SCATTERING THEORY FOR N-BODY QUANTUM SYSTEMS IN A TIME-PERIODIC ELECTRIC FIELD

Tadayoshi Adachi

Department of Mathematics, Faculty of Sciences, Kobe University
1-1, Rokkodai, Nada-ku, Hyogo 657-8501, Japan
e-mail: adachi@math.kobe-u.ac.jp
and
Centre de Mathématiques, École Polytechnique
F-91128 Palaiseau Cedex, France

§1. Introduction

In this paper, we study the scattering theory for N-body quantum systems in a time-periodic electric field. First we shall give the notations in the N-body scattering theory for describing our results. We consider a system of N particles moving in a given time-periodic electric field $\mathcal{E}(t) \in \boldsymbol{R}^d$, $\mathcal{E}(t) \not\equiv 0$. Let m_j, e_j and $r_j \in \boldsymbol{R}^d$, $1 \le j \le N$, denote the mass, charge and position vector of the j-th particle, respectively. We suppose that the particles under consideration interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \le j < k \le N$. We assume that these pair potentials are independent of time t. Then the total Hamiltonian for the system is given by

$$\tilde{H}(t) = \sum_{1 \le j \le N} \left\{ -\frac{1}{2m_j} \Delta_{r_j} - e_j \langle \mathcal{E}(t), r_j \rangle \right\} + V,$$

where $\langle \xi, \eta \rangle = \sum_{j=1}^d \xi_j \eta_j$ for ξ, $\eta \in \boldsymbol{R}^d$ and

$$V = \sum_{1 \le j < k \le N} V_{jk}(r_j - r_k).$$

We shall consider the Hamiltonian $\tilde{H}(t)$ in the center-of-mass frame. We equip $\boldsymbol{R}^{d \times N}$ with the metric $r \cdot \tilde{r} = \sum_{j=1}^N m_j \langle r_j, \tilde{r}_j \rangle$ for $r = (r_1, \dots, r_N)$ and $\tilde{r} = (\tilde{r}_1, \dots, \tilde{r}_N) \in \boldsymbol{R}^{d \times N}$ and denote $|r| = \sqrt{r \cdot r}$. Let X and X_{cm} be the configuration spaces for the inner motion of the particles and the center of mass motion, respectively:

$$X = \left\{ r \in \boldsymbol{R}^{d \times N} \;\middle|\; \sum_{1 \le j \le N} m_j r_j = 0 \right\},$$
$$X_{\mathrm{cm}} = \left\{ r \in \boldsymbol{R}^{d \times N} \;\middle|\; r_j = r_k \text{ for } 1 \le j < k \le N \right\}.$$

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X and $X_{\rm cm}$ are mutually orthogonal and $\boldsymbol{R}^{d\times N} = X \oplus X_{\rm cm}$. We denote by $\pi : \boldsymbol{R}^{d\times N} \to X$ and $\pi_{\rm cm} : \boldsymbol{R}^{d\times N} \to X_{\rm cm}$ the orthogonal projections onto X and $X_{\rm cm}$, respectively, and write $x = \pi r$ and $x_{\rm cm} = \pi_{\rm cm} r$ for $r \in \boldsymbol{R}^{d\times N}$, and

$$E(t) = \pi\left(\frac{e_1}{m_1}\mathcal{E}(t), \dots, \frac{e_N}{m_N}\mathcal{E}(t)\right), \quad E_{\rm cm}(t) = \pi_{\rm cm}\left(\frac{e_1}{m_1}\mathcal{E}(t), \dots, \frac{e_N}{m_N}\mathcal{E}(t)\right).$$

Then $\tilde{H}(t)$ is decomposed into

$$\tilde{H}(t) = H(t) \otimes Id + Id \otimes T_{\rm cm}(t) \qquad \text{on } L^2(X) \otimes L^2(X_{\rm cm}),$$

where Id are the identity operators,

$$\begin{aligned} H(t) &= -\frac{1}{2}\Delta - E(t)\cdot x + V \qquad \text{on } L^2(X), \\ T_{\rm cm}(t) &= -\frac{1}{2}\Delta_{\rm cm} - E_{\rm cm}(t)\cdot x_{\rm cm} \qquad \text{on } L^2(X_{\rm cm}), \end{aligned}$$

and Δ (resp. $\Delta_{\rm cm}$) is the Laplace-Beltrami operator on X (resp. $X_{\rm cm}$). Throughout this paper, we assume that there exists at least one pair (j,k) such that $e_j/m_j \neq e_k/m_k$. Under this assumption, when $\mathcal{E}(t) \neq 0$, we have $|E(t)| \neq 0$. We shall consider the Hamiltonian $H(t)$ which satisfies this assumption in this paper.

A non-empty subset of the set $\{1, \dots, N\}$ is called a cluster. Let C_j, $1 \le j \le m$, be clusters. If $\cup_{1\le j\le m} C_j = \{1, \dots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \le j < k \le m$, $a = \{C_1, \dots, C_m\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. Let $\mathcal{A}$ be the set of all cluster decompositions. Suppose $a, b \in \mathcal{A}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. Any a is regarded as a refinement of itself. We shall identify the pair $\alpha = (j,k)$ with the $(N-1)$-cluster decomposition $\{(j,k), (1), \dots, \widehat{(j)}, \dots, \widehat{(k)}, \dots, (N)\}$.

Next, for $a \in \mathcal{A}$, the two subspaces X^a and X_a of X are defined by

$$\begin{aligned} X^a &= \left\{ r \in X \;\middle|\; \sum_{j\in C} m_j r_j = 0 \text{ for each cluster } C \text{ in } a \right\}, \\ X_a &= \left\{ r \in X \;\middle|\; r_j = r_k \text{ for each pair } \alpha = (j,k) \subset a \right\}. \end{aligned}$$

In particular, for $\alpha = (j,k)$, X^α is the configuration space for the relative position of j-th and k-th particles. Hence we can write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$. These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed into the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ the orthogonal projections onto X^a and X_a, respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for $x \in X$. The intercluster potential I_a is defined by

$$I_a(x) = \sum_{\alpha \not\subset a} V_\alpha(x^\alpha),$$

and the cluster Hamiltonian

$$H_a(t) = -\frac{1}{2}\Delta - E(t)\cdot x + V^a, \quad V^a(x) = V^a(x^a) = \sum_{\alpha\subset a} V_\alpha(x^\alpha),$$

governs the motion of the system broken into non-interacting clusters of particles. Let $E^a(t) = \pi^a E(t)$ and $E_a(t) = \pi_a E(t)$. Then the cluster Hamiltonian $H_a(t)$ acting on $L^2(X)$ is decomposed into

$$H_a(t) = H^a(t)\otimes Id + Id\otimes T_a(t) \qquad \text{on } \ L^2(X^a)\otimes L^2(X_a),$$

where $H^a(t)$ is the subsystem Hamiltonian defined by

$$H^a(t) = -\frac{1}{2}\Delta^a - E^a(t)\cdot x^a + V^a \qquad \text{on } \ L^2(X^a),$$

$T_a(t)$ is the free Hamiltonian defined by

$$T_a = -\frac{1}{2}\Delta_a - E_a(t)\cdot x_a \qquad \text{on } \ L^2(X_a),$$

and Δ^a (resp. Δ_a) is the Laplace-Beltrami operator on X^a (resp. X_a).

Now we shall state the assumptions on the time-periodic electric field $\mathcal{E}(t)$ and the pair potentials. We suppose that $\mathcal{E}(t)$ is a $\boldsymbol{R}^d$-valued continuous function on $\boldsymbol{R}$, has its period $T > 0$, that is, $\mathcal{E}(t+T) = \mathcal{E}(t)$ for any $t \in \boldsymbol{R}$, and its average $\mathcal{E}$ in time is non-zero, i.e.

$$\mathcal{E} = \frac{1}{T}\int_0^T \mathcal{E}(t)\, dt \neq 0.$$

We shall denote

$$E = \pi\left(\frac{e_1}{m_1}\mathcal{E}, \dots, \frac{e_N}{m_N}\mathcal{E}\right).$$

By the assumption that there exists at least one pair (j,k) such that $e_j/m_j \neq e_k/m_k$, we see that $E \neq 0$. We let c be a maximal element of the set $\{a \in \mathcal{A} \mid E^a = 0\}$ with respect to the relation $\subset$, where $E^a = \pi^a E$. Such a cluster decomposition uniquely exists and it follows that $E^\alpha = 0$ if $\alpha \subset c$, and $E^\alpha \neq 0$ if $\alpha \not\subset c$. Thus the potential V_α with $\alpha \not\subset c$ (resp. $\alpha \subset c$) describes the pair interaction between two particles with $e_j/m_j \neq e_k/m_k$ (resp. $e_j/m_j = e_k/m_k$). If, in particular, $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then c becomes the N-cluster decomposition. We shall put different assumptions on V_α according as $\alpha \not\subset c$ or $\alpha \subset c$. We shall consider the following conditions on the pair potentials:

$(V)_{c,S}$ $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$, $\alpha \subset c$, is a real-valued function and has the decay property

$$|\partial_{x^\alpha}^\beta V_\alpha(x^\alpha)| \leq C_\beta \langle x^\alpha\rangle^{-(\rho'+|\beta|)},$$

with $\rho' > 1$.

