EINSTEIN RELATION FOR A CLASS OF INTERFACE MODELS

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Abstract. A class of SOS interface models which can be seen as simplified stochastic Ising model interfaces is studied. In the absence of an external field the long-time fluctuations of the interface are shown to behave as Brownian motion with diffusion coefficient $(\sigma^{\mathrm{GK}})^2$ given by a Green-Kubo formula. When a small external field h is applied, it is shown that the shape of the interface converges exponentially fast to a stationary distribution and the interface moves with an asymptotic velocity $v(h)$. The mobility is shown to exist and to satisfy the Einstein relation: $(dv/dh)(0) = \beta(\sigma^{\mathrm{GK}})^2$, where β is the inverse temperature.

1. Introduction

1.1. Motivation and overview. In this paper we are concerned with the movement of an interface between two phases of a system subject to a small external field which favors one of them. This is a well known problem, with a large number of applications (see, e.g., [Spo], [RK] and references therein). In particular, our interest on this issue arose from the analysis of the nucleation phenomenon in the context of stochastic Ising models below the critical temperature, and under a small external field h (see, [SS], [RTMS] and references therein). In this regime, assuming $h > 0$, droplets of the $(+)$-phase which are sufficiently large, grow and invade a background given by the $(-)$-phase. The order of magnitude of the velocity $v(h)$ with which the linear dimensions of such a droplet grow is of great interest. In particular information on this is needed in order to estimate the time required for the system to relax from the metastable $(-)$-phase to the stable $(+)$-phase (see references quoted above, and especially Section 1.5 of [SS]). In two dimensions, at any subcritical temperature, and in higher dimensions above the roughening transition temperature, it is believed that $v(h)$ is asymptotically linear in h for small values of h. In contrast, in higher dimensions and below the roughening transition temperature, there does not seem to be a consensus on what to expect. Some papers (see e.g., [RTMS] and

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references therein) suggest that $v(h)$ is still asymptotically linear in h for small values of h. Other papers (see, e.g., Sect. 8 of [Spo] and references therein) propose that in this regime $v(h)$ goes to 0 substantially faster than linearly in h (actually as $\exp(-C/h^{d-2})$), as $h \to 0$, due to the rigidity of the interface, with nucleation on the interface playing a major role in its movement.

The rigorous results in [SS] on the growth of the (+)-phase into the (−)-phase in the two-dimensional case were sufficient to precisely estimate the relaxation time at the level of its logarithmic dependence on $1/h$, as $h \to 0$. Nevertheless, the estimate in that paper on the behavior of the velocity $v(h)$ in this limit, was quite modest. That estimate basically stated that for arbitrary $\epsilon > 0$, $v(h) > \exp(-\epsilon/h)$, for small $h > 0$. (No precise definition of $v(h)$ was provided or needed in that paper, but the estimates on droplet growth, in Section 3.2 of that paper, imply this bound for any reasonable definition of $v(h)$.) While such a bound is very far from a conjectured linear behavior, it seems very hard to improve it much in the context of stochastic Ising models.

Here we will follow a common route in Mathematical Physics. Having as motivation the problem discussed above, and not being able to solve it, we will analyze some simplified interface models, in the hope that their behavior and the techniques used may shed light on the original problem. The idealized models to be considered correspond to finite interfaces, confined to move inside tubes. The movement will be described by a reversible Markov process, obtained by considering that the interface separates spins -1 and $+1$ and that these spins flip like a stochastic Ising model, but with suppression of flips that would create more than one interface, or violate a restriction on the interface. This restriction will be that the interface is of the SOS type, i.e., described by a function on a set of sites, perpendicular to the direction of movement. This restriction separates the idealized models further from the original models, but should be of lesser relevance at low temperatures.

While the SOS interface models in a tube that we will consider are not new (see, e.g., Sect. 6 of [Spo]), our results are, to the best of our knowledge, new. We will relate the velocity with which the interface moves under a small external field h to the fluctuations of the same interface in the absence of an external field. The specific relation that will be derived, and which is stated in the abstract above, is a sort of Einstein or Einstein-Green-Kubo relation (see, e.g., [LR] and references therein for an introduction to this relation and rigorous results in a different context). In the context of interface models, such a relation was proposed based on non-rigorous arguments (for more ambitious models) in Sect. 3 of [Spo]. In this connection, we point out that in [But] rigorous results on the Einstein relation for interfaces in certain interacting particle systems were obtained. That paper is, nevertheless, completely different from the present one; there a Kac-type interaction was considered in infinite volume, and the definitions of the quantities appearing in the Einstein relation were different from those here.

One corollary to the Einstein relation is that in our models the interface moves with velocity $v(h)$ which is asymptotically linear in a small external field h. In other words, the mobility $\mathcal{M} = (dv/dh)(0)$ is positive and finite. While this may not be a surprise, since we are studying an interface confined to a tube of finite cross section, we could only prove it using some rather involved arguments. This is so because the tube itself is infinite. In particular, it does not seem obvious that even a small external field will not produce a major change in the behavior of the interface. To rule out this possibility, we considered the Markov chain that describes the shape of the interface, and showed that when $h = 0$ it is reversible and has positive spectral gap. This then allowed us to treat the case with small h as a perturbation. As a by-product, we learned that also under a small $h \neq 0$, the Markov chain which describes the shape of the interface, in spite of no longer being reversible, still has a single invariant distribution, to which it converges exponentially fast.

Our results then raise the question of how the mobility $\mathcal{M}$ depends on the size of the interface, and how the answer to this question depends on the dimension of the interface and on the temperature and the corresponding rigidity or roughness of the interface in the absence of external field. These are fundamental questions, which should help clarify the issues discussed at the end of the first paragraph of this introduction. Since their solution seems to require subtantial additional techniques on top of those used in the current paper, we are postponing these issues to a further investigation.

1.2. The models and basic notation. For reasons that will become clear soon, our interface models will be described as functions from the vertices of a finite graph to the integers. For this purpose, let $G = (V, E)$ be a finite graph, with vertex set V and edge set E. For a technical reason, we will suppose that $|V| \geq 3$. (The case $|V| = 1$ is trivial, and the case $|V| = 2$ can be treated by elementary methods.) In order to present some basic examples, we recall first some terminology from graph theory. Given two graphs, $G' = (V', E')$ and $G'' = (V'', E'')$, the Cartesian product $G' \times G''$ is defined as the graph $G = (V, E)$ which has as vertex set $V = V' \times V'' = \{\{v', v''\} \,:\, v' \in V', v'' \in V''\}$, and as edge set $E = \{\{(v', v''), (u', u'')\} \,:\, (\{v', u'\} \in E' \text{ and } v'' = u'') \text{ or } (v' = u' \text{ and } \{v'', u''\} \in E'')\}$. Up to relabeling, the Cartesian product is associative and commutative.

The basic examples of graphs on which our interface models are defined are:

(1) *1 dimensional interface, free boundary condition:* $G_1^{\mathrm{fr}}(l) = (V_1^{\mathrm{fr}}(l), E_1^{\mathrm{fr}}(l))$, where

$$V_1^{\mathrm{fr}}(l) = \{1, 2, ..., l\}, \qquad E_1^{\mathrm{fr}}(l) = \{\{1, 2\}, \{2, 3\}, ..., \{l-1, l\}\}.$$

(2) *1 dimensional interface, periodic boundary conditions:* $G_1^{\mathrm{per}}(l) = (V_1^{\mathrm{per}}(l), E_1^{\mathrm{per}}(l))$, where

$$V_1^{\mathrm{per}}(l) = \{1, 2, ..., l\}, \qquad E_1^{\mathrm{per}}(l) = \{\{1, 2\}, \{2, 3\}, ..., \{l-1, l\}, \{l, 1\}\}.$$

(3) *d dimensional interface, free boundary conditions:*

$$G_d^{\mathrm{fr}}(l_1, l_2, ..., l_d) \;=\; G_1^{\mathrm{fr}}(l_1) \times G_1^{\mathrm{fr}}(l_2) \times ... \times G_1^{\mathrm{fr}}(l_d).$$

(4) *d dimensional interface, periodic boundary conditions:*

$$G_d^{\mathrm{per}}(l_1, l_2, ..., l_d) \;=\; G_1^{\mathrm{per}}(l_1) \times G_1^{\mathrm{per}}(l_2) \times ... \times G_1^{\mathrm{per}}(l_d).$$

An SOS interface is defined as a function $\eta : V \to \mathbb{Z}$, and the set of all such functions will be denoted, as usual, by $\mathbb{Z}^V$. Of particular importance will be the flat interfaces, ϕ_i, $i \in \mathbb{Z}$, defined by

$$\phi_i(v) = i, \quad \text{for each } v \text{ in } V.$$

For $\eta \in \mathbb{Z}^V$ and $i \in \mathbb{Z}$, we will denote by $\eta + i$ the interface obtained by shifting η by i, i.e.,

$$(\eta + i)(x) = \eta(x) + i, \quad \text{for each } x \text{ in } V.$$

We will write $\eta \leq \zeta$ if $\eta(v) \leq \zeta(v)$, for each v in V (and similarly for $\geq$). We will write $\eta < \zeta$ if $\eta \leq \zeta$ and $\eta(v) < \zeta(v)$ for some v in V (and similarly for $>$).

To motivate the definition of our interface models, we describe now Ising models on the set of sites:

$$\bar{V} = V \times \mathbb{Z}.$$

Their configurations lie in the set

$$\Sigma^{\text{Ising}} = \{-1, +1\}^{\bar{V}}.$$

In order to define the Hamiltonian, we declare sites in $\bar{V}$ to be neighbors if they have the first coordinate in common and their second coordinates differ by one unit, or if they have the second coordinate in common and their first coordinates are endpoints of an edge in E. The set of pairs of neighbors of $\bar{V}$ will be denoted by $\bar{E}$. In other words, $(\bar{V}, \bar{E}) = (V, E) \times (\mathbb{Z}, E_{\mathbb{Z}})$, in the sense of Cartesian product of graphs, where $E_{\mathbb{Z}} = \{\{i, i+1\} \ : \ i \in \mathbb{Z}\}$.

The formal Ising model Hamiltonian is defined by

$$\mathcal{H}_h^{\text{Ising}}(\sigma) = -\frac{1}{2} \sum_{\{x,y\} \in \bar{E}} \sigma(x)\sigma(y) - \frac{h}{2} \sum_{x \in \bar{V}} \sigma(x), \tag{1.1}$$

where $h \in \mathbb{R}$ is the external field.

