# HIGH DISORDER LOCALIZATION FOR RANDOM SCHRÖDINGER OPERATORS THROUGH EXPLICIT FINITE VOLUME CRITERIA

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*Dedicated to Leonid Pastur on the occasion of his 65th birthday*

Abstract. We apply finite volume criteria for localization to random Schrödinger operators. These provide explicit conditions, depending on the parameters of the model, for starting the bootstrap multiscale analysis. The criteria are used to study localization of Anderson Hamiltonians on the continuum at the bottom of the spectrum at high disorder.

## 1. Introduction

In this article we illustrate the use (and need) of explicit finite volume criteria for proving localization of waves in random media. We consider an Anderson Hamiltonian in the continuum,

$$H_{\lambda,\omega} = -\Delta + V_{\mathrm{per}} + \lambda V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{1.1}$$

where $V_{\mathrm{per}}$ is a periodic potential with period 1 (by rescaling we can always take the period to be one), $\lambda > 0$ is the disorder parameter and the random potential $V_\omega$ is of the form

$$V_\omega(x) = \sum_{i \in \frac{1}{q}\mathbb{Z}^d} \omega_i \, u(x - i), \, , \tag{1.2}$$

with $q \in \mathbb{N}$; $\omega = \{\omega_i;\ i \in \frac{1}{q}\mathbb{Z}^d\}$ a family of independent, identically distributed random variables taking values in the bounded interval $[M_-, M_+]$, whose common probability distribution has a bounded density $g$ with $g > 0$ near $M_-$; $u(x)$ a measurable function with compact support, $0 \le u \in \mathrm{L}^p(\mathbb{R}^d, \mathrm{d}x)$ with $p \in [1, \infty]$ if $d = 1$ or $M_- \ge 0$ , $p \in (\frac{d}{2}, \infty]$ if $d \ge 2$ and $M_- < 0$, and $0 < U_- \le U(x)$ where $U(x) = \sum_{i \in \frac{1}{q}\mathbb{Z}^d} u(x - i)$.

This model was studied by Combes and Hislop [CH1] and Kirsch [Ki] with $M_- = 0$ and $V_{\mathrm{per}} = 0$, who proved that for any fixed energy $E_1 > 0$ we have Anderson localization in the interval $[0, E_1]$ for sufficiently large disorder. (In this case the almost-sure spectrum of $H_{\lambda,\omega}$ is $[0, \infty]$ for all $\lambda > 0$.)

But the case $M_- \ne 0$ cannot be treated in the same way. If $E_\lambda^{\mathrm{inf}}$ denotes the bottom of the spectrum of $H_{\lambda,\omega}$, we now have $E_\lambda^{\mathrm{inf}} \to \pm\infty$ as $\lambda \to \infty$ if $M_- \gtrless 0$. Thus even for a fixed interval at the bottom of the spectrum (i.e., of the form $[E_\lambda^{\mathrm{inf}}, E_\lambda^{\mathrm{inf}} + \delta]$ with a fixed $\delta > 0$), both the constant in Wegner's estimate and the constant in the Simon-Lieb inequality increase as the disorder $\lambda$ increases, so as we

2000 *Mathematics Subject Classification.* Primary 82B44; Secondary 47B80, 60H25.

A.K. was supported in part by NSF Grant DMS-0200710.

increase $\lambda$ the initial length scale for the multiscale analysis also increases. Thus we cannot fix a length scale and satisfy the initial condition for the multiscale analysis at the disorder dependent energy interval by taking the disorder large enough; the initial length scale is a moving target.

To deal with such difficulties, we developped in [GK4] explicit finite volume criteria for proving localization in situations where the crucial quantities of the model that enter the multiscale analysis (e.g., the constant in Wegner's estimate, the constant in the Simon-Lieb inequality) depend on the parameters of the model (e.g., the disorder parameter, the energy where localization is to be proven, the strength of the magnetic field). The emphasis was on providing explicit conditions, depending on the various parameters of the model, for starting the bootstrap multiscale analysis [GK1]. These criteria thus yield Anderson localization, strong dynamical localization, SULE, etc. (See also [GK3] for a discussion of the consequences of the bootstrap multiscale analysis; see [DRJLS] for a discussion of SULE.) These criteria are reviewed in Section 2.

In Section 3 we apply our explicit criteria to Anderson Hamiltonians $H_{\lambda,\omega}$ as in (1.1), proving localization at high disorder in an interval of size proportional to the disorder at the bottom of the spectrum. To do so we derive explicit bounds on the location of the bottom of the spectrum, $E_\lambda^{\inf}$, as a function of the disorder $\lambda$. (See Theorem 3.1.) Using these bounds and our explicit criteria, we prove localization in the interval $[E_\lambda^{\inf}, E_\lambda^{\inf} + c\lambda]$ for large disorder $\lambda$, where $c$ is a constant independent of $\lambda$. In addition, we show that for large disorder eigenfunctions with eigenvalues in the interval $[E_\lambda^{\inf}, E_\lambda^{\inf} + c\lambda]$ decay exponentially with a rate $\geq c'\sqrt{\lambda}$ for some constant $c'$. (See Theorem 3.2.) We had previously obtained such results when $M_- = 0$ and $u$ bounded [GK4].

Localization for continuous random operators has been usually proven by a multiscale analysis. (But note that the fractional moment method [AM, ASFH] has just been extended to the continuum [AENSS].) The multiscale analysis is a technique, initially developed by Fröhlich and Spencer [FS] and Fröhlich, Martinelli, Spencer and Scoppolla [FMSS], and simplified by von Dreifus [vD] and von Dreifus and Klein [vDK], for the purpose of proving Anderson localization. i.e., pure point spectrum and exponential decay of eigenfunctions. (See also [HM, CKM, Sp, KLS, vDK2, K, CH1, Klo1, FK1, FK2, KSS1, KSS2, CHT, FLM, Wa, St, GK1, U, KK1, KK2].) It was later shown to also yield dynamical localization (non spreading of the wave packets) [GDB, Ge, DS, GK1]. Explicit finite volume criteria for applying the multiscale analysis were provided in [GK4].

## 2. Explicit finite volume criteria for localization

In this article a *random Schrödinger operator* will be a random operator of the form

$$H_\omega = -\Delta + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{2.1}$$

where $\Delta$ is the $d$-dimensional Laplacian operator and $V_\omega$ is a random potential, i.e., $\{V_\omega(x);\ x \in \mathbb{R}^d\}$ is a real valued measurable process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that:

**(i):** $V_\omega = V_\omega^{(1)} + V_\omega^{(2)}$, where $\{V_\omega^{(i)}(x);\ x \in \mathbb{R}^d\}$, $i = 1, 2$, are real valued measurable processes on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for $\mathbb{P}$-a.e. $\omega$ we have:

**(i$_1$):** $0 \leq V_\omega^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$.

**(i$_2$):** $V_\omega^{(2)}$ is relatively form-bounded with respect to $-\Delta$ with relative bound $< 1$.

**(ii):** There is an ergodic family $\{\tau_y;\ y \in \mathbb{Z}^d\}$ of measure preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $V^{(i)}_{\tau_y\omega}(x) = V^{(i)}_\omega(x-y)$ for $i = 1, 2$ and all $y \in \mathbb{Z}^d$.