$(V)_{c,L}$ $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$, $\alpha \subset c$, is a real-valued function and has the decay property

$$|\partial_{x^\alpha}^\beta V_\alpha(x^\alpha)| \le C_\beta \langle x^\alpha \rangle^{-(\rho'+|\beta|)},$$

with $\sqrt{3}-1 < \rho' \le 1$.

$(V)_{\bar{c}}$ $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$, $\alpha \not\subset c$, is a real-valued function and has the decay property

$$|\partial_{x^\alpha}^\beta V_\alpha(x^\alpha)| \le C_\beta \langle x^\alpha \rangle^{-(\rho+|\beta|)/2},$$

with $\rho > 1$.

Under these assumptions, all the Hamiltonians defined above are essentially self-adjoint on C_0^∞. We denote their closures by the same notations. If V_α, $\alpha \subset c$, satisfies the condition $(V)_{c,S}$, then V_α is called a short-range potential. And, if V_α, $\alpha \subset c$, satisfies the condition $(V)_{c,L}$, then V_α is called a long-range potential. We note that if V_α, $\alpha \not\subset c$, satisfies the condition $(V)_{\bar{c}}$, then V_α should be called a "Stark short-range" potential.

To formulate the obtained results precisely, we shall define the usual and the modified wave operators. Let $U(t,s)$ and $\bar{U}_a(t,s)$, $a \subset c$, be the unitary propagators generated by the time-dependent Hamiltonians $H(t)$ and $T_a(t)$, respectively, whose existence and uniqueness may be guaranteed by virtue of the result of Yajima [Ya2] and the Avron-Herbst formula [CFKS]. Here the unitary propagator $U(t,s)$ generated by the time-dependent Hamiltonian $H(t)$ means the family of unitary operators $\{U(t,s)\}_{t,s\in \boldsymbol{R}}$ on $L^2(X)$ with the following properties:

(1) $(t,s) \mapsto U(t,s)$ is strongly continuous.
(2) $U(t,s) = U(t,r)U(r,s)$ holds for any r, s, $t \in \boldsymbol{R}$.
(3) $U(t+T, s+T) = U(t,s)$ holds for any s, $t \in \boldsymbol{R}$.
(4) For $\psi \in \mathcal{D}$,

$$\frac{d}{dt}U(t,s)\psi = -iH(t)U(t,s)\psi, \quad \frac{d}{ds}U(t,s)\psi = iU(t,s)H(s)\psi$$

hold, where $\mathcal{D}$ is the common domain of $H(t)$.

Here we should note that for $a \subset c$, $H^a(t)$ is independent of time t because of $E^a(t) \equiv 0$. Then we write it as H^a, and we put

$$U_a(t,s) = e^{-i(t-s)H^a} \otimes \bar{U}_a(t,s). \tag{1.1}$$

Under the assumptions $(V)_{c,S}$ and $(V)_{\bar{c}}$, we define the usual wave operators $W_a^\pm(s)$, $a \subset c$ and $s \in \boldsymbol{R}$, by

$$W_a^\pm(s) = \mathrm{s}\text{-}\lim_{t\to\pm\infty} U(t,s)^* U_a(t,s)(P^a \otimes Id), \tag{1.2}$$

where $P^a : L^2(X^a) \to L^2(X^a)$ is the eigenprojection associated with H^a. On the other hand, we suppose that the assumptions $(V)_{c,L}$ and $(V)_{\bar{c}}$ are satisfied. We put

$$U_{a,D}(t,s) = U_a(t,s)e^{-i\int_s^t I_a^c(p_a u)\,du} \tag{1.3}$$

for $a \subset c$. Here $I_a^c = I_a - I_c$ and $p_a = -i\nabla_a$ is the velocity operator on $L^2(X_a)$. Now we define the modified wave operators $W_{a,D}^\pm(s)$, $a \subset c$, by

$$W_{a,D}^\pm(s) = \mathrm{s}\text{-}\lim_{t\to\pm\infty} U(t,s)^* U_{a,D}(t,s)(P^a \otimes Id). \tag{1.4}$$

The main results of this paper are the following two theorems:

Theorem 1.1. *Assume that $(V)_{c,S}$ and $(V)_{\bar{c}}$ are fulfilled. Let c be as above. Then the usual wave operators $W_a^{\pm}(s)$, $a \subset c$ and $s \in \boldsymbol{R}$, exist, and are asymptotically complete*

$$L^2(X) = \sum_{a \subset c} \oplus Ran\, W_a^{\pm}(s).$$

Theorem 1.2. *Assume that $(V)_{c,L}$ and $(V)_{\bar{c}}$ are fulfilled. Let c be as above. Then the modified wave operators $W_{a,D}^{\pm}(s)$, $a \subset c$ and $s \in \boldsymbol{R}$, exist, and are asymptotically complete*

$$L^2(X) = \sum_{a \subset c} \oplus Ran\, W_{a,D}^{\pm}(s).$$

Remark. As it follows from the proof below, we may be allowed to include the time-periodicity with the same period T as the one of the electric field $\mathcal{E}(t)$ in the pair potentials V_α with $\alpha \not\subset c$. But we shall not consider such cases here.

The problem of the asymptotic completeness for N-body quantum systems has been studied by many mathematicians and they have achieved a great success. For N-body Schrödinger operators, this problem was first solved by Sigal-Soffer [SS] for a large class of short-range potentials, and some alternative proofs appeared (e.g. Graf [Gr2] and Yafaev [Y]). On the other hand, for the long-range case, Dereziński [D] solved this problem with arbitrary N for the class of potentials decaying like $O(|x^\alpha|^{-\rho})$ with some $\rho > \sqrt{3} - 1$ (see also Zielinski [Z]). Also the case of potentials decaying more slowly has been dealt with (see the references in [DG]). Also for N-body Stark Hamiltonians, satisfactory results of this problem has been obtained (see e.g. [AT1,AT2] and [HMS1,HMS2]). For other systems, see [DG]. These results are concerned with time-independent Hamiltonians.

On the other hand, for time-dependent Hamiltonians, the lack of energy conservation is a barrier in studying this problem. For instance, in [Gr1], the time-boundedness of the kinetic energy was the key fact for studying the charge transfer model. Howland [Ho1] proposed the stationary scattering theory for time-dependent Hamiltonians, whose formalization was the quantum analogue to the procedure in the classical mechanics in order to 'recover' the conservation of energy. Yajima [Ya1] applied this Howland method to the two-body quantum systems with time-periodic potentials and studied the problem of the asymptotic completeness for the systems (see also [Ho2]). His result was extended to the three-body case by Nakamura [N] later. As for the scattering theory in a time-periodic electric field, for instance, Kitada-Yajima [KY] dealt with the so-called AC Stark case for two-body quantum systems. Recently Møller [Mø] studied the scattering theory for two-body quantum systems in a time-periodic electric field whose average in time is non-zero, by using the so-called Howland-Yajima method. In his work, it seems to be important that he used the so-called Avron-Herbst formula (see [CFKS]) in order to remove the oscillating part of the electric field, and reduced the problem to the scattering problem for two-body Stark Hamiltonians with time-periodic potentials. This point of view motivates partly us to study the present problem.

The plan of this paper is as follows: In §2, we shall reduce the present problem to the one which it is easier to deal with. The Howland-Yajima method plays an important role, combining the notion of the asymptotic clustering developed by ourselves and Tamura [AT1,AT2] for N-body Stark Hamiltonians. In §3, we shall give some lemmas for proving the absence of bound states for the Floquet Hamiltonian associated with this problem. In §4, we will prove the absence of bound states for the Hamiltonian. These two sections are closely related to the work of Herbst-Møller-Skibsted [HMS1]. In §5, we get the Mourre estimate for the Floquet Hamiltonian as a consequence of the results in §3 and §4, and derive some propagation estimates for the evolution of the Hamiltonian, which are based on the Mourre estimate. The contents of §5 are closely related to our previous work [A2]. Finally, in §6, we shall prove Theorems 1.1 and 1.2.

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§2. Reduction of the problem

In this section, we shall reduce the problem under consideration to the one which it is easier to deal with.

