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Abstract. We correct a mistake in [9] and prove that in the case of long range single site perturbations, the density of states of a continuous Anderson model exhibits Lifshitz tails at all open gap edges, and that the Lifshitz exponent is determined by the decay rate of the single site perturbation.

Résumé. Nous corrigeons une erreur survenue dans [9] et démontrons que, dans le cas du potentiel de simple site à longue portée, la densité d'états d'un modèle d'Anderson continu admet des asymptotiques de Lifshitz au bord de chaque lacune ouverte, et que l'exposant de Lifshitz est déterminé par le taux de décroissance du potentiel de simple site.

0. Introduction

In the case of long range single site perturbations (assumption (H.2bis.1)), the result stated in [9] is false. Indeed, one has a much stronger result : namely the asymptotic given for this case in Theorem 2.1 in [9] is correct without any assumptions on the non-degeneracy of the underlying periodic potential. This is what we will prove in the present note. In the case of short range single site perturbations (assumption (H.2bis.2)), it is known that the behavior of the Floquet eigenvalues generating the band edges plays a crucial role for the determination of the Lifshitz exponent (see [9] and in dimension 2, see [8, 7] for general results).

0.1. **The result.** To state our result and corrections to [9], we will need to modify slightly assumption (H.2bis) (concerning the behavior of V at infinity)
(H.2bis)

- there exists $0 \leq g_- \leq g_+$, $g_+ \in L^p(\mathbb{R}^d)$ (p is taken as in (H.2)), $\text{supp} g_\pm \subset C_0$ and $g_- > 0$ on some open set such that,
  1. either, for some $\nu \in (d, d+2]$, one has, for any $\gamma \in \Gamma$ and almost every $x \in C_0$,
  $$g_-(x) \leq V(x+\gamma) \cdot (1+ \mid \gamma \mid)^\nu \leq g_+(x),$$
  2. or, there exists $\nu > d+2$ such that, one has, for any $\gamma \in \Gamma$ and almost every $x \in C_0$,
  $$V > 0 \text{ on some open set and } 0 \leq V(x+\gamma) \cdot (1+ \mid \gamma \mid)^\nu \leq g_+(x).$$

We prove

Theorem 0.1. *Let H and H_ω be constructed as above, and assume that (H.1), (H.2) and (H.3) hold. Then, we show that,*

- *if V satisfies (H.2bis.1), then, we have*
  - *if (H.4) hold then, we have*
  $$\liminf_{E \to 0^+} \frac{\log |\log(N(E) - N(0^-))|}{\log E} \geq -\frac{d}{\nu - d}$$

– *if (H.4s) holds then, 0 is a continuity point for N and we have*

$$\lim_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} = -\frac{d}{\nu-d}. \tag{0.1}$$

- *if V satisfies (H.2bis.2), then, we have*
  – *if (H.4) hold then, we have*

$$\liminf_{E\to 0^+} \frac{\log|\log(N(E)-N(0^-))|}{\log E} \geq -\frac{d}{2}$$

and

$$\limsup_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} < \frac{d}{2} \Longrightarrow \liminf_{E\to 0^+} \frac{\log|\log(N(E)-N(0^-))|}{\log E} > -\frac{d}{2},$$

  – *if (H.4s) holds then, 0 is a continuity point for N and we have*

$$\lim_{E\to 0^+} \frac{\log|\log(N(E)-N(0))|}{\log E} = -\frac{d}{2} \Longleftrightarrow \lim_{E\to 0^+} \frac{\log(n(E)-n(0))}{\log E} = \frac{d}{2},$$

The proofs corresponding to the case when V satisfies (H.2bis.2) are given in [9]. So from now on we assume that V satisfies (H.2bis.1).