**(iii):** There exists $\varrho > 0$ such that for any bounded subsets $B_1, B_2$ of $\mathbb{R}^d$ with $\operatorname{dist}(B_1, B_2) > \varrho$ the processes $\{V_\omega(x);\ x \in B_1\}$ and $\{V_\omega(x);\ x \in B_2\}$ are independent.

It follows, using the ergodicity, that there are nonnegative constants $\Theta_1 < 1$ and $\Theta_2$ such that for all $\psi \in \mathcal{D}(\nabla)$ we have

$$\left|\left\langle \psi, V_\omega^{(2)}\psi\right\rangle\right| \le \Theta_1\|\nabla\psi\|^2 + \Theta_2\|\psi\|^2 \quad \text{for } \mathbb{P}\text{-a.e. } \omega\,, \tag{2.2}$$

and hence $H_\omega$ is defined as a semi-bounded self-adjoint operator for $\mathbb{P}$-a.e. $\omega$, with

$$H_\omega \ge -\Theta_2 \quad \text{for } \mathbb{P}\text{-a.e. } \omega\,. \tag{2.3}$$

Moreover, $H_\omega$ is a random operator, i.e., the mappings $\omega \to f(H_\omega)$ are strongly measurable for all bounded measurable functions on $\mathbb{R}$. Thus there exists a nonrandom set $\Sigma$ such that $\sigma(H_\omega) = \Sigma$ with probability one, and that the decomposition of $\sigma(H_\omega)$ into pure point spectrum, absolutely continuous spectrum, and singular continuous spectrum is also independent of the choice of $\omega$ with probability one. (See [KM]and the discussion in [GK3].)

In this article we are concerned with localization, from both the spectral and dynamical point of views.

**Definition 2.1.** *The random Schrödinger operator $H_\omega$ exhibits exponential localization in the open interval $I$ if it has pure point spectrum in $I$, and for $\mathbb{P}$-a.e. $\omega$ the eigenfunctions of $H_\omega$ with eigenvalue in $I$ decay exponentially in the $L^2$-sense. The exponential localization region $\Sigma_{\mathrm{EL}}$ for the random Schrödinger operator $H_\omega$ is the set of $E \in \Sigma$ for which there exists some open interval $I \ni E$ such that $H_\omega$ exhibits exponential localization in $I$.*

**Definition 2.2.** *The random Schrödinger operator $H_\omega$ exhibits strong HS-dynamical localization in the open interval $I$ if for all $0 \le \mathcal{X} \in C^\infty$ with support in $I$ we have*

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}}\left\|\langle X\rangle^{\frac{n}{2}}\mathrm{e}^{-itH_\omega}\mathcal{X}(H_\omega)\chi_0\right\|_2^2\right\} < \infty \quad \textit{for all } n \ge 0\,, \tag{2.4}$$

*where $\langle X\rangle$ denotes the operator given by multiplication by the function $\langle x\rangle = \sqrt{1+|x|^2}$, and $\chi_x$ is the characteristic function of the cube of side $1$ centered at $x \in \mathbb{R}^d$. The strong insulator region $\Sigma_{\mathrm{SI}}$ for the random Schrödinger operator $H_\omega$ is the set of $E \in \Sigma$ for which there exists some open interval $I \ni E$ such that $H_\omega$ exhibits strong HS-dynamical localization in $I$.*

To discuss criteria for localization, we need to consider the restriction of the random Schrödinger operator $H_\omega$ to a finite box. Throughout this paper we use the sup norm in $\mathbb{R}^d$:

$$|x| = |x|_\infty = \max\{|x_i|,\ i = 1, \dots, d\}.$$

By $\Lambda_L(x)$ we denote the open box (or cube) of side $L > 0$:

$$\Lambda_L(x) = \left\{y \in \mathbb{R}^d;\ \ |y - x| < \tfrac{L}{2}\right\}, \tag{2.5}$$

and by $\overline{\Lambda}_L(x)$ the closed box. The characteristic function of a set $\Lambda \subset \mathbb{R}^d$ is denoted by $\chi_\Lambda$; we set

$$\chi_{x,L} = \chi_{\Lambda_L(x)}\,, \quad \text{with} \quad \chi_x = \chi_{x,1}\,. \tag{2.6}$$

*In this article we will take boxes centered at sites $x \in \mathbb{Z}^d$ with side $L \in 2\mathbb{N}$.* For such a box we set

$$\Gamma_{x,L} = \sum_{y \in \Upsilon_L(x)} \chi_y\,, \quad \text{where} \quad \Upsilon_L(x) = \left\{y \in \mathbb{Z}^d;\ |y-x| = \tfrac{L}{2} - 1\right\}\,. \tag{2.7}$$

The finite volume random Schrödinger operator $H_{\omega,x,L}$ is defined as the restriction of $H_\omega$, either to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition. (We consistently work with either Dirichlet or periodic boundary condition.) To see that $H_{\omega,x,L}$ is well defined as a semi-bounded self-adjoint operator on $\mathrm{L}^2(\Lambda_L(x), \mathrm{d}y)$, note that if $\nabla_{x,L}$ is the gradient operator restricted to either to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition, then it follows from (2.2) that for all $\psi \in \mathcal{D}(\nabla_{x,L})$ we have

$$\left|\left\langle \psi, V_\omega^{(2)}\psi \right\rangle\right| \le \Theta_1 \|\nabla_{x,L}\psi\|^2 + \Theta_2\|\psi\|^2 \quad \text{for } \mathbb{P}\text{-a.e. } \omega\,. \tag{2.8}$$

(For Dirichlet boundary condition (2.8) follows immediately from (2.2) with the same $\Theta_1$ and $\Theta_2$ as in (2.2). For periodic boundary condition (2.8) follows from (2.2) by using a smooth partition of the identity on the torus, with the same $\Theta_1$ but with $\Theta_2$ enlarged by a finite constant depending only on the dimension $d$, so we can modify $\Theta_2$ in (2.2) so (2.2) and (2.8) hold with the same $\Theta_1$ and $\Theta_2$ for all boxes $\Lambda_L(x)$.) We write $R_{\omega,x,L}(z) = (H_{\omega,x,L} - z)^{-1}$ for the resolvent.

A random Schrödinger operator satisfies all the requirements for the bootstrap multiscale analysis of [GK1] with the possible exception of a Wegner estimate, i.e., it satisfies Assumptions SLI, EDI, IAD, NE, and SGEE of [GK1] in any bounded interval [GK3, Theorem A.1]. We refer to [GK1, GK3] for a discussion of all these assumptions. For our purposes [GK3, Theorem A.1] can be restated as follows. (Note that (2.12) below is given in [GK3, Eq. (A.7)].)

