$$

$A_{\omega,k}$ is $(2k+1)\mathbb{Z}^d$-periodic and essentially self adjoint operator. Let $\mathbb{T}_k^* = (\mathbb{R}^d)/2(2k+1)\pi\mathbb{Z}^d$. We define $N_{\omega,k}$ the IDS of $A_{\omega,k}$ by

$$N_{\omega,k}(E) = \frac{1}{(2\pi)^d} \sum_{n\in\mathbb{N}} \int_{\{\theta\in\mathbb{T}_k^*,\ E_{\omega,k,n}(\theta)\le E\}} d\theta. \tag{2.10}$$

Let $dN_{\omega,k}$ the derivative of $N_{\omega,k}$ in the distribution sense. As $N_{\omega,k}$ is increasing, $dN_{\omega,k}$ is a positive measure; it is the density of states of $A_{\omega,k}$. We denote by dN the density of states of A_ω. For all $\varphi \in C_0^\infty(\mathbb{R})$, $dN_{\omega,k}$ verifies (see [23])

$$\begin{aligned}\langle \varphi, dN_{\omega,k}\rangle &= \frac{1}{(2\pi)^d}\int_{\theta\in\mathbb{T}_k^*} \mathrm{tr}_{\mathcal{H}_\theta}\big(\varphi(A_{\omega,k,\theta})\big)d\theta, \\ &= \frac{1}{\mathrm{vol}(C_k)}\mathrm{tr}\Big(\chi_{C_k}\varphi(A_{\omega,k})\chi_{C_k}\Big), \end{aligned} \tag{2.11}$$

where for $\Lambda \subset \mathbb{R}^d$, χ_Λ will design the characteristic function of Λ and $\mathrm{tr}(A)$ is the trace of A (we index by $\mathcal{H}_\theta$ if the trace is taken in $\mathcal{H}_\theta$). The proof of (2.11) is given in [23].

Theorem 2.1 *[23, 28] For any $\varphi \in C_0^\infty(\mathbb{R})$ and for almost all $\omega \in \Omega$ we have*

$$\lim_{k\to\infty}\langle \varphi, dN_{\omega,k}\rangle = \langle \varphi, dN\rangle.$$

In what follow we prove that the IDS of A_ω is exponentially well approximated by the expectation of the IDS of the periodic operators $A_{\omega,k}$ when k is polynomial in ε^{-1}. More precisely we prove

Lemma 2.2 *For any $\eta_0 > 0$, there exists $\nu_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$ and $k \geq k_1 = \varepsilon^{-\nu_0}$, we have*

$$\begin{aligned}\mathbb{E}[N_{\omega,k}(E+\varepsilon/2) - N_{\omega,k}(E-\varepsilon/2)] - e^{\varepsilon^{-\eta_0}} \\ \leq N(E+\varepsilon) - N(E) \\ \leq \mathbb{E}[N_{\omega,k}(E+2\varepsilon) - N_{\omega,k}(E-2\varepsilon)] + e^{-\varepsilon^{-\eta_0}}.\end{aligned} \tag{2.12}$$

Proof. The last result is well known for operators with compact single sit potentials [24, 26]. For this we need to define another operator. More precisely a function f on $\mathbb{R}^d$, one define $f^\varepsilon(x) = f(x)\chi_{\{\varepsilon\cdot|x|\leq 1\}}$, f^ε is compactly supported. we define the following random operator:

$$A_{\varepsilon,\omega} = \nabla \cdot a_{\varepsilon,\omega} \cdot \nabla,$$

where $a_{\varepsilon,\omega} = (a_{\varepsilon,i,j,\omega})_{1\leq i,j\leq d}$ is the matrix given by

$$a_{\varepsilon,i,j,\omega}(\cdot) = a_{i,j,0}^\varepsilon(\cdot) + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma^{i,j} u_{i,j}^\varepsilon(\cdot - \gamma).$$

The periodic approximations of $A_{\varepsilon,\omega}$ it is defined analogously to $A_{\omega,k}$ and will be denoted by $A_{\varepsilon,\omega,k}$. Let N_ε (respectively $N_{\varepsilon,\omega,k}$) be the IDS of $A_{\varepsilon,\omega}$ (respectively of $A_{\varepsilon,\omega,k}$). Notice that from the decaying assumption (1.4) and the fact that the random variables are bounded uniformly in k and ω, we get that there exits $C > 0$ such that for any $\Psi = (-1-\Delta)^{-1}\varphi$, where $\varphi \in L^2(\mathbb{R}^d)$

$$0 \le \langle A_\omega \Psi, \Psi\rangle - \langle A_{\varepsilon,\omega}\Psi, \Psi\rangle \le C \cdot \varepsilon^{\nu-d} \cdot \|\Psi\|^2 .$$

The same inequality holds for the periodic approximations. This yields that, uniformly in k and ω and locally uniformly in the energy E, we have

$$N_\varepsilon(E - C\cdot \varepsilon^{\nu-d}) \le N(E) \le N_\varepsilon(E + C\cdot\varepsilon^{\nu-d}), \tag{2.13}$$

and

$$N_{\varepsilon,\omega,k}(E - C\cdot \varepsilon^{\nu-d}) \le N_{\omega,k}(E) \le N_{\varepsilon,\omega,k}(E + C\cdot\varepsilon^{\nu-d}). \tag{2.14}$$

This tells us that the IDS of A_ω is well approximated by that of $A_{\varepsilon,\omega}$. But for the last operator the singles sites potentials are compactly supported and so many techniques and results are available [24]. We need the following lemma

Lemma 2.3 *[24] We assume that the single sit potential is supported in a ball of radius R_ε. Let I, a relatively compact open interval in $\mathbb{R}$. For any $\beta \in (0,1)$, there exists $C > 1$ and $\rho > 0$ such that , for any $\varphi \in C_0^\infty(I)$, for $n \in \mathbb{N}^*$ and $k > R_\varepsilon$, we have*

$$\begin{aligned}|\mathbb{E}(\langle\varphi, dN_{\omega,k}\rangle) - \langle \varphi, dN\rangle| \\ \le C\cdot|k-R_\varepsilon|^{-(1-\beta)k}\cdot n^n \cdot \sup_{x\in\mathbb{R},0\le j\le n+\rho}\left|(|x|+1)^{\rho+n}\varphi^{(j)}(x)\right|.\end{aligned} \tag{2.15}$$

Remark 2.4 *This lemma is proven in [24] for the Schrödinger case. It is still true for our case. The proof is based on the Helffer Sjöstrand formula and the resolvent equation with the exponential decay of the resolvent kernels (the Combes-Thomas argument).*

Let φ be a Gevrey class function, of Gevrey exponent $\alpha > 1$ (see [15]) such that φ is supported in $[-1, 2]$, $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $[0, 1]$. For $0 < \varepsilon < 1$, we set

$$\varphi_{E_+,\varepsilon}(\cdot) = \varphi\Big(\frac{\cdot - E_+}{\varepsilon}\Big).$$

From Lemma 2.3 and properties of Gevrey class functions, we deduce that there exists $C > 1$ such that for all $k > R_\varepsilon$, $n \geq 1$ and $0 < \varepsilon < 1$ we have

$$\left|\mathbb{E}(\langle \varphi_{E_+,\varepsilon}, dN_{\omega,k}\rangle) - \langle \varphi_{E_+,\varepsilon}, dN\rangle\right| \\ \leq \varepsilon^{-n-\rho}(n+\rho)^{2\alpha(n+\rho)}(k-R_\varepsilon)^{-(1-\beta)n}.$$

We pick $n \leq (k-R_\varepsilon)^{(1-\beta)/4\alpha} - \rho$. For $k - R_\varepsilon$ large, we get that, there exists $C > 1$ such that for $k > R_\varepsilon + 2$, $n > \rho$ and $0 < \varepsilon < 1$, we have

$$\left|\mathbb{E}(\langle \varphi_{E_+,\varepsilon}, dN_{\omega,k}\rangle) - \langle \varphi_{E_+,\varepsilon}, dN\rangle\right| \leq \left(\varepsilon^{-1}(k-R_\varepsilon)^{(1-\beta)/4}\right)^{(k-R_\varepsilon)^{(1-\beta)/4\alpha}}.$$

As $R_\varepsilon \sim \varepsilon^{1/(d-\nu)}$, for $\eta_0 > 1$ such that $\alpha \cdot \eta_0 > 1$ and $k = k_1 = \varepsilon^{-\nu_0} > \varepsilon^{-\eta_0 4\alpha/(1-\beta)} + R_\varepsilon$, (it suffice to take $\nu_0 > \sup(\eta_0 4\alpha/(1-\beta), \frac{1}{d-\nu})$) we get that there exist $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, we have

$$\left|\mathbb{E}(\langle \varphi_{E_+,\varepsilon}, dN_{\omega,k_\varepsilon}\rangle) - \langle \varphi_{E_+,\varepsilon}, dN\rangle\right| \leq e^{-\varepsilon^{-\eta_0}}. \tag{2.16}$$

As dN_ε and $dN_{\varepsilon,\omega,k}$ are positive measures and by the definition of φ, we have

$$\mathbb{E}\big(N_{\varepsilon,\omega,k}(E+\varepsilon) - N_{\varepsilon,\omega,k}(E-\varepsilon)\big) \\ \leq \mathbb{E}\big(\langle \varphi_{E,\varepsilon}, dN_{\varepsilon,\omega,k}\rangle\big) \\ \leq \mathbb{E}\big(N_{\varepsilon,\omega,k}(E+2\varepsilon) - N_{\varepsilon,\omega,k}(E-2\varepsilon)\big)$$

and

$$N_\varepsilon(E+\varepsilon) - N_\varepsilon(E-\varepsilon) \leq \langle \varphi_{E,\varepsilon}, dN\rangle \leq N_\varepsilon(E+2\varepsilon) - N_\varepsilon(E-2\varepsilon).$$

This, gives (2.12), for $A_{\varepsilon,\omega}$. To get (2.12) for A_ω, it suffices to pick $\varepsilon = \varepsilon^{1/\nu - d}$ and take into account (2.13), (2.14) and (2.16). □

Now we turn to the prof of the lower bound.