Following the idea of Møller [Mø], we shall remove the oscillating part of the electric field, and reduce the present problem to the scattering problem for so-called N-body Stark Hamiltonians with time-periodic potentials. In removing the oscillating part of the field, we will use a version of the Avron-Herbst formula:

We define C^1 periodic functions on $\boldsymbol{R}$

$$\begin{aligned} b(t) &= \int_0^t (E(s) - E)\, ds - b_0, \qquad b_0 = \frac{1}{T}\int_0^T \int_0^t (E(s)-E)\, ds dt, \\ c(t) &= \int_0^t b(s)\, ds - c_0, \qquad c_0 = \frac{1}{T}\int_0^T \left(-\frac{1}{2}|b(t)|^2 + \int_0^t E\cdot b(s)\, ds\right) dt \frac{E}{|E|^2}, \\ a(t) &= \int_0^t \left(\frac{1}{2}|b(s)|^2 - E\cdot c(s)\right) ds, \end{aligned} \tag{2.1}$$

where $b(t)$, $c(t) \in X$ and $a(t) \in \boldsymbol{R}$, and a strongly continuous periodic family of unitary operators on $L^2(X)$ by

$$\mathcal{T}(t) = e^{-ia(t)} e^{ib(t)\cdot x} e^{-ic(t)\cdot p}, \tag{2.2}$$

where $p = -i\nabla$ is the velocity operator on $L^2(X)$. Moreover we define the time-dependent Hamiltonian $H^S(t)$ by

$$H^S(t) = -\frac{1}{2}\Delta - E\cdot x + V(x + c(t)), \tag{2.3}$$

and define the time-independent Hamiltonian H_c^S by

$$H_c^S = -\frac{1}{2}\Delta - E\cdot x + V^c(x). \tag{2.4}$$

We should note that the time-periodic potential $V(x + c(t))$ in the definition of the Hamiltonian $H^S(t)$ are written as

$$V(x + c(t)) = V^c(x) + I_c(x + c(t)), \tag{2.5}$$

because $c(t) \in X_c$ by definition and $V^c(x) = V^c(x^c)$ is independent of $x_c \in X_c$ also by definition. Let $\tilde{U}(t,s)$ be the unitary propagator generated by the Hamiltonian $H^S(t)$, whose existence and uniqueness may be guaranteed by the result of Yajima [Ya2] and the Avron-Herbst formula [CFKS]. Then the Avron-Herbst formula which we shall use here is that

$$U(t,s) = \mathcal{T}(t)\tilde{U}(t,s)\mathcal{T}(s)^*, \qquad U_c(t,s) = \mathcal{T}(t)e^{-i(t-s)H_c^S}\mathcal{T}(s)^*, \tag{2.6}$$

where we used the relationships (2.1) and (2.2).

In order to prove Theorems 1.1 and 1.2, we shall claim that the following theorem holds:

Theorem 2.1. (The Asymptotic Clustering) *Assume that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_{\bar{c}}$ are fulfilled. Let $s \in \boldsymbol{R}$. Then the strong limits*

$$\hat{W}_c^{\pm}(s) = \text{s -}\lim_{t\to\pm\infty} U(t,s)^* U_c(t,s) \tag{2.7}$$

exist and are unitary on $L^2(X)$.

This property played an important role to prove the asymptotic completeness for N-body Stark Hamiltonians in the works of ourselves and Tamura [AT1,AT2] (see also [A1] and [HMS2]). As for the present problem, since the propagator $U_c(t,s)$ can be decomposed into

$$U_c(t,s) = e^{-i(t-s)H^c} \otimes \bar{U}_c(t,s), \tag{2.8}$$

we have only to study the scattering theory for the many body Schrödinger operator H^c by virtue of Theorem 2.1. Thus Theorems 1.1 and 1.2 can be proved, if we see that Theorem 2.1 holds. Now, using the above Avron-Herbst formula (2.6), Theorem 2.1 can be translated into the following theorem:

Theorem 2.2. (The Asymptotic Clustering) *Assume that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_{\bar{c}}$ are fulfilled. Let $s \in \boldsymbol{R}$. Then the strong limits*

$$\tilde{W}_c^{\pm}(s) = \text{s -}\lim_{t\to\pm\infty} \tilde{U}(t,s)^* e^{-i(t-s)H_c^S} \tag{2.9}$$

exist and are unitary on $L^2(X)$.

Therefore the end of this paper is to show that Theorem 2.2 holds. In order to prove Theorem 2.2, we shall follow the argument of Yajima [Ya1] (see also Howland [Ho1,Ho2]).

We let $\boldsymbol{T} = \boldsymbol{R}/(T\boldsymbol{Z})$ be the torus and introduce $\mathcal{H} = L^2(\boldsymbol{T}; L^2(X)) \cong L^2(\boldsymbol{T}) \otimes L^2(X)$. We define two families of operators $\{\hat{U}(\sigma)\}_{\sigma\in\boldsymbol{R}}$ and $\{\hat{U}_c(\sigma)\}_{\sigma\in\boldsymbol{R}}$ on $\mathcal{H}$ by

$$(\hat{U}(\sigma)f)(t) = U(t, t-\sigma)f(t-\sigma), \tag{2.10}$$

$$(\hat{U}_c(\sigma)f)(t) = e^{-i\sigma H_c^S} f(t-\sigma), \tag{2.11}$$

for $f \in \mathcal{H}$. Then $\{\hat{U}(\sigma)\}_{\sigma\in\boldsymbol{R}}$ and $\{\hat{U}_c(\sigma)\}_{\sigma\in\boldsymbol{R}}$ form strongly continuous unitary groups on $\mathcal{H}$. Now we may write

$$\hat{U}(\sigma) = e^{-i\sigma K}, \quad \hat{U}_c(\sigma) = e^{-i\sigma K_c}, \tag{2.12}$$

where K and K_c are self-adjoint operators on $\mathcal{H}$. We call these self-adjoint operators K and K_c the Floquet Hamiltonians associated with the Hamiltonians $H^S(t)$ and H_c^S, respectively. From now on we denote the norm and scalar product in $\mathcal{H}$ by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. We also denote the operator norm on $\mathcal{H}$ by $\|\cdot\|$.

Proving Theorem 2.2 is equivalent to showing the following theorem, by virtue of the argument of Yajima [Ya1]:

Theorem 2.3. (The Asymptotic Clustering) *Assume that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_{\bar{c}}$ are fulfilled. Then the strong limits*

$$\mathcal{W}_c^{\pm} = \text{s -} \lim_{\sigma\to\pm\infty} e^{i\sigma K} e^{-i\sigma K_c} \tag{2.13}$$

exist and are unitary on $\mathcal{H}$.

Now by assuming that Theorem 2.3 holds and the wave operators $\tilde{W}_c^{\pm}(s)$, $s \in \boldsymbol{R}$, exist, we shall prove the unitarity of them in Theorem 2.2. The existence of $\tilde{W}_c^{\pm}(s)$ may be guaranteed by the argument similar to the ones of ourselves and Tamura [AT1,AT2], Herbst-Møller-Skibsted [HMS2] and ourselves [A2].

Proof under the assumption mentioned above.

First we note that the wave operators $\mathcal{W}_c^{\pm}$ are the multiplication operators by $\tilde{W}_c^{\pm}(t)$. Let $\mathcal{V}$ and $\mathcal{V}_c$ be unitary operators on $\mathcal{H}$ defined by

$$(\mathcal{V}f)(t) = \tilde{U}(t,s)f(t), \quad (\mathcal{V}_c f)(t) = e^{-i(t-s)H_c^S} f(t)$$

for $f \in \mathcal{H}$. By the unitarity of $\mathcal{V}$, we have

$$\mathcal{H} = \mathcal{V}\mathcal{H} = \mathcal{V}L^2(\boldsymbol{T}; L^2(X)). \tag{2.14}$$

On the other hand, letting $\hat{\mathcal{W}}_c^{\pm}$ be the multiplication operator by $\tilde{W}_c^{\pm}(s)$, we see that

$$\begin{aligned} Ran\, \mathcal{W}_c^{\pm} &= Ran\, \mathcal{V}\hat{\mathcal{W}}_c^{\pm}\mathcal{V}_c^* \\ &= \mathcal{V} Ran\, \hat{\mathcal{W}}_c^{\pm} \\ &= \mathcal{V}L^2(\boldsymbol{T}; Ran\, \tilde{W}_c^{\pm}(s)). \end{aligned} \tag{2.15}$$

By virtue of Theorem 2.3, comparing (2.14) with (2.15), we have

$$Ran\, \tilde{W}_c^{\pm}(s) = L^2(X),$$

which implies the unitarity of $\tilde{W}_c^{\pm}(s)$. $\square$

Therefore we shall study the scattering theory for the pair of the Floquet Hamiltonians K and K_c below.

§3. Preliminaries

In this section, we shall give some lemmas which are needed for proving the absence of bound states of the Floquet Hamiltonian K associated with the Hamiltonian $H^S(t)$. The results obtained in this and next sections are related closely to the results of the sophisticated paper by Herbst-Møller-Skibsted [HMS1], in which they studied the spectral properties of N-body Stark Hamiltonians. As one can imagine, the results here are similar to their results. However, we may not use the relative boundedness of, for example, $\langle x\rangle^{-1/2}p$ with respect to the Hamiltonian under consideration. For the N-body Stark Hamiltonians, this operator $\langle x\rangle^{-1/2}p$ is relative bounded with respect to them, and this fact was often used in studying the spectral and scattering theory for N-body Stark Hamiltonians (see e.g. [AT1,AT2], [HMS1,HMS2] and so on). At this point, the circumstances of the Floquet Hamiltonian K and N-body Stark Hamiltonians are slightly different. Nevertheless we shall follow the idea of [HMS1], which will work well for this Hamiltonian K with slight modifications.