**Theorem 2.3.** *Let $H_\omega$ be a random Schrödinger operator. Then $H_\omega$ satisfies Assumptions SLI, EDI, IAD, NE, and SGEE of* [GK1]*. In particular, Assumptions SLI and NE hold in the folowing form:*

**SLI:** *Given $L, \ell', \ell'' \in 2\mathbb{N}$, $x, y, y' \in \mathbb{Z}^d$ with $\Lambda_{\ell''}(y) \subset \Lambda_{\ell'-3}(y')$ and $\Lambda_{\ell'}(y') \subset \Lambda_{L-3}(x)$, then for $\mathbb{P}$-a.e. $\omega$, if $E \notin \sigma(H_{\omega,x,L}) \cup \sigma(H_{\omega,y',\ell'})$, we have*

$$\|\Gamma_{x,L}R_{\omega,x,L}(E)\chi_{y,\ell''}\| \le \gamma_E \,\|\Gamma_{y',\ell'}R_{\omega,y',\ell'}(E)\chi_{y,\ell''}\|\, \|\Gamma_{x,L}R_{\omega,x,L}(E)\Gamma_{y',\ell'}\|\,, \tag{2.9}$$

*with*

$$\gamma_E = 6\sqrt{\tfrac{2d}{1-\Theta_1}}\sqrt{\max\{E,0\} + \Theta_2 + \tfrac{(1+\Theta_1)100d}{1-\Theta_1}}\,. \tag{2.10}$$

*In the special case $\Theta_1 = 0$ we may take*

$$\gamma_E = 6\sqrt{2d}\,\sqrt{\max\{E,0\} + \Theta_2 + 50d}\,. \tag{2.11}$$

**NE:** *For $\mathbb{P}$-a.e. $\omega$ and all $E \in \mathbb{R}$ we have*

$$\operatorname{tr}\left\{\chi_{(-\infty,E)}(H_{\omega,x,L})\right\} \le C_E L^d \quad \textit{for all } x \in \mathbb{Z}^d \textit{ and } L \in 2\mathbb{N}\,, \tag{2.12}$$

*with*

$$C_E = c_d \left( \frac{\max\{E + \Theta_2\,, 0\}}{1 - \Theta_1} \right)^{\frac{d}{2}} , \tag{2.13}$$

$c_d$ *being a finite constant depending on* $d$ *only.*

Given a bounded interval $I$ we set

$$\gamma_I = \sup_{E \in I} \gamma_E\ , \quad C_I = \sup_{E \in I} C_E = C_{\sup_{E \in I} E}\ . \tag{2.14}$$

Note that Assumption EDI also holds at an energy $E$ with the same constant $\gamma_E$ as in (2.10) and (2.11).

In this article a Wegner estimate in an open interval (Assumption W in [GK1]) will be an explicit hypothesis.

**Definition 2.4.** *The random Schrödinger operator* $H_\omega$ *satisfies a Wegner estimate in an open interval* $\mathcal{I}$ *if for some* $b \geq 1$ *there exists a constant* $Q_{\mathcal{I}}$*, such that*

$$\mathbb{P}\{\operatorname{dist}(\sigma(H_{\omega,x,L}), E) \leq \eta\} \leq Q_{\mathcal{I}} \eta L^{bd}\ , \tag{2.15}$$

*for all* $E \in \mathcal{I}$*,* $0 < \eta \leq 1$*,* $x \in \mathbb{Z}^d$*, and* $L \in 2\mathbb{N}$*. The Wegner region* $\Sigma_{\mathrm{W}}$ *for the random Schrödinger operator* $H_\omega$ *is the set of* $E \in \Sigma$ *for which there exists some open interval* $I \ni E$ *such that* $H_\omega$ *satisfies a Wegner estimate in* $I$*.*

Wegner estimates have been proven for a large variety of random Schrödinger operators (e.g., [We, HM, CKM, CH1, Klo1, CHM, Ki, KSS1, St, CHN, CHKN, HK]), under some assumptions on the random potentials. Usually $b = 1$ or 2. In this paper, we shall use (2.15) as stated, the modifications in our methods required for the other forms of (2.15) being obvious. (For a discussion of possible modifications see [GK1, Remark 2.4].)

We will look for localization by studying the decay of the finite volume resolvent from the center of a box $\Lambda_L(x)$ to its boundary as measured by

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\|\,. \tag{2.16}$$

We use the convention that $\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\| = \infty$ if $E \in \sigma(H_{\omega,x,L})$.

We start with two deterministic (i.e., for a given $\omega$, which is omitted from the notation) definitions.

**Definition 2.5.** *Given* $\theta > 0$*,* $E \in \mathbb{R}$*,* $x \in \mathbb{Z}^d$*, and* $L \in 6\mathbb{N}$*, we say that the box* $\Lambda_L(x)$ *is* $(\theta, E)$*-suitable if*

$$\|\Gamma_{x,L} R_{x,L}(E) \chi_{x,L/3}\| \leq \frac{1}{L^\theta}\ . \tag{2.17}$$

**Definition 2.6.** *Given* $m > 0$*,* $E \in \mathbb{R}$*,* $x \in \mathbb{Z}^d$*, and* $L \in 6\mathbb{N}$*, we say that the box* $\Lambda_L(x)$ *is* $(m, E)$*-regular if*

$$\|\Gamma_{x,L} R_{x,L}(E) \chi_{x,L/3}\| \leq \mathrm{e}^{-m\frac{L}{2}}. \tag{2.18}$$

We define the multiscale analysis region $\Sigma_{\mathrm{MSA}} \subset \Sigma$ by requiring the conclusions of the bootstrap multiscale analysis [GK1, Theorem 3.4]. We use the notation

$$[K]_{6\mathbb{N}} = \max\{L \in 6\mathbb{N};\ L \leq K\}. \tag{2.19}$$

**Definition 2.7.** *The multiscale analysis region $\Sigma_{\mathrm{MSA}}$ for the random Schrödinger operator $H_\omega$ is the set of $E \in \Sigma_{\mathrm{W}}$ for which there exists some open interval $I \ni E$ such that, given any $\zeta$, $0 < \zeta < 1$, and $\alpha$, $1 < \alpha < \zeta^{-1}$, there is a length scale $L_0$ and a mass $m_\zeta > 0$, so if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \ldots$, we have*

$$\mathbb{P}\left\{R\left(m_\zeta, L_k, I, x, y\right)\right\} \geq 1 - \mathrm{e}^{-L_k^\zeta} \tag{2.20}$$

*for all $k = 0, 1, \ldots$, and $x, y \in \mathbb{Z}^d$ with $|x - y| > L_k + \varrho$, where*

$$\begin{aligned} &R(m, L, I, x, y) = \\ &\qquad \{\textit{for every } E' \in I, \ \textit{either } \Lambda_L(x) \textit{ or } \Lambda_L(y) \textit{ is } (m, E')\textit{-regular}\} . \end{aligned} \tag{2.21}$$

On $\Sigma_{\mathrm{MSA}}$ we have all desired properties of localization, including exponential localization and strong Hilbert-Schmidt dynamical localization [GK1]. In particular,

$$\Sigma_{\mathrm{MSA}} \subset \Sigma_{\mathrm{EL}} \cap \Sigma_{\mathrm{SI}} . \tag{2.22}$$

On the other hand, we proved in [GK3] that

$$\Sigma_{\mathrm{SI}} \cap \Sigma_{\mathrm{W}} \subset \Sigma_{\mathrm{MSA}} , \tag{2.23}$$

and hence we have:

**Theorem 2.8.** *Let $H_\omega$ be a random Schrödinger operator. Then*

$$\Sigma_{\mathrm{MSA}} = \Sigma_{\mathrm{SI}} \cap \Sigma_{\mathrm{W}} = \Sigma_{\mathrm{EL}} \cap \Sigma_{\mathrm{SI}} \cap \Sigma_{\mathrm{W}} . \tag{2.24}$$

In [GK4] we gave explicit finite volume criteria for $E \in \Sigma_{\mathrm{MSA}}$. The first criterion works for a *prescribed* value of the initial length scale $L_0$ [GK4, Theorem 2.4].