2.2 The lower bound

The lower bound is proven in the same way as in [28] and consists on proving the following theorem.

Theorem 2.5 *Let A_ω, the operator defined by (1.1). We assume that* **(A.1)**, **(A.2)** *and* **(A.3)** *hold. Then, we have*

$$\liminf_{\varepsilon\to 0^+} \frac{\log\left|\log\Big(N(E_+ + \varepsilon) - N(E_+)\Big)\right|}{\log\varepsilon} \geq -\frac{d}{\nu - 2}. \tag{2.17}$$

Proof. As by assumption, there is a spectral gap below E_+ of length at least $\delta' > 0$. Thus, for $\varepsilon < \delta'$ we have

$$N(E_+ + \varepsilon) - N(E_+) = N(E_+ + \varepsilon) - N(E_+ - \varepsilon).$$

To prove Theorem 2.5, we will lower bound $N(E_+ + \varepsilon) - N(E_+ - \varepsilon)$. So, for N large, we will show that A_{ω,Λ_N} (A_{ω,Λ_N} is A_ω restricted to Λ_N with Dirichlet boundary conditions) has a large number of eigenvalues in $[E_+ - \varepsilon, E_+ + \varepsilon]$ with large a probability. For this we will construct a family of approximate eigenvectors associated to approximate eigenvalues of A_{ω,Λ_N} in $[E_+ - \varepsilon, E_+ + \varepsilon]$. These functions will be constructed from an eigenvector of A_0 associated to E_+. Locating this eigenvector in θ, one obtains an approximate eigenfunction of A_{ω,Λ_N}. Locating this eigenfunction in x in several disjointed places, we get several eigenfunctions two by two orthogonal.
In order to simplify the notations, in what follows we assume that $\theta^0 = 0$ is a point where $E_1(\theta)$ reaches E_+. From [23, 28] there exists $C > 1$, V a neighborhood of 0 and $f : \theta \in V \to f(\cdot,\theta)$ a real analytic function such that, $||f(\cdot,\theta)||_{L^2(C_0)} = 1$ and

$$||(A_0(\theta) - E_+)f(\cdot,\theta)||_{L^2(C_0)} \leq C|\theta|^2. \tag{2.18}$$

Let $0 < \xi < 1$ be a small constant. Let $\chi \in C_0^\infty(\mathbb{R})$ such that it is positif, supported in $[\frac{\xi}{2}, \xi]$ and $\displaystyle\int_{[\frac{\xi}{2},\xi]} \chi(t)^2 dt = 2$.
For $\varepsilon > 0$, we define

$$\mathcal{W}_\varepsilon(\theta) = \varepsilon^{-d/4} \prod_{j=1}^{d} \chi(\varepsilon^{-\frac{1}{2}}\theta_j) \in L^2(\mathbb{T}^*)$$

and

$$\mathcal{W}_\varepsilon^f(\cdot,\theta) = \mathcal{W}_\varepsilon(\theta) \cdot f(\cdot,\theta).$$

In [29, 30] it's proved that if we choose ξ small enough we have

$$||(A_0 - E_+)\mathcal{W}_\varepsilon^f||^2_{\mathcal{H}} \leq \frac{\varepsilon^2}{8}. \tag{2.19}$$

For $\beta \in \mathbb{Z}^d$, we define

$$\mathcal{W}^f_{\varepsilon,\beta}(\cdot,\theta) = e^{-i\beta\cdot\theta}\mathcal{W}^f_{\varepsilon}(\cdot,\theta) \text{ and } \mathcal{W}^{\mathrm{f}}_{\alpha,\varepsilon,\beta,\zeta}(\cdot,\theta) = \mathrm{e}^{-\mathrm{i}\beta\cdot\theta}(\Pi_{\Lambda_\alpha(\zeta)}\mathcal{W}^{\mathrm{f}}_{\varepsilon})(\cdot,\theta),$$

where $\Lambda_\alpha(\zeta)$ is the cube defined by

$$\Lambda_\alpha(\zeta) = \{\gamma \in \mathbb{Z}^d;\ \text{for } 1 \leq j \leq d;\ |\gamma_j| \leq \zeta^{-(\frac{1}{\nu-d}+\alpha)}\},$$

and $\Pi_{\Lambda_\alpha(\zeta)}$ is the orthogonal projection on $\Lambda_\alpha(\zeta)$.
We set

$$\mathcal{U}^f_{\varepsilon,\beta}(x) = \int_{\mathbb{T}^*}\mathcal{W}^f_{\varepsilon,\beta}(x,\theta)d\theta \text{ and } \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x) = \int_{\mathbb{T}^*}\mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta}(x,\theta)d\theta.$$

For N large and well chosen β and $(\omega^{i,j}_\gamma)_{\gamma\in\mathbb{Z}^d,1\leq i,j\leq d}$, $\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}$ will be an approximate eigenfunction of A_{ω,Λ_N} associated with an approximate eigenvalue in the interval $[E_+ - \varepsilon, E_+ + \varepsilon]$.
As in [23] using the non-stationary phase we see that $\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}$ and $\mathcal{U}^f_{\varepsilon,\beta}$ are close to each others. More precisely, for any $n \in \mathbb{N}$ and $\beta \in \mathbb{Z}^d$, there exists $C_n > 0$ such that

$$(\mathrm{vol}(\mathbb{T}^*))\cdot||\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta} - \mathcal{U}^f_{\varepsilon,\beta}||_{L^2(\mathbb{R})} = ||\mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta} - \mathcal{W}^f_{\varepsilon,\beta}||_{\mathcal{H}} \leq C_n\varepsilon^{-n/2}\zeta^{n(\frac{1}{\nu-d}+\alpha)}. \tag{2.20}$$

So, for $\zeta = \varepsilon$ small enough, we get

$$||\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}||_{L^2(\mathbb{R}^d)} \geq 1.$$

Now we have to look to the conditions under which we have

$$\Big|\Big|\Big(A_\omega - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 \leq \varepsilon^2.$$

Notice that

$$\begin{aligned}\Big|\Big|\Big(A_{\omega,\Lambda_N} - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 &\leq \Big|\Big|\Big(A_\omega - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 \\ &\leq 2\Big|\Big|\Big(A_0 - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 + 2\Big|\Big|\Delta A_\omega\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2.\end{aligned} \tag{2.21}$$

Equations (2.19) and (2.20) give the bound on the first member of (2.21), it just remains to us to control the second term. We have

$$\begin{aligned}\Big|\Big|\Big(\Delta A_\omega\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 \leq 2\Big|\Big|&\sum_{\gamma\in\mathbb{Z}^d,1\leq i,j\leq d}\omega^{i,j}_\gamma(\partial_{x_i}u_{i,j})(\cdot-\gamma)\cdot\Big(\partial_{x_j}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big)\Big|\Big|^2 + \\ &2\Big|\Big|\sum_{\gamma\in\mathbb{Z}^d,1\leq i,j\leq d}\omega^{i,j}_\gamma u_{i,j}(\cdot-\gamma)\partial^2_{x_ix_j}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2.\end{aligned} \tag{2.22}$$

To estimate (2.22), one needs the following lemmas.