We denote the N-body free Stark Hamiltonian by

$$H_0^S = -\frac{1}{2}\Delta - E\cdot x \tag{3.1}$$

on $L^2(X)$, and the Floquet Hamiltonian associated with H_0^S by K_0. Similarly, we denote the cluster Hamiltonians $H_a^S(t)$ by

$$H_a^S(t) = -\frac{1}{2}\Delta - E\cdot x + V^a(x+c(t)) \tag{3.2}$$

on $L^2(X)$, and the Floquet Hamiltonians associated with $H_a^S(t)$ by K_a. Moreover we denote the subsystem Hamiltonians $H^{a,S}(t)$ by

$$H^{a,S}(t) = -\frac{1}{2}\Delta^a - E^a\cdot x^a + V^a(x^a+c^a(t)) \tag{3.3}$$

on $L^2(X^a)$, and the Floquet Hamiltonians associated with $H^{a,S}(t)$ by K^a acting on $L^2(\boldsymbol{T};L^2(X^a))$. We should note that if 0 is identified with the N-cluster decomposition $\{(1),\ldots,(N)\}$, $H_0^S = H_0^S(t)$, which is independent of time t. We should also notice that for $a\subset c$, the cluster Hamiltonian $H_a^S(t)$ is independent of time t, because the potential $V^a(x)$ is independent of $x_a\in X_a$, and $c(t)\in X_c\subset X_a$.

Formally one may consider, for instance, the Floquet Hamiltonian K as the operator sum $D_t\otimes Id + Id\otimes H^S(t)$. Here $D_t = -i\partial_t$ is a self-adjoint operator on $L^2(\boldsymbol{T})$ with its domain $\mathcal{D}(D_t) = AC^2(\boldsymbol{T})$, which is the space of absolutely continuous functions on $\boldsymbol{T}$ with their derivatives being square integrable (following the notation in Reed-Simon [RS]). We should note that $AC^2(\boldsymbol{T})\cong\big\{f\in AC^2([0,T])\bigm| f(T)=f(0)\big\}$. We shall consider an appropriate common core of them on which they may be identified.

Lemma 3.1. *Let $\mathcal{D}_0 = AC^2(\boldsymbol{T})\otimes C_0^\infty(X)$. Then $\mathcal{D}_0$ is a core for the Floquet Hamiltonian K_0. Moreover, the operator $D_t\otimes Id + Id\otimes H_0^S$ defined on $\mathcal{D}_0$ by $((D_t\otimes Id + Id\otimes$*

H_0^S))$f(t) = -if'(t) + H_0^S f(t)$, is essentially self-adjoint on $\mathcal{D}_0$ and its closure is equal to K_0, where $f'(t)$ is the derivative of $f(t)$ with respect to t.

Proof. First we should note that $e^{-i\sigma K_0}\mathcal{D}_1 \subset \mathcal{D}_1$. Here $\mathcal{D}_1 = AC^2(\boldsymbol{T}) \otimes \mathcal{S}(X)$ and $\mathcal{S}(X)$ is the Schwartz space on X. This fact follows from the fact that $e^{-i\sigma H_0^S}\mathcal{S}(X) \subset \mathcal{S}(X)$. Thus we see that the restriction of K_0 on $\mathcal{D}_1$ is essentially self-adjoint on $\mathcal{D}_1$, and its closure is equal to K_0. On the other hand, the operator sum $K_1 = D_t \otimes Id + Id \otimes H_0^S$ defined on $\mathcal{D}_1$ in the way similar to the one in the statement of the lemma is essentially self-adjoint on $\mathcal{D}_1$ by virtue of Theorem VIII.33 in [RS] because $\mathcal{S}(X)$ is a core for H_0^S, and is equal to the restriction of K_0 on $\mathcal{D}_1$, which may be easily seen. That is, the closure of K_1 is equal to K_0. Moreover the restriction of K_1 on $\mathcal{D}_0$ is essentially self-adjoint on $\mathcal{D}_0$ by virtue of Theorem VIII.33 in [RS] because $C_0^\infty(X)$ is a core for H_0^S, and its closure is equal to the closure of K_1. These facts imply the lemma. $\square$

Since the pair potentials under consideration are bounded, by virtue of the Kato-Rellich theorem (see e.g. [RS]), the Floquet Hamiltonians K and K_a are self-adjoint on $\mathcal{D}(K_0)$, and essentially self-adjoint on any core for K_0. We shall keep these facts in mind for the computations below. Moreover we shall abbreviate, for instance, the operator sum $D_t \otimes Id + Id \otimes H^S(t)$ as $D_t + H^S(t)$ below.

Now we shall get the results similar to the ones in [HMS1] as the preparation for showing the absence of bound states of the Floquet Hamiltonian K.

We shall use the following convention for smooth cut-off functions F with $0 \le F \le 1$, which is often used throughout the discussion below. For sufficiently small $\delta > 0$, we define

$$\begin{aligned} F(s \le d) &= 1 \quad \text{for} \quad s \le d - \delta, \quad &= 0 \quad \text{for} \quad s \ge d, \\ F(s \ge d) &= 1 \quad \text{for} \quad s \ge d + \delta, \quad &= 0 \quad \text{for} \quad s \le d, \end{aligned}$$

and $F(d_1 \le s \le d_2) = F(s \ge d_1)\, F(s \le d_2)$. The choice of $\delta > 0$ does not matter to the argument below.

The following lemma is Lemma 3.3 in [HMS1]. We omit the proof.

Lemma 3.2. *Let $R > 0$ and $\mu \in \boldsymbol{R} \setminus \{0\}$. Then we have*

$$\|F(|x| \le R)(H_0^S - \lambda - i\mu)^{-1} F(|x| \le R)\|_{B(L^2(X))} \to 0$$

as $|\lambda| \to \infty$.

The following lemma may be a consequence of Lemma 3.4 in [HMS1], by diagonalizing the part D_t of K_0. We should note that the statement of Lemma 3.4 in [HMS1] says the estimate holds uniformly in the energy λ. We omit the proof. For the proof, see [HMS1] and [He].

Lemma 3.3. *Let $0 < \gamma_1 < \gamma_2 \leq 1/2$, $\mu \in \boldsymbol{R} \setminus \{0\}$ and $\Pi : X \to X$ is an orthogonal projection. Then there exists $C > 0$ such that*

$$\|\langle \Pi x\rangle^{\gamma_1}(K_0 - \lambda - i\mu)^{-1}\langle \Pi x\rangle^{-\gamma_2}\| \leq C$$

holds uniformly in $\lambda \in \boldsymbol{R}$.

We introduce a family of functions $\{j_a\}_{\#(a)\neq 1}$ on X like [HMS1]: Each j_a is smooth and homogeneous of degree 0 outside a ball $\{x \in X \mid |x| < R_0\}$. Moreover they satisfy

$$\sum_a j_a(x) = 1 \tag{3.4}$$

for $|x| > R_1$ with some $R_1 > 0$,

$$\frac{\langle x\rangle}{\langle x^b\rangle} \text{ is bounded on } \operatorname{supp} j_a \text{ for any } b \not\subset a. \tag{3.5}$$

In particular, for any $R > 0$ and $b \not\subset a$, there exists $r_{R,a,b} > 0$ such that

$$F(|x^b| \leq R)j_a(x) = 0 \tag{3.6}$$

for $|x| > r_{R,a,b}$. For simplicity, we put $a_{\max} = \{(1, \dots, N)\}$, that is, $a_{\max}$ is the 1-cluster decomposition, and

$$j_{a_{\max}}(x) = 1 - \sum_a j_a(x).$$

We shall consider $\{j_a\}_{\#(a)\geq 1}$ below.

We also need more notations. For $\epsilon > 0$, let η_ϵ be a function $\eta_\epsilon \in C_0^\infty(\boldsymbol{R})$ such that $0 \leq \eta_\epsilon \leq 1$, $\eta_\epsilon(t) = 1$ for $|t| \leq \epsilon$ and $\eta_\epsilon(t) = 0$ for $|t| \geq 2\epsilon$.

The following lemma should be compared with Lemma 3.5 (2) in [HMS1]. In [HMS1], because of the boundedness of the potentials, using the Neumann series, they reduced the problem under consideration to the analysis for the resolvent of the free Hamiltonian H_0^S, without using the relative boundedness with respect to the Hamiltonian of some operators as mentioned above. Therefore we may obtain the following result in the same way as in [HMS1]. This observation seems to be important for the present problem. Thus we shall not omit the proof, although the proof is quite similar to the one in [HMS1].

Lemma 3.4. *Let $\epsilon > 0$. Then we have*

$$\|F(|x| \geq R)\{\eta_\epsilon(K - \lambda) - \eta_\epsilon(K_a - \lambda)\}j_a\| \to 0$$

as $R \to \infty$ uniformly in $\lambda \in \boldsymbol{R}$, for $a \neq a_{\max}$.