1. Proof of the lower bound if (H.4) holds

We will use the same ideas and notations as in section 6 of [9]. The mistake made there was to chose $\Lambda_\alpha(\varepsilon)$ not sufficiently large so as to accommodate for the slow decay of the single site potential. First, we set $s=1$ for s defined on page 371 of [9]. We pick

$$\Lambda_\alpha(\varepsilon) = \left\{\gamma\in\Gamma;\ \text{for } 1\leq j\leq d,\ |\gamma_j| \leq \varepsilon^{-(1/(\nu-d)+\alpha)}\right\}.$$

Then, equation (6.6) in [9] becomes

$$\sum_{\eta\notin\Lambda_\alpha(\varepsilon)} (1+|\eta|)^{-\nu} \leq C\varepsilon^{1+\alpha(\nu-d)}.$$

The computations done in pages 376–378 give

$$\liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{\log|\log(N(\varepsilon)-N(0^-))|}{\log\varepsilon} \geq -\frac{\nu}{\nu-d} - (d-1)\alpha$$

2. Proof of (0.1) when (H.4s) holds

First note that to prove (0.1) we only need to prove an upper bound on the density of states N that coalesces with the already proven lower bound. Therefore we will use a method different from the one developed in [9]; it will be related to the method used in [5, 6]. The proof goes roughly as follows. First we will prove that the periodic approximations introduced in section 5.1 of [9] are exponentially close to the true density of states. Then, we will study the density of states of these approximations.

2.1. **Periodic approximations.** Let $H_{\omega,n}$ be the periodic approximation defined in [9] and $N_{\omega,n}$ be their integrated density of state. Then, we prove

Theorem 2.1. *Assume (H.1), (H.3) and (H.2bis.1). Pick $I \subset \mathbb{R}$ a compact interval. Then, for any $\eta_0 > 0$, there exists $\nu_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, $E \in I$ and $n \geq \varepsilon^{-\nu_0}$, we have*

$$\begin{aligned} &\mathbb{E}(N_{\omega,n}(E+\varepsilon/2) - N_{\omega,n}(E-\varepsilon/2)) - e^{-\varepsilon^{-\eta_0}} \\ &\qquad \leq N(E+\varepsilon) - N(E) \\ &\qquad\qquad \leq \mathbb{E}(N_{\omega,n}(E+2\varepsilon) - N_{\omega,n}(E-2\varepsilon)) + e^{-\varepsilon^{-\eta_0}}. \end{aligned} \tag{2.1}$$

Theorem 2.1 states that the density of states of the random Hamiltonian H_ω is exponentially well approximated by the expectation of the densities of states of the periodic Hamiltonians $H_{\omega,n}$ even for n not very large (i.e. for n of size roughly inverse polynomial of the approximation parameter).

Proof. Theorem 2.1 is a consequence of a similar result for random operators with a compactly supported single site potential. More precisely, define $V_\varepsilon(x) = V(x)\mathbf{1}_{\varepsilon\cdot|x|\geq 1}$ and the random operator

$$H_{\varepsilon,\omega} = H + V_{\varepsilon,\omega} = H + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V_\varepsilon(\cdot - \gamma).$$

Let N_ε be the integrated density of states of $H_{\varepsilon,\omega}$. We define $H_{\varepsilon,\omega,n}$, the periodic approximations of $H_{\varepsilon,\omega}$ in the same way as for H_ω (just replace V by V_ε). Let $N_{\varepsilon,\omega,n}$ be their integrated density of states.

We first notice that using the decay assumption (H.2bis.1) and the fact that the random variables $(\omega_\gamma)_\gamma$ are bounded, uniformly in n and ω, we have

$$\|(1-\Delta)^{-1}(H_{\varepsilon,\omega} - H_\omega)\| + \|(1-\Delta)^{-1}(H_{\varepsilon,\omega,n} - H_{\omega,n})\| \leq C\varepsilon^{\nu-d}.$$

This implies that, uniformly in n and ω, and locally uniformly in E, one has

$$\begin{aligned} N_\varepsilon(E - C\varepsilon^{\nu-d}) \leq N(E) \leq N_\varepsilon(E + C\varepsilon^{\nu-d}) \\ N_{\varepsilon,\omega,n}(E - C\varepsilon^{\nu-d}) \leq N_{\omega,n}(E) \leq N_{\varepsilon,\omega,n}(E + C\varepsilon^{\nu-d}). \end{aligned} \tag{2.2}$$