Lemma 2.6 *[29, 30]There exists $K > 0$, such that for any $1 \le i, j \le d$ we have*

$$\Big|\Big|\sum_{\gamma\in\mathbb{Z}^d,}\omega_\gamma^{i,j}(\partial_{x_i}u_{i,j})(\cdot-\gamma)\cdot\Big(\partial_{x_j}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big)\Big|\Big|^2 \le \varepsilon^4+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\substack{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)\\ 1\le i,j\le d}}\omega_\gamma^{i,j}\Big)^2. \tag{2.23}$$

Lemma 2.7 *[29, 30]There exists $K > 0$, such that for any $1 \le i, j \le d$ we have*

$$\Big|\Big|\sum_{\gamma\in\mathbb{Z}^d}\omega_\gamma^{i,j}u_{i,j}(\cdot-\gamma)\partial^2_{x_ix_j}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 \le \varepsilon^4+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\substack{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)\\ 1\le i,j\le d}}\omega_\gamma^{i,j}\Big)^2. \tag{2.24}$$

Now, combining (2.23), (2.24) and taking (2.22) into account we get that there exists $K > 0$ such that

$$\Big|\Big|(\Delta A_\omega)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon}\Big|\Big|^2 \le \varepsilon^3+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\substack{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)\\ 1\le i,j\le d}}\omega_\gamma^{i,j}\Big)^2. \tag{2.25}$$

By (2.19),(2.20) and (2.25), it follows that:

$$\Big|\Big|\Big(A_\omega-E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon}\Big|\Big|^2 \le \frac{\varepsilon^2}{2}+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\substack{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)\\ 1\le i,j\le d}}\omega_\gamma^{i,j}\Big)^2. \tag{2.26}$$

Now, for N large, we may divide Λ_N into $N(\varepsilon)$ disjoints cubes of size $2\Lambda_\alpha(\varepsilon)$. One has

$$N(\varepsilon) \simeq \frac{(2N)^d}{\varepsilon^{-d(\frac{1}{\nu-d}+\alpha)}}; \tag{2.27}$$

and

$$\cup_{j=1}^{N(\varepsilon)}(\beta_j+\Lambda_\alpha(\varepsilon)) \subset \Lambda_N$$

and for $j \ne j'$,

$$(\beta_j+2\Lambda_\alpha(\varepsilon)) \cap (\beta_{j'}+2\Lambda_\alpha(\varepsilon)) = \emptyset.$$

This implies that for $j \ne j'$, $\mathcal{U}^f_{\alpha,\varepsilon,\beta_j,\varepsilon}$ and $\mathcal{U}^f_{\alpha,\varepsilon,\beta_{j'},\varepsilon}$ are orthogonal.
We denote the counting function of the eigenvalues of A_{ω,Λ_N} below E by $\Theta_{\Lambda_N}(E)$, then

$$\mathbb{E}\Big(\Theta_{\Lambda_N}(E+\varepsilon)-\Theta_{\Lambda_N}(E-\varepsilon)\Big) = \mathbb{E}\Big(\#\Big\{\text{eigenvalues of } \Pi_{\rm N}{\rm A}_\omega\Pi_{\rm N} \text{ in } [{\rm E}_+-\varepsilon,{\rm E}_++\varepsilon]\Big\}\Big)$$

$$\geq \mathbb{E}\Big(\#\Big\{1 \leq j \leq N(\varepsilon); ||(A_\omega - E_+)\mathcal{U}^f_{\alpha,\varepsilon,\beta_j,\varepsilon}||_{L^2(\mathbb{R}^d)} \leq \varepsilon\Big\}\Big) \geq \mathbb{E}\Big(\sum_{j=1}^{N(\varepsilon)} B_j(\omega)\Big), \tag{2.28}$$

where

$$B_j(\omega) = \begin{cases} 1 & \text{if} \quad K\left(\varepsilon^{s\alpha(\nu-d)} \cdot \varepsilon + \sup\limits_{\substack{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)\\ 1\leq i,j\leq d}} \omega_\gamma^{i,j}\right)^2 \leq \dfrac{\varepsilon^2}{2}. \\ 0 & \text{if} \quad \text{not.} \end{cases}$$

The $(B_j)_{1\leq j\leq N(\varepsilon)}$ are independent, identically distributed, Bernoulli random variables. So equations (2.27) and (2.28), imply that there exists $C > 0$, such that one has

$$\begin{aligned} &N_{\Lambda_N}(E+\varepsilon) - N_{\Lambda_N}(E-\varepsilon) = \\ &\quad \frac{1}{((2N+1))^d}\mathbb{E}\Big(\#\Big\{\text{eigenvalues of } \Pi_\mathrm{N}\mathrm{A}_\omega\Pi_\mathrm{N} \text{ in } [\mathrm{E}_+ - \varepsilon, \mathrm{E}_+ + \varepsilon]\Big\}\Big) \\ &\qquad\qquad \geq \frac{N(\varepsilon)}{(2N+1)^d}\mathbb{P}(B_1 = 1) \geq \frac{1}{C}\varepsilon^{d(1/(\nu-d)+\alpha}\mathbb{P}(B_1 = 1). \end{aligned}$$

Hence, taking the limit $N \to \infty$, we get that, for $\varepsilon > 0$ small, we obtain

$$N(E_+ + \varepsilon) - N(E_+ - \varepsilon) \geq \frac{1}{C}\varepsilon^d\mathbb{P}(B_1 = 1). \tag{2.29}$$

It just remains to estimate $\mathbb{P}(B_1 = 1)$. If for $1 \leq j \leq N(\varepsilon)$, and any $\gamma \in \beta_j + \Lambda_\alpha(\varepsilon); 1 \leq i, j \leq d$, one has $\omega_\gamma^{i,j} \leq \dfrac{\varepsilon}{2K}$, then for ε rather small

$$K\Big(\varepsilon^{\alpha(\nu-d)} \cdot \varepsilon + \sup_{\substack{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)\\ 1\leq i,j\leq d}} \omega_\gamma^{i,j}\Big)^2 \leq \varepsilon^2 \cdot K\Big(\varepsilon^{\alpha(\nu-d)} + \frac{1}{2K}\Big)^2 \leq \frac{\varepsilon^2}{2}.$$

As the random variables are independent identically distributed, one has the estimate

$$\mathbb{P}(B_j = 1) \geq \Big(\mathbb{P}(\omega_0^{1,1} \leq \frac{\varepsilon}{2K})\Big)^{d^2\#\Lambda_\alpha(\varepsilon)}.$$

So, taking the double logarithm of (2.29), using assumption **(A.3)** and the fact that $\#\Lambda_\alpha(\varepsilon) = \varepsilon^{-d(\frac{1}{\nu-d}+\alpha)}$, we get that

$$\lim_{\varepsilon\to 0^+} \frac{\log\Big|\log\Big(N(E_+ + \varepsilon) - N(E_+)\Big)\Big|}{\log\varepsilon} \geq -\frac{d}{\nu - d} - d\alpha. \tag{2.30}$$

The equation (2.30) is true for any $\alpha > 0$, by taking α small we end the proof of Theorem 2.5 and hence the subsection. $\square$

2.3 The upper bound

The proof of the upper bound in [29] is given in the case of the non degeneracy of the IDS of the back ground operator which is relaxed here.
From Lemma 2.2 and for $\eta_0 > 1/(\nu - d)$ and $k \sim \varepsilon^{-\delta}$ such that $\delta > \nu_0$ the proof of the upper bound is reduced to prove that

$$\limsup_{\varepsilon \to 0^+} \frac{\log|\log(\mathbb{E}(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)))|}{\log \varepsilon} \leq -\frac{d}{\nu - d}. \tag{2.31}$$

Lemma 2.8 *Let $k \sim \varepsilon^{-\rho}$ with $\rho > 1/(\nu - d)$. Define the event,*

$$\mathbf{E}_{\varepsilon,\omega} = \{\omega; \Delta A_{\omega,k} \geq -\varepsilon\Delta = -\varepsilon \sum_{i=1}^{d} \partial_{x_i}^2\}.$$

Then $\mathbf{E}_{\varepsilon,\omega}$ has a probability at least $1 - \mathbb{P}_\varepsilon$ where $\mathbb{P}_\varepsilon$ satisfies

$$\limsup_{\varepsilon \to 0^+} \frac{\log|\log(\mathbb{P}_\varepsilon)|}{\log \varepsilon} \leq -\frac{d}{\nu - d}. \tag{2.32}$$

Proof. Using equation (1.4) we get that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} \langle \Delta A_{\omega,k}\varphi, \varphi\rangle &= \sum_{\gamma \in \mathbb{Z}^d; 1 \leq i,j \leq d} \omega_{\widetilde{\gamma}}^{i,j} \langle u_{i,j}(\cdot - \gamma)\partial_{x_j}\varphi, \partial_{x_i}\varphi\rangle \ \text{ where } \ \widetilde{\gamma} = \gamma \mod (2k+1)\mathbb{Z}^d \\ &\geq g_- \sum_{\alpha \in \mathbb{Z}^d} A_\alpha(\omega)|\nabla\varphi|^2. \end{aligned}$$

Here $A_\alpha(\omega) = \displaystyle\sum_{\gamma \in \mathbb{Z}^d; 1 \leq i,j \leq d} \omega_{\widetilde{\gamma}}^{i,j}(1 + |\alpha - \gamma|)^{-\nu}$.
Notice that for any $1 \leq i, j \leq d$; $\omega_{\widetilde{\gamma}}^{i,j}$ is $(2k+1)\mathbb{Z}^d$ periodic so is $A_\alpha(\omega)$. We set $\mathbb{Z}^d_{2k+1} = \{\gamma \in \mathbb{Z}^d; |\gamma| < k\}$. Then we have

$$\begin{aligned} \mathbb{P}(\{\Delta A_{\omega,k} \geq -\varepsilon\Delta\}) &\geq \mathbb{P}(\{\forall\gamma \in \mathbb{Z}^d_{2k+1}; g_- \sum_{\alpha \in \mathbb{Z}^d} A_\alpha \geq \varepsilon\}) \\ &\geq 1 - \sum_{\gamma \in \mathbb{Z}^d_{2k+1}} \mathbb{P}(\{A_\gamma(\omega) \leq \varepsilon\}). \end{aligned}$$