Proof. We follow the argument of [HMS1]. First we take

$$\mu > \sum_\alpha \sup |V_\alpha(x^\alpha)|. \tag{3.7}$$

Applying the Stone-Weierstrass theorem, we have only to prove that

$$\begin{aligned}\|F(|x|\geq R)\{(K-\lambda-i\mu_1)^{-1}\cdots(K-\lambda-i\mu_n)^{-1}\\ -(K_a-\lambda-i\mu_1)^{-1}\cdots(K_a-\lambda-i\mu_n)^{-1}\}j_a\|\to 0\end{aligned} \tag{3.8}$$

as $R\to\infty$ uniformly in $\lambda\in\boldsymbol{R}$, where $\mu_j\in\boldsymbol{R}$ such that $|\mu_j|=\mu$ for $j\leq n\in\boldsymbol{N}$. In order to prove this, we should recall that the Neumann series

$$(K-\lambda-i\mu_j)^{-1}=(K_0-\lambda-i\mu_j)^{-1}\sum_{m=0}^{\infty}\{-V(x+c(t))(K_0-\lambda-i\mu_j)^{-1}\}^m,$$
$$(K_a-\lambda-i\mu_j)^{-1}=(K_0-\lambda-i\mu_j)^{-1}\sum_{m=0}^{\infty}\{-V^a(x+c(t))(K_0-\lambda-i\mu_j)^{-1}\}^m$$

converge by virtue of (3.7). Thus, noting that

$$(K-\lambda-i\mu_j)^{-1}-(K_a-\lambda-i\mu_j)^{-1}=-(K-\lambda-i\mu_j)^{-1}I_a(x+c(t))(K_a-\lambda-i\mu_j)^{-1},$$

truncating the series for each j as the finite sum and substituting the truncated one for the original one in (3.8), we may reduce the problem to showing that

$$\|F(|x|\geq R)B_1(\lambda)V_\alpha(x^\alpha+c^\alpha(t))B_2(\lambda)j_a\|\to 0$$

as $R\to\infty$ uniformly in $\lambda\in\boldsymbol{R}$, where $\alpha\not\subset a$, $B_l(\lambda)$, $l=1,2$, are the finite products of bounded functions and $(K_0-\lambda-i\mu_j)^{-1}$. By inserting $1=F(|x^\alpha|\leq R')+(1-F(|x^\alpha|\leq R'))$ with sufficiently large $R'>0$, we may assume that the support of V_α is compact in X^α. We put

$$A_1(\lambda,R)=F(|x|\geq R)B_1(\lambda)V_\alpha(x^\alpha+c^\alpha(t))B_2(\lambda)j_aF(|x|\geq R''),$$
$$A_2(\lambda,R)=F(|x|\geq R)B_1(\lambda)V_\alpha(x^\alpha+c^\alpha(t))B_2(\lambda)j_a(1-F(|x|\geq R'')),$$

with $R''\geq R_1$, and shall estimate them. Noticing (3.5), it is sufficient to estimate the following $A_1'(\lambda,R)$ instead of $A_1(\lambda,R)$:

$$A_1'(\lambda,R)=F(|x|\geq R)B_1(\lambda)V_\alpha(x^\alpha+c^\alpha(t))B_2(\lambda)\left(\frac{\langle x^\alpha\rangle}{\langle x\rangle}\right)^{1/4}j_aF(|x|\geq R'').$$

By Lemma 3.3, $\|V_\alpha(x^\alpha+c^\alpha(t))B_2(\lambda)\langle x^\alpha\rangle^{1/4}\|$ is bounded uniformly in $\lambda\in\boldsymbol{R}$. Thus we have $\|A_1'(\lambda,R)\|=O((R'')^{-1/4})$ uniformly in $\lambda\in\boldsymbol{R}$ and $R>0$. Then we fix R'' sufficiently large. By the same argument we may write $A_2(\lambda,R)=F(|x|\geq R)\langle x\rangle^{-1/4}A_2'(\lambda)$, where $\|A_2'(\lambda)\|$ is bounded uniformly in $\lambda\in\boldsymbol{R}$. Therefore we have $\|A_2(\lambda,R)\|=O(R^{-1/4})$ uniformly in $\lambda\in\boldsymbol{R}$. This implies the lemma. □

We shall show one of the important properties of the Floquet Hamiltonians.

Lemma 3.5. *Let $R > 0$. Then the operator $F(|x| \leq R)$ is K-compact.*

Proof. By the boundedness of the potentials, we have only to prove that $F(|x| \leq R)$ is K_0-compact, that is, $F(|x| \leq R)(K_0 + i)^{-1}$ is a compact operator on $\mathcal{H}$. We should note that $F(|x| \leq R)(H_0^S + (2\pi/T)n + i)^{-1}$, $n \in \boldsymbol{Z}$, are compact operators on $L^2(X)$. Moreover, using Lemma 3.2 and the first resolvent equation, we see that $\|F(|x| \leq R)(H_0^S + (2\pi/T)n + i)^{-1}\|_{B(L^2(X))} \to 0$ as $|n| \to \infty$. Thus, by diagonalizing the part D_t of K_0, we have the lemma (see also [Ya1]). $\square$

Combining this with Lemma 3.4, we have the following:

Corollary 3.6. *For $a \neq a_{\max}$, $\{\eta_\epsilon(K - \lambda) - \eta_\epsilon(K_a - \lambda)\}j_a$ is a compact operator on $\mathcal{H}$.*

The next lemma should be compared with Lemma 3.6 in [HMS1], which was proved by applying the correspondence of Lemma 3.4 here without using the relative boundedness with respect to the Hamiltonian of some operators as mentioned above. This lemma will be an important role in the induction argument for proving the absence of bound states of K in the next section. Although the proof is similar to the one in [HMS1], we shall not omit the proof.

Lemma 3.7. *Let $R > 0$ and b be such that $\#(b) \neq N$ and $b \subset d$. Suppose that $\sigma_{pp}(K^a) = \emptyset$ for all a such that $b \subset a \subset d$. Then we have*

$$\|\eta_\epsilon(K^d - \lambda)F(|x^b| \leq R)\| \to 0$$

as $\epsilon \to +0$ uniformly in $\lambda \in \boldsymbol{R}$.

Proof. We follow the argument of [HMS1]. We shall prove the lemma by induction in $\#(d)$. The starting point of the induction is the case $\#(d) = N - 1$. For the convenience of notations, we assume that the statement for any d such that $\#(d) > 1$ holds and take care of the case $d = a_{\max}$ under this assumption.

First we suppose that b is also equal to $a_{\max}$. We should note that for any $n \in \boldsymbol{Z}$

$$\|\eta_\epsilon(K - \lambda)F(|x| \leq R)\| = \|\eta_\epsilon(K - \lambda - (2\pi/T)n)F(|x| \leq R)\|.$$

Thus, in order to prove the uniformity with respect to $\lambda \in \boldsymbol{R}$, we have only to consider the case $\lambda \in [0, 2\pi/T]$. Now by Lemma 3.5 and the assumption $\sigma_{pp}(K) = \emptyset$, we have for any $\delta > 0$ there exists $\epsilon_1(\lambda, \delta) > 0$ such that for $0 < \epsilon < \epsilon_1(\lambda, \delta)$

$$\|\eta_\epsilon(K - \lambda)F(|x| \leq R)\| < \delta.$$

Then by virtue of this, we should notice that for any $\lambda_1 \in [0, 2\pi/T]$

$$\|\eta_\epsilon(K - \lambda)F(|x| \leq R)\| < \delta$$

with $\epsilon = \epsilon_1(\lambda_1, \delta)/4$ and $\lambda \in [\lambda_1 - \epsilon_1(\lambda_1, \delta)/2, \lambda_1 + \epsilon_1(\lambda_1, \delta)/2]$. Therefore we may get the uniformity in $\lambda \in \boldsymbol{R}$ by a compactness argument.