**Theorem 2.9.** *Let $H_\omega$ be a random Schrödinger operator. Fix a length scale $L_0 \in 6\mathbb{N}$, $L_0 \geq \max\{6, 3\varrho, \}$ . Let $E_0 \in \Sigma_{\mathrm{W}}$, with $H_\omega$ satisfying a Wegner estimate in the open bounded interval $\mathcal{I} \ni E_0$, and suppose*

$$\mathbb{P}\left\{ D_{\mathcal{I}} L_0^{\left(\frac{5}{3}+b\right)d} \|\Gamma_{0,L_0} R_{\omega,0,L_0}(E_0) \chi_{0,L_0/3}\| < 1 \right\} \geq 1 - \tfrac{2}{368^d} , \tag{2.25}$$

*with*

$$D_{\mathcal{I}} = 39^{(3+b)d} \max\left\{16 \cdot 60^d Q_{\mathcal{I}}, 1\right\} \gamma_{\mathcal{I}}^2 . \tag{2.26}$$

*Then $E_0 \in \Sigma_{\mathrm{MSA}}$.*

The second criterion is for *large* initial length scale $L_0$ and *weak* initial decay: any rate that is faster than the volume (if $b = 1$ in Wegner) or the volume squared (if $b = 2$) is allowed [GK4, Theorem 2.5]. It provides a precise estimate on how large $L_0$ has to be, depending on the parameters $Q_{\mathcal{I}}$ and $\gamma_{\mathcal{I}}$ of the model, and on the prescribed rate of decay of the resolvent.

**Theorem 2.10.** *Let $H_\omega$ be a random Schrödinger operator. Let $E_0 \in \Sigma_{\mathrm{W}}$, with $H_\omega$ satisfing a Wegner estimate in the open bounded interval $\mathcal{I} \ni E_0$. Fix $s > bd$, and set*

$$\mathcal{L} = \max\left\{3\varrho, 42, 3\left(\tfrac{107^d}{2}\right)^{\frac{2}{s-bd}}, \tfrac{1}{37}\left(16 \cdot 60^d Q_{\mathcal{I}}\right)^{\frac{2}{s-bd}}\right\} . \tag{2.27}$$

*Suppose that for some $L_0 \geq \mathcal{L}$, $L_0 \in 6\mathbb{N}$, we have*

$$\mathbb{P}\left\{90^d \gamma_{\mathcal{I}}^2 (37 L_0)^s \|\Gamma_{0,L_0} R_{\omega,0,L_0}(E_0) \chi_{0,L_0/3}\| < 1\right\} \geq 1 - \tfrac{2}{344^d} . \tag{2.28}$$

*Then $E_0 \in \Sigma_{\mathrm{MSA}}$.*

The next criterion is an analog of Theorem 2.10 but for the second multiscale analysis of the bootstrap scheme of [GK1], i.e., for the multiscale analysis with exponential decay of the resolvent and polynomial decay of the probabilities as in von Dreifus and Klein [vDK], modified as in Figotin and Klein [FK1, Theorem 32] to allow the mass in the starting hypotheses to decrease as the initial length scale increases [GK4, Theorem 2.6].

Theorem 2.11 gives an estimate of the exponential rate of decay of the eigenfunctions in terms of the constants of the problems; the price is paid in stronger hypotheses than Theorem 2.10. If the probability estimate in the initial length scale is sufficiently good, one may use Theorem 2.11 to start the bootstrap multiscale analysis, instead of applying first Theorem 2.9 or Theorem 2.10 and then bootstrapping to Theorem 2.11 to obtain the exponential rate of decay of eigenfunctions.

**Theorem 2.11.** *Let $H_\omega$ be a random Schrödinger operator. Let $E_0 \in \Sigma_{\mathrm{W}}$, with $H_\omega$ satisfing a Wegner estimate in the open bounded interval $\mathcal{I} \ni E_0$. Fix $p > 0$, $\alpha = 1 + \frac{p/2}{p+2d}$, $s > 2p + (b+2)d$, and $\theta > 4\alpha s$, and set*

$$\mathcal{L} = \max \Big\{ 3\varrho, 417^{\frac{1}{\alpha - 1}}, \operatorname{dist}(E_0, \mathbb{R}\backslash\mathcal{I})^{-\frac{1}{s}}, Q_{\mathcal{I}}^{\frac{1}{s-p-bd}}, \qquad (2.29)$$
$$\left(16 \cdot 36^d 2^{s-2p} C_{\mathcal{I}} Q_{\mathcal{I}}\right)^{\frac{1}{\alpha(s-2p-(b+2)d)}}, \left(69^d \gamma_{\mathcal{I}}^2\right)^{\frac{4}{\theta - 4\alpha s}}, \left(3^d \gamma_{\mathcal{I}}\right)^{\frac{16\alpha}{\theta(\alpha-1)}} \Big\}.$$

*If for some $L_0 \geq \mathcal{L}$, $L_0 \in 6\mathbb{N}$, we have*

$$\mathbb{P}\{\Lambda_{L_0}(0) \textit{ is } (\theta, E_0)\textit{-suitable}\} > 1 - \frac{1}{L_0^p}, \qquad (2.30)$$

*then there exists an open interval $I = I(\theta, s, L_0)$, with $E_0 \in I \subset \mathcal{I}$, such that if we set $m_0 = 2\theta \frac{\log L_0}{L_0}$, and $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $k = 0, 1, \ldots$, we have*

$$\mathbb{P}\{\Lambda_{L_k}(0) \textit{ is } (\tfrac{m_0}{4}, E)\textit{-regular}\} \geq 1 - \frac{2}{L_k^p} \quad \textit{for all } E \in I, \qquad (2.31)$$

*and*

$$\mathbb{P}\left[R\left(\tfrac{m_0}{4}, L_k, I, x, y\right)\right] \geq 1 - \frac{4}{L_k^{2p}} \textit{ for } x, y \in \mathbb{Z}^d, |x-y| > L_k + \varrho, \qquad (2.32)$$

*for all $k = 0, 1, \ldots$, where*

$$R(m, L, I, x, y) = \qquad (2.33)$$
$$\{\textit{for every } E \in I, \textit{ either } \Lambda_L(x) \textit{ or } \Lambda_L(y) \textit{ is } (m, E)\textit{-regular}\}.$$

*Moreover,*

**(i):** $I \cap \Sigma \subset \Sigma_{\mathrm{MSA}}$.

**(ii):** *For almost every $\omega$, an eigenfunction $\varphi_{\omega,E}$ of $H_\omega$ with eigenvalue $E \in I$ decays exponentially (in the $L^2$-sense) with a rate $\geq \frac{\theta}{2}\frac{\log L_0}{L_0}$, i.e.,*

$$\liminf_{|x| \to \infty} -\frac{\log \|\chi_x \varphi_{\omega,E}\|}{|x|} \geq \frac{\theta \log L_0}{2L_0}. \qquad (2.34)$$

## 3. Anderson Hamiltonians

In this section we apply our explicit criteria to Anderson Hamiltonians in the continuum. These are the most studied random Schrödinger operators in the continuum (e.g., [HM, CH1, Klo1, CHM, Ki, KSS1, CHN, St, Klo2, GK4]).