So to prove Theorem 2.1, we only need to prove it for $H_{\omega,\varepsilon}$ and replace 'ε by $\varepsilon^{1/(\nu-d)}$. The gain is that, for the new operators, the single site potential is compactly supported. The proof of Theorem 2.1 in this case relies upon the

Lemma 2.1. *Assume that the single site potential V is compactly supported in a ball of radius R centered at 0. Pick I, a relatively compact open interval in $\mathbb{R}$ .For any $\beta \in (0,1)$, there exists $C > 1$ and $\rho > 0$ (depending only on d) such that, for any $\varphi \in \mathcal{C}_0^\infty(I)$, for $k \in \mathbb{N}^*$ and $n \in \mathbb{N}^*$, $n > R$, we have*

$$|\mathbb{E}((\varphi, dN_{\omega,n})) - (\varphi, dN)| \leq C|n-R|^{-(1-\beta)k} \sup_{\substack{x\in\mathbb{R} \\ 0\leq j\leq k+\rho}} \left|(|x|+C)^{\rho+k}\frac{d^j\varphi}{d^jx}(x)\right|. \tag{2.3}$$

Proof. This lemma was proved in [5, 6, 10] in a different setting but essentially the same arguments apply in our case too. The proof relies upon the following estimate on the difference between the resolvents of $H_{\omega,n}$ and H_ω at an energy $z \in \mathbb{C}\setminus\mathbb{R}$:

$$\begin{aligned} &\left\|\mathbf{1}_{C(0,1)}\left((z-H_{\omega,n})^{-1} - (z-H_\omega)^{-1}\right)\mathbf{1}_{C(0,1)}\right\| = \\ &\qquad = \left\|\mathbf{1}_{C(0,1)}\left((z-H_{\omega,n})^{-1}(V_{\omega,n} - V_\omega)(z-H_\omega)^{-1}\right)\mathbf{1}_{C(0,1)}\right\| \leq Ce^{-\frac{1+|z|}{C\cdot \mathrm{Im}z}|n-R|}. \end{aligned} \tag{2.4}$$

Estimate (2.4) is a consequence of the computation done to prove Theorem 5.1 in [9] (pp 358–360), of the fact that, by construction, $(V_{\omega,n} - V_\omega)$ vanishes in the cube centered at 0 of side length $n - R$ (as V is supported in a ball of size R), and of the exponential decay of the resolvent kernels $(z - H_\omega)^{-1}$ and $(z - H_{\omega,n})^{-1}$ (this is a consequence of Combes-Thomas estimates (see e.g. [1, 4])). To obtain Lemma 2.1 from (2.4), one then uses the representation formula (5.9) in [9] for $\varphi(H)$ in terms of almost analytic extensions of φ, and the standard estimates for almost analytic extension (5.8) in [9]. $\square$

Let us now derive Theorem 2.1 from Lemma 2.1. Let φ be a Gevrey class function of Gevrey exponent $\alpha > 1$ (see [2]); assume moreover that φ has compact support in $(-2, 2)$, that $0 \leq \varphi \leq 1$ and that $\varphi \equiv 1$ on $[-1, 1]$. Let $E \in \mathbb{R}$ and set

$$\varphi_{E,\varepsilon}(\cdot) = \varphi\left(\frac{\cdot - E}{\varepsilon}\right).$$

Fix $I \subset \mathbb{R}$ compact. Then, by Lemma 2.1 and the Gevrey estimates on the derivatives of φ, we get that, there exist $C > 1$ such that, for $E \in I$, $n > R$, $k \geq 1$ and $0 < \varepsilon < 1$, we have

$$|\mathbb{E}((\varphi_{E,\varepsilon}, dN_\omega^n)) - (\varphi_{E,\varepsilon}, dN)| \leq \varepsilon^{-k-\rho}(k+\rho)^{\alpha(k+\rho)}(n-R)^{-(1-\beta)k}$$