As the random variables are i.i.d, we have

$$\mathbb{P}(\{\Delta A_{\omega,k} \geq -\varepsilon\Delta\}) \geq 1-(2k+1)^d \mathbb{P}(\{A_0(\omega) \leq \varepsilon\}). \tag{2.33}$$

To estimate $\mathbb{P}(\{A_0(\omega) \leq \varepsilon\})$ it suffices to follow the same computation done in [24] and based on the Markov's inequality, and the Taylor expansion of e^{-x} to get that

$$\mathbb{P}(\{A_0(\omega) \leq \varepsilon\}) \leq e^{-\frac{1}{C}\varepsilon^{-\frac{d}{\nu-d}}}. \tag{2.34}$$

This complete the proof of Lemma 2.8. $\square$

Lemma 2.9 *There exists $C>0$ and $\varepsilon_0>0$ (uniform in k and ω) such that, if $0<\varepsilon<\varepsilon_0$ and ω satisfies $\Delta_{\omega,k} \geq -C\varepsilon\Delta$, for $k \in \mathbb{N}$, one has*

$$N_{\omega,k}(E_+) = N_{\omega,k}(E_+ + \varepsilon).$$

Lemma 2.9 says that if $\Delta A_{\omega,k} \geq -C\varepsilon\Delta$, then the spectrum of $A_{\omega,k}$ does not intersect $(E_+, E_+ + \varepsilon)$ for ε small . This lemma will be proved in the appendix.
Now estimate (2.31) is an immediate consequence of Lemma 2.8 and 2.9. Indeed, picking C as in Lemma 2.9; one computes

$$\begin{aligned}
\mathbb{E}\Big(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)\Big) &= \mathbb{E}\Big([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega;\Delta A_{\omega,k} \geq -C\varepsilon\Delta\}}\Big) \\
&+ \mathbb{E}\Big([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega;\Delta A_{\omega,k} < -C\varepsilon\Delta\}}\Big) \\
&\leq C_0\mathbb{P}(\{\omega; \Delta A_{\omega,k} < -C\varepsilon\Delta\}) \\
&= C_0(1-\mathbb{P}(\mathbf{E}_{C\cdot\varepsilon,\omega})) = C_0\mathbb{P}_{C\cdot\varepsilon}.
\end{aligned}$$

Here, we have used the fact that $N_{\omega,k}$ is bounded, locally uniformly in energy, uniformly in ω, k by C_0. Taking (2.32) into account, we end the proof of (2.31), and consequently the proof of the upper bound. $\square$

3 The proof of Theorem 1.1.

All the proofs of localization, except in the discrete case [1] for the multi-dimensional case, use the method of the multiscale analysis. This method was used for the first time by Fröhlich and Spencer, at the beginning of the eighties [12] and it knew many extensions and simplifications to lead to the form described in the section below. This analysis makes possible to obtain information on the operator in the hole space, starting from information on the operator restricted to cubes of finished size [7]. For sake of completeness in the following section we briefly recall the technique of multiscale analysis. We refer the reader to [7, 17, 20, 34] for detailed description of results mentioned below.

3.1 Multiscale analysis .

We start by introducing some notations. Let $\Lambda_k(x)$, be the cube of center $x \in \mathbb{Z}^d$ and with side length $2k$. The restriction of A_ω on $\Lambda_k(x)$ with Dirichlet boundary condition is noted by $A_{x,k}$. Each $A_{x,k}$ is a positive operator with compact resolvent $R_{x,k}(z) = (A_{x,k} - z)^{-1}$. The norm on $L^2(\Lambda_k(x))$ is noted by $\|\cdot\|_{x,k}$. The operator of multiplication by φ (where $\varphi \in L^\infty(\Lambda)$) on $L^2(\Lambda)$ will be simply indicated by φ. In what follows we consider $k \in 3\mathbb{N}$.
Let

$$\Upsilon_k(x) = \Big\{ y \in \Gamma;\ ||y - x|| = \frac{k}{3} - 1 \Big\}.$$

We set

$$\Gamma_{x,k} = \sum_{y \in \Upsilon_k(x)} \chi_y.$$

where χ_y is the characteristic function of the unit cube centered in y. Note that $|\Upsilon_k(x)| \le cdk^{d-1}$.
For $\Lambda \subset \mathbb{Z}^d$ we consider the following probability space $([0,1]^{d^2 \cdot \Lambda}, \mathcal{F}', \mathbb{P}_\Lambda)$ when $A \in \mathcal{F}'$ we set $A' = A \otimes_{j \notin \Lambda} [0,1]^{d^2} \in \mathcal{F}$, and take

$$\mathbb{P}_\Lambda(A) = \mathbb{P}(A').$$

By $\mathbb{E}_\Lambda$ we denote the expectation corresponding to $\mathbb{P}_\Lambda$ and when $\Lambda \subset \mathbb{R}^d$ we identify Λ to $\Lambda \cap \mathbb{Z}^d$.

Definition 3.1 *Let $m > 0, E > 0, x \in \Gamma$. We said that $\Lambda_k(x)$ is (m, E)-regular if $E \notin \sigma(A_{x,k})$ and*

$$||\Gamma_{x,k} R_{x,k}(E) \chi_x||_{x,k} \le e^{-m\frac{k}{2}}. \tag{3.35}$$

The multiscale analysis is based on two fundamental assumptions, namely:
Initial estimate: **(P1)** (k_0, m_0, p) for $E_0 \in [E_+, E_+ + \varepsilon_0], p > d$ and $m_0 > 0$,

there exists $k_0 \in \mathbb{N}$ such that

$$\mathbb{P}_{\Lambda_{k_0}}\Big\{\Lambda_{k_0}(0) \text{ is } (m_0, E_0) - \text{regular}\Big\} \geq 1 - \frac{1}{k_0^p}.$$

uniform Wegner estimate: **(P2)** there exists $\varepsilon_0 > 0, C_w > 0$ such that for all E in $[E_+, E_+ + \varepsilon_0/2]$, for all $\varepsilon < \varepsilon_0/2$ we have

$$\mathbb{P}_{\Lambda_k}\Big\{\omega | d(\sigma(A_{0,k}), E) \leq \varepsilon\Big\} \leq C_w \cdot \varepsilon \cdot |\Lambda_k|^2.$$

The multiscale analysis is a method of recurrence [7] on a succession of scale k_j, defined by recurrence as follow $k_{j+1} = [k_j^\zeta]_3$ where $\zeta \in]1, 2[$ and $[t]_3$ is the greatest multiple of 3 which is smaller than t. For each scale k_j one shows that $R_{0,k_j}(z)$ decay exponentially with probability which tends to 1 when j tends to the infinity. Such operation consists in proving that if we have **(P2)** then

$$\textbf{(P1)}\ (k_j, m_j, p) \Rightarrow \textbf{(P1)}\ (k_{j+1}, m_{j+1}, p'). \tag{3.36}$$

The decay rate m_{j+1} and the scale k_{j+1} are such: for all $\xi > 0$ there exists c_1, c_2, c_3 such that

$$m_{j+1} \geq m_j\Big(1 - \frac{4k_j}{k_{j+1}}\Big) - \frac{c_1}{k_j} - c_2\frac{\log k_{j+1}}{k_{j+1}}, \tag{3.37}$$

$$k_{j+1}^{p'} \leq c_3\Big(\frac{k_{j+1}}{k_j}\Big)^{2d} k_j^{2p} + \frac{k_{j+1}^{-\xi}}{2}. \tag{3.38}$$

Roughly, the idea of the proof of (3.36) is to divide a large cube of side k_{j+1} into small cubes of side k_j and variable centers x and then to develop $R_{0,k_{j+1}}(z)$ according to $R_{x,k_j}(z)$. Such operation is based on the inequality called of Simon-Lieb (SLI); see Lemma 26 of [8]. One points out this inequality, the proof is given in [8].

Lemma 3.2 (SLI) *For all $k, k' \in 3\mathbb{N}$, such that $k \geq 4$*

$$||\Gamma_{x,k'} R_{x,k'}(z)\chi_y||_{x,k'} \leq C_z k^{d-1} ||\Gamma_{y,k} R_{y,k}(E)\chi_y||_{y,k} ||\Gamma_{x,k'} R_{x,k'}(z)\chi_{y'}||_{x,k'}.$$

For some $y' \in \Upsilon_k(y)$ and $C_z = C(z, d, a_+, a_-)$.