To each interface $\eta \in \mathbb{Z}^V$ we associate the spin configuration $\sigma[\eta] \in \Sigma^{\text{Ising}}$, given by

$$(\sigma[\eta])(v, i) = \begin{cases} +1 & \text{if} \quad i \leq \eta(v), \\ -1 & \text{if} \quad i > \eta(v). \end{cases}$$

The set of Ising model configurations which correspond to SOS interfaces will be denoted by

$$\Sigma^{\text{SOS}} = \{\sigma[\eta] : \eta \in \mathbb{Z}^V\}.$$

Note that $\sigma[\cdot]$ establishes a one-to-one correspondence between $\mathbb{Z}^V$ and Σ^{SOS}. In terms of the SOS interface, the formal Hamiltonian (1.1) restricted to Σ^{SOS} can be rewritten, up to a constant (infinite) term as

$$\mathcal{H}_h^{\text{Int}}(\eta) = \mathcal{H}_h^{\text{Ising}}(\sigma[\eta]) = \sum_{\{v,u\} \in E} |\eta(v) - \eta(u)| - h \sum_{v \in V} \eta(v). \tag{1.2}$$

Note that $\mathcal{H}_h^{\mathrm{Int}}(\cdot)$ is not just a formal Hamiltonian, but rather a well defined function $\mathbb{Z}^V \to \mathbb{R}$. For later use, note also that for $i \in \mathbb{Z}$,

$$\mathcal{H}_h^{\mathrm{Int}}(\eta + i) = \mathcal{H}_h^{\mathrm{Int}}(\eta) - |V|hi. \tag{1.3}$$

We introduce the unnormalized Gibbs measure at inverse temperature $\beta \in (0, \infty)$ on $\mathbb{Z}^V$ by

$$\mu_{\beta,h}^{\mathrm{Int}}(\eta) \;=\; \exp\left(-\beta \mathcal{H}_h^{\mathrm{Int}}(\eta)\right).$$

Note that since $\mathbb{Z}^V$ is countable, this defines a measure on this set. But

$$\sum_{\eta \in \mathbb{Z}^V} \mu_{\beta,h}^{\mathrm{Int}}(\eta) \;\geq\; \sum_{i \in \mathbb{Z}} \mu_{\beta,h}^{\mathrm{Int}}(\phi_i) \;=\; \infty,$$

so that this measure cannot be normalized to a probability measure.

We will now introduce stochastic dynamics on the interface space $\mathbb{Z}^V$. These stochastic dynamics will be introduced in discrete time, in order to simplify certain arguments later on, but they are motivated by continuous time spin flip dynamics for the underlying Ising model (stochastic Ising models), in which we suppress flips which would move the configuration outside of Σ^{SOS}.

Given $\eta \in \mathbb{Z}^V$ and $v \in V$, define the interfaces $\eta^{v,\uparrow}$ and $\eta^{v,\downarrow}$, via

$$\eta^{v,\uparrow}(u) = \begin{cases} \eta(u) & \text{if} \quad u \neq v, \\ \eta(u) + 1 & \text{if} \quad u = v. \end{cases}$$

$$\eta^{v,\downarrow}(y) = \begin{cases} \eta(u) & \text{if} \quad u \neq v, \\ \eta(u) - 1 & \text{if} \quad u = v. \end{cases}$$

We will write $\eta \sim \zeta$ if $\eta = \zeta^{v,\updownarrow}$, for some $v \in V$ and $\updownarrow \in \{\uparrow, \downarrow\}$. Note that this is a symmetric relation and that if $\eta \sim \zeta$ then $\eta < \zeta$ or $\eta > \zeta$.

Suppose given a function $F : \mathbb{R} \to (0, 1]$ which satisfies

$$F(-r) = e^r\, F(r), \quad \text{for all } r \in \mathbb{R}. \tag{1.4}$$

For instance, we can have

$$F(r) = \exp(-r^+), \quad \text{where } r^+ = r \text{ for } r \geq 0 \text{ and } r^+ = 0, \text{ for } r < 0. \tag{1.5}$$

Or,

$$F(r) = \frac{1}{1 + e^r}. \tag{1.6}$$

Set

$$\tilde{p}_{\beta,h}(\eta, \zeta) \;=\; F\left(\beta\left(\mathcal{H}_h^{\mathrm{Int}}(\zeta) - \mathcal{H}_h^{\mathrm{Int}}(\eta)\right)\right).$$

Condition (1.4) was imposed in order for the detailed balance (reversibility) condition to hold for $\tilde{p}_{\beta,h}(\cdot,\cdot)$:

$$\mu^{\text{Int}}_{\beta,h}(\eta)\ \tilde{p}_{\beta,h}(\eta,\zeta) \ = \ \mu^{\text{Int}}_{\beta,h}(\zeta)\ \tilde{p}_{\beta,h}(\zeta,\eta), \tag{1.7}$$

for every $\eta, \zeta \in \mathbb{Z}^V$.

Our discrete time dynamics on $\mathbb{Z}^V$ can now be defined as follows. Suppose that at time n the process is in the configuration η. Then at time $n+1$ a vertex $v \in V$ is randomly chosen, and then, independently, one of the two symbols $\uparrow$ or $\downarrow$ is also chosen at random; denote by $\updownarrow$ the chosen one. Given these choices, the state of the process at time $n+1$ will be either $\eta^{v,\updownarrow}$ or η, the former case happening with probability $\tilde{p}_{\beta,h}(\eta,\eta^{v,\updownarrow})$ independently of anything else. In other words, we are defining a Markov chain $(\eta_n)_{n=0,1,\ldots}$ on $\mathbb{Z}^V$, with transition probabilities, when $\eta \neq \zeta$, given by

$$p^{\text{Int}}_{\beta,h}(\eta,\zeta) \ = \ \begin{cases} \dfrac{1}{2|V|}\,\tilde{p}_{\beta,h}(\eta,\zeta) & \text{if } \eta \sim \zeta, \\ 0 & \text{otherwise}, \end{cases} \tag{1.8}$$

while

$$p^{\text{Int}}_{\beta,h}(\eta,\eta) \ = \ 1 \ - \ \sum_{\zeta\neq\eta} p^{\text{Int}}_{\beta,h}(\eta,\zeta). \tag{1.9}$$

$\mathbb{P}^{\text{Int}}_{\beta,h;\mu}$ will denote a probability measure under which $(\eta_n)_{n=0,1,\ldots}$ is such a Markov chain, started from η_0 which has probability distribution μ. The corresponding expectation will be denoted by $\mathbb{E}^{\text{Int}}_{\beta,h;\mu}$. When μ is concentrated on a single configuration ζ, we will also write ζ instead of μ in this notation.

When $F(\cdot)$ is given by (1.5), $(\eta_n)_{n=0,1,\ldots}$ evolves according to Metropolis dynamics, while when $F(\cdot)$ is given by (1.6), it evolves according to the Gibbs sampler dynamics.

For each β and h,

$$\left\{\beta\left(\mathcal{H}^{\text{Int}}_h(\zeta) - \mathcal{H}^{\text{Int}}_h(\eta)\right) \ : \ \eta \sim \zeta\right\} \tag{1.10}$$

is a finite set. Therefore there exists $\widetilde{C}(\beta,h) > 0$ such that

$$p^{\text{Int}}_{\beta,h}(\eta,\zeta) \ > \ \widetilde{C}(\beta,h) \quad \text{for all } \eta\sim\zeta. \tag{1.11}$$

Lemma 1.1. *For every $\beta \in (0,\infty)$, $h \in \mathbb{R}$ and $\eta \in \mathbb{Z}^V$,*

$$p^{\text{Int}}_{\beta,h}(\eta,\eta) \ \geq \ \frac{1-e^{-\beta}}{2|V|} \tag{1.12}$$

Proof. Suppose, with no loss of generality, that $h \geq 0$. Let $v \in V$ be a point of minimum of $\eta(v)$. Then $\mathcal{H}^{\text{Int}}_h(\eta^{v,\downarrow}) = \mathcal{H}^{\text{Int}}_h(\eta) + \text{Deg}_G(v) + h$, where $\text{Deg}_G(v) = |\{u \in V : \{v,u\} \in E\}|$ is the degree of v in the graph G. So

$$\tilde{p}_{\beta,h}(\eta,\eta^{v,\downarrow}) \ = \ F(\beta(\text{Deg}_G(v)+h)) \ = \ e^{-\beta(\text{Deg}_G(v)+h)}\,F(-\beta(\text{Deg}_G(v)+h)) \ \leq \ e^{-\beta},$$

since F takes values in $(0,1]$ and $\mathrm{Deg}_G(v) \geq 1$. Therefore,

$$p^{\mathrm{Int}}_{\beta,h}(\eta,\eta) \;\geq\; \frac{1}{2|V|}\,\big(1-\tilde{p}_{\beta,h}(\eta,\eta^{v,\downarrow})\big) \;\geq\; \frac{1-e^{-\beta}}{2|V|},$$

where the expression in the middle is the probability that the vertex v and the symbol $\downarrow$ are chosen, but the jump from η to $\eta^{v,\downarrow}$ does not occur. ■

For later convenience we introduce the notation $\eta \approx \zeta$ when $\eta = \zeta$ or $\eta \sim \zeta$. From (1.11) and (1.12), we learn that there exists $C(\beta,h) > 0$ such that

$$p^{\mathrm{Int}}_{\beta,h}(\eta,\zeta) \;>\; C(\beta,h) \quad \text{for all } \eta \approx \zeta. \tag{1.13}$$

From (1.7) and the definition of $p^{\mathrm{Int}}_{\beta,h}$, it follows that the Markov chain $(\eta_n)_{n=0,1,\ldots}$ is reversible with respect to $\mu^{\mathrm{Int}}_{\beta,h}$, i.e.,

$$\mu^{\mathrm{Int}}_{\beta,h}(\eta)\; p^{\mathrm{Int}}_{\beta,h}(\eta,\zeta) \;=\; \mu^{\mathrm{Int}}_{\beta,h}(\zeta)\; p^{\mathrm{Int}}_{\beta,h}(\zeta,\eta), \tag{1.14}$$

for every $\eta,\zeta \in \mathbb{Z}^V$. It is clear that $(\eta_n)_{n=0,\ldots}$ is irreducible, and from (1.12) it is also aperiodic. Since $\mu^{\mathrm{Int}}_{\beta,h}$ has infinite mass, $(\eta_n)_{n=0,\ldots}$ is either transient or null recurrent.

It is natural to describe an interface $\eta \in \mathbb{Z}^V$ through its shape and its position. This "decomposition" will be crucial in this paper. In the next two subsections, we introduce each one of these "components" in technically convenient fashions.

1.3. The shape Markov chain. To define the shape of an interface, first define the relation

$$\eta \cong \zeta \quad \text{if} \quad \eta = \zeta + i, \text{ for some } i \in \mathbb{Z}.$$

Note that this is an equivalence relation, and define the shape of $\eta \in \mathbb{Z}^V$, denoted by $\mathrm{Sh}[\eta]$, as the equivalence class to which η belongs. The set of all shapes will be denoted by

$$\mathcal{S} = \{\mathrm{Sh}[\eta] : \eta \in \mathbb{Z}^V\}.$$

The class of the flat interfaces will be denoted by

$$\phi \;=\; \{\phi_i : i \in \mathbb{Z}\}.$$

We will write $s' \sim s''$ (*resp.* $s' \approx s''$) if $s' = \mathrm{Sh}[\eta']$, $s'' = \mathrm{Sh}[\eta'']$, for some $\eta',\eta'' \in \mathbb{Z}^V$, $\eta' \sim \eta''$ (*resp.* $\eta' \approx \eta''$).