In this article an Anderson Hamiltonian will be a random Schrödinger operator of the form

$$H_{\lambda,\omega} = -\Delta + V_{\mathrm{per}} + \lambda V_\omega \,, \tag{3.1}$$

where

**(a):** $V_{\mathrm{per}}$ is a periodic potential with period 1 (by rescaling we can always take the period to be one), such that $V_{\mathrm{per}} = V_{\mathrm{per}}^{(1)} + V_{\mathrm{per}}^{(2)}$, with $V_{\mathrm{per}}^{(i)}$, $i = 1, 2$, periodic with period one, $0 \le V_{\mathrm{per}}^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$, and $V_{\mathrm{per}}^{(2)}$ relatively form-bounded with respect to $-\Delta$ with relative bound $< 1$: there are nonnegative constants $\Theta_{\mathrm{per},1} < 1$ and $\Theta_{\mathrm{per},2}$ such that for all $\psi \in \mathcal{D}(\nabla)$ we have

$$\left|\left\langle \psi, V_{\mathrm{per}}^{(2)} \psi \right\rangle\right| \le \Theta_{\mathrm{per},1} \|\nabla \psi\|^2 + \Theta_{\mathrm{per},2} \|\psi\|^2 \,. \tag{3.2}$$

**(b):**

$$V_\omega(x) = \sum_{i \in \frac{1}{q}\mathbb{Z}^d} \omega_i \, u(x - i), \tag{3.3}$$

where

**(b1):** $q \in \mathbb{N}$.

**(b2):** $\omega = \{\omega_i;\ i \in \frac{1}{q}\mathbb{Z}^d\}$ a family of independent, identically distributed random variables taking values in the bounded interval $[M_-, M_+]$, whose common probability distribution $\mu$ has a bounded density $g$ with $g > 0$ a.e. in $[M_-, M_- + a)$ for some $a > 0$.

**(b3):** $u(x)$ is a real valued measurable function with compact support, say $\operatorname{supp} u \subset \Lambda_\varrho(0)$, $0 \le u \in \mathrm{L}^p(\mathbb{R}^d, \mathrm{d}x)$ with $p \in [1, \infty]$ if $d = 1$ or $M_- \ge 0$, $p \in (\frac{d}{2}, \infty]$ if $d \ge 2$ and $M_- < 0$, and $0 < U_- \le U(x)$ where $U(x) = \sum_{i \in \frac{1}{q}\mathbb{Z}^d} u(x - i)$.

**(c):** $\lambda > 0$ is the disorder parameter.

$H_{\lambda,\omega}$ is a random Schrödinger operator for each $\lambda > 0$. ( If $M_- < 0$, $V_\omega$ is a potential in Kato class for $\mathbb{P}$-a.e. $\omega$ (see [Si, Example E, p. 457]). Its nonrandom spectrum will be denoted by $\Sigma_\lambda$, i.e., $\Sigma_\lambda = \sigma(H_{\lambda,\omega})$ for $\mathbb{P}$-a.e. $\omega$. We let $E_\lambda^{\inf}$ denote the bottom of the spectrum, i.e.,

$$E_\lambda^{\inf} = \inf \Sigma_\lambda \,. \tag{3.4}$$

**Theorem 3.1.** *Let $H_{\lambda,\omega}$ be an Anderson Hamiltonian. Then*

$$E_\lambda^{\inf} = \inf \sigma(H_{\lambda, M_-}) \,, \tag{3.5}$$

*where*

$$H_{\lambda,M_-} = H_{\lambda, \{\omega_i = M_-;\ i \in \frac{1}{q}\mathbb{Z}^d\}} = -\Delta + V_{\mathrm{per}} + \lambda M_- U \,. \tag{3.6}$$

*We have the upper bound*

$$E_\lambda^{\inf} \le \int_{\Lambda_1(0)} V_{\mathrm{per}}(x)\, \mathrm{d}x + \lambda M_- \int_{\Lambda_1(0)} U(x)\, \mathrm{d}x \,, \tag{3.7}$$

*so, in particular,*

$$E_\lambda^{\inf} \leq \int_{\Lambda_1(0)} V_{\mathrm{per}}(x)\,\mathrm{d}x + \lambda M_- U_- \quad \text{if } M_- < 0\,. \tag{3.8}$$

*Moreover, if $M_- < 0$ we have*

$$\left|\left\langle \psi, \left(V_{\mathrm{per}}^{(2)} + \lambda M_- U\right)\psi\right\rangle\right| \leq \tfrac{1}{2}(1+\Theta_{\mathrm{per},1})\|\nabla\psi\|^2 + \left(\Theta_{\mathrm{per},2} + \beta\lambda^{\frac{2p}{2p-d}}\right)\|\psi\|^2 \tag{3.9}$$

*for all $\psi \in \mathcal{D}(\nabla)$, where*

$$\beta = \left(C_{d,q,\varrho,p}\,|M_-|\,\|u\|_p\right)^{\frac{2p}{2p-d}}\left(\frac{2}{1-\Theta_{\mathrm{per},1}}\right)^{\frac{d}{2p-d}}, \tag{3.10}$$

*with $C_{d,q,\varrho,p}$ a constant depending only on the indicated parameters. Thus we have the lower bound*

$$E_\lambda^{\inf} \geq \begin{cases} -\Theta_{\mathrm{per},2} + \lambda M_- U_- & \text{if } M_- \geq 0 \\ -\Theta_{\mathrm{per},2} - \beta\lambda^{\frac{2p}{2p-d}} & \text{if } M_- < 0 \end{cases}. \tag{3.11}$$

We give a proof for Theorem 3.1 in Appendix A.

**Theorem 3.2.** *Let $H_{\lambda,\omega}$ be an Anderson Hamiltonian, and set*

$$L_0 = \min\{L \in 6\mathbb{N};\ L \geq \max\{3\varrho, 3(2+\sqrt{d})\}\}\,. \tag{3.12}$$

*Then there exists $\lambda^*$, depending only on $d$, $U_-$, $\|g\|_\infty$, $\varrho$, $p$, $\|u\|_p$, $M_-$, $\Theta_{\mathrm{per},1}$, and $\Theta_{\mathrm{per},2}$, such that for any $\lambda \geq \lambda^*$ we have*

$$\left[E_\lambda^{\inf}, E_\lambda^{\inf} + c\lambda\right] \cap \Sigma_\lambda \subset \Sigma_{\mathrm{MSA}} \quad \text{with } c = \frac{U_-}{(368L_0)^d\|g\|_\infty}\,. \tag{3.13}$$

*Moreover there exists $c' > 0$, depending on $d$, $U_-$, $\|g\|_\infty$, and $\varrho$, but not on $\lambda \geq \lambda^*$, such that if $\lambda \geq \lambda^*$, then for a.e. $\omega$, if $\varphi_{\lambda,\omega,E}$ is an eigenfunction of $H_{\lambda,\omega}$ with eigenvalue $E \in \left[E_\lambda^{\inf}, E_\lambda^{\inf} + c\lambda\right]$, then*

$$\liminf_{|x|\to\infty} -\frac{\log\|\chi_x\varphi_{\omega,E}\|}{|x|} \geq c'\sqrt{\lambda}. \tag{3.14}$$

**Remark 3.3.** *In the weak disorder regime, localization of Anderson Hamiltonians in an interval of size $\lambda$ at the bottom of the spectrum has been proved by Klopp* [Klo2] *using Lifschitz tails. As pointed out in* [Klo2, p. 728], *in this case the minimal length scale for the initial length scale estimate grows polynomially in $\frac{1}{\lambda}$, since the constant in the Wegner estimate is proportional to $\frac{1}{\lambda}$, but, because of the strong estimate given in* [Klo2, Proposition 3.2], *the usual proofs of the multiscale analysis may be modified to yield localization in this situation. Theorem 2.11 may be applied directly with Klopp's estimates to prove localization in this weak disorder regime.*

*Proof of Theorem 3.2.* We follow the same strategy as in [GK1, Proof of Theorem 3.4], but taking into account the fact that $E_\lambda^{\inf} \to \pm\infty$ as $\lambda \to \infty$ if $M_- \gtrless 0$, with $|E_\lambda^{\inf}|$ growing polynomially in $\lambda$.