In order to deal with a general b as in the lemma, we shall use the partition of unity $\{j_a\}$. By the statement with $b = a_{\max}$ just proved, (3.4) and (3.6), we have only to show that for any $a \neq a_{\max}$ with $b \subset a$,

$$\|\eta_\epsilon(K-\lambda)j_a F(|x^b| \leq R)\| \to 0$$

as $\epsilon \to +0$ uniformly in $\lambda \in \boldsymbol{R}$. We consider

$$\begin{aligned}\eta_\epsilon(K-\lambda)j_a =& \eta_\epsilon(K-\lambda)\{\eta_{\epsilon'}(K-\lambda) - \eta_{\epsilon'}(K_a-\lambda)\}j_a \\ &+ \eta_\epsilon(K-\lambda)\eta_{\epsilon'}(K_a-\lambda)j_a \end{aligned} \tag{3.9}$$

with $\epsilon' \geq 2\epsilon$. As for the second term of (3.9), we may use the fact

$$\|\eta_{\epsilon'}(K_a-\lambda)F(|x^b| \leq R)\| \leq \sup_{\nu\in\boldsymbol{R}} \|\eta_{\epsilon'}(K^a+\nu-\lambda)F(|x^b| \leq R)\|. \tag{3.10}$$

By the induction assumption, we have as $\epsilon \to +0$

$$\|\eta_{\epsilon'}(K^a-\tilde{\lambda})F(|x^b| \leq R)\| \to 0 \tag{3.11}$$

uniformly in $\tilde{\lambda} = \lambda - \nu \in \boldsymbol{R}$. We shall fix ϵ' so small that we may use (3.11) to make the left-hand side of (3.10) small. Now we shall take care of the first term of (3.9). We put

$$\begin{aligned} A_1(\lambda,\epsilon) &= \eta_\epsilon(K-\lambda)F(|x| \geq R')\{\eta_{\epsilon'}(K-\lambda) - \eta_{\epsilon'}(K_a-\lambda)\}j_a F(|x^b| \leq R), \\ A_2(\lambda,\epsilon) &= \eta_\epsilon(K-\lambda)(1-F(|x| \geq R'))\{\eta_{\epsilon'}(K-\lambda) - \eta_{\epsilon'}(K_a-\lambda)\}j_a F(|x^b| \leq R). \end{aligned}$$

By Lemma 3.4, we may make $\|A_1(\lambda,\epsilon)\|$ very small uniformly in $\epsilon > 0$ and $\lambda \in \boldsymbol{R}$, if we will take $R' > 0$ sufficiently large. We fix such an R'. Then we have

$$\|\eta_\epsilon(K-\lambda)(1-F(|x| \geq R'))\| \to 0$$

as $\epsilon \to +0$ uniformly in $\lambda \in \boldsymbol{R}$, which was proved at the first step of this proof. Therefore we may make $\|A_2(\lambda,\epsilon)\|$ small uniformly in $\lambda \in \boldsymbol{R}$, if we will take $\epsilon > 0$ small enough. This implies the lemma. $\square$

§4. The absence of bound states

In this section we prove the absence of bound states of the Floquet Hamiltonian K. By induction in $\#(a)$, we will show that $\sigma_{pp}(K^a) = \emptyset$ for any a such that $E^a \neq 0$. The starting point is $\#(a) = N-1$. For this case, it was proved by Møller [Mø]. But in the following proof, we may also obtain the result for such a two-body case. For the convenience of notations, we shall assume the statement for $\#(a) > 1$ holds, and consider the case $\#(a) = 1$ under this assumption.

Let $\omega = E/|E|$, $A = \omega \cdot p$ and $z = \omega \cdot x$. We take a real-valued smooth function $\tilde{f}_1$ on $\boldsymbol{R}$ with bounded derivatives such that $\tilde{f}_1(u) = 0$ for $u \leq 1$, $\tilde{f}_1(u) = 1$ for $u \geq 2$, $0 \leq \tilde{f}_1 \leq 1$ and $\tilde{f}_1' \geq 0$. Then we put $f_1(u) = \tilde{f}_1(u)^2$, $\tilde{f}_R(u) = \tilde{f}(u/R)$, $f_R(u) = f_1(u/R)$

and $g_R(u) = R + \int_0^u f_R(s)\,ds$ for $R > 0$. We note that $g_R(u) = R$ for $u \le R$, $g_R(u) = u + R(\int_1^2 f_1(s)\,ds - 1)$ for $u \ge 2R$, and $g_R \ge R$. We put

$$A_f = \frac{1}{2}(f_R(z)A + Af_R(z)).$$

For some real-valued smooth function θ on $\boldsymbol{R}$, we write

$$\vartheta = \theta(g_R(z)), \quad \vartheta' = \theta'(g_R(z)), \quad \vartheta'' = \theta''(g_R(z)),$$

where we shall abbreviate the dependence on R for the simplicity of notations, and chose two kinds of functions as θ later (in Lemmas 4.1 and 4.2). The difference between this work and [HMS1] is in the choice of the "weight" g_R. Although they chose g_R as $g_R(x) = z + R\langle x\rangle$, their choice seems not to work well for this problem. Our choice is inspired by the work of Sigal [S], who proved the absence of bound states for some multielectron systems with the Stark effect (see also [Mø]). Our choice will also be convenient for some limiting procedures later.

By following the idea of Froese-Herbst [FH] for proving the absence of positive bound states of N-body Schrödinger operators, we compute the following commutator (see also e.g. [HMS1]).

$$i[K, e^{\vartheta}A_f e^{\vartheta}] = e^{\vartheta}\{T_1 + T_2 + T_3 + T_4 + T_5 + T_6\}e^{\vartheta},$$

$$T_1 = 2A_f\vartheta' A_f, \quad T_2 = Af'A, \quad T_3 = |E|f, \quad T_4 = -(\omega\cdot\nabla V(x + c(t)))f,$$

$$T_5 = \frac{f}{2}(f^2((\vartheta')^2 - \vartheta''))', \quad T_6 = -\frac{1}{2}f'''.$$

Moreover the following formula holds:

$$\left(K - \lambda - \frac{1}{2}((\vartheta')^2 - \vartheta'')f^2 + i\vartheta' A_f\right)e^{\vartheta} = e^{\vartheta}(K - \lambda). \tag{4.1}$$

We shall take θ with the following properties:

$$\frac{d}{du}\theta(u) \ge 0, \quad \left(\frac{d^n}{du^n}\right)\theta(u) = O(u^{1-n}) \text{ as } u \to \infty, \quad n \ge 0.$$

Hence we have

$$T_1 \ge 0. \tag{4.2}$$

Moreover, by noticing that f is bounded, $f' = O(R^{-1})$ and $g \ge R$, and by using the above properties of θ, we see

$$|(f^2((\vartheta')^2 - \vartheta''))'| = O(R^{-1}).$$

Thus taking R sufficiently large, we may estimate the term T_5 as

$$T_5 \ge -\frac{|E|}{6}f. \tag{4.3}$$

Also noting that $f' \ge 0$, we have

$$T_2 \ge 0. \tag{4.4}$$

Finally we have the following estimate by (4.2)–(4.4):

$$i[K, e^{\vartheta}A_f e^{\vartheta}] \ge e^{\vartheta}\left(T_1 + \frac{5|E|}{6}f + T_4 + T_6\right)e^{\vartheta}. \tag{4.5}$$

In the argument below, we will use some specific functions θ which are dependent on some parameters. The following two lemmas are analogous results to the ones in [HMS1].

Lemma 4.1. *Let $\lambda \in \boldsymbol{R}$, $s > 0$ and*

$$\theta(u) = s \log \left(u \left(1 + \frac{u}{\mu} \right)^{-1} \right), \quad u > 0$$

for $\mu > 1$. Then there exist constants $C > 0$, $r_0 > 0$ and a compact operator $\mathcal{K}$ such that for any $R > r_0$, $\mu > 1$ and $\phi \in \mathcal{D}_0$

$$(i[(K-\lambda), e^{\vartheta} A_f e^{\vartheta}]\phi, \phi) + C\|e^{\vartheta}(K-\lambda)\phi\|^2 \geq \frac{1}{C}\|\tilde{f} e^{\vartheta}\phi\|^2 - \|\mathcal{K} e^{\vartheta}\phi\|^2 - C\|\chi e^{\vartheta}\phi\|^2,$$

where χ is the characteristic function of the interval $[R, 2R]$.

Lemma 4.2. *Let $\lambda \in \boldsymbol{R}$, $\kappa_0 \geq 0$ and*

$$\theta(u) = \mu \log \left(1 + \frac{u}{R\mu} \right) + \kappa_0 u, \quad u > 0$$

for $\mu > 1$. Then there exist constant $C > 0$, $r_0 > 0$ and a compact operator $\mathcal{K}$ such that for any $R > r_0$, $\mu > 1$ and $\phi \in \mathcal{D}_0$

$$(i[(K-\lambda), e^{\vartheta} A_f e^{\vartheta}]\phi, \phi) + C\|e^{\vartheta}(K-\lambda)\phi\|^2 \geq \frac{1}{C}\|\tilde{f} e^{\vartheta}\phi\|^2 - \|\mathcal{K} e^{\vartheta}\phi\|^2 - C\|\chi' e^{\vartheta}\phi\|^2,$$

where χ' is the characteristic function of the interval $(-\infty, 2R]$.

Proof of Lemmas 4.1 and 4.2. We will follow the argument of [HMS1]. As for the properties of θ, we use

$$0 \leq \theta'(u) = \frac{s\mu}{u(\mu+u)} \leq \frac{s}{u}, \quad |\theta''(u)| \leq \frac{s}{u^2}, \quad |\theta'''(u)| \leq \frac{2s}{u^3} \tag{4.6}$$

in case of Lemma 4.1, and

$$0 \leq \theta'(u) - \alpha_0 = \frac{\mu}{R\mu + u} \leq \frac{1}{R}, \quad |\theta''(u)| \leq \frac{1}{Ru}, \quad |\theta'''(u)| \leq \frac{2}{Ru^2} \tag{4.7}$$

in case of Lemma 4.2.

At first, noting that the support of f''' is in $[R, 2R]$ and $|f'''| = O(R^{-3})$, we have

$$T_6 \geq -C\chi^2. \tag{4.8}$$

For the convenience of notations, we put $\kappa_0 = 0$ in case of Lemma 4.1, and set $\bar{\lambda} = \lambda + (\kappa_0)^2/2$.