The interface Markov chain $(\eta_n)_{n=0,1,\ldots}$, defined above, gives rise to the process

$$(S_n)_{n=0,1,\ldots} \;=\; (\mathrm{Sh}[\eta_n])_{n=0,1,\ldots}.$$

It is not hard to see that this process is also a Markov chain, since

$$p^{\mathrm{Int}}_{\beta,h}(\eta,\zeta) \;=\; p^{\mathrm{Int}}_{\beta,h}(\eta+i,\zeta+i),$$

for any $\eta,\zeta \in \mathbb{Z}^V$, $i \in \mathbb{Z}$, thanks to (1.3). We will call the process $(S_n)_{n=0,1,\ldots}$ the shape Markov chain. Its transition probabilities will be denoted by $p^{\mathrm{Sh}}_{\beta,h}(\cdot,\cdot)$. In order to express $p^{\mathrm{Sh}}_{\beta,h}(\cdot,\cdot)$ in terms of $p^{\mathrm{Int}}_{\beta,h}(\cdot,\cdot)$, we state the following simple fact.

Lemma 1.2. *Suppose* $\mathrm{Sh}[\eta'] = \mathrm{Sh}[\zeta']$, $\mathrm{Sh}[\eta''] = \mathrm{Sh}[\zeta'']$, $\eta' \sim \eta''$, *and* $\zeta' \sim \zeta''$. *Then there is* $i \in \mathbb{Z}$ *such that* $\zeta' = \eta' + i$, $\zeta'' = \eta'' + i$. *In particular, if* $\eta' = \zeta'$, *then* $\eta'' = \zeta''$.

Proof. Since $\eta' \sim \eta''$, η' and η'' differ at a single vertex $v \in V$. Since $\mathrm{Sh}[\eta'] = \mathrm{Sh}[\zeta']$ and $\mathrm{Sh}[\eta''] = \mathrm{Sh}[\zeta'']$, there are $i', i'' \in \mathbb{Z}$ such that $\eta' = \zeta' + i'$, $\eta'' = \zeta'' + i''$. If it were the case that $i' \neq i''$, then ζ' and ζ'' would differ at every vertex were η' and η'' take the same value, i.e., at every vertex $u \in V \backslash \{v\}$. Since $|V| \geq 3$, this is in contradiction with the hypothesis that $\zeta' \sim \zeta''$. ■

This lemma is the only place in this paper were the assumption $|V| \geq 3$ plays a role. It is easy to see that the lemma would be false without this assumption.

It is not hard to see that Lemma 1.2 implies that for $s' \approx s''$,

$$p^{\mathrm{Sh}}_{\beta,h}(s', s'') = p^{\mathrm{Int}}_{\beta,h}(\eta', \eta'') \quad \text{if} \ \ s' = \mathrm{Sh}[\eta'], \ \ s'' = \mathrm{Sh}[\eta''], \ \ \eta' \approx \eta'', \tag{1.15}$$

while, for $s' \not\approx s''$,

$$p^{\mathrm{Sh}}_{\beta,h}(s', s'') = 0. \tag{1.16}$$

In particular, it follows from (1.13) that

$$p^{\mathrm{Sh}}_{\beta,h}(s', s'') \ > \ C(\beta, h) \quad \text{for all } s' \approx s''. \tag{1.17}$$

The shape Markov chain $(S_n)_{n=0,1,\ldots}$ was defined above on the same probability space where the interface Markov chain $(\eta)_{n=0,1,\ldots}$ was defined. For certain purposes in this paper, we will be interested in considering the shape Markov chain with transition probabilities given by (1.15) and (1.16) without making reference to the interface Markov chain. $\mathbb{P}^{\mathrm{Sh}}_{\beta,h;\nu}$ will denote a probability measure under which $(S_n)_{n=0,1,\ldots}$ is such a Markov chain, started from S_0 which has probability distribution ν. The corresponding expectation will be denoted by $\mathbb{E}^{\mathrm{Sh}}_{\beta,h;\nu}$. When ν is concentrated on a single configuration s, we will also write s instead of ν in this notation.

When $h = 0$, (1.3) states that $\mathcal{H}^{\mathrm{Int}}_h(\eta)$ only depends on η through its shape. This allows us to define, for each $s \in \mathcal{S}$,

$$\mathcal{H}^{\mathrm{Sh}}_0(s) \ = \ \mathcal{H}^{\mathrm{Int}}_0(\eta),$$

where η is such that $\mathrm{Sh}(\eta) = s$. The unique element of $\mathcal{S}$ which minimizes $\mathcal{H}^{\mathrm{Sh}}_0(\cdot)$ is the flat shape ϕ.

Lemma 1.3. *For any* $\beta \in (0, \infty)$,

$$Z^{\mathrm{Sh}}_{\beta,0} \ = \ \sum_{s \in \mathcal{S}} \exp(-\beta \mathcal{H}^{\mathrm{Sh}}_0(s)) \ < \ \infty.$$

Proof. For each $s \in \mathcal{S}$ choose a representative $\mathrm{Rep}[s] \in \mathbb{Z}^V$ as follows. $\mathrm{Rep}[s]$ is the unique $\eta \in \mathbb{Z}^V$ that satisfies

$$\mathrm{Sh}[\eta] = s, \qquad \min_{v \in V} \eta(v) = 0.$$

Define

$$\text{Range}(s) = \max_{v \in V}(\text{Rep}[s])(v).$$

Note that $\mathcal{H}_0^{\text{Sh}}(s) \geq \text{Range}(s)$. Therefore

$$Z_{\beta,0}^{\text{Sh}} \;\leq\; \sum_{k=0}^{\infty} \sum_{\substack{s \in \mathcal{S} \\ \text{Range}(s)=k}} e^{-\beta k} \;\leq\; \sum_{k=0}^{\infty} (k+1)^{|V|}\, e^{-\beta k} \;<\; \infty.$$

■

We can now introduce the $h = 0$ Gibbs probability distribution on $\mathcal{S}$ as

$$\nu_{\beta,0}(s) \;=\; \frac{\exp(-\beta \mathcal{H}_0^{\text{Sh}}(s))}{Z_{\beta,0}^{\text{Sh}}}.$$

From (1.17), we learn that for any value of β and h, the shape Markov chain $(S_n)_{n=0,1,...}$ is irreducible and aperiodic. From (1.14), (1.15) and (1.16), we learn that , when $h = 0$, $(S_n)_{n=0,1,...}$ is also reversible with respect to $\nu_{\beta,0}$, i.e.,

$$\nu_{\beta,0}(s')\; p_{\beta,h}^{\text{Sh}}(s', s'') \;=\; \nu_{\beta,0}(s'')\; p_{\beta,h}^{\text{Sh}}(s'', s'), \tag{1.18}$$

for every $s', s'' \in \mathcal{S}$. In particular, $\nu_{\beta,0}$ is then the unique invariant probability distribution for the shape Markov chain, which is, therefore positive recurrent in this case.

In contrast, when $h \neq 0$, the shape Markov chain is not reversible. To see this, one can use a cycle argument (see Theorem 4.2 of Chapter 5 of [Dur], p. 303). (To show that the cycle condition is violated, one can consider a cycle of length $|V|$, starting and ending at ϕ, obtained by modifying the shape at each $v \in V$ in exactly one of the steps. Since this negative result will not be needed here, the simple computation is left for the interested reader. Interestingly, this depends on the assumption $|V| \geq 3$; one can also check that if $|V| = 2$, then even when $h \neq 0$, the shape Markov chain is reversible.)

1.4 The position of the interface. It will be convenient to determine the position of an interface $\eta \in \mathbb{Z}^V$ by means of $\sum_{x \in V} \eta(v)$, so that the displacement of the interface process $(\eta_n)_{n=0,...}$ from time 0 to time n will be determined by

$$M(n) = \sum_{v \in V} (\eta_n(v) - \eta_0(v)). \tag{1.19}$$

This quantity is proportional to the variation in magnetization from time 0 to time n in the spin process $(\sigma[\eta_n])_{n=0,...}$, since

$$M(n) = \frac{1}{2} \sum_{x \in \bar{V}} \left\{ (\sigma[\eta_n])(x) - (\sigma[\eta_0])(x) \right\}.$$

We will also consider the mean displacement of the interface,

$$m(n) = \frac{M(n)}{|V|}.$$

Set

$$J_n = M(n+1) - M(n),$$

so that

$$M(n) = J_0 + ... + J_{n-1}. \tag{1.20}$$

Thanks to Lemma 1.2, for any $0 \le n' < n''$, the sequence of shapes $\mathrm{Sh}[\eta_{n'}], ..., \mathrm{Sh}[\eta_{n''}]$ determines the sequence of interfaces $\eta_{n'}, ..., \eta_{n''}$ up to a common additive constant. Therefore this sequence of shapes determines the sequence $J_{n'}, ..., J_{n''-1}$ in a time invariant fashion, i.e., for $n = n', ..., n'' - 1$,

$$J_n = g\big(\mathrm{Sh}[\eta_n], \mathrm{Sh}[\eta_{n+1}]\big), \tag{1.21}$$

for a fixed function $g : \mathcal{S}^2 \to \{-1, 0, 1\}$.

Given the shape Markov chain $(S_n)_{n=0,1,...}$, we can use (1.21) to define $(J_n)_{n=0,1,...}$ via

$$J_n = g\big(S_n, S_{n+1}\big), \tag{1.22}$$

and then use (1.20) to define $(M(n))_{n=0,1,...}$ (with $M(0) = 0$). The processes defined in this way have the same law as the ones defined in terms of the interface Markov chain $(\eta_n)_{n=0,1,...}$.

For values of β and h for which the shape Markov chain $(S_n)_{n=0,...}$ is recurrent, it is natural to look at the return times to the flat shape ϕ. For this purpose set

$$\begin{aligned} N_0 &= 0, \\ N_{k+1} &= \inf\{n > N_k : S_n = \phi\}, \quad k = 1, 2, ... \end{aligned}$$

From the observation in the last paragraph above, $(S_n)_{n=0,...}$ determines also the sequence of random variables $M(N_k)$, $k = 0, 1, ...$ Set

$$(\Delta N)_k = N_{k+1} - N_k, \quad \text{and} \quad (\Delta M)_k = M(N_{k+1}) - M(N_k).$$

Note that, regardless of the distribution of S_0, the sequences

$$(\Delta N)_1, (\Delta N)_2, ... \qquad \text{and} \qquad (\Delta M)_1, (\Delta M)_2, ...$$

are both i.i.d.. For the sake of notation, it is natural to introduce two random variables, ΔN and ΔM on a probability space with probability measure denoted by $\mathbb{P}_{\beta,h}$, and corresponding expectation denoted by $\mathbb{E}_{\beta,h}$, so that the law of ΔN coincides with the law of each of the $(\Delta N)_k$ in the first sequence above, while the law of ΔM coincides with the law of each of the $(\Delta M)_k$ in the second sequence above. Note that ΔM is stochastically bounded above by ΔN: For any $k \in \mathbb{Z}$,

$$\mathbb{P}_{\beta,h}(\Delta M > k) \; = \; \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(M(N_1) > k) \; \le \; \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(N_1 > k) \; = \; \mathbb{P}_{\beta,h}(\Delta N > k). \tag{1.23}$$

1.5 Results. Suppose that β and h are such that the shape Markov chain $(S_n)_{n=0,\ldots}$ has a unique invariant probability distribution $\nu_{\beta,h}$ (as we already know to be the case when $h=0$). We will say that a probability distribution μ on the set of interfaces $\mathbb{Z}^V$ is shape-stationary, if

$$\sum_{\eta\,:\,\mathrm{Sh}[\eta]=s} \mu(\eta) \;=\; \nu_{\beta,h}(s),$$

for each $s \in \mathcal{S}$. Note that if the distribution μ of η_0 is shape-stationary, then the distribution of $\mathrm{Sh}[\eta_n]$ is $\nu_{\beta,h}$ for any time n.