We pick $k = (n-R)^{(1-\beta)/2\alpha}$ and get that, there exist $C > 1$ such that, for $E \in I$, $n > R + 2$ and $0 < \varepsilon < 1$, we have

$$|\mathbb{E}((\varphi_{E,\varepsilon}, dN_\omega^n)) - (\varphi_{E,\varepsilon}, dN)| \leq \left(\frac{(n-R)^{(1-\beta)/3}}{\varepsilon^2}\right)^{(n-R)^{(1-\beta)/2\alpha}}.$$

Now, in the case we are interested in $R = R_\varepsilon \sim \varepsilon^{1/(d-\nu)}$. Hence, we choose $n \geq \varepsilon^{-3-\eta} + R_\varepsilon$, we get that there exist $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, we have

$$|\mathbb{E}((\varphi_{E,\varepsilon}, dN_\omega^{n_\varepsilon})) - (\varphi_{E,\varepsilon}, dN)| \leq e^{-\varepsilon^{-\eta\cdot(1-\beta)/3\alpha}}. \tag{2.5}$$

By the definition of φ, as $dN_{\varepsilon,\omega,n}$ and dN_ε are positive measures, we have

$$\mathbb{E}(N_{\varepsilon,\omega,n}(E+\varepsilon) - N_{\varepsilon,\omega,n}(E-\varepsilon)) \leq \mathbb{E}((\varphi_{E,\varepsilon}, dN_{\varepsilon,\omega,n})) \leq \mathbb{E}(N_{\varepsilon,\omega,n}(E+2\varepsilon) - N_{\varepsilon,\omega,n}(E-2\varepsilon)),$$
$$N_\varepsilon(E+\varepsilon) - N_\varepsilon(E-\varepsilon) \leq (\varphi_{E,\varepsilon}, dN) \leq N_\varepsilon(E+2\varepsilon) - N_\varepsilon(E-2\varepsilon).$$

This, equation (2.2) and (2.5) complete the proof of Theorem 2.1 for $H_{\varepsilon,\omega}$ hence for H_ω. $\square$

2.2. **Proof of the upper bound.** We pick $\eta_0 > \nu/(\nu - d)$ in Theorem 2.1 and $n \sim \varepsilon^{-\eta}$ for $\eta > \nu_0$ (ν_0 is given in Theorem 2.1 and η will be chosen later on). So to prove (0.1), we only need to prove that

$$\limsup_{\varepsilon\to 0^+} \frac{\log|\log(\mathbb{E}(N_{\omega,n}(\varepsilon) - N_{\omega,n}(0)))|}{\log\varepsilon} \leq -\frac{d}{\nu - d}. \tag{2.6}$$

Indeed, by assumption (H.4s) and the computations done in section 5.2 in [9], we know that ,for ε small enough, for all ω and n, one has $N_{\omega,n}(0) = N_{\omega,n}(-\varepsilon)$.

We prove

Lemma 2.2. *Define the potential G by $G(\cdot) = \sum_{\gamma\in\Gamma} g_-(\cdot - \gamma)$. Then, if $n \sim \varepsilon^{-\rho}$ with $\rho > \frac{1}{\nu-d}$, the event $\{\omega;\ V_{\omega,n} \geq \varepsilon G\}$ has probability at least $1 - P_\varepsilon$ where P_ε satisfies*

$$\limsup_{\varepsilon\to 0^+} \frac{\log|\log(P_\varepsilon)|}{\log\varepsilon} \leq -\frac{d}{\nu - d}. \tag{2.7}$$

and

Lemma 2.3. *There exists $C > 0$ and $\varepsilon_0 > 0$ (independent of n and ω) such that, if $0 < \varepsilon < \varepsilon_0$ and $V_{\omega,n} \geq C\varepsilon G$ then, for $n \in \mathbb{N}$, one has $N_{\omega,n}(0) = N_{\omega,n}(\varepsilon)$.*