To make this development successfully one needs two main ingredients:
1) A succession of regular cubes $\Lambda_{k_j}(x)$ which join the edges of $\Lambda_{k_{j+1}}(0)$. It

is shown that this is the case with a large probability.
2) A fundamental estimate of $R_{0,k_{j+1}}(E)$ for each $E \in [E_+, E_+ + \varepsilon_0]$, with a sufficiently large probability. This is a consequence of the uniform Wegner estimate **(P2)**.
Then, we obtain an estimate on the decay of $R_{0,k_{j+1}}(z)$ with a rate slightly smaller compared to that of R_{0,k_j}, but with a higher probability.
Here we use the *uniform* Wegner estimate [17] instead of the Wegner estimate [8], to overcome the difficulty caused by the long range potential. Notice that the crucial probabilistic estimates in [7] (See Lemma 4.1) can be done using the fact that for any Λ_1, Λ_2 in $\mathbb{Z}^2$, we have

$$\mathbb{P}_{\Lambda_2}(A) \leq \mathbb{P}_{\Lambda_1}(A) \quad \text{if} \quad \Lambda_1 \subset \Lambda_2$$

and

$$\mathbb{P}_{\Lambda_1 \cup \Lambda_2}(A \cap B) \leq \mathbb{P}_{\Lambda_1}(A) \cdot \mathbb{P}_{\Lambda_2}(B) \;\; \text{if} \quad \Lambda_1 \cap \Lambda_2 = \emptyset.$$

3.2 Localization for the divergence operator

Our strategy, in proving localization, consists in proving that from Theorem 1.2 we are able to obtain the necessary standard ingredients for recurrence procedure and, thus, to start the machinery of multiscale analysis, and prove the existence located states on edge of the almost sure spectrum $\Sigma \backslash \{0\}$. Indeed, points (i) and (ii) of Theorem 1.1 are proven in [8] and in [34] as a consequence of **(P1)** and **(P2)**. For the strong dynamical localization (point (iii) of Theorem 1.1); Damanik and Stollmann in [6], show that this can be seen as a result of Anderson localization (hypothesis **EDI** in [34]); which is guaranteed by points (i) and (ii) of Theorem 1.1.
Recently, Strong dynamical localization up to any order, is proved by Germinet and Klein in [13]. It is given for Schrödinger operators, but it is also true for our model.
From what it was said previously to prove Theorem 1.1, it is enough to check the necessary assumptions to start the multiscale analysis i.e; **(P1)** and **(P2)**.

3.3 Initial estimate

We start by verifying **(P1)**. This already has been done in [3, 4, 6, 19, 20] for Schrödinger operators and in [34] for the model given by the form of (1.1) under the assumption of a special mode of disorder. More precisely their approach requires a sufficiently fast decay of the probability distribution μ at the approach to the edges of the support. Indeed, they assumed that there exists some $\tau > 0$ ($\tau > \frac{3d}{2}$ in [3]; $\tau > d$ in [19] and $\tau > \frac{d}{2}$ in [20]): such that $\forall \varepsilon > 0$,

$$\mathbb{P}_0([0,\varepsilon]) = \int_0^{\varepsilon} g(s)(s)ds \leq \varepsilon^{\tau},$$

or

$$\mathbb{P}_0([1-\varepsilon,1]) = \int_{1-\varepsilon}^{1} g(s)ds \leq \varepsilon^{\tau}.$$

According to whether one considers the higher or lower edges of the gap. Those conditions are relaxed here.
In what follows, to get **(P1)** one will adapt an other approach which uses the fact that at the internal band edges, the IDS of A_ω exhibits Lifshitz tails. See [21, 36, 29, 30].
Let us consider the following assumption
(P' 1) $\mathbb{P}_{\Lambda_{k_0}}\left\{d(\sigma(A_{0,k_0}),[E_+,E_++\varepsilon_0]) < \frac{1}{2}k_0^{-\alpha}\right\} \leq \frac{1}{k_0^p}$,
with $\alpha \in]0,1[$ and $\varepsilon_0 > 0$.
In the appendix we prove the following statement:

Lemma 3.3 **(P'1)** *implies* **(P1)** *for certain* $m_0 \geq Ck_0^{-2}$.

So the proof of the Theorem 1.1 is reduced to the proof of:

Proposition 3.4 *Under the assumptions of Theorem 1.1,* **(P'1)** *holds.*

3.3.1 The proof of the Proposition 3.4.

To get **(P'1)**, in an interval of energy which contains E_+ one will compare the number of states of A_ω to that of $A_{0,k}$. This will be done by a comparison between $N(A_\omega, E)$ and $N(A_{0,k}, E)$. At the bottom of the spectrum, such technique was used with the Čebiševe inequality by Klopp, in [21], and by Kirsch, Stollman and Stolz in [19]. In the medium of the spectrum one refers to the work of Veselić in [27, 36]. All results quoted here are given in the Schrödinger case.
Notice that (1.9) is a limit when one approaches the band edges. We ask for an estimate to hold on an interval. This is the purpose of the following computation.

Lemma 3.5 *For all $n \in \mathbb{N}$ and $\alpha \in]0,1[$; there exists $k_2 \in \mathbb{N}$ such that if $k \geq k_2$ we have*

$$\mathbb{E}_{\Lambda_k}\Big(N_{\omega,k}(E_+ + k^{-\alpha}) - N_{\omega,k}(E_+)\Big) \leq k^{-n}. \tag{3.39}$$

Proof. By the use of the result of Theorem 1.2, and Lemma 2.2; for $\varepsilon = k^{-\alpha}$ and $k > k_1$ (k_1 of Lemma 2.2) we get that for all $n \in \mathbb{N}$, there exists $k_2 > 0$ such that if $k \geq k_2 > k_1$, we have

$$\mathbb{E}_{\Lambda_k}\Big(N_{\omega,k}(E_+ + k^{-\alpha}) - N_{\omega,k}(E_+)\Big) \leq k^{-n}. \tag{3.40}$$

So the proof of Lemma 3.5 is ended □
We need also the following lemma.

Lemma 3.6 *[36] For all $\varepsilon > 0$, we have*

$$\int_{\mathbb{T}_k^*} \mathbb{P}_{\Lambda_k}\Big(\{\omega | \sigma(A_{\omega,k}(\theta)) \cap [E_+, E_+ + \varepsilon] \neq \emptyset\}\Big) d\theta \leq (2\pi)^d \mathbb{E}_{\Lambda_k}\Big(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)\Big).$$

Let us note the j^{th} eigenvalue of $A_{\omega,k}(\theta)$by $E_j(A_{\omega,k}(\theta))$. In the appendix one shows the following lemma.

Lemma 3.7 *For all $\theta, \theta' \in \mathbb{T}_k^*$ and $j \in \mathbb{N}$ there exists $M_{j,k} > 0$ such that*

$$|E_j(A_{\omega,k}(\theta) - E_j(A_{\omega,k}(\theta')| \leq M_{j,k}|\theta - \theta'|. \tag{3.41}$$

Now using Lemma 3.6 and Lipshitz continuity in θ of the Floquet eigenvalues, we will be able to estimate the probability to find an eigenvalue of $A_{0,k}(\theta^0)$ in the interval $[E_+, E_+ + \varepsilon]$. Here $A_{0,k}(\theta^0) = A_{0,k}|_{\Lambda_k}^{\theta^0}$ is the operator $A_{0,k}$ restricted to Λ_k with θ^0-periodic conditions. More precisely we have

Lemma 3.8 *Let $\theta^0 \in \mathbb{T}_k^*$. For all $\varepsilon > 0$ we have*

$$\mathbb{P}_{\Lambda_k}\Big(\{\omega | \sigma(A_{\omega,k}(\theta^0)) \cap [E_+, E_+ + \varepsilon] \neq \emptyset\}\Big)$$
$$\leq \frac{(2\pi)^d}{\mathrm{vol}(\mathbb{T}_k^*)} \mathbb{E}_{\Lambda_k}\Big(N_{\omega,k}(E_+ + \varepsilon + ck^{-1}) - N_{\omega,k}(E_+)\Big). \tag{3.42}$$

Proof. From Lemma 3.7 we know that there exists $M_{j,k} > 0$ such that for all $j \in \mathbb{N}$ we have

$$|E_j(A_{0,k}(\theta)) - E_j(A_{0,k}(\theta'))| \leq M_{j,k}|\theta - \theta'|.$$

As we are interested only in the eigenvalues in $[E_+, E_+ + \varepsilon]$, the $M_{j,k}$ can be chosen uniformly in j and k. So there exists $M > 0$ such that

$$M \geq M_{j,k}, \quad \forall j, k \in \mathbb{N}.$$

Now we will use this to estimate

$$\mathbb{P}_{\Lambda_k}(\{\omega|\sigma(A_{0,k}(\theta^0)) \cap [E_+, E_+ + \varepsilon] \neq \emptyset\}).$$

For this it is enough to estimate

$$\mathbb{P}_{\Lambda_k}(\{\omega|\exists\ j \in \mathbb{N};\ E_j(A_{0,k}(\theta^0)) \in [E_+, E_+ + \varepsilon]\}) =$$

$$\frac{1}{\mathrm{vol}(\mathbb{T}_k^*)} \int_{\mathbb{T}_k^*} \mathbb{P}_{\Lambda_k}(\{\omega|\exists\ j \in \mathbb{N};\ E_j(A_{0,k}(\theta^0)) \in [E_+, E_+ + \varepsilon]\})d\theta. \tag{3.43}$$