Theorem 1.1 (Long-time fluctuations of the inteface when $h=0$). *For any $\beta \in (0,\infty)$, if $h=0$ and the distribution μ of η_0 is shape-stationary, then*

$$\frac{M(n\cdot)}{\sqrt{n}} \;\to\; \sigma_\beta^{\mathrm{GK}} B(\cdot), \tag{1.24}$$

in distribution, as $n\to\infty$, where $M(\cdot):[0,\infty)\to\mathbb{R}$ is obtained from $M(n)$, $n\in\{0,1,\ldots\}$ by linear interpolation, $(B(t))_{t\geq 0}$ is standard Brownian motion started from 0, and

$$\begin{aligned}\left(\sigma_\beta^{\mathrm{GK}}\right)^2 \;&=\; \mathbb{E}^{\mathrm{Sh}}_{\beta,0;\nu_{\beta,0}}((J_0)^2) + 2\sum_{n=1}^{\infty} \mathbb{E}^{\mathrm{Sh}}_{\beta,0;\nu_{\beta,0}}(J_0\,J_n)\\ &=\; \mathbb{E}^{\mathrm{Int}}_{\beta,0;\mu}((J_0)^2) + 2\sum_{n=1}^{\infty} \mathbb{E}^{\mathrm{Int}}_{\beta,0;\mu}(J_0\,J_n)\\ &=\; \frac{\mathbb{E}_{\beta,0}((\Delta M)^2)}{\mathbb{E}_{\beta,0}(\Delta N)} \;\in\; (0,\infty).\end{aligned} \tag{1.25}$$

The superscript GK above refers to the fact that the first two expressions above for $\sigma_\beta^{\mathrm{GK}}$ in terms of the J_n are sometimes referred to as Green-Kubo formulas (see, e.g., Section 3 of [Spo]).

We turn now to the case when possibly $h\neq 0$. We want to consider the case in which $|h|$ is small as a perturbation of the case $h=0$. This can only make sense if we add one more condition to our assumptions on the dynamics. We will suppose that $F(\cdot)$ is continuous. This implies that the transition probabilities $p^{\mathrm{Int}}_{\beta,h}(\eta,\zeta)$ and $p^{\mathrm{Sh}}_{\beta,h}(s',s'')$ are continuous functions of h. This extra assumption is verified for the examples (1.5) and (1.6).

Recall the definition of the total variation distance between two probability measures ν' and ν'' on $\mathcal{S}$:

$$||\nu'-\nu''||_{\mathrm{TV}} \;=\; \sum_{s\in\mathcal{S}} |\nu'(s)-\nu''(s)|.$$

Theorem 1.2 (The shape Markov chain under a small external field). *Suppose that F is continuous. For each $\beta\in(0,\infty)$ there is $h(G,\beta)>0$ such that the following statements hold.*

(a) *There is $\gamma_\beta>0$ such that for all $s\in\mathcal{S}$ there exists $C_{\beta,s}\in(0,\infty)$ such that for all $|h|<h(G,\beta)$,*

$$\mathbb{P}^{\mathrm{Sh}}_{\beta,h;s}(N_1>n) \;\leq\; C_{\beta,s}\,e^{-\gamma_\beta n}, \tag{1.26}$$

for $n = 0, 1, \ldots$. In particular, when $|h| < h(G,\beta)$ the shape Markov chain $(S_n)_{n=0,\ldots}$ is positive recurrent, with unique invariant probability distribution denoted by $\nu_{\beta,h}$.

(b) *For each $\beta \in (0,\infty)$ and $|h| < h(G,\beta)$, $(S_n)_{n=0,1,\ldots}$ converges exponentially fast to equilibrium in the sense that there is $\gamma_{\beta,h} > 0$ such that for each $s \in \mathcal{S}$ there is $C_{\beta,h,s} < \infty$ such that*

$$||\mathbb{P}^{\mathrm{Sh}}_{\beta,h;s}(S_n \in \cdot) - \nu_{\beta,h}(\cdot)||_{\mathrm{TV}} \;\leq\; C_{\beta,h,s}\, e^{-\gamma_{\beta,h} n},$$

for $n = 0, 1, \ldots$.

(c) *For each $\beta \in (0,\infty)$, the distribution $\nu_{\beta,h}$ is weakly continuous in $h \in (-h(G,\beta), h(G,\beta))$, i.e., for all $s \in \mathcal{S}$,*

$$\lim_{h' \to h} \nu_{\beta,h'}(s) \;=\; \nu_{\beta,h}(s).$$

(d) *For each $\beta \in (0,\infty)$ and uniformly in $h \in (-h(G,\beta), h(G,\beta))$, $\nu_{\beta,h}$ has exponentially decaying tails, in the sense that there are $\bar{C}_\beta, \bar{\gamma}_\beta \in (0,\infty)$ such that*

$$\nu_{\beta,h}(\{s : \mathit{Range}(s) \geq k\}) \;\leq\; \bar{C}_\beta\, e^{-\bar{\gamma}_\beta k}, \tag{1.27}$$

for $k = 0, 1, \ldots$.

Theorem 1.3 (Movement of the interface under a small external field). *Suppose that F is continuous. If $|h| < h(G,\beta)$, then for any initial configuration η_0 of the interface process $(\eta_n)_{n=0,1,\ldots}$,*

$$\lim_{n\to\infty} \frac{M(n)}{n} = v^M_\beta(h) \quad a.s., \tag{1.28}$$

where

$$v^M_\beta(h) \;=\; \sum_{s\in\mathcal{S}} \nu_{\beta,h}(s)\, \mathbb{E}^{\mathrm{Sh}}_{\beta,h;s}(J_0) \;=\; \frac{\mathbb{E}_{\beta,h}(\Delta M)}{\mathbb{E}_{\beta,h}(\Delta N)} \;=\; \frac{\mathbb{E}_{\beta,h}(|\Delta M|\, \tanh(\beta\, h\, |\Delta M|\,/\,2))}{\mathbb{E}_{\beta,h}(\Delta N)}. \tag{1.29}$$

Moreover

$$\frac{dv^M_\beta}{dh}(0) \;=\; \frac{\beta}{2}\,(\sigma^{\mathrm{GK}}_\beta)^2. \tag{1.30}$$

Also,

$$\lim_{n\to\infty} \frac{\max_{v\in V}\{\eta_n(v)\}}{n} \;=\; \lim_{n\to\infty} \frac{\min_{v\in V}\{\eta_n(v)\}}{n} \;=\; \frac{v^M_\beta(h)}{|V|} \quad a.s.. \tag{1.31}$$

The quantity $v^M_\beta(h)$ is the long-time average velocity with which $M(n)$ changes with time n, while $(\sigma^{\mathrm{GK}}_\beta)^2$ is the diffusion coefficient which gives the long-time fluctuations of the interface in equilibrium in the absence of an external field. Therefore (1.30), which shows that for small $|h|$ this average velocity is roughly linear in h, with proportionality constant given by $(\beta/2)(\sigma_\beta^{\mathrm{GK}})^2$, is a version of the "Einstein relation" (see, e.g., [LR] and references therein).

There is an alternative way to express (1.24) and (1.30), which may seem more natural. Recall that while the interface process $(\eta_n)_{n=0,\dots}$ was introduced in discrete time for later convenience, it is related to a continuous time process in which spins on $\bar V$ try to flip at rate 1, but flips out of Σ^{SOS} are suppressed. Because there are exactly $2|V|$ spins that are allowed to flip at any time, if we define $t = n/(2|V|)$, then t approximates the time in the continuous time process. Note that (1.24) can be restated as

$$\frac{M(n\cdot)}{\sqrt{|V|}\,\sqrt{n/(2|V|)}} \;\to\; \sqrt{2}\,\sigma_\beta^{\mathrm{GK}} B(\cdot) \;=\; \sigma_\beta B(\cdot),$$

in distribution, as $n \to \infty$. The factor $\sqrt{n/(2|V|)}$ corresponds to the transformation of n into t and the extra factor $\sqrt{|V|}$ which appears now makes sense, in that the fluctuations of an interface of size of order $|V|$ should grow with $\sqrt{|V|}$, as $|V|$ grows.

As for the velocity of the interface, it is natural to locate the interface at $m(n) = M(n)/|V|$, and define its velocity then, when time is measured by t, as the following a.s. limit:

$$v_\beta(h) \;=\; \lim_{n\to\infty} \frac{m(n)}{n/(2|V|)} \;=\; 2\,v_\beta^M(h).$$

(Thanks to (1.31), the position of the interface in this definition could also be taken as $\max_{v\in V}\{\eta_n(v)\}$ or $\min_{v\in V}\{\eta_n(v)\}$, rather than $m(n)$.) Therefore (1.30) yields the Einstein relation for $v_\beta(h)$:

$$\mathcal{M}_\beta \;=\; \frac{dv_\beta}{dh}(0) \;=\; \beta\,(\sigma_\beta^{\mathrm{GK}})^2 \;=\; \frac{\beta}{2}\,\sigma_\beta^2.$$

$\mathcal{M}_\beta$ is called the mobility of the interface. Note that from Theorem 1.1 we learn that $\mathcal{M}_\beta \in (0,\infty)$.

Theorem 1.1 is proved in Section 2, while Theorem 1.2 and Theorem 1.3 are proved in Section 3.

2. The interface in the absence of external field

2.1. L^2 theory of the shape Markov chain when $h = 0$. When $h = 0$, the shape Markov chain $(S_n)_{n=0,1,\dots}$ is reversible and therefore it is natural to exploit its L^2 theory. For an introduction to the use of such techniques, see, e.g., [ST2] and references threrein. Define

$$L^2(\mathcal{S},\nu_{\beta,0}) \;=\; \left\{ f : \mathcal{S} \to \mathbb{C} \;:\; \sum_{s\in\mathcal{S}} |f(s)|^2 \nu_{\beta,0}(s) < \infty \right\},$$

where $\mathbb{C}$ is the set of complex numbers and for $z \in \mathbb{C}$, $|z|$ denotes its modulus. $L^2(\mathcal{S},\nu_{\beta,0})$ is a Hilbert space with inner product

$$\langle f, g\rangle \;=\; \sum_{s\in\mathcal{S}} \bar f(s) g(s) \nu_{\beta,0}(s),$$

where for $z \in \mathbb{C}$, $\bar{z}$ denotes its complex conjugate. For $f \in L^2(\mathcal{S}, \nu_{\beta,0})$, set

$$(\mathcal{P}_{\beta,0} f)(s) \;=\; \sum_{s' \in \mathcal{S}} p^{\mathrm{Sh}}_{\beta,0}(s,s')\, f(s').$$

Then $\mathcal{P}_{\beta,0}$ is a self-adjoint contraction, so that in particular its spectrum lies in the real number interval $[-1,1]$. Since the shape Markov chain is irreducible, the number 1 is a simple eigenvalue, corresponding to the eigenspace of constant functions. Define

$$(\Pi_{\beta,0} f)(s) \;=\; \sum_{s' \in \mathcal{S}} f(s) \nu_{\beta,0}(s).$$

Then $\Pi_{\beta,0}$ is a self-adjoint projection on the space of constant functions. The operator $\mathcal{P}_{\beta,0} - \Pi_{\beta,0}$ is identically 0 on the space of constant functions and is identical to $\mathcal{P}_{\beta,0}$ on the orthogonal complement of that space. Set

$$R_\beta \;=\; \sup\{|r| : r \in \mathrm{spec}(\mathcal{P}_{\beta,0}) \backslash \{1\}\} \;=\; \sup\{|r| : r \in \mathrm{spec}(\mathcal{P}_{\beta,0} - \Pi_{\beta,0})\},$$

where $\mathrm{spec}(\mathcal{Q})$ denotes the spectrum of the operator $\mathcal{Q}$.