Now, estimate (2.6) is an immediate consequence of Lemmas 2.2 and 2.3. Indeed, picking C as in Lemma 2.3, one computes

$$\begin{aligned}
\mathbb{E}(N_{\omega,n}(\varepsilon) - N_{\omega,n}(0)) &= \mathbb{E}([N_{\omega,n}(\varepsilon) - N_{\omega,n}(0)]\mathbf{1}_{\{\omega;\ V_{\omega,n} \geq C\varepsilon G\}}) \\
&\quad + \mathbb{E}([N_{\omega,n}(\varepsilon) - N_{\omega,n}(0)]\mathbf{1}_{\{\omega;\ V_{\omega,n} \not\geq C\varepsilon G\}}) \\
&= \mathbb{E}([N_{\omega,n}(\varepsilon) - N_{\omega,n}(0)]\mathbf{1}_{\{\omega;\ V_{\omega,n} \not\geq C\varepsilon G\}}) \\
&\leq C_0 \mathbb{P}(\{\omega;\ V_{\omega,n} \not\geq C\varepsilon G\}) \\
&= C_0(1 - \mathbb{P}(\{\omega;\ V_{\omega,n} \geq C\varepsilon G\})) = C_0 P_{C\varepsilon}
\end{aligned}$$

and we conclude using (2.7) and taking $(\nu - d)\eta > 1$. Here, we have used the fact that the integrated density of states of $H_{\omega,n}$ is bounded locally uniformly in energy, uniformly in ω and n.

2.3. **The proof of Lemma 2.3.** We will use the notations and results of section 5.2 in [9]. The operator $H + \varepsilon G$ is Γ-periodic. Hence, it is also as $(2n+1)\Gamma$-periodic. When doing the Floquet reduction using this period, we will call it $(H + \varepsilon G)_n$, and $(H + \varepsilon G)_{n,\theta}$ denotes this operator considered with quasi-periodic boundary conditions given by θ (see the introduction of [9] for more details).

Recall that $\vartheta((H + \varepsilon G)_n, E)$ denotes the number of Floquet eigenvalues of $(H + \varepsilon G)_{n,\theta}$ below energy E for Floquet parameter θ. Standard perturbation theory and the results of section 5.2 in [9] (see (5.24)) tell us that, for $\varepsilon > 0$ sufficiently small,

$$\vartheta((H + \varepsilon G)_{n,\theta}, 0) = \vartheta((H)_{n,\theta}, 0) = \vartheta(H_{\omega,n,\theta}, 0^-). \tag{2.8}$$

Hence, if $V_{\omega,n} \geq \varepsilon G$, this immediately implies that, for $E \geq 0$, one has

$$0 \leq N_{\omega,n}(E) - N_{\omega,n}(0) \leq N_\varepsilon(E) - N_\varepsilon(0)$$

where N_ε is the density of states of the periodic operator $H + \varepsilon G$. So Lemma 2.2 will be proved if we prove that there exists $C > 0$ (independent of ε) such that, for ε sufficiently small, one has $N_\varepsilon(\varepsilon/C) - N_\varepsilon(0) = 0$. This is a consequence of

Lemma 2.4. *Let H be a Γ-periodic Schrödinger operator, and G be a non negative Γ-periodic potential that is locally uniformly in $L^p(\mathbb{R}^d)$ (where p is chosen as in assumption (H.2)) and such that $G > 0$ on some open set. Assume $(-\delta, 0)$ does not intersect the spectrum of H (for some $\delta > 0$).*

Then, there exists $C > 0$ and $\varepsilon_0 > 0$ such that, for $0 \leq \varepsilon \leq \varepsilon_0$, $(-\delta/2, \varepsilon/C)$ does not intersect the spectrum of $H + \varepsilon G$.