Thus if $E_j(A_{0,k}(\theta^0)) \in [E_+, E_+ + \varepsilon]$, the Lipshitz continuity implies that $E_j(A_{0,k}(\theta)) \in [E_+, E_+ + \varepsilon + M\mathrm{diam}(\mathbb{T}_k^*)]$, $\forall\theta \in \mathbb{T}_k^*$ with $\mathrm{diam}(\mathbb{T}_k^*) \leq ck^{-1}$. So we get that

$$\begin{aligned}
(3.43) \quad &\leq \frac{1}{\mathrm{vol}(\mathbb{T}_k^*)} \int_{\mathbb{T}_k^*} \mathbb{P}_{\Lambda_k}(\{\omega|\exists\ j \in \mathbb{N};\ E_j(A_{0,k}(\theta)) \in [E_+, E_+ + \varepsilon + ck^{-1}]\})d\theta \\
&= \frac{1}{\mathrm{vol}(\mathbb{T}_k^*)} \int_{\mathbb{T}_k^*} \mathbb{P}_{\Lambda_k}(\{\omega|;\sigma(A_{0,k}(\theta)) \cap [E_+, E_+ + \varepsilon + ck^{-1}] \neq \emptyset\})d\theta \\
&\leq \frac{(2\pi)^d}{\mathrm{vol}(\mathbb{T}_k^*)} \mathbb{E}_{\Lambda_k}\Big[N_{\omega,k}(E_+ + \varepsilon + ck^{-1}) - N_{\omega,k}(E_+)\Big].
\end{aligned}$$

This proves Lemma 3.8. □

For $\alpha < 1$, (α given in Lemma 3.5) and $\varepsilon = k^{-\alpha} > 0$ there exists $k_3 \in \mathbb{N}$ such that if $k > k_3$, we have $k^{-\alpha} + ck^{-1} \leq 2k^{-\alpha}$ where k_3 depends on α and c. By using equation (3.40) we obtain that for $k > k_3$;

$$\mathbb{E}_{\Lambda_k}\Big(N_{\omega,k}(E_+ + 2k^{-\alpha}) - N_{\omega,k}(E_+)\Big) \leq ck^{-n}.$$

So Lemma 3.8 gives:

$$\mathbb{P}_{\Lambda_k}\Big(\{\omega|\sigma(A_{0,k}(\theta^0))\cap[E_+,E_++k^{-\alpha}]\neq\emptyset\}\Big)\leq ck^{-n+d}.$$

Let $p>d$ and n given in Lemma 3.5 such that $n>p+d$. There exists $k_4\in\mathbb{N}$ such that if $k\geq k_4$ we have $ck^{-n+d}\leq k^{-p}$. This implies that for $k>k_4$ we have

$$\mathbb{P}_{\Lambda_k}\Big(\{\omega|\sigma(A_{0,k}(\theta^0))\cap[E_+,E_++k^{-\alpha}]\neq\emptyset\}\Big)\leq k^{-p}. \tag{3.44}$$

This ends the proof of the Proposition 3.4 **(P'1)**. □

3.4 Wegner estimate

As it was mentioned previously the estimate we are aiming at is one of the basic ingredients of the multiscale analysis. Roughly speaking it is an upper bound on the probability that the spectrum of the local operator $A_{x,k}$ lies within an ε-neighborhood of a given energy. This probability vanishes as the size of the energy neighborhood shrinks to zero. Such an estimate is proved by Klopp, in [21], for energies at the bottom of the spectrum, which applies to Schrödinger operators with Anderson perturbation. Kirsch Stollman and Stolz proved in [20] a Wegner estimate with minimal decay conditions on the single site potential. For random divergence operators, a Wegner estimate is proved in [34]. In [3] the authors obtain a Wegner estimate with $|\Lambda_k|$-dependence. We refer to [22] for more details and explanation on this subject.

Proposition 3.9 *There exists c_w, ε_0 such that for all $E\in[E_+,E_++\varepsilon_0]$*

$$\mathbb{P}_{\Lambda_k}\{dist(\sigma(A_{0,k}),E)\leq\varepsilon\}\leq C_w\cdot\varepsilon\cdot|\Lambda_k|^2. \tag{3.45}$$

Proof. The proof given here is classic and very similar to those in [17, 34, 36], and it is based on the study of the mobility of the eigenvalues of $A_{0,k}$ as function of ω. For this it is not strange to use the derivative with respect to ω. Let ρ a monotone C^1function : $\mathbb{R}\longrightarrow[0,1]$ which is 1 on $[\varepsilon,+\infty)$ and 0 on $(-\infty, -\varepsilon]$. Let us estimate

$$\begin{aligned}
\mathbb{P}_{\Lambda_k}\{dist(\sigma(A_{0,k}), E) \leq \varepsilon\} &\leq \mathbb{E}_{\Lambda_k}\{tr\chi_{[E-\varepsilon, E+\varepsilon]}(A_{0,k})\} \\
&\leq \mathbb{E}_{\Lambda_k}\{tr[\rho(A_{0,k} - E + 2\varepsilon) - \rho(A_{0,k} - E - 2\varepsilon)]\} \\
&\leq \mathbb{E}_{\Lambda_k}\{tr[\int_{-2\varepsilon}^{2\varepsilon} \rho'(A_{0,k} - E + t)dt]\} \\
&= \sum_n \mathbb{E}_{\Lambda_k}(\int_{-2\varepsilon}^{2\varepsilon} \rho'(E_n(\omega) - E + t)dt) \\
&= \sum_n \int_{-2\varepsilon}^{2\varepsilon} \mathbb{E}_{\Lambda_k}(\rho'(E_n(\omega) - E + t)dt). \qquad (3.46)
\end{aligned}$$

Here

$$\mathbb{E}_{\Lambda_k}\{\rho'(E_n(\omega) - E + t)\} = \int_0^1 \cdots \int_0^1 \rho'(E_n(\omega) - E + t) \prod_{\gamma\in\Lambda, 1\leq i,j\leq d} \mathbb{P}_0(d\omega_\gamma^{i,j});$$

and the sum is over the eigenvalues of $A_{0,k}$ in $[E_+ - \varepsilon, E_+ + \varepsilon]$.
Let us notice that for any eigenvalue $E_n(\omega)$ associated an eigenfunction Ψ_n and for $\gamma' \in \Lambda_k$ and $1 \leq i', j' \leq d$ we have

$$\begin{aligned}
\frac{\partial E_n(\omega)}{\partial \omega_{\gamma'}^{i',j'}} &= \frac{\partial}{\partial \omega_{\gamma'}^{i',j'}} \Big(\sum_{\substack{\gamma\in\Lambda_k \\ 1\leq i,j\leq d}} \omega_\gamma^{i,j} \langle u_{i,j}(\cdot - \gamma)\nabla\Psi_n, \nabla\Psi_n\rangle \Big) \\
&= \langle u_{i',j'}(\cdot - \gamma')\nabla\Psi_n, \nabla\Psi_n\rangle.
\end{aligned}$$

This yields to

$$\begin{aligned}
\sum_{\substack{\gamma'\in\Lambda_k \\ 1\leq i',j'\leq d}} \frac{\partial E_n(\omega)}{\partial \omega_{\gamma'}^{i',j'}} &= \sum_{\substack{\gamma'\in\Lambda \\ 1\leq i',j'\leq d}} \langle u_{i',j'}(\cdot - \gamma)\nabla\Psi_n, \nabla\Psi_n\rangle \\
&\geq g_- \cdot \langle \nabla\Psi_n, \nabla\Psi_n\rangle \\
&\geq g_- \cdot a_+^{-1} \cdot \langle A\,(a_\omega)\,\Psi_n, \Psi_n\rangle \\
&\geq g_- \cdot a_+^{-1} \cdot E_n(\omega) > 0.
\end{aligned}$$

As we are interested in $E > E_+ > 0$, we can get that

$$\sum_{\substack{\gamma'\in\Lambda_k \\ 1\leq i',j'\leq d}} \frac{\partial E_n(\omega)}{\partial \omega_{\gamma'}^{i',j'}} \geq g_- \cdot a_+^{-1} \cdot E_+,$$

and consequently we have

$$1 \leq \frac{1}{g_- \cdot a_+^{-1} \cdot E_+} \cdot \sum_{\substack{\gamma' \in \Lambda_k \\ 1 \leq i', j' \leq d}} \frac{\partial E_n(\omega)}{\partial \omega_{\gamma'}^{i',j'}}.$$

Since ρ is monotone, we get

$$\begin{aligned} \rho'(E_n(\omega)) &\leq \frac{1}{g_- \cdot a_+^{-1} \cdot E_+} \cdot \sum_{\substack{\gamma' \in \Lambda_k \\ 1 \leq i', j' \leq d}} \frac{\partial E_n(\omega)}{\partial \omega_{\gamma}^{i,j}} \rho'(E_n(\omega)) \\ &= \frac{1}{g_- \cdot a_+^{-1} \cdot E_+} \cdot \sum_{\substack{\gamma' \in \Lambda_k \\ 1 \leq i', j' \leq d}} \frac{\partial \rho(E_n(\omega))}{\partial \omega_{\gamma'}^{i',j'}}. \end{aligned} \tag{3.47}$$