The main technical step in this paper will be the proof of the following.

Theorem 2.1. *For any $\beta > 0$,*

$$R_\beta < 1.$$

Proof. The separation of the spectrum from -1 is related to the aperiodicity of the shape Markov chain in a standard way. Thanks to Lemma 1.1, we can write

$$\mathcal{P}_{\beta,0} = \alpha I + (1-\alpha)\mathcal{P}'_{\beta,0},$$

where I is the identity operator, $\mathcal{P}'_{\beta,0}$ is also a self-adjoint contraction (so that its spectrum is also contained in $[-1,+1]$) and $0 < \alpha \le 1$. Therefore

$$\inf \mathrm{spec}(\mathcal{P}_{\beta,0}) \;>\; -1. \tag{2.1}$$

Set

$$\mathrm{gap}(\mathcal{P}_{\beta,0}) \;=\; \sup\left(\mathrm{spec}(\mathcal{P}_{\beta,0}) \backslash \{1\}\right).$$

$\mathrm{gap}(\mathcal{P}_{\beta,0})$ is usually referred to as the "spectral gap" of $\mathcal{P}_{\beta,0}$. Thanks to (2.1), the proof of the theorem will be complete once we show that

$$\mathrm{gap}(\mathcal{P}_{\beta,0}) > 0. \tag{2.2}$$

For this purpose we use the following Cheeger type inequality from [LS] (Theorem 2.1 and remark on first paragraph of p.560 about the discrete time case):

$$\mathrm{gap}(\mathcal{P}_{\beta,0}) \;\ge\; C\, (\mathcal{C}_\beta)^2\,, \tag{2.3}$$

where C is a positive constant and

$$\mathcal{C}_\beta \;=\; \inf_{\substack{A\subset\mathcal{S}\\ 0<\nu_{\beta,0}(A)<1}} \frac{\sum_{\substack{s'\in A\\ s''\in A^c}} \nu_{\beta,0}(s')\,p^{\mathrm{Sh}}_{\beta,0}(s',s'')}{\nu_{\beta,0}(A)\,\nu_{\beta,0}(A^c)}$$

is usually referred to as the "Cheeger constant". To estimate $\mathcal{C}_\beta$, we will use a technique from Jerrum and Sinclair based on choosing paths in $\mathcal{S}$ (see, e.g., the proof of Theorem 4.2 of [JS], or Proposition 7 of [DS]). This will take us on a long sequence of definitions and estimates.

A path in $\mathcal{S}$ from s' to s'' is a sequence $(s_1, s_2, ..., s_n)$ of elements of $\mathcal{S}$, with $s_i \sim s_{i+1}$ for $i = 1, ..., n-1$, $s_1 = s'$, $s_n = s''$. Given a path $\pi = (s_1, s_2, ..., s_n)$, we will denote by $\overleftarrow{\pi} = (s_n, s_{n-1}, ..., s_1)$ the reversed path. Given paths $\pi' = (s'_1, s'_2, ..., s'_m)$ and $\pi'' = (s''_1, s''_2, ..., s''_n)$ with $s'_m = s''_1$, we will denote by $\pi'\circ\pi'' = (s'_1, s'_2, ..., s'_{m-1}, s'_m, s''_2, s''_3, ...s''_m)$ the concatenated path.

Set

$$\mathcal{E} \;=\; \{\{s', s''\} \;:\; s', s'' \in \mathcal{S},\, s' \sim s''\}.$$

Given a path $\pi = (s_1, s_2, ..., s_n)$ and $e \in \mathcal{E}$, we will, in an abuse of notation, write $e \in \pi$ in case $e = \{s_i, s_{i+1}\}$ for some $i \in \{1, ..., n-1\}$. Given $\{s', s''\} \in \mathcal{E}$, set

$$Q(\{s', s''\}) \;=\; \nu_{\beta,0}(s')\; p^{\mathrm{Sh}}_{\beta,h}(s', s'') \;=\; \nu_{\beta,0}(s'')\; p^{\mathrm{Sh}}_{\beta,h}(s'', s'),$$

where the second equality is the reversibility equation (1.18).

Suppose that for any pair of distinct $s', s'' \in \mathcal{S}$, we have chosen a path $\pi_{s',s''}$, from s' to s'', with the only restriction that $\pi_{s'',s'} = \overleftarrow{\pi}_{s',s''}$. For $e \in \mathcal{E}$ set then

$$w(e) \;=\; \sum_{\{s',s''\}\,:\,e\in\pi_{s',s''}} \nu_{\beta,0}(s')\,\nu_{\beta,0}(s'').$$

(In this sum each unordered pair s', s'' is taken only once. Note that the condition $e \in \pi_{s',s''}$ is equivalent to $e \in \pi_{s'',s'}$.) Define now

$$\mathcal{J}_\beta \;=\; \sup_{e\in\mathcal{E}} \frac{w(e)}{Q(e)}.$$

We will show next that

$$\mathcal{C}_\beta \;\geq\; \frac{1}{\mathcal{J}_\beta}. \tag{2.4}$$

Indeed, for $A \subset \mathcal{S}$, set $\partial A = \{\{s', s''\} \in \mathcal{E} : s' \in A, s'' \in A^c\}$, and write

$$\begin{aligned}\nu_{\beta,0}(A)\,\nu_{\beta,0}(A^c) \;&=\; \sum_{\substack{s'\in A\\ s''\in A^c}} \nu_{\beta,0}(s')\,\nu_{\beta,0}(s'') \;\leq\; \sum_{e\in\partial A}\; \sum_{\substack{s',s'':\\ e\in\pi_{s',s''}}} \nu_{\beta,0}(s')\,\nu_{\beta,0}(s'') \\ &=\; \sum_{e\in\partial A} w(e) \;\leq\; \mathcal{J}_\beta \sum_{e\in\partial A} Q(e) \;=\; \mathcal{J}_\beta \sum_{\substack{s'\in A\\ s''\in A^c}} \nu_{\beta,0}(s')\,p^{\mathrm{Sh}}_{\beta,0}(s', s''),\end{aligned}$$

which implies (2.4).

To use (2.4), we will choose the paths $\pi_{s',s''}$ in a convenient way, that we present next.

Recall the definitions of Rep$[s]$ and of Range(s), from the proof of Lemma 1.3, and define also the volume

$$\mathrm{Vol}(s) \;=\; \sum_{v\in V} (\mathrm{Rep}[s])(v).$$

Note that for each $s \in \mathcal{S}$, $\mathrm{Vol}(s) \geq 0$, and that $\mathrm{Vol}(s) = 0$ iff $s = \phi$. We will define a tranformation $D : \mathcal{S}\backslash\{\phi\} \to \mathcal{S}$ which decreases the volume of a shape by one unit "moving it towards ϕ" in a convenient fashion. (D is for "down".) For this purpose, order the vertices in V in an arbitrary fashion. Given $s \in \mathcal{S}$, set

$$V^{\mathrm{top}}(s) \;=\; \{v \in V \;:\; (\mathrm{Rep}[s])(v) = \mathrm{Range}(s)\},$$

and let $v_1(s)$ be the first element of $V^{\mathrm{top}}(s)$. Define now

$$D(s) \;=\; \mathrm{Sh}\left[(\mathrm{Rep}[s])^{v_1(s),\downarrow}\right].$$

Observe that for any $s \in \mathcal{S}\backslash\{\phi\}$,

$$\mathrm{Vol}(D(s)) = \mathrm{Vol}(s) - 1, \qquad D(s) \sim s.$$

Therefore

$$\pi_{s,\phi} \;=\; (s, D(s), D^2(s), ..., D^{\mathrm{Vol}(s)}(s))$$

defines a path in $\mathcal{S}$ from s to ϕ. Define also

$$\pi_{\phi,s} \;=\; \overleftarrow{\pi}_{s,\phi}.$$

And for each pair of distinct $s', s'' \in \mathcal{S}\backslash\{\phi\}$, define

$$\pi_{s',s''} \;=\; \pi_{s',\phi} \circ \pi_{\phi,s''}.$$

Set

$$\mathcal{E}^D \;=\; \{\{s', s''\} \in \mathcal{E} :\; s'' = D(s')\},$$

and observe that $w(e) = 0$, unless $e \in \mathcal{E}^D$. Therefore,

$$\mathcal{J}_\beta \;=\; \sup_{e\in\mathcal{E}^D} \frac{w(e)}{Q(e)}. \tag{2.5}$$

Given $e = \{s', s''\} \in \mathcal{E}^D$, such that $s'' = D(s')$, set $e^\uparrow = s'$ and $e^\downarrow = s''$. (Since D reduces the volume, there is no possibility that also $s' = D(s'')$ here.) Define also

$$\mathcal{A}_e \;=\; \{s \in \mathcal{S} \;:\; D^k(s) = e^\uparrow \text{ for some } k \geq 0\}.$$

(We are using the standard notation in which D^0 is the identity. $\mathcal{A}$ is for "above".) Note that if $e \in \pi_{s',s''}$, then s' or s'' must be in $\mathcal{A}_e$. Hence, for $e \in \mathcal{E}^D$,

$$w(e) \;=\; \sum_{\{s',s''\}\,:\,e\in\pi_{s',s''}} \nu_{\beta,0}(s')\,\nu_{\beta,0}(s'') \;\leq\; \sum_{\substack{s'\in\mathcal{A}_e\\ s''\in\mathcal{S}}} \nu_{\beta,0}(s')\,\nu_{\beta,0}(s'') \;=\; \nu_{\beta,0}(\mathcal{A}_e). \quad (2.6)$$

It follows from (1.17) that for some $C(\beta) > 0$,

$$Q(e) \;\geq\; C(\beta)\,\nu_{\beta,0}(e^{\uparrow}),$$

for all $e \in \mathcal{E}^D$. Combining this bound with (2.5) and (2.6), we obtain

$$\mathcal{J}_\beta \;\leq\; \frac{1}{C(\beta)} \sup_{e\in\mathcal{E}^D} \frac{\nu_{\beta,0}(\mathcal{A}_e)}{\nu_{\beta,0}(e^{\uparrow})}. \quad (2.7)$$