Proof. Let us first assume that 0 is not in the spectrum of H; then, there exists $\delta' > 0$ such that $(-\delta, \delta')$ does not intersect the spectrum of H, and regular perturbation theory tells us that, for ε sufficiently small, $(-\delta/2, \delta'/2)$ does not intersect the spectrum of $H + \varepsilon G$. This completes the proof of Lemma 2.4 in this case.

We now assume that 0 belongs to the spectrum of H. Let $(E_n(\varepsilon, \theta))_{1 \leq n}$ be the Floquet eigenvalues of $(H + \varepsilon G)_\theta$ ordered increasingly; they are continuous in ε and θ. Hence, there exists $\varepsilon_0 > 0$ and $n_0 \geq 1$ such that, for $0 \leq \varepsilon \leq 0$, one has that $E_n(\varepsilon, \theta) < E_0$ if and only if $n < n_0$. Then, to prove Lemma 2.4, we need to prove that, there exists $C > 0$ and $\varepsilon_0 > 0$ such that, for $0 \leq \varepsilon \leq \varepsilon_0$, one has

$$E_{n_0}(\varepsilon, \theta) \geq \varepsilon/C. \tag{2.9}$$

Define $\mathcal{S} = \{\theta \in \mathbb{T}^*;\ E_{n_0}(0,\theta) = 0\}$; $\mathcal{S}$ is compact. Outside of a neighborhood of $\mathcal{S}$, (2.9) is certainly satisfied as, there, one has $E_{n_0}(\varepsilon,\theta) \geq E_{n_0}(0,\theta) \geq \delta'$ (for some $\delta' > 0$). So we need only to prove (2.9) in a neighborhood of $\mathcal{S}$.
Therefore, pick $\theta_0 \in \mathcal{S}$. Let $p(\theta_0)$ be the multiplicity of 0 as an eigenvalue of $H_\theta = (H+\varepsilon G)_{\theta\,|\varepsilon=0}$; it is finite. For θ close to θ_0 and ε small, define the Riesz projector

$$\Pi_{\varepsilon,\theta} = \oint_{\mathcal{C}_0} (z - (H+\varepsilon G)_\theta)^{-1} dz$$

where $\mathcal{C}_0 = \{|z| = \delta'\}$ is a small loop in the complex plane. $\Pi_{\varepsilon,\theta}$ is the projector onto the spectral space associated to the interval $(-\delta', \delta')$ and the operator $(H+\varepsilon G)_\theta$. It is real analytic in θ and ε; hence of constant rank. Moreover, $E_{n_0}(\varepsilon,\theta)$ is the smallest eigenvalue of the finite rank operator $O_{\varepsilon,\theta} = \Pi_{\varepsilon,\theta}(H+\varepsilon G)_\theta \Pi_{\varepsilon,\theta}$ (acting on $\Pi_{\varepsilon,\theta} L^2_{\mathrm{loc}}(R^d)$).
The operator $O_{\varepsilon,\theta}$ is real analytic in ε and θ; differentiating it in ε, we obtain

$$\frac{dO_{\varepsilon,\theta}}{d\varepsilon} = \Pi_{\varepsilon,\theta} G \Pi_{\varepsilon,\theta} \quad \text{and} \quad \left(\frac{dO_{\varepsilon,\theta}}{d\varepsilon}\right)_{|\varepsilon=0} = \Pi_{0,\theta} G \Pi_{0,\theta}. \tag{2.10}$$

One proves

Lemma 2.5. *For some $C > 0$ and θ close to θ_0, one has*

$$\Pi_{0,\theta} G \Pi_{0,\theta} \geq 1/C \Pi_{0,\theta}. \tag{2.11}$$

This and (2.10) implies that, for ε positive, sufficiently small, one has

$$O_{\varepsilon,\theta} \geq \frac{\varepsilon}{C} \Pi_{\varepsilon,\theta} \tag{2.12}$$

as, for any ε, ε' sufficiently small and θ, θ' sufficiently close to θ_0, one has

$$\Pi_{\varepsilon,\theta} \Pi_{\varepsilon,\theta'} \Pi_{\varepsilon,\theta} \geq \frac{1}{2} \Pi_{\varepsilon,\theta}. \tag{2.13}$$