So by the use of equation (3.46) and (3.47) we get

$$\mathbb{P}_{\Lambda_k}\{dist(\sigma(A_{0,k}), E) \leq \varepsilon\} \leq \frac{1}{g_- \cdot a_+^{-1} \cdot E_+} \cdot \sum_n \int_{-2\varepsilon}^{2\varepsilon} dt \cdot \int_0^1 \cdots \int_0^1 \sum_{\substack{\gamma' \in \Lambda_k \\ 1 \leq i', j' \leq d}} \frac{\partial \rho(E_n\left(\omega_{\gamma}^{i,j}\right) - E + t)}{\partial \omega_{\gamma'}^{i',j'}} \prod_{\gamma \in \Lambda, 1 \leq i,j \leq d} \mathbb{P}_0(d\omega_{\gamma}^{i,j}).$$

But for any fixed $\gamma' \in \Lambda_k$, and $1 \leq i', j' \leq d$, we have

$$\int_0^1 \cdots \int_0^1 \frac{\partial \rho(E_n\left(\omega_{\gamma}^{i,j}\right) - E + t)}{\partial \omega_{\gamma'}^{i',j'}} \prod_{\gamma \in \Lambda, 1 \leq i,j \leq d} \mathbb{P}_0(d\omega_{\gamma}^{i,j}) =$$

$$\int_0^1 \cdots \int_0^1 \prod_{\substack{\gamma \in \Lambda 1 \leq i,j \leq d \\ \gamma \neq \gamma', i \neq i', j \neq j'}} g(\omega_{\gamma}^{i,j}) d\omega_{\gamma}^{i,j} \times \int_0^1 g(\omega_{\gamma'}^{i',j'}) \frac{\partial \rho(E_n\left(\omega_{\gamma}^{i,j}\right) - E + t)}{\partial \omega_{\gamma'}^{i',j'}} d\omega_{\gamma'}^{i',j'};$$

and

$$\int_0^1 g(\omega_{\gamma'}^{i',j'}) \frac{\partial \rho(E_n\left(\omega_{\gamma}^{i,j}\right) - E + t)}{\partial \omega_{\gamma'}^{i',j'}} d\omega_{\gamma'}^{i',j'} = \rho(E_n(\omega_{\gamma}^{i,j}, \omega_{\gamma'}^{i',j'} = 1) - E + t) - \rho(E_n(\omega_{\gamma}^{i,j}, \omega_{\gamma'}^{i',j'} = 0) - E + t).$$

Here $E_n(\omega_\gamma^{i,j}, \omega_{\gamma'}^{i',j'} = \alpha)$ is the n-th eigenvalue of $A_{0,k}$ when $\omega_{\gamma'}^{i',j'}$ is changed by α. So

$$\begin{aligned} &\mathbb{P}_{\Lambda_k}\{dist(\sigma(A_{0,k}), E) \le \varepsilon\} \le \\ &\qquad C \cdot \|g\|_\infty \sum_n \sum_{\substack{\gamma' \in \Lambda_k \\ 1 \le i',j' \le d}} \int_{-2\varepsilon}^{2\varepsilon} dt \int_0^1 \cdots \int_0^1 \prod_{\substack{\gamma \in \Lambda, 1 \le i,j \le d \\ \gamma \ne \gamma', i \ne i', j \ne j'}} g(\omega_\gamma^{i,j}) d\omega_\gamma^{i,j} \\ &\qquad \times \{\rho(E_n(\omega_\gamma^{i,j}, \omega_{\gamma'}^{i',j'} = 1) - E + t) - \rho(E_n(\omega_\gamma^{i,j}, \omega_{\gamma'}^{i',j'} = 0) - E + t)\}. \end{aligned}$$

For our choice of ρ we get that

$$\mathbb{P}_{\Lambda_k}\{dist(\sigma(A_{0,k}), E) \le \varepsilon\} \le C \cdot \sum_n \sum_{\substack{\gamma' \in \Lambda_k \\ 1 \le i',j' \le d}} \int_{-2\varepsilon}^{2\varepsilon} dt.$$

Notice that the number of terms in the sum over γ, i and j can be bounded by $d^2|\Lambda_k|$ and by the Weyl's law we know that there are at most $C\,|\Lambda_k|$ eigenvalues of $A_{0,k}$ below some fixed energy. This ends the proof of the Proposition 3.9. $\square$

4 Appendix .

The proof of Lemma 2.9. Let us take $\Delta A_{\omega,k} > -C\varepsilon\Delta$. Let $\varphi \in \mathcal{D}(A_{\omega,k})$ such that $\langle A_{\omega,k}\varphi, \varphi\rangle > E_+$, then

$$\begin{aligned} \langle A_{\omega,k}\varphi, \varphi\rangle &= \langle A_{0,k}\varphi, \varphi\rangle + \langle \Delta A_{\omega,k}\varphi, \varphi\rangle \\ &\ge \langle A_{0,k}\varphi, \varphi\rangle + C\varepsilon|\nabla\varphi|^2 \\ &> \langle A_{0,k}\varphi, \varphi\rangle + C\varepsilon a_+^{-1}E_+. \end{aligned} \tag{4.48}$$

But by assumption **(A.1)**, below the energy E_+ there is a gap in the spectrum of A_0 of length at least $\delta > 0$; taking (1.2) into account, we get that for $< 0\varepsilon < \varepsilon_0 = \frac{\delta a_+}{E_+ C}$; A_ω has no spectrum in $(E_+, E_+ + C\varepsilon E_+ a_+^{-1})$. So the proof of Lemma 2.9 is ended if we choose $C = \frac{a_+}{E_+} > 0$. $\square$

The proof of Lemma 3.3. The proof is based on the Combes-Thomas argument [8], leading that there exists $c, c' > 0$ such that

$$\|\chi_y R_{x,k}(E)\chi_x\|_{x,k} \le \frac{c}{d(E, \sigma(A_{x,k}))} \cdot e^{-c' d(E,\sigma(A_{x,k}))\|x-y\|}.$$

We have

$$\begin{aligned}||\Gamma_{x,k} R_{x,k}(E)\chi_x||_{x,k} &\leq \sum_{y\in\Upsilon(x)} \|\chi_y R_{x,k}(E)\chi_x||_{x,k} \\ &\leq cdk^{d-1}\frac{1}{d(E,\sigma(A_{x,k}))}\cdot e^{-c'd(E,\sigma(A_{x,k}))(\frac{k}{3}-1)}.\end{aligned}$$

Let us notice that

$$\mathbb{P}_{\Lambda_{k_0}}\Big\{\Lambda_{k_0}(0) \text{ is } (m_0,E_0)-\text{regular }\Big\} \geq 1-\frac{1}{k_0^p}$$
$$\Leftrightarrow \mathbb{P}_{\Lambda_{k_0}}\Big\{\Lambda_{k_0}(0) \text{ is } \text{ not } (m_0,E_0)-\text{regular}\Big\} \leq \frac{1}{k_0^p}$$

and

$$\mathbb{P}_{\Lambda_{k_0}}\Big\{\Lambda_{k_0}(0) \text{ is not } \ (m_0,E_0)-\text{regular}\Big\} \leq$$
$$\mathbb{P}_{\Lambda_{k_0}}\Big\{cdk_0^{d-1}\frac{1}{d(E,\sigma(A_{0,k_0}))}\cdot e^{-c'd(E,\sigma(A_{0,k_0}))(\frac{k_0}{3}-1)} > e^{-m_0\frac{k_0}{2}}\Big\}.$$

however

$$\mathbb{P}_{\Lambda_{k_0}}\Big\{d\Big(\sigma(A_{0,k_0}),[E_+,E_++\varepsilon_0]\Big) < \frac{1}{2}k_0^{-\alpha}\Big\} \leq \frac{1}{k_0^p} \Leftrightarrow$$

$$\mathbb{P}_{\Lambda_{k_0}}\Big\{cdk_0^{d-1}\frac{1}{d(E,\sigma(A_{0,k_0}))}\cdot e^{-c'd(\sigma(A_{0,k_0}),E)(\frac{k_0}{3}-1)} >$$
$$cdk_0^{(d-1)+\alpha}e^{-\frac{k_0^{-\alpha}}{2}(\frac{k_0}{3}-1)}\Big\} \leq \frac{1}{k_0^p}.$$

Let $m_0 \geq ck_0^{-2}$ such that $e^{-m_0\frac{k_0}{2}} > cdk_0^{(d-1)+\alpha}e^{-\frac{k_0^{-\alpha}}{2}(\frac{k_0}{3}-1)}$. Thus we obtain that if **(P'1)** hold we have

$$\mathbb{P}_{\Lambda_{k_0}}\Big\{cdk_0^{d-1}\frac{1}{d(E,\sigma(A_{0,k_0}))}\cdot e^{-c'd(E,\sigma(A_{0,k_0}))(\frac{k_0}{2}-1)} > e^{-m_0\frac{k_0}{2}}\Big\} \leq \frac{1}{k_0^p}.$$