To estimate

$$\nu_{\beta,0}(\mathcal{A}_e) \;=\; \frac{1}{Z^{\mathrm{Sh}}_{\beta,0}} \sum_{s\in\mathcal{A}_e} \exp\left(-\beta\mathcal{H}_0^{\mathrm{Int}}(\mathrm{Rep}[s])\right), \quad (2.8)$$

we will show that for any $s \in \mathcal{A}_e$,

$$\mathcal{H}_0^{\mathrm{Int}}(\mathrm{Rep}[s]) \;\geq\; \mathcal{H}_0^{\mathrm{Int}}(\mathrm{Rep}[e^{\uparrow}]) - |E| + \mathrm{Range}(s) - \mathrm{Range}(e^{\uparrow}). \quad (2.9)$$

(Recall that E is the edge set of $G = (V, E)$.) To this end, we first observe that if $s \in \mathcal{A}_e$, then for each $v \in V$,

$$(\mathrm{Rep}[s])(v) \;=\; e^{\uparrow}(v) \quad \text{if} \quad 0 \leq e^{\uparrow}(v) \leq \mathrm{Range}(e^{\uparrow}) - 2, \quad (2.10)$$

and

$$(\mathrm{Rep}[s])(v) \;\in\; \{e^{\uparrow}(v), ..., \mathrm{Range}(s)\} \quad \text{if} \quad e^{\uparrow}(v) \geq \mathrm{Range}(e^{\uparrow}) - 1. \quad (2.11)$$

For an arbitrary $\eta \in \mathbb{Z}^V$, write

$$\mathcal{H}_0^{\mathrm{Int}}(\eta) \;=\; \sum_{i\in\mathbb{Z}} \mathcal{H}_0^{\mathrm{Int},i}(\eta),$$

where

$$\mathcal{H}_0^{\mathrm{Int},i}(\eta) \;=\; \sum_{\{v',v''\}\in E} \mathbb{1}_{\{\eta(v')<i\leq\eta(v'')\ \text{or}\ \eta(v'')<i\leq\eta(v')\}}.$$

(Here $\mathbb{1}_{\cdot}$ is the indicator function.) Using (2.10) and (2.11), we see that

$$\mathcal{H}_0^{\mathrm{Int},i}(\mathrm{Rep}[s]) = \mathcal{H}_0^{\mathrm{Int},i}(e^{\uparrow}) \quad \text{if} \quad i \leq \mathrm{Range}(e^{\uparrow}) - 1 \text{ or } i \geq \mathrm{Range}(s) + 1,$$

and

$$\mathcal{H}_0^{\mathrm{Int},i}(\mathrm{Rep}[s]) \geq 1 = \mathcal{H}_0^{\mathrm{Int},i}(e^{\uparrow}) + 1 \quad \text{if} \quad \mathrm{Range}(e^{\uparrow}) + 1 \leq i \leq \mathrm{Range}(s).$$

The inequality (2.9) follows from these two statements and the trivial bound

$$\mathcal{H}_0^{\mathrm{Int},i}(\mathrm{Rep}[s]) \geq \mathcal{H}_0^{\mathrm{Int},i}(e^{\uparrow}) - |E| \quad \text{if} \quad i = \mathrm{Range}(e^{\uparrow}).$$

From (2.8) and (2.9), we obtain

$$\begin{aligned} \nu_{\beta,0}(\mathcal{A}_e) &\leq \nu_{\beta,0}(e^{\uparrow})\ e^{\beta|E|} \sum_{s\in\mathcal{A}_e} \exp\left(-\beta(\mathrm{Range}(s) - \mathrm{Range}(e^{\uparrow}))\right) \\ &= \nu_{\beta,0}(e^{\uparrow})\ e^{\beta|E|} \sum_{k=0}^{\infty} e^{-\beta k} \left|\{s \in \mathcal{A}_e : \mathrm{Range}(s) - \mathrm{Range}(e^{\uparrow}) = k\}\right| \\ &\leq \nu_{\beta,0}(e^{\uparrow})\ e^{\beta|E|} \sum_{k=0}^{\infty} e^{-\beta k}(k+1)^{|V|} \ \leq\ C(|V|,|E|,\beta)\ \nu_{\beta,0}(e^{\uparrow}), \end{aligned} \tag{2.12}$$

where $C(|V|,|E|,\beta) < \infty$. Here we used that (2.10) and (2.11) imply

$$\begin{aligned} &\{s \in \mathcal{A}_e : \mathrm{Range}(s) - \mathrm{Range}(e^{\uparrow}) = k\} \\ &\quad \subset\ \{s \in \mathcal{S} : (\mathrm{Rep}[s])(v) \in \{(\mathrm{Rep}[e^{\uparrow}])(v), ..., (\mathrm{Rep}[e^{\uparrow}])(v) + k\} \text{ for all } v \in V\}, \end{aligned}$$

and the cardinality of that latter set is clearly bounded above by $(k+1)^{|V|}$.

Combining (2.3), (2.4), (2.7) and (2.12), we obtain (2.2), finishing the proof. ■

It may be of interest to point out that for certain contour models which were also motivated by stochastic Ising model interfaces, it was shown in [ST1] that the analogue of Theorem 2.1 fails. The main reason for this difference is the fact that our interfaces are constrained to move inside the "tube" $\bar{V}$ of constant "width" $|V|$, while the contours of [ST1] can grow in all directions.

Denote the $L^2(\mathcal{S},\nu_{\beta,0})$-norm of $f \in L^2(\mathcal{S},\nu_{\beta,0})$ by

$$||f||_{L^2(\mathcal{S},\nu_{\beta,0})} \ =\ \sqrt{\langle f,f\rangle} \ =\ \sqrt{\sum_{s\in\mathcal{S}} |f(s)|^2 \nu_{\beta,0}(s)}.$$

Then for arbitrary n,

$$\begin{aligned} \left|\left|(\mathcal{P}_{\beta,0}^{\mathrm{Sh}})^n(f) - \sum_{s\in\mathcal{S}} f(s)\nu_{\beta,0}(s)\right|\right|_{L^2(\mathcal{S},\nu_{\beta,0})} &= \left|\left|(\mathcal{P}_{\beta,0}^{\mathrm{Sh}} - \Pi_{\beta,0})^n \left(f - \sum_{s\in\mathcal{S}} f(s)\nu_{\beta,0}(s)\right)\right|\right|_{L^2(\mathcal{S},\nu_{\beta,0})} \\ &\leq \left|\left|f - \sum_{s\in\mathcal{S}} f(s)\nu_{\beta,0}(s)\right|\right|_{L^2(\mathcal{S},\nu_{\beta,0})} (R_\beta)^n. \end{aligned} \tag{2.13}$$

Therefore, Theorem 2.1 implies that the shape Markov chain converges to equilibrium exponentially fast in L^2. It is known (see [RT] and references therein) that this is equivalent to exponentially fast convergence to equilibrium in the total variation norm, i.e., there is $\gamma'_\beta \in (0,\infty)$ such that for all $s \in \mathcal{S}$ there is $C'_{\beta,s} \in (0,\infty)$ such that

$$\left|\left|\mathbb{P}^{\mathrm{Sh}}_{\beta,0;s}(S_n \in \cdot) - \nu_{\beta,0}(\cdot)\right|\right|_{\mathrm{TV}} \le C'_{\beta,s}\, e^{-\gamma'_\beta n}, \tag{2.14}$$

for $n = 0, 1, \ldots.$

Furthermore, it is also known (see Proposition 2.11 of [Twe], or Theorem 15.0.1 in [MT] for a more general result) that (2.14) is equivalent to the statement that the time of first visit to an arbitrary state of the shape Markov chain has exponentially decaying tails. More precisely, applied to the state ϕ, this statement guarantees that there is $\gamma''_\beta \in (0,\infty)$ such that for all $s \in \mathcal{S}$ there is $C''_{\beta,s} \in (0,\infty)$ such that

$$\mathbb{P}^{\mathrm{Sh}}_{\beta,0;s}(N_1 > n) \le C''_{\beta,s}\, e^{-\gamma''_\beta n}, \tag{2.15}$$

for $n = 0, 1, \ldots.$ (For later use, we point out that this equivalence does not require reversibility.)

2.2. Proof of Theorem 1.1. Let $(S_n)_{n\in\mathbb{Z}}$ be a two-sided stationary version of the shape Markov chain, so that, in particular, S_n has law $\nu_{\beta,0}$ for each $n \in \mathbb{Z}$. We will denote by $\mathbb{P}^{\mathrm{Sh}}_{\beta,0}$ the law of this two-sided process, and by $\mathbb{E}^{\mathrm{Sh}}_{\beta,0}$ the corresponding expectation.

Because the shape Markov chain is irreducible and aperiodic, the stationary sequence $(S_n)_{n\in\mathbb{Z}}$ is ergodic (see [Dur], Example 4.2 of Chapter 6, p.352). Note that if we define J_n, $n \in \mathbb{Z}$, via (1.22), then $(\ldots, J_{-1}, J_0, J_1, \ldots)$ is also stationary and ergodic (see [Dur], Theorem 1.3 of Chapter 6, p. 340), and, by symmetry, $\mathbb{E}^{\mathrm{Sh}}_{\beta,0}(J_n) = 0$. Set $\mathcal{F}_n = \sigma(\ldots, S_n, S_{n+1})$. Then for each n, J_n is $\mathcal{F}_n$-measurable. We want to use Theorem 7.6 of Chapter 7 of [Dur] (p.419) applied to the sequence $(\ldots, J_{-1}, J_0, J_1, \ldots)$ in order to obtain a functional central limit theorem for $M(n) = J_1 + \ldots + J_n$. For this purpose we need to show that

$$\sum_{n\ge1}\sqrt{E^{\mathrm{Sh}}_{\beta,0}\left(E^{\mathrm{Sh}}_{\beta,0}(J_0|\mathcal{F}_{-n})\right)^2} < \infty. \tag{2.16}$$

Set $f(s) = E^{\mathrm{Sh}}_{\beta,0;s}(J_0)$. We can use the Markov property of $(S_n)_{n\in\mathbb{Z}}$ to write

$$\begin{aligned} E^{\mathrm{Sh}}_{\beta,0}(J_0|\mathcal{F}_{-n}) &= E^{\mathrm{Sh}}_{\beta,0}\left(E^{\mathrm{Sh}}_{\beta,0}(J_0|\mathcal{F}_{-1})|\mathcal{F}_{-n}\right) = E^{\mathrm{Sh}}_{\beta,0}\left(E^{\mathrm{Sh}}_{\beta,0}(J_0|S_0)|\mathcal{F}_{-n}\right) \\ &= E^{\mathrm{Sh}}_{\beta,0}\left(f(S_0)|S_{-n+1}\right) = \left(\left(\mathcal{P}^{\mathrm{Sh}}_{\beta,0}\right)^{n-1}(f)\right)(S_{-n+1}). \end{aligned}$$

Since S_{-n+1} has law $\nu_{\beta,0}$,

$$\sum_{n\ge1}\sqrt{E^{\mathrm{Sh}}_{\beta,0}\left(E^{\mathrm{Sh}}_{\beta,0}(J_0|\mathcal{F}_{-n})\right)^2} = \sum_{n\ge1}\left|\left|(\mathcal{P}^{\mathrm{Sh}}_{\beta,0})^{n-1}(f)\right|\right|_{L^2(\mathcal{S},\nu_{\beta,0})} \le \sum_{n\ge1}(R_\beta)^{n-1},$$

from (2.13), since $\sum_{s\in\mathcal{S}} f(s)\nu_{\beta,0}(s) = 0$ and $||f||_{L^2(\mathcal{S})} \le 1$. Therefore (2.16) follows from Theorem 2.1.