Equation (2.12) then immediately implies (2.9) for ε sufficiently small and θ sufficiently close to θ_0. We the obtain (2.9) for a neighborhood of $\mathcal{S}$ as $\mathcal{S}$ is compact. This completes the proof of Lemma 2.4. $\square$

Proof of Lemma 2.5. First, we note that

$$\Pi_{0,\theta} G \Pi_{0,\theta} = \Pi_{0,\theta_0} G \Pi_{0,\theta_0} - \Pi_{0,\theta} \left(\Pi_{0,\theta_0} G \Pi_{0,\theta_0} - \Pi_{0,\theta} G \Pi_{0,\theta}\right) \Pi_{0,\theta}.$$

Hence, as $\theta \mapsto \Pi_{0,\theta} G \Pi_{0,\theta}$ is continuous, by (2.13), we just need to prove (2.11) for $\theta = \theta_0$. Let $(\psi_j)_{1\leq j\leq p(\theta_0)}$ be a basis of eigenvector in the kernel of H_{θ_0}. Then $\Pi_{0,\theta} = \sum_{1\leq j\leq p(\theta_0)} \psi_j \otimes \psi_j$. Hence, (2.11) will be satisfied if and only if the matrix $((\langle G\psi_i, \psi_j\rangle))_{i,j}$ is positive definite. It is non negative. If 0 were an eigenvalue of this matrix, there would exist $\psi \in \mathrm{Span}((\psi_j)_{1\leq j\leq p(\theta_0)})$ such that $\displaystyle\int_{C_0} G(x)|\psi(x)|^2 dx = 0$. Hence, as $G > 0$ on some open set, ψ would vanish on this open set: this is not possible by the unique continuation principle as ψ satisfies $H\psi = 0$ (see e.g. [3, 11]). $\square$

2.4. **The proof of Lemma 2.2.** Let us write out $V_{\omega,n}$ and use assumption (H.2bis.1). One computes

$$\begin{aligned} V_{\omega,n}(x) &= \sum_{\gamma\in\Gamma}\omega_{[\gamma]}V(x-\gamma) \text{ where } [\gamma]=\gamma \bmod (2n+1)\Gamma \\ &\geq \sum_{\gamma\in\Gamma}\omega_{[\gamma]}\sum_{\alpha\in\Gamma} g_-(x-\gamma-\alpha)\cdot(1+\mid\alpha\mid)^{-\nu} \\ &= \sum_{\alpha\in\Gamma}A_\alpha(\omega)g_-(x-\alpha) \text{ where } A_\alpha(\omega)=\sum_{\gamma\in\Gamma}\omega_{[\gamma]}(1+\mid\alpha-\gamma\mid)^{-\nu}. \end{aligned}$$

One checks that $A_\alpha(\omega)=A_{\alpha+\beta}(\omega)$ for $\beta\in(2n+1)\Gamma$. Define $\Gamma_{2n+1}=\Gamma/(2n+1)\Gamma=\{\gamma\in\Gamma;|\gamma|\leq n\}$. Then, we have

$$\mathbb{P}(\{V_{\omega,n}\geq\varepsilon G\})\geq\mathbb{P}(\{\forall\gamma\in\Gamma_{2n+1};\ A_\gamma(\omega)\geq\varepsilon\})\geq 1-\sum_{\gamma\in\Gamma_{2n+1}}\mathbb{P}(\{A_\gamma(\omega)\leq\varepsilon\}).$$

Note that, as the $(\omega_\gamma)_\gamma$ are stationary, the distribution of the random variable $A_\gamma(\omega)$ is independent of γ. Hence

$$\mathbb{P}(\{V_{\omega,n}\geq\varepsilon G\})\geq 1-(2n+1)^d\mathbb{P}(\{A_0(\omega)\leq\varepsilon\}). \tag{2.14}$$