This completes the proof of Lemma 3.2. $\square$

The proof of Lemma 3.7. The proof is based on the variational formula. Indeed from the Theorem XIII.2 of [33] for $\varphi \in C_0^{+\infty}(\mathbb{R}^d)$ we have

$$E_j(A_{\omega,k}(\theta)) = \sup_{\psi_1,\dots,\psi_{j-1}} \inf_{\substack{\varphi\in(vect\langle\psi_1,\dots,\psi_{j-1}\rangle)^\perp \\ \|\varphi\|=1}} \langle A_{\omega,k}(\theta)\varphi,\varphi\rangle. \tag{4.49}$$

As

$$\langle A_{\omega,k}(\theta)\varphi,\varphi\rangle = \langle a_{\omega,k}(\nabla - i\theta)\varphi, (\nabla - i\theta)\varphi\rangle.$$

Then

$$\begin{aligned}
\langle A_{\omega,k}(\theta)\varphi,\varphi\rangle &- \langle A_{\omega,k}(\theta')\varphi,\varphi\rangle \\
&= \langle a_{\omega,k}(\nabla - i\theta)\varphi, (\nabla - i\theta)\varphi\rangle - \langle a_{\omega,k}(\nabla - i\theta')\varphi, (\nabla - i\theta')\varphi\rangle \\
&= \sum_{1\le i,j\le d} \langle a_{\omega,k,i,j}(\theta_i\theta_j - \theta'_i, \theta'_j)\varphi,\varphi\rangle + 2\Re e\Big(\langle a_{\omega,k}\nabla\cdot\varphi, -i(\theta-\theta')\cdot\varphi\rangle\Big) \\
&\le C\Big(\sum_{1\le i,j\le d} |\theta_i\theta_j - \theta'_i\theta'_j|\cdot||\varphi||^2 + |\theta-\theta'|\cdot||\nabla\varphi||^2\Big) \\
&\le C\Big(\sum_{1\le i\le d} |\theta_i - \theta'_i|^2\cdot||\varphi||^2 + |\theta-\theta'|\cdot||\nabla\varphi||^2\Big) \\
&\le C|\theta-\theta'|\Big(||\varphi||^2 + ||\nabla\varphi||^2\Big).
\end{aligned}$$

So

$$\langle A_{\omega,k}(\theta)\varphi,\varphi\rangle \le \langle A_{\omega,k}(\theta')\varphi,\varphi\rangle + C|\theta-\theta'|\Big(||\varphi||^2 + ||\nabla\varphi||^2\Big). \tag{4.50}$$

Now we take the inf on φ in the orthogonal to spaces of dimension $j-1$ in the left side of equation (4.50) then in the right side. By applying the sup on all spaces of dimension $j-1$ in the right-hand side then in the left-hand side and taking into account (1.2) and (4.49), we get that

$$E_j(A_{\omega,k}(\theta)) - E_j(A_{\omega,k}(\theta')) \le |\theta-\theta'|(1+E_k(\theta)).$$

As it is symmetric in θ we obtain that

$$E_j(A_{\omega,k}(\theta')) - E_j(A_{\omega,k}(\theta)) \le |\theta-\theta'|(1+E_k(\theta)).$$

This achieve the proof of Lemma 3.7. $\square$

References

[1] M. Aizenman and Molchanov: *Localization at large disorder and at extreme energies: an elementary derivation.* Commu. Math. Phys. **157** (1993) p 245-278.

[2] P.W. Anderson: *Absence of diffusion in certain random lattices.* Physical Reviews **109** (1959) p 1492-1501.

[3] J. M. Barbaroux, J. M Combes and P.D Hislop : *Localization near band edges for random Schrödinger operators.* Helv. phys. Acta. **70** (1997) N° 1-2 p 16-43.

[4] J. M. Combes and P.D. Hislop: *Localization for some continuous random Hamiltonian in d-dimensions.* Jour. Funt. Anal. **124** (1994) p 149-180.

[5] J. M. Combes P.D. Hislop and A. Tip: *Band edge localization and the density of sates for acoustic and electromagnetic waves in random media.* Ann. I.H.P **70** (1999) N° 4 p 381-428.

[6] D. Damanik and P. Stollmann: *Multiscale Analysis implies Strong Dynamical Localization.* Geom. Funct. Anal. **26** (2001) N° 3 p 411-428.

[7] H. von Dreifus and A. Klein: *A new proof of localization in the Anderson tight binding model.* Commu. Math. Phys. **124** (1989) p 285-299.

[8] A. Figotin and A. Klein: *Localization of Classical Waves I : Acoustic Waves.* Commu. Math. Phys. **180** (1996) p 439-482.

[9] A. Figotin and A. Klein: *Localization of Classical Waves II : electromagnetic Waves.* Commu. Math. Phys. **184** N°2 (1997) p 411-441.

[10] A. Figotin and A. Klein: *Localization of Classical Waves created by defects.* J. Stat. Phys. **86** (1997) p 165-177.

[11] A. Figotin and A. Klein: *Localization of light in lossless inhomogeneous dielectrics.* J. Opt. Soc. Amer. **A 15** N° 6 (1998) p 1748-1773.

[12] J. Fröhlich and T. Spencer: *Absence of diffusion in the Anderson tight binding model.* Commu. Math. Phys. (1983) **88** p 151-184.

[13] F. Germinet and A. Klein: *Bootstrap Multiscale Analysis and Localization in random media.* Commu. Math. Phys. N° 2 **222** (2001) p 415-448.

[14] F. Germinet and S. De Bièvre: *Dynamical localization for discrete and continuous random Schrödinger operators.* Commu. Math. Phys. **194** (1998) p 323-341.

[15] L. Hörmander: *The Analysis of Linear Partial Differential Operators.* Springer Verlag Heidelberg, 1983.

[16] W. Kirsch: *Random Schrödinger operators* A Course Lecture Notes In Phys. **345** (1989)Springer-Verlag, Berlin p 264-370.

[17] W. Kirsch: *Wegner estimates and Anderson localization for alloy-type potentials.* Math. Z. **221** (1996) p 507-512.

[18] W. Kirsch and F. Martinelli: *On the the Spectrum Of Schrödinger operators with a Random Potential.* Commu. Math. Phys. **85** (1982) p 329-350.

[19] W. Kirsch, P. Stollmann and G. Stolz: *Localization for random perturbation of periodic Schrodinger operators.* Random Oper. Stochastic Equations **6** (1998) p 241-268.

[20] W. Kirsch, P. Stollmann and G. Stolz: *Anderson localization for random Schrodinger operators with long range interactions.* Commu. Math. Phys. **195** (1998) p 495-507.

[21] F. Klopp: *Localization for some continus Random Schrödinger operators.* Commu. Math. Phys. **167** (1995) p 553-569.

[22] F. Klopp: *A short introduction to the theory of random Schrödinger operators.* Vietnam Jour. Math. **26** (1998) N∘ 3 p 189-216.

[23] F. Klopp: *Internal Lifshits Tails For Random Perturbations Of Periodic Schrödinger Operators.* Duke Math. Jour. **98** (1999) N° 2 p 335-396.

[24] F. Klopp: *Internal Lifshits Tails For long range single site potentials.* Jour. Math. Phys. **43** (2002) N° 6 p 2948-2958.

[25] F. Klopp: *Weak disorder localization and Lifshitz Tails: continuous hamiltonian.* Anna. I.H.P **3** (2002) N° 4 p 711-735.

[26] F. Klopp and L. Pastur: *Lifshits Tails For Random Schrödinger operators with negative singular Poisson potential.* Commu. Math. Phys. **206** (1999) p 57-103.

[27] F. Klopp and T. Wolff: *Lifshitz tails for 2-dimensional random Schrodinger operators.* to appear.

[28] H. Najar: *Asymptotique de la densité d'états intégrée des opérateurs acoustiques aléatoires.* C. R. Acad. Sci. Paris, **333** I, p 191-194 (2001).

[29] H. Najar: *Lifshitz tails for random acoustic operators.* Jour. Math. Phys. **44** N° 4 (2003)

[30] H. Najar: *Asymptotic of the integrated density of states of acoustic operator with long range random perturbations.*, 2002 submitted .

[31] H. Najar: *2-Dimensional loclisation of acoustic waves in random perturbation of periodic media.* in preparation (2003).

[32] L. Pastur and A. Figotin: *Spectra of Random and Almost-Periodic Operators.* Springer-Verlag.

[33] M. Reed and B. Simon: *Methods of Modern Mathematical vol IV: Analysis of Operators.* Academic Press, New York, 1978.

[34] P. Stollmann: *Localization for acoustic waves in random perturbation of periodic media.* Israel Jour. Math. **107** (1998) p 125-139.

[35] J. Sjöstrand: *Microlocal analysis for the periodic magnetic Schrödinger equation and related questions.* Lecture Notes in Math 1495, Springer-Verlag, Berlin (1991) p 237-332.

[36] I. Veselić: *Localization for random perturbation of periodic Schrödinger operators with regular Floquet eigenvalues.* Technical report Univ. Bochum.