We can now apply the aforementioned Theorem 7.6 of Chapter 7 of [Dur], combined with the remark in the paragraph with display (1.22), to conclude that (1.24) holds, with

$$\begin{aligned}(\sigma_\beta^{\mathrm{GK}})^2 &= \sum_{n\in\mathbb{Z}} \mathbb{E}^{\mathrm{Sh}}_{\beta,0}(J_0\,J_n) = \mathbb{E}^{\mathrm{Sh}}_{\beta,0;\nu_{\beta,h}}((J_0)^2) + 2\sum_{n=1}^{\infty} \mathbb{E}^{\mathrm{Sh}}_{\beta,0;\nu_{\beta,h}}(J_0\,J_n) \\ &= \mathbb{E}^{\mathrm{Int}}_{\beta,0;\mu}((J_0)^2) + 2\sum_{n=1}^{\infty} \mathbb{E}^{\mathrm{Int}}_{\beta,0;\mu}(J_0\,J_n) < \infty.\end{aligned}$$

To prove that also

$$(\sigma_\beta^{\mathrm{GK}})^2 = \frac{\mathbb{E}_{\beta,0}((\Delta M)^2)}{\mathbb{E}_{\beta,0}(\Delta N)} \tag{2.17}$$

it is now enough to show that

$$\frac{M(n)}{\sqrt{n}} \to \sqrt{\frac{\mathbb{E}_{\beta,0}((\Delta M)^2)}{\mathbb{E}_{\beta,0}(\Delta N)}}\, Z, \tag{2.18}$$

in distribution, as $n \to \infty$, where Z has a standard normal distribution. To derive (2.18), recall the definitions of N_k, $(\Delta N)_k$ and $(\Delta M)_k$, from Subsection 1.4, set $K(n) = \sup\{k : N_k \le n\}$, and write

$$n = (\Delta N)_0 + (\Delta N)_1 + ... + (\Delta N)_{K(n)-1} + (n - N_{K(n)}),$$

$$M(n) = (\Delta M)_0 + (\Delta M)_1 + ... + (\Delta M)_{K(n)-1} + (M(n) - M(N_{K(n)})).$$

Because the shape Markov chain is recurrent, $|(\Delta M)_0| \le (\Delta N)_0 < \infty$ a.s.. Recall that

$$(\Delta N)_1, (\Delta N)_2, ... \quad \text{and} \quad (\Delta M)_1, (\Delta M)_2, ...$$

are both i.i.d. sequences. In the former the common law is that of ΔN and in the latter that of ΔM. From (2.15) and (1.23), we learn that ΔN and ΔM have finite second moments. By symmetry, $\mathbb{E}_{\beta,0}(\Delta M) = 0$. The proof of (2.18) from these facts is standard, being based on a random index central limit theorem and on the estimates

$$|M(n) - M(N_{K(n)})| \le n - N_{K(n)} \le \sup_{k=0}^{n} (\Delta N)_k,$$

and

$$\frac{\sup_{k=0}^{n}(\Delta N)_k}{\sqrt{n}} \to 0,$$

in probability, as $n \to \infty$. (See exercise 5.6 of Chapter 5 of [Dur], p.324, for details).

Finally note that since, clearly, $\mathbb{E}_{\beta,0}((\Delta M)^2) > 0$, the claim that $\sigma_\beta^{\mathrm{GK}} > 0$ follows from (2.17) and the finiteness of $\mathbb{E}_{\beta,0}(\Delta N)$.

3. The interface under a small external field

3.1 Proof of Theorem 1.2. First we claim that for fixed $\beta \in (0,\infty)$ and $h \in \mathbb{R}$

$$\lim_{h' \to h} p^{\mathrm{Int}}_{\beta,h'}(\eta,\zeta) \;=\; p^{\mathrm{Int}}_{\beta,h}(\eta,\zeta), \tag{3.1}$$

uniformly over $\eta,\zeta \in \mathbb{Z}^V$. To see this, first note that for each $\eta,\zeta \in \mathbb{Z}^V$ (3.1) is a consequence of the assumed continuity of F. Uniformity over $\eta \sim \zeta$ follows from the finiteness of the set in display (1.10). Uniformity over all pairs $\eta,\zeta \in \mathbb{Z}^V$ now follows from the fact that $p^{\mathrm{Int}}_{\beta,h}(\eta,\zeta) = 0$, unless $\eta \sim \zeta$ or $\eta = \zeta$, from (1.9), and from the fact that for each $\eta \in \mathbb{Z}^V$, $|\{\zeta \in \mathbb{Z}^V : \zeta \sim \eta\}| \;=\; 2|V|$.

Thanks to (1.15) and (1.16), a similar statement holds for the shape Markov chain: For fixed $\beta \in (0,\infty)$ and $h \in \mathbb{R}$,

$$\lim_{h' \to h} p^{\mathrm{Sh}}_{\beta,h'}(s',s'') \;=\; p^{\mathrm{Sh}}_{\beta,h}(s',s''), \tag{3.2}$$

uniformly over $s', s'' \in \mathcal{S}$.

As a consequence of (3.2), we have for every $n \in \{1,2,...\}$ and $A \subset \mathcal{S}^n$,

$$\begin{aligned}\lim_{h' \to h} \mathbb{P}^{\mathrm{Sh}}_{\beta,h';s}((S_1,...,S_n) \in A) \;&=\; \lim_{h' \to h} \sum_{(s^1,...,s^n) \in A} p^{\mathrm{Sh}}_{\beta,h'}(s,s^1)\, p^{\mathrm{Sh}}_{\beta,h'}(s^1,s^2) \cdots p^{\mathrm{Sh}}_{\beta,h'}(s^{n-1},s^n) \\ &=\; \mathbb{P}^{\mathrm{Sh}}_{\beta,h;s}((S_1,...,S_n) \in A),\end{aligned} \tag{3.3}$$

where we used the fact that the sum over $(s^1,...,s^n)$ is a finite sum (with at most $(2|V|+1)^n$ terms).

Also, from (3.2) and (1.17),

$$\lim_{h' \to h} \frac{p^{\mathrm{Sh}}_{\beta,h'}(s',s'')}{p^{\mathrm{Sh}}_{\beta,h}(s',s'')} \;=\; 1, \tag{3.4}$$

uniformly, over $s' \approx s''$.

Given $\epsilon > 0$, (3.4) and (1.16) guarantee that for small enough $|h|$,

$$\begin{aligned}\mathbb{P}^{\mathrm{Sh}}_{\beta,h;s}(N_1 > n) \;&=\; \sum_{s^1,...,s^n \neq \phi} p^{\mathrm{Sh}}_{\beta,h}(s,s^1)\, p^{\mathrm{Sh}}_{\beta,h}(s^1,s^2) \cdots p^{\mathrm{Sh}}_{\beta,h}(s^{n-1},s^n) \\ &\le\; \sum_{s^1,...,s^n \neq \phi} (1+\epsilon)^n\, p^{\mathrm{Sh}}_{\beta,0}(s,s^1)\, p^{\mathrm{Sh}}_{\beta,0}(s^1,s^2) \cdots p^{\mathrm{Sh}}_{\beta,0}(s^{n-1},s^n) \\ &=\; (1+\epsilon)^n\, \mathbb{P}^{\mathrm{Sh}}_{\beta,0;s}(N_1 > n) \;\le\; (1+\epsilon)^n\, C''_{\beta,s}\, e^{-\gamma''_\beta n},\end{aligned}$$

where, in the last step, we used (2.15). A proper choice of ϵ leads to (1.26), proving the statement in (a).

As pointed out when (2.15) was derived, it is known that (1.26) implies the statement in (b).

To prove the weak convergence claimed in (c), we use a "cycle trick" representation of $\nu_{\beta,h}$ (see, e.g., Sect. 5.4 of [Dur]). For each $s \in \mathcal{S}$,

$$\nu_{\beta,h}(s) \;=\; \frac{\sum_{n=0}^{\infty} \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(S_n = s, N_1 > n)}{\sum_{n=0}^{\infty} \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(N_1 > n)}. \tag{3.5}$$

From (3.3) we know that each term in the series in the numerator or the denominator is continuous in h. From (1.26) we learn that these series converge uniformly over $h \in (-h(G,\beta), h(G,\beta))$. Therefore numerator and denominator are continuous functions of $h \in (-h(G,\beta), h(G,\beta))$, and since the denominator is bounded below by 1 and hence does not vanish, we conclude that $\nu_{\beta,h}(s)$ is also continuous in $h \in (-h(G,\beta), h(G,\beta))$, as claimed.

To prove (1.27), we use (3.5) again. Note that if $S_0 = \phi$, then $\mathrm{Range}(S_n) \geq k$ can only occur when $n \geq k$. Hence, using (1.26),

$$\begin{aligned} \nu_{\beta,h}(\{s : \mathrm{Range}(s) \geq k\}) \;&=\; \frac{\sum_{n=k}^{\infty} \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(\mathrm{Range}(S_n) \geq k, N_1 > n)}{\sum_{n=0}^{\infty} \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(N_1 > n)} \\ &\leq\; \sum_{n=k}^{\infty} \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(N_1 > n) \;\leq\; \sum_{n=k}^{\infty} C_{\beta,\phi}\, e^{-\gamma_\beta n} \;\leq\; \bar{C}_\beta\, e^{-\bar{\gamma}_\beta k}. \end{aligned}$$

3.2 Proof of Theorem 1.3. The claim (1.28), with

$$v^M_\beta(h) \;=\; \frac{\mathbb{E}_{\beta,h}(\Delta M)}{\mathbb{E}_{\beta,h}(\Delta N)},$$

can be proved in a standard way, by methods similar to, but simpler than those used to prove (2.18) (see exercise 5.5 of Chapter 5 of [Dur], p. 323).