So we only need to estimate $\mathbb{P}(\{A_0(\omega)\leq\varepsilon\})$. Therefore, we compute

$$\begin{aligned} A_0(\omega) &= \sum_{\beta\in(2n+1)\Gamma}\ \sum_{\gamma\in\Gamma_{2n+1}}\omega_\gamma(1+\mid\beta+\gamma\mid)^{-\nu} \\ &= \sum_{\gamma\in\Gamma_{2n+1}}\omega_\gamma u_\gamma \text{ where } u_\gamma=\sum_{\beta\in(2n+1)\Gamma}(1+\mid\beta+\gamma\mid)^{-\nu} \end{aligned}$$

Using Markov's inequality, for any $\lambda>0$, one obtains

$$\begin{aligned} \mathbb{P}(A_0(\omega)\leq\varepsilon) &\leq \mathbb{E}\left(e^{\lambda(\varepsilon-\sum_{\gamma\in\Gamma_{2n+1}}\omega_\gamma u_\gamma)}\right) \\ &\leq e^{\lambda\varepsilon}\prod_{\gamma\in\Gamma_{2n+1}}\mathbb{E}\left(e^{-\lambda\omega_\gamma u_\gamma}\right). \end{aligned} \tag{2.15}$$

Moreover, there exists $\eta>0$ such that, if $\lambda u_\gamma<\eta$, then

$$\mathbb{E}\left(e^{-\lambda\omega_\gamma u_\gamma}\right)\leq e^{-\overline{\omega_0}\lambda u_\gamma/2} \text{ where } \overline{\omega_0}=\mathbb{E}(\omega_0)>0 \text{ by assumption (H.3).} \tag{2.16}$$

Indeed, using the Taylor expansion of $x\mapsto e^{-x}$, as the random variables $(\omega_\gamma)_\gamma$ are bounded, for λu_γ sufficiently small, one obtains

$$\mathbb{E}\left(e^{-\lambda\omega_\gamma u_\gamma}\right)\leq 1-\mathbb{E}(\omega_\gamma)\lambda u_\gamma+C(\lambda u_\gamma)^2\leq 1-\mathbb{E}(\omega_\gamma)\lambda u_\gamma(1-C\eta)\leq e^{-\overline{\omega_0}\lambda u_\gamma/2}.$$

Plugging (2.16) into (2.15), we compute

$$\log(\mathbb{P}(A_0(\omega)\leq\varepsilon))\leq\lambda\varepsilon-\frac{\lambda\overline{\omega_0}}{2}\cdot\sum_{\substack{\gamma\in\Gamma_{2n+1}\\ \lambda u_\gamma<\eta}}u_\gamma. \tag{2.17}$$

Assume that $n>>\lambda^{1/\nu}$; then, the definition of u_γ implies that, for some $C>0$, one has

$$\sum_{\substack{\gamma\in\Gamma_{2n+1}\\ \lambda u_\gamma<\eta}}u_\gamma\geq\sum_{\substack{\gamma\in\Gamma\\ C\lambda^{1/\nu}\leq|\gamma|\leq n-\lambda^{1/\nu}}}(1+|\gamma|)^{-\nu}\geq\frac{1}{C}\lambda^{\frac{d-\nu}{\nu}}.$$

Plugging this back into (2.17) and picking $\lambda = \rho\varepsilon^{\frac{\nu}{d-\nu}}$, we obtain

$$\log(\mathbb{P}(A_0(\omega) \leq \varepsilon)) \leq -\rho(\overline{\omega_0}\rho^{d/\nu}/C - 1)\varepsilon^{-\frac{d}{\nu-d}} \leq \frac{1}{C}\varepsilon^{-\frac{d}{\nu-d}} \tag{2.18}$$

for ρ sufficiently large.
So that, if we pick e.g. $n \sim \varepsilon^{-\rho}$ with $\rho > \frac{1}{\nu-d}$, and plug estimate (2.18) into (2.14), we complete the proof of Lemma 2.2. □

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