The finite volume operators $H_{\lambda,\omega,x,L}$ are taken as the restriction of $H_{\lambda,\omega}$, either to the open box $\Lambda_L(x)$ with Dirichlet boundary condition, or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary condition. The random potential $V_{\mathrm{per}} + \lambda V_\omega$ satisfies the finite volume condition (2.8) with, if $M_- \geq 0$,

$$\Theta_{\lambda,1} = \Theta_{\mathrm{per},1}\,, \tag{3.15}$$

$$\Theta_{\lambda,2} = \Theta_{\mathrm{per},2} + c_d\,, \tag{3.16}$$

and, if $M_- < 0$, in view of (3.9),

$$\Theta_{\lambda,1} = \tfrac{1}{2}(1+\Theta_{\mathrm{per},1}), \tag{3.17}$$

$$\Theta_{\lambda,2} = \Theta_{\mathrm{per},2} + \beta\lambda^{\frac{2p}{2p-d}} + c_d \, , \tag{3.18}$$

where $c_d$ is a constant depending only on $d$ (see the remark following (2.8); $c_d = 0$ for Dirichlet boundary coundition). For a given bounded interval $\mathcal{I}$, the constants $\gamma_{\lambda,\mathcal{I}}$ and $C_{\lambda,\mathcal{I}}$ can then be read from (2.14), (2.10) and (2.13).

The Wegner estimate (2.15) can be derived as in [Ki, Proposition 1] or [FK1, Theorem 2.3] with $b=2$, $\eta_{\mathcal{I}} = 1$, and

$$Q_{\lambda,\mathcal{I}} = \sup_{E\in\mathcal{I}} Q_{\lambda,E}\, , \text{ with } Q_{\lambda,E} = \frac{C_d\|g\|_\infty}{\lambda U_-}\left(\frac{\max\{E+\Theta_{\lambda,2}+1\, , 0\}}{1-\Theta_{\lambda,1}}\right)^{\frac{d}{2}}, \tag{3.19}$$

with $C_d$ a constant depending only on $d$.

We now derive an elementary probability estimate: if $\delta > 0$ and $L_0 \in 2\mathbb{N}$, we have

$$\begin{aligned} &\mathbb{P}\{\omega_i > M_- + \delta \text{ for all } i \in \Lambda_{L_0}(0)\} \\ &\quad = 1 - \mathbb{P}\{\omega_i \in [M_-, M_- + \delta] \text{ for some } i \in \Lambda_{L_0}(0)\} \\ &\quad \geq 1 - \delta\|g\|_\infty L_0^d. \end{aligned} \tag{3.20}$$

We fix $\delta$ and $L_0$, and set

$$K_\lambda = \tfrac{1}{2}\delta U_- \lambda\, , \tag{3.21}$$

$$E_\lambda = E_\lambda^{\inf} + K_\lambda\, . \tag{3.22}$$

It follows that

$$\mathbb{P}\{A_\lambda\} \geq 1 - \delta\|g\|_\infty L_0^d\, , \tag{3.23}$$

where $A_\lambda$ denotes the event

$$A_{\lambda,L} = \{\inf\sigma(H_{\lambda,\omega,L_0}) \geq E_\lambda^{\inf} + 2K_\lambda\}\, . \tag{3.24}$$

We now use Theorem 2.9. To maximize $\delta$, we need to minimize $L_0$; hence we pick $L_0$ as in (3.12) and choose $\delta$ by matching the right hand sides of (3.23) and (2.25), i.e.,

$$\delta\|g\|_\infty L_0^d = \tfrac{2}{368^d}\, . \tag{3.25}$$

If $\omega \in A_\lambda$, it follows that

$$\operatorname{dist}(E, \sigma(H_{\lambda,\omega,L})) \geq K_\lambda \;\text{ for any }\; E \in [E_\lambda^{\inf}, E_\lambda]\, , \tag{3.26}$$

so we can use the Combes-Thomas estimate to get the decay of the resolvent for any $E \in [E_\lambda^{\inf}, E_\lambda]$. The exact dependency of the exponential decay rate in the Combes-Thomas estimate in terms of the energy parameter and the distance to the spectrum is crucial in our argument, since we deal with large energies and large distances from the spectrum. Such a precise estimate is provided in [GK2, Eq. (19) in Theorem 1], and adapted to the finite volume case in [GK1, Proof of Theorem 3.4]: In a box $\Lambda_{L_0}(0)$, with either Dirichlet or periodic boundary condition, the same estimates as in [GK2, Eq. (19) in Theorem 1] hold for $x, y \in \bar{\Lambda}_{L_0-2}(0)$ (i.e., at a distance $\geq 1$ from the boundary) but the exponential rates of decay get divided by $1+2\sqrt{d}L_0$. Thus, if $\omega \in A_\lambda$ and $E \in [E_\lambda^{\inf}, E_\lambda]$, we have that for any $x,y \in \mathbb{Z}^d \cap \Lambda_{L_0}(0) = \mathbb{Z}^d \cap \bar{\Lambda}_{L_0-2}(0)$, with $|x-y| \geq \sqrt{d}$,

$$\|\chi_x R_{\lambda,\omega,L_0}(E)\chi_y\| \quad \leq \quad \tfrac{2}{K_\lambda}\, \mathrm{e}^{-\sqrt{2K_\lambda}\left(1+2\sqrt{d}L_0\right)^{-1}\left(|x-y|-\sqrt{d}\right)}\,. \tag{3.27}$$

Summing over the support of $\Gamma_{L_0}$ and of $\chi_{L_0/3}$, noting that if $x \in \Lambda_{L_0/3}(0)$, $y \in \Upsilon_{L_0}(0)$ we have $|x-y| \geq \frac{L_0}{3} - 1$, and using (3.12) yields

$$\|\Gamma_{L_0} R_{\lambda,\omega,L_0}(E)\chi_{L_0/3}\| \leq \tfrac{2d}{K_\lambda}\, L_0^{2d-1} \mathrm{e}^{-\sqrt{2K_\lambda}\left(1+2\sqrt{d}L_0\right)^{-1}}\,. \tag{3.28}$$