Since $|M(n)| \leq n$, (1.28), the dominated convergence theorem and (1.20) now imply

$$\begin{aligned} v^M_\beta(h) \;&=\; \lim_{n\to\infty} \mathbb{E}^{\mathrm{Sh}}_{\beta,h;\nu_{\beta,h}}\left(\frac{M(n)}{n}\right) \;=\; \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}^{\mathrm{Sh}}_{\beta,h;\nu_{\beta,h}}(J_k) \\ &=\; \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{s\in\mathcal{S}} \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\nu_{\beta,h}}(S_k = s)\, \mathbb{E}^{\mathrm{Sh}}_{\beta,h;s}(J_0) \\ &=\; \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{s\in\mathcal{S}} \nu_{\beta,h}(s)\, \mathbb{E}^{\mathrm{Sh}}_{\beta,h;s}(J_0) \;=\; \sum_{s\in\mathcal{S}} \nu_{\beta,h}(s)\, \mathbb{E}^{\mathrm{Sh}}_{\beta,h;s}(J_0). \end{aligned}$$

To obtain the last equality in (1.29), it suffices to show

$$\mathbb{E}_{\beta,h}(\Delta M) \;=\; \mathbb{E}_{\beta,h}(|\Delta M|\, \tanh(\beta\, h\, |\Delta M| \,/\, 2)). \tag{3.6}$$

Note that the l.h.s. of (3.6) can be written as

$$\mathbb{E}_{\beta,h}(\Delta M) \;=\; \sum_{k\in\mathbb{Z}} k\,\mathbb{P}_{\beta,h}(\Delta M = k) \;=\; \sum_{k\geq 1} k\,\{\mathbb{P}_{\beta,h}(\Delta M = k) - \mathbb{P}_{\beta,h}(\Delta M = -k)\}, \tag{3.7}$$

since this series converges absolutely, thanks to (1.23) and (1.26). Obviously $\mathbb{P}_{\beta,h}(\Delta M = k) \neq 0$ only if $k = j|V|$ for some $j \in \mathbb{Z}$. In this case we can use (1.14) to write:

$$\begin{aligned}
\mu^{\rm Int}_{\beta,h}(\phi_0)\;\mathbb{P}_{\beta,h}(\Delta M = j|V|) \;&=\; \mu^{\rm Int}_{\beta,h}(\phi_0)\;\mathbb{P}^{\rm Int}_{\beta,h;\phi_0}(M(N_1) = j|V|)\\
&=\; \mu^{\rm Int}_{\beta,h}(\phi_0)\;\sum_{n=1}^{\infty}\mathbb{P}^{\rm Int}_{\beta,h;\phi_0}(N_1 = n,\, M(n) = j|V|)\\
&=\; \sum_{n=1}^{\infty}\;\sum_{\eta^1,\dots,\eta^{n-1}\not\ni\phi}\mu^{\rm Int}_{\beta,h}(\phi_0)\;p^{\rm Int}_{\beta,h}(\phi_0,\eta^1)p^{\rm Int}_{\beta,h}(\eta^1,\eta^2)\cdots p^{\rm Int}_{\beta,h}(\eta^{n-1},\phi_j)\\
&=\; \sum_{n=1}^{\infty}\;\sum_{\eta^1,\dots,\eta^{n-1}\not\ni\phi}p^{\rm Int}_{\beta,h}(\eta^1,\phi_0)\mu^{\rm Int}_{\beta,h}(\eta^1)\;p^{\rm Int}_{\beta,h}(\eta^1,\eta^2)\cdots p^{\rm Int}_{\beta,h}(\eta^{n-1},\phi_j)\\
&=\; \sum_{n=1}^{\infty}\;\sum_{\eta^1,\dots,\eta^{n-1}\not\ni\phi}p^{\rm Int}_{\beta,h}(\eta^1,\phi_0)p^{\rm Int}_{\beta,h}(\eta^2,\eta^1)\mu^{\rm Int}_{\beta,h}(\eta^2)\;\cdots p^{\rm Int}_{\beta,h}(\eta^{n-1},\phi_j)\\
&=\;\cdots\\
&=\; \sum_{n=1}^{\infty}\;\sum_{\eta^1,\dots,\eta^{n-1}\not\ni\phi}p^{\rm Int}_{\beta,h}(\eta^1,\phi_0)p^{\rm Int}_{\beta,h}(\eta^2,\eta^1)\cdots p^{\rm Int}_{\beta,h}(\phi_j,\eta^{n-1})\;\mu^{\rm Int}_{\beta,h}(\phi_j)\\
&=\; \sum_{n=1}^{\infty}\;\sum_{\eta^1,\dots,\eta^{n-1}\not\ni\phi}\mu^{\rm Int}_{\beta,h}(\phi_j)\;p^{\rm Int}_{\beta,h}(\phi_j,\eta^{n-1})p^{\rm Int}_{\beta,h}(\eta^{n-1},\eta^{n-2})\cdots p^{\rm Int}_{\beta,h}(\eta^2,\eta^1)p^{\rm Int}_{\beta,h}(\eta^1,\phi_0)\\
&=\; \mu^{\rm Int}_{\beta,h}(\phi_j)\;\sum_{n=1}^{\infty}\mathbb{P}^{\rm Int}_{\beta,h;\phi_j}(N_1 = n,\, M(n) = -j|V|)\\
&=\; \mu^{\rm Int}_{\beta,h}(\phi_j)\;\mathbb{P}^{\rm Int}_{\beta,h;\phi_j}(M(N_1) = -j|V|) \;=\; \mu^{\rm Int}_{\beta,h}(\phi_j)\;\mathbb{P}_{\beta,h}(\Delta M = -j|V|).
\end{aligned}$$

Therefore, using (1.3),

$$\mathbb{P}_{\beta,h}(\Delta M = j|V|) \;=\; \frac{\mu^{\rm Int}_{\beta,h}(\phi_j)}{\mu^{\rm Int}_{\beta,h}(\phi_0)}\;\mathbb{P}_{\beta,h}(\Delta M = -j|V|) \;=\; e^{\beta j|V|h}\;\mathbb{P}_{\beta,h}(\Delta M = -j|V|).$$

In case $j \geq 0$, the combination of the last display with

$$\mathbb{P}_{\beta,h}(|\Delta M| = j|V|) \;=\; \mathbb{P}_{\beta,h}(\Delta M = j|V|) \;+\; \mathbb{P}_{\beta,h}(\Delta M = -j|V|)$$

leads to

$$\mathbb{P}_{\beta,h}(\Delta M = j|V|) \;=\; \frac{e^{\beta j|V|h/2}}{e^{\beta j|V|h/2} + e^{-\beta j|V|h/2}}\;\mathbb{P}_{\beta,h}(|\Delta M| = j|V|),$$

and

$$\mathbb{P}_{\beta,h}(\Delta M = -j|V|) \;=\; \frac{e^{-\beta j|V|h/2}}{e^{\beta j|V|h/2} + e^{-\beta j|V|h/2}} \, \mathbb{P}_{\beta,h}(|\Delta M| = j|V|).$$

Applying these two equalities to (3.7), yields (3.6), and completes the proof of (1.29).

We can now use (1.29) to write

$$\begin{aligned}
\frac{d\nu_\beta^M}{dh}(0) \;&=\; \lim_{h\to 0} \frac{\mathbb{E}_{\beta,h}\left(|\Delta M| \, \frac{\tanh(\beta\, h\, |\Delta M|\, /\, 2)}{h}\right)}{\mathbb{E}_{\beta,h}(\Delta N)} \\
&=\; \lim_{h\to 0} \frac{\sum_{k\geq 1} \mathbb{P}_{\beta,h}(|\Delta M| = k)\, k\, \frac{\tanh(\beta\, h\, k\, /\, 2)}{h}}{\sum_{n\geq 1} \mathbb{P}_{\beta,h}(\Delta N = n)\, n} \\
&=\; \lim_{h\to 0} \frac{\sum_{n\geq 1} \left\{\sum_{k=1}^{n} \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(N_1 = n, |M(N_1)| = k)\, k\, \frac{\tanh(\beta\, h\, k\, /\, 2)}{h}\right\}}{\sum_{n\geq 1} \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\phi}(N_1 = n)\, n}.
\end{aligned}$$

And since $|\tanh(\beta hk/2)/h| \leq \beta k/2$, one can use the same argument as in the proof of Part(c) of Theorem 1.2, based on (1.26) and (3.3), to conclude that

$$\begin{aligned}
\frac{d\nu_\beta^M}{dh}(0) \;&=\; \frac{\sum_{n\geq 1} \left\{\sum_{k=1}^{n} \mathbb{P}^{\mathrm{Sh}}_{\beta,0;\phi}(N_1 = n, |M(N_1)| = k)\, k\, \lim_{h\to 0} \frac{\tanh(\beta\, h\, k\, /\, 2)}{h}\right\}}{\sum_{n\geq 1} \mathbb{P}^{\mathrm{Sh}}_{\beta,0;\phi}(N_1 = n)\, n} \\
&=\; \frac{\sum_{n\geq 1} \left\{\sum_{k=1}^{n} \mathbb{P}^{\mathrm{Sh}}_{\beta,0;\phi}(N_1 = n, |M(N_1)| = k)\, k^2\beta/2)\right\}}{\sum_{n\geq 1} \mathbb{P}^{\mathrm{Sh}}_{\beta,0;\phi}(N_1 = n)\, n} \\
&=\; \frac{\sum_{k\geq 1} \mathbb{P}_{\beta,0}(|\Delta M| = k)\, k^2\beta/2}{\sum_{n\geq 1} \mathbb{P}_{\beta,0}(\Delta N = n)\, n} \;=\; \frac{\beta}{2}\, \frac{\mathbb{E}_{\beta,0}\left((\Delta M)^2\right)}{\mathbb{E}_{\beta,0}(\Delta N)} \;=\; \frac{\beta}{2}\, \left(\sigma_\beta^{\mathrm{GK}}\right)^2,
\end{aligned}$$

where in the last step we used (1.25).

Finally, to prove (1.31), set $R_n = \mathrm{Range}(\mathrm{Sh}[\eta_n])$. We will first prove that

$$\lim_{n\to\infty} \frac{R_n}{n} \;=\; 0 \quad \text{a.s.}. \tag{3.8}$$

Given $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}^{\mathrm{Int}}_{\beta,h;\eta_0}(R_n > \epsilon n) \;&=\; \mathbb{P}^{\mathrm{Sh}}_{\beta,h;\mathrm{Sh}[\eta_0]}(\mathrm{Range}(S_n) > \epsilon n) \;\leq\; \frac{\mathbb{P}^{\mathrm{Sh}}_{\beta,h;\nu_{\beta,h}}(\mathrm{Range}(S_n) > \epsilon n)}{\nu_{\beta,h}(\mathrm{Sh}[\eta_0])} \\
&=\; \frac{\nu_{\beta,h}(\{s : \mathrm{Range}(s) > \epsilon n\})}{\nu_{\beta,h}(\mathrm{Sh}[\eta_0])} \;\leq\; \frac{\bar{C}_\beta\, e^{-\bar{\gamma}_\beta \epsilon n}}{\nu_{\beta,h}(\mathrm{Sh}[\eta_0])},
\end{aligned}$$

by (1.27). The claim (3.8) now follows from Borel-Cantelli and the arbitrariness of ϵ.

The claim (1.31) follows from (1.28), (3.8), and the bounds

$$\frac{M(n)}{|V|} \;\leq\; \max_{v\in V}\{\eta_n(v)\} \;-\; \min_{v\in V}\{\eta_0(v)\} \;\leq\; \frac{M(n)}{|V|} + R_n + R_0,$$

and

$$\frac{M(n)}{|V|} - R_n - R_0 \;\le\; \min_{v\in V}\{\eta_n(v)\} \;-\; \max_{v\in V}\{\eta_0(v)\} \;\le\; \frac{M(n)}{|V|}.$$

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