We consider the term T_4. First we should note that

$$\omega \cdot \nabla V(x + c(t)) = \sum_{\alpha \not\subset c} \omega^{\alpha} \cdot \nabla_{x^{\alpha}} V_{\alpha}(x^{\alpha} + c^{\alpha}(t))$$

because if $\alpha \subset c$, $\omega^\alpha = 0$. For the simplicity of notations, we put $h_\alpha(t,x) = \omega^\alpha \cdot \nabla_{x^\alpha} V_\alpha(x^\alpha + c^\alpha(t))$. Noting the boundedness of $c(t)$, inserting $1 = F(|x^\alpha| \le R') + (1 - F(|x^\alpha| \le R'))$ and taking R' sufficiently large, we see

$$-\sum_{\alpha \not\subset c} h_\alpha(t,x) \ge -\sum_{\alpha \not\subset c} F(|x^\alpha| \le R') h_\alpha(t,x) - \frac{|E|}{6}. \tag{4.9}$$

In order to treat the first term in the right-hand side of (4.9), we shall use the idea of [HMS1]. First we rewrite $F(|x^\alpha| \le R')$ as

$$\begin{aligned} &F(|x^\alpha| \le R') = \eta_\epsilon(K - \bar\lambda) F(|x^\alpha| \le R') + \mathcal{R}_\alpha, \\ &\mathcal{R}_\alpha = (K - \bar\lambda)\zeta_\epsilon(K - \bar\lambda) F(|x^\alpha| \le R'), \quad \zeta_\epsilon(u) = u^{-1}(1 - \eta_\epsilon(u)). \end{aligned} \tag{4.10}$$

Here $\epsilon > 0$ will be chosen small enough to deal with the first term later.

We look at the first term in (4.10). By inserting $1 = \sum_a j_a$ and using the properties of $\{j_a\}$, we may write $\eta_\epsilon(K - \bar\lambda)$ as

$$\eta_\epsilon(K - \bar\lambda) F(|x^\alpha| \le R') = \sum_{\alpha \subset a \ne a_{\max}} \eta_\epsilon(K - \bar\lambda) F(|x^\alpha| \le R') j_a + \mathcal{K}_1, \tag{4.11}$$

where $\mathcal{K}_1$ is compact and we used Lemma 3.5. Each term $\eta_\epsilon(K - \bar\lambda) F(|x^\alpha| \le R') j_a$ may be written as

$$\begin{aligned} \eta_\epsilon(K - \bar\lambda) F(|x^\alpha| \le R') j_a = &\{\eta_\epsilon(K - \bar\lambda) - \eta_\epsilon(K_a - \bar\lambda)\} F(|x^\alpha| \le R') j_a \\ &+ \eta_\epsilon(K_a - \bar\lambda) F(|x^\alpha| \le R') j_a. \end{aligned}$$

For the last term, we first observe

$$\|\eta_\epsilon(K_a - \bar\lambda) F(|x^\alpha| \le R')\| \le \sup_{\nu \in \boldsymbol{R}} \|\eta_\epsilon(K^a + \nu - \bar\lambda) F(|x^\alpha| \le R')\|.$$

Since $\alpha \subset a \ne a_{\max}$, we may use the induction assumption and get by virtue of Lemma 3.7

$$\sum_{\alpha \subset c} \sum_{\alpha \subset a \ne a_{\max}} \|\eta_\epsilon(K_a - \bar\lambda) F(|x^\alpha| \le R') j_a h_\alpha\| \le \frac{|E|}{6}$$

uniformly in $\lambda \in \boldsymbol{R}$ for sufficiently small $\epsilon > 0$. Here we fix such an ϵ. The first term is compact by virtue of Corollary 3.6. Finally we get

$$-\mathrm{Re} \sum_{\alpha \subset c} \eta_\epsilon(K - \bar\lambda) F(|x^\alpha| \le R') h_\alpha \ge -\frac{|E|}{6} - \mathcal{K}, \tag{4.12}$$

where $\mathcal{K}$ is compact.

On the other hand, for fixed ϵ above we can estimate as follows:

$$
\begin{aligned}
&|(\mathcal{R}_\alpha h_\alpha e^{\vartheta}\phi, e^{\vartheta}\phi)| \\
\le & C\|(K-\bar{\lambda})e^{\vartheta}\phi\| \times \|e^{\vartheta}\phi\| \\
\le & C'\|(K-\bar{\lambda})e^{\vartheta}\phi\|^2 + \epsilon'\|e^{\vartheta}\phi\|^2 \\
\le & 2C'\left\|\left(K-\lambda-\frac{1}{2}((\vartheta')^2-\vartheta'')f^2+i\vartheta' A_f\right)e^{\vartheta}\phi\right\|^2 \\
& + 2C'\left\|\left\{\frac{1}{2}(\kappa_0)^2-\frac{1}{2}((\vartheta')^2-\vartheta'')f^2+i\vartheta' A_f\right\}e^{\vartheta}\phi\right\|^2 + \epsilon'\|e^{\vartheta}\phi\|^2,
\end{aligned}
$$

where $\epsilon' > 0$ and C' is dependent on ϵ and ϵ'. As for the first term we may use the formula (4.1). Noting that

$$
\|i\vartheta' A_f e^{\vartheta}\phi\|^2 \le \frac{1}{2}(\sup \vartheta')(T_1 e^{\vartheta}\phi, e^{\vartheta}\phi) \le C(T_1 e^{\vartheta}\phi, e^{\vartheta}\phi)
$$

for some $C > 0$ independent of $R > 1$, we have

$$
\begin{aligned}
&|(\mathcal{R}_\alpha h_\alpha e^{\vartheta}\phi, e^{\vartheta}\phi)| \\
\le & 2C'\|e^{\vartheta}(K-\lambda)\phi\|^2 + C''\left\|\left\{\frac{1}{2}(\kappa_0)^2-\frac{1}{2}((\vartheta')^2-\vartheta'')f^2\right\}e^{\vartheta}\phi\right\|^2 \\
& + \epsilon''(T_1 e^{\vartheta}\phi, e^{\vartheta}\phi) + \epsilon'\|e^{\vartheta}\phi\|^2,
\end{aligned} \tag{4.13}
$$

where $\epsilon'' > 0$ and C'' is dependent on ϵ, ϵ' and ϵ''. In order to estimate the second term, we first note that

$$
0 \le \vartheta' - \kappa_0 \le \frac{C}{R}, \quad |\vartheta''| \le \frac{C}{R^2}.
$$

This implies

$$
\left|\frac{1}{2}(\kappa_0)^2 - \frac{1}{2}((\vartheta')^2 - \vartheta'')\right| \le \frac{C}{R}.
$$

Also we should notice that $1 - f^2$ is supported in the interval $(-\infty, 2R]$. Finally we get

$$
\begin{aligned}
& -\operatorname{Re}\sum_{\alpha\not\subset c}(\mathcal{R}_\alpha h_\alpha e^{\vartheta}\phi, e^{\vartheta}\phi) \\
\ge & -\frac{|E|}{6}\|e^{\vartheta}\phi\|^2 - C\|e^{\vartheta}(K-\lambda)\phi\|^2 - \frac{1}{2}(T_1 e^{\vartheta}\phi, e^{\vartheta}\phi) - C\|\chi' e^{\vartheta}\phi\|^2
\end{aligned} \tag{4.14}
$$

for sufficiently large R. Here we should notice that C is independent of R. Combining (4.5), (4.8), (4.9), (4.12) and (4.14), the lemmas may be proved. $\square$

Lemma 4.3. *Suppose that $(K-\lambda)\psi = 0$ for $\lambda \in \boldsymbol{R}$ and $\psi \in \mathcal{D}(K)$. Then for any $\kappa \geq 0$,*

$$e^{\kappa z_+}\psi \in \mathcal{H}$$

holds. Here $z_+ = \max\{z, 1\}$.

Proof. First, using Lemma 4.1, we shall show that for any $s \geq 0$

$$(z_+)^s \psi \in \mathcal{H}. \tag{4.15}$$

First we introduce the following family of bounded operators $\{P_S\}_{S>0}$ on $\mathcal{H}$, which was used by Møller [Mø] for studying the two-body case:

$$P_S = i\left(\frac{A}{S} + i\right)^{-1}. \tag{4.16}$$

We should note that they satisfy

$$\text{s - }\lim_{S\to\infty} P_S = Id \tag{4.17}$$

and

$$\text{s - }\lim_{S\to\infty} \frac{A}{S} P_S = 0. \tag{4.18}$$

We may compute the commutator $i[K, P_S]$ as

$$i[K, P_S] = \frac{i}{S} P_S \omega \cdot (E - \nabla V(x + c(t))) P_S, \tag{4.19}$$

which is bounded. Since $\psi \in \mathcal{D}(K)$, we see $P_S\psi \in \mathcal{D}(K) \cap \mathcal{D}(e^{\vartheta} A_f e^{\vartheta})$. Here we should note that e^{ϑ} in this case is smooth with bounded derivatives in z. Thus we may replace ϕ in the inequality of Lemma 4.1 by $P_S\psi$. We shall claim that by taking the limit as $S \to \infty$, we have

$$0 \geq \frac{1}{C}\|\tilde{f} e^{\vartheta}\psi\|^2 - \|\mathcal{K} e^{\vartheta}\psi\|^2 - C\|\chi e^{\vartheta}\psi\|^2. \tag{4.20}$$

Noting that

$$\|P_S \omega \cdot (E - \nabla V(x + c(t))) P_S\| \leq C,$$

where C is independent of S, we have by (4.18)

$$\lim_{S\to\infty} |([K, P_S]\psi, e^{\vartheta} A_f e^{\vartheta} P_S \psi)| = 0.$$

Thus the first term goes to 0 as $S \to \infty$. As for other terms, they may be treated more easily than the first term. Thus we obtain (4.20).