We will show that (2.25) of Theorem 2.9 is satisfied for all $E \in [E_\lambda^{\inf}, E_\lambda]$ for large disorder. Let $\mathcal{I}_\lambda = (-\Theta_{\lambda,2} - 1, E_\lambda + K_\lambda - 1)$, we have, using (2.26), (2.14), (2.10), (3.7), (3.15)-(3.18), (3.19), (3.21), and (3.28),

$$\begin{aligned} & D_{\lambda,\mathcal{I}_\lambda} L_0^{\left(\frac{5}{3}+b\right)d} \|\Gamma_{L_0} R_{\lambda,\omega,L_0}(E)\chi_{L_0/3}\| && (3.29)\\ &\quad = \; 39^{5d} \max\left\{16 \cdot 60^d Q_{\lambda,\mathcal{I}_\lambda}, 1\right\} \gamma_{\lambda,\mathcal{I}_\lambda}^2 L_0^{\frac{11}{3}d} \|\Gamma_{L_0} R_{\lambda,\omega,,L_0}(E)\chi_{L_0/3}\| \\ &\quad \leq \; c_1 \lambda^\tau \mathrm{e}^{-c_2\sqrt{\lambda}} && (3.30)\\ &\quad < \; 1\,, && (3.31) \end{aligned}$$

with

$$\tau = \begin{cases} \max\left\{\frac{d}{2} - 1, 0\right\} & \text{if } M_- \geq 0 \\ \max\left\{\frac{2p}{2p-d}\frac{d}{2} - 1, 0\right\} + \frac{2p}{2p-d} - 1 & \text{if } M_- < 0 \end{cases}\,, \tag{3.32}$$

where (3.30) holds for $\omega \in A_\lambda$ with $c_1$ and $c_2$ constants depending only on $d$, $U_-$, $\|g\|_\infty$, $\varrho$, $p$, $\|u\|_p$, $M_-$, $\Theta_{\mathrm{per},1}$, and $\Theta_{\mathrm{per},2}$ (note that we *fixed* $L_0$ and $\delta$ in (3.12) and (3.25)). We conclude that there exists $\lambda^*$, depending only on $d$, $U_-$, $\|g\|_\infty$, $\varrho$, $p$, $\|u\|_p$, $M_-$, $\Theta_{\mathrm{per},1}$, and $\Theta_{\mathrm{per},2}$, such that we have (3.31) for all $\lambda > \lambda^*$ and $\omega \in A_\lambda$.

Thus condition (2.25) holds for $\lambda > \lambda^*$ and $E \in [E_\lambda^{\inf}, E_\lambda]$ by (3.23), hence Theorem 2.9 implies that $[E_\lambda^{\inf}, E_\lambda] \cap \Sigma_\lambda \subset \Sigma_{\mathrm{MSA}}$.

It remains to prove the estimate (3.14) on the the rate of the exponential decay of the eigenfunctions with energies in $[E_\lambda^{\inf}, E_\lambda]$. To do so we use the criterion given in Theorem 2.11. We start by defining $\theta_\lambda$ by

$$L_0^{-\theta_\lambda} = \tfrac{2d}{K_\lambda}\, L_0^{2d-1} \mathrm{e}^{-\sqrt{2K_\lambda}\left(1+2\sqrt{d}L_0\right)^{-1}}\,. \tag{3.33}$$

Then (3.28) says that for $\omega \in A_\lambda$ the box $\Lambda_{L_0}(0)$ is $(\theta_\lambda, E)$-suitable for any $E \in [E_\lambda^{\inf}, E_\lambda]$. Moreover, it follows from (3.33) that

$$\theta_\lambda \geq c_0\sqrt{\lambda} \quad \text{for all} \quad \lambda \geq \lambda^*\,, \tag{3.34}$$

where $c_0$ is some constant and $\lambda^*$ is taken large enough (both depending only on $d$, $U_-$, $\|g\|_\infty$, and $\varrho$).

As in [GK1, Proof of Theorem 3.4], we bootstrap from [GK4, Theorem 2.7] (of which Theorem 2.9 is a special case) to Theorem 2.11. It follows, as in [GK4, Proof of Theorem 3.1], that if we set $L_k = 39^k L_0$, $k = 0, 1, 2, \ldots$, where $L_0$ is as in (3.12), then for all $\lambda > \lambda^*$ we have

$$\mathbb{P}\{\Lambda_{L_k}(0) \text{ is } (\theta_\lambda, E)\text{-suitable}\} \geq 1 - \frac{1}{L_k^{\frac{5}{3}}}\,, \tag{3.35}$$

for all $E \in [E_\lambda^{\inf}, E_\lambda]$ and $k \geq \mathcal{K}$, where $\mathcal{K} = \mathcal{K}(d, \varrho) < \infty$ is a constant depending only on $d$ and $\varrho$. A key fact is that $\mathcal{K}$ does not depend on $\lambda$.

We now feed (3.35) into the hypotheses of Theorem 2.11. We have already fixed $p = \frac{5}{3}$, and hence $\alpha$ is fixed. On the other hand for each $\lambda$ we have $\theta = \theta_\lambda$ as in (3.33), and $Q_{\lambda,\mathcal{I}_\lambda}$, $C_{\lambda,\mathcal{I}_\lambda}$, and $\gamma_{\lambda,\mathcal{I}_\lambda}$ grow polynomially with $\lambda$. To control $\mathcal{L}_\lambda$ as given in (2.29) with all this dependence in $\lambda$, we pick the remaining parameter, $s$, to also depend on $\lambda$ by $s_\lambda = \log \lambda$. By taking $\lambda^*$ sufficiently large, as before depending only on $d$, $U_-$, $\|g\|_\infty$, $\varrho$, $p$, $\|u\|_p$, $M_-$, $\Theta_{\mathrm{per},1}$, and $\Theta_{\mathrm{per},2}$, and using (3.34), we can also guarantee $\theta_\lambda > 4\alpha s_\lambda$ and $s_\lambda > 2p + 4d$ for all $\lambda > \lambda^*$. It follows from the explicit form of (2.29) that

$$\mathcal{L}_\infty = \sup_{\lambda > \lambda^*} \sup_{E \in [E_\lambda^{\inf}, E_\lambda]} \mathcal{L}_\lambda(E) < \infty\,, \tag{3.36}$$

where $\mathcal{L}_\lambda(E)$ is given by (2.29) with $E_0 = E$, $\mathcal{I} = \mathcal{I}_\lambda$, $p = \frac{5}{3}$, $\theta = \theta_\lambda$, and $s = s_\lambda$. Note that $\mathcal{L}_\infty$ depends only on $d$, $U_-$, $\|g\|_\infty$, $\varrho$, $p$, $\|u\|_p$, $M_-$, $\Theta_{\mathrm{per},1}$, and $\Theta_{\mathrm{per},2}$.

We now fix $\kappa$ to be that smallest $k \geq \mathcal{K}$ such that $L_k = 39^k L_0 \geq \mathcal{L}_\infty$. It follows from Theorem 2.11 that if $\lambda > \lambda^*$, then for almost every $\omega$, the eigenfunctions $\varphi_{\lambda,\omega,E}$ with energy $E \in [E_\lambda^{\inf}, E_\lambda]$ decay exponentially (in the $L^2$-sense) with

$$\liminf_{|x| \to \infty} -\frac{\log \|\chi_x \varphi_{\lambda,\omega,E}\|}{|x|} \geq \frac{\theta_\lambda \log L_\kappa}{2L_\kappa}\,. \tag{3.37}$$

so (3.14) follows from (3.34). □

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## Appendix A. Proof of Theorem 3.1

The identity (3.5) follows from ergodicity and hypothesis (b2), as in [KM, Theorem 3]. Note that

$$H_{\lambda,\omega} = H_{\lambda,M_-} + \lambda \sum_{i \in \frac{1}{q}\mathbb{Z}^d} (\omega_i - M_-)\, u(x-i) \ \text{ with } \ \omega_i - M_- \geq 0. \tag{A.1}$$

To obtain the upper bound (3.7), let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$, and set $\varphi_n(x) = \frac{1}{n^{d/2}}\varphi(\frac{1}{n}x)$ for $n \in \mathbb{N}$; note $\|\varphi_n\|_2 = 1$. It follows from (3.5) that

$$\begin{aligned} E_\lambda^{\inf} &\leq \langle \varphi_n, H_{\lambda,M_-}\varphi_n\rangle \\ &= \frac{1}{n^2}\langle \varphi, -\Delta\varphi\rangle + \int_{\mathbb{R}^d} (V_{\mathrm{per}}(nx) + \lambda M_- U(nx))\, |\varphi(x)|^2 \,\mathrm{d}x\,. \end{aligned} \tag{A.2}$$

Letting $n \to \infty$ we get (3.7).

We now turn to (3.9) and the lower bound (3.11). The lower bound for $M_- \geq 0$ is obvious. For $M_- < 0$ it follows from (3.9), which will follow from the following lemma.

**Lemma A.1.** *Let $W$ be a real valued measurable function on $\mathbb{R}^d$ such that that $W \in \mathrm{L}_u^p(\mathbb{R}^d, \mathrm{d}x)$, i.e.,*

$$\|W\|_{p,u} = \sup_{x \in \mathbb{Z}^d} \|\chi_x W\|_p < \infty\,, \tag{A.3}$$

*with $p \in (\frac{d}{2}, \infty]$ if $d \geq 2$, and $p \in [1, \infty]$ if $d = 1$. Then for every $m > 0$ we have*

$$|\langle \psi, W\psi\rangle| \leq C_{d,p}\, m^{-2+\frac{d}{p}} \|W\|_{p,u} \left(\|\nabla\psi\|^2 + m^2\|\psi\|^2\right) \ \textit{for all } \psi \in \mathcal{D}(\nabla), \tag{A.4}$$

*where $C_{d,p}$ is a finite constant depending only on $d$ and $p$.*

*Proof.* It suffices to prove

$$\left(-\Delta+m^2\right)^{-\frac{1}{2}}|W|\left(-\Delta+m^2\right)^{-\frac{1}{2}} \leq C_{d,p}\, m^{-2+\frac{d}{p}}\|W\|_{p,u}\,, \tag{A.5}$$

or, equivalently,

$$|W|^{\frac{1}{2}}\left(-\Delta+m^2\right)^{-1}|W|^{\frac{1}{2}} \leq C_{d,p}\, m^{-2+\frac{d}{p}}\|W\|_{p,u}\,. \tag{A.6}$$

We recall (e.g., [GJ, Proposition 7.2.1]) that for $m>0$ the resolvent $\left(-\Delta+m^2\right)^{-1}$ has an integral kernel $K_m(x-y)$, such that

$$K_m(x)=m^{d-2}K_1(mx)\,, \tag{A.7}$$

and

$$0<K_1(x)\leq\begin{cases} C_1\,\mathrm{e}^{-|x|} & \text{if } d=1\\ C_2\,(|\ln|x||+1)\,\mathrm{e}^{-|x|} & \text{if } d=2\\ C_3\,|x|^{-1}\mathrm{e}^{-|x|} & \text{if } d=3\\ C_d\,|x|^{-d+2}\,\mathrm{e}^{-\frac{1}{2}|x|} & \text{if } d\geq 4\end{cases}, \tag{A.8}$$

where $C_d$ is a constant depending only on the dimension. If $p>\frac{d}{2}$ if $d\geq 2$, and $p\geq 1$ if $d=1$, we have

$$\|K_m\|_{\frac{p}{p-1}} \leq C_{d,p}\, m^{-2+\frac{d}{p}}\,, \tag{A.9}$$

where $C_{d,p}<\infty$ depends only on $d$ and $p$

If $\psi\in \mathrm{L}^2(\mathbb{R}^d,\mathrm{d}x)$, we have,

$$\begin{aligned}\left\langle\psi,|W|^{\frac{1}{2}}\left(-\Delta+m^2\right)^{-1}|W|^{\frac{1}{2}}\psi\right\rangle &= \left\langle\psi,|W|^{\frac{1}{2}}K_m*|W|^{\frac{1}{2}}\psi\right\rangle & \text{(A.10)}\\ &\leq \sum_{x,y\in\mathbb{Z}^d}\left|\left\langle\psi,|W|^{\frac{1}{2}}\chi_x K_m*\chi_y|W|^{\frac{1}{2}}\psi\right\rangle\right| \\ &\leq \sum_{x,y\in\mathbb{Z}^d}\|W\|_{p,u}\|K_m\|_{\frac{p}{p-1}}\|\chi_x\psi\|_2\|\chi_y\psi\|_2 & \text{(A.11)}\\ &\leq \|W\|_{p,u}\|K_m\|_{\frac{p}{p-1}}\|\psi\|_2^2\,, & \text{(A.12)}\end{aligned}$$

where we used Holder's and Young's inequalities to get (A.11), and the Cauchy inequality to obtain (A.12).

The estimate (A.6) follows. from (A.12) and (A.9). ☐

We can now prove (3.9). Note that there exists a finite constant $c_{d,q,\varrho}$, depending only on $d$ and $\varrho$, such that

$$\|U\|_{p,u}=\|\chi_0\, U\|_p\leq c_{d,q,\varrho}\|u\|_p<\infty \tag{A.13}$$

It thus follows from (A.4) that for all $m>0$ we have

$$|\langle\psi,\lambda M_- U\psi\rangle|\leq\alpha\lambda\left(m^{-2+\frac{d}{p}}\|\nabla\psi\|^2+m^{\frac{d}{p}}\|\psi\|^2\right) \text{ for all } \psi\in\mathcal{D}(\nabla), \tag{A.14}$$

with

$$\alpha=c_{d,q,\varrho}\, C_{d,p}\,|M_-|\,\|u\|_p\,. \tag{A.15}$$

To obtain (3.9), we choose $m$ depending on $\lambda$ by

$$\alpha\lambda m^{-2+\frac{d}{p}}=\tfrac{1}{2}(1-\Theta_{\mathrm{per},1})\,, \tag{A.16}$$

i.e.,

$$m = \left( \frac{2\alpha\lambda}{1 - \Theta_{\text{per},1}} \right)^{\frac{p}{2p-d}} , \tag{A.17}$$

and we obtain

$$|\langle \psi, \lambda M_{-} U \psi \rangle| \leq \tfrac{1}{2}(1 - \Theta_{\text{per},1}) \|\nabla \psi\|^2 + \beta \lambda^{\frac{2p}{2p-d}} \|\psi\|^2 \text{ for all } \psi \in \mathcal{D}(\nabla), \tag{A.18}$$

with

$$\beta = \alpha^{\frac{2p}{2p-d}} \left( \frac{2}{1 - \Theta_{\text{per},1}} \right)^{\frac{d}{2p-d}} . \tag{A.19}$$

The estimate (3.9) now follows from (3.2) and (A.18). □

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