Then suppose that there is a $s > 0$ such that $(z_+)^s\psi \notin \mathcal{H}$. We put

$$\psi_\mu = \frac{e^{\vartheta}\psi}{\|e^{\vartheta}\psi\|},$$

which satisfy

$$\|\psi_\mu\| = 1, \quad \text{w - }\lim_{\mu\to\infty} \psi_\mu = 0.$$

Noticing that

$$\lim_{\mu\to\infty} \|\chi\psi_\mu\| = 0, \quad \lim_{\mu\to\infty} \|\tilde{f}\psi_\mu\| = 1$$

and $\mathcal{K}$ is a compact operator, we may get a contradiction, by virtue of (4.20). Thus (4.15) holds for any $s \geq 0$.

Now let

$$\kappa_1 = \sup\left\{\kappa \geq 0 \;\middle|\; e^{\kappa z_+}\psi \in \mathcal{H}\right\}.$$

We want to show that $\kappa_1 = \infty$. Then suppose this statement is false. What we first should mention is that if $e^{(\kappa_0+\epsilon)z_+}\psi \in \mathcal{H}$ with $\epsilon > 0$,

$$0 \geq \frac{1}{C}\|\tilde{f}e^{\vartheta}\psi\|^2 - \|\mathcal{K}e^{\vartheta}\psi\|^2 - C\|\chi' e^{\vartheta}\psi\|^2 \tag{4.21}$$

holds by virtue of Lemma 4.2. But only for the case $\kappa_0 = 0$, we consider ψ such that $(z_+)^s\psi \in \mathcal{H}$ for any $s \geq 0$. Since it is easier than for the case $\kappa_0 > 0$ to prove (4.21) for the case $\kappa_0 = 0$, we shall take care of the case $\kappa_0 > 0$ only below. For proving (4.21), we first note that Lemma 4.5 holds. In order not to stop the flow of the proof, we shall mention its statement and proof after the proof of Theorem 4.4. By Lemma 4.5 with $\kappa = \kappa_0$, we have

$$\|e^{\kappa_0 G_\infty(z)}(Id - P_S)\psi\| \leq \|(Id - P_S)e^{\kappa_0 G_\infty(z)}\psi\| + \frac{\kappa_0}{S - \kappa_0}\|e^{\kappa_0 G_\infty(z)}\psi\|,$$

where

$$G_\infty(z) = \int_0^z F(s \geq 1)\, ds.$$

Using (4.17) and that $e^{\kappa_0 G_\infty(z)}\psi \in \mathcal{H}$ by assumption, this implies

$$\lim_{S\to\infty} e^{\kappa_0 G_\infty(z)} P_S\psi = e^{\kappa_0 G_\infty(z)}\psi.$$

Noting that

$$\frac{A}{S}P_S = i(Id - P_S),$$

this implies

$$\lim_{S\to\infty} e^{\kappa_0 G_\infty(z)}\frac{A}{S}P_S\psi = 0.$$

Thus we can take the limiting procedure similar to the one in order to get (4.20), by virtue of Lemma 4.5. Therefore we get (4.21).

If $\kappa_1 = 0$, using Lemma 4.2 with $\kappa_0 = 0$, we may obtain a contradiction by letting $\mu \to \infty$ in the same way as above. If $\kappa_1 > 0$, we have only to take κ_0 in Lemma 4.2 as $\kappa_0 < \kappa_1 < \kappa_0 + (1/R)$. Here we should note that we may fix some $R > r_0$ and take $\epsilon = (\kappa_1 - \kappa_0)/2$. Thus we have a contradiction by virtue of (4.20) and see the lemma holds. $\square$

The end of this section is showing the following theorem:

Theorem 4.4. (The Absence of Bound States) *Suppose that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_{\bar{c}}$ are fulfilled. Then the pure point spectrum $\sigma_{pp}(K)$ of the Floquet Hamiltonian K is empty.*

Proof. We first consider the free Floquet Hamiltonian K_0. We may obtain the following formula for $\phi \in \mathcal{D}_0$ like as the case of N-body Stark Hamiltonians (see [HMS1]):

$$\begin{aligned}\|e^{\kappa z}(K_0-\lambda)\phi\|^2 &= \left\|\left(K_0-\lambda-\frac{1}{2}\kappa^2\right)e^{\kappa z}\phi\right\|^2 \\ &\quad + \kappa^2\|Ae^{\kappa z}\phi\|^2+\kappa|E|\|e^{\kappa z}\phi\|^2 \\ &\geq \kappa|E|\|e^{\kappa z}\phi\|^2 \end{aligned} \tag{4.22}$$

for $\lambda \in \boldsymbol{R}$ and $\kappa \geq 0$. By virtue of Lemma 4.3, we may replace ϕ in (4.22) by $\psi \in \mathcal{D}(K)$ such that $(K-\lambda)\psi = 0$, by taking the limiting procedure as in the proof of Lemma 4.3. Thus we obtain

$$\|e^{\kappa z}(K_0-\lambda)\psi\|^2 \geq \kappa|E|\|e^{\kappa z}\psi\|^2,$$

which implies

$$\|e^{\kappa z}V(x+c(t))\psi\|^2 \geq \kappa|E|\|e^{\kappa z}\psi\|^2,$$

for any $\kappa \geq 0$. In particular, taking κ as $\kappa|E| > \sum_\alpha \sup|V_\alpha(x^\alpha)|$, we have $\psi = 0$. $\square$

Here we shall state the facts used in the proof of Lemma 4.3. Some of them were proved by Møller [Mø] for two-body case (see Lemma 5.2 in [Mø]).

Lemma 4.5. *For $\gamma > 1$, we write $F_\gamma(r) = F(r \geq 1)F(r/\gamma \leq 1)$ and define*

$$G_\gamma(r) = \int_0^r F_\gamma(s)\,ds, \quad G_\infty(r) = \int_0^r F(s \geq 1)\,ds.$$

Let $\kappa > 0$ and $S > 2\kappa$. Then
(1) for any $\phi \in \mathcal{H}$ we have

$$\begin{aligned}&\|e^{\kappa G_\gamma(z)}P_S\phi\| \leq \frac{S}{S-\kappa}\|e^{\kappa G_\gamma(z)}\phi\|, \\ &\|e^{\kappa G_\gamma(z)}(Id-P_S)\phi\| \leq \|(Id-P_S)e^{\kappa G_\gamma(z)}\phi\| + \frac{\kappa}{S}\|e^{\kappa G_\gamma(z)}P_S\phi\|.\end{aligned}$$

(2) Let $\epsilon > 0$ and $\psi \in \mathcal{H}$ be such that $e^{(\kappa+\epsilon)z_+}\psi \in \mathcal{H}$. Then we have

$$e^{\kappa z_+}P_S\psi \in \mathcal{H}, \quad \lim_{\gamma\to\infty} e^{\kappa G_\gamma(z)}P_S\psi = e^{\kappa G_\infty(z)}P_S\psi.$$

Moreover, we see that $e^{\kappa z_+}P_Se^{-(\kappa+\epsilon)z_+}$ is bounded on $\mathcal{H}$ and

$$\sup_{S>2\kappa}\|e^{\kappa z_+}P_Se^{-(\kappa+\epsilon)z_+}\| < \infty.$$

(3) Let $\epsilon > 0$ and $\psi \in \mathcal{D}(K)$ be such that $e^{(\kappa+\epsilon)z_+}\psi$, $e^{(\kappa+\epsilon)z_+}K\psi \in \mathcal{H}$. Then we have $e^{\kappa z_+}KP_S\psi \in \mathcal{H}$.

Proof. The proof is similar to the one by Møller [Mø], but we need more precise estimate as for (2). (1) may been seen by computing the commutator

$$i[e^{\kappa G_\gamma(z)}, P_S] = \frac{-i\kappa}{S} P_S F_\gamma(z) e^{\kappa G_\gamma(z)} P_S. \tag{4.23}$$

In fact, we have

$$\begin{aligned}
\|e^{\kappa G_\gamma(z)} P_S \phi\| &\le \|e^{\kappa G_\gamma(z)}\phi\| + \frac{\kappa}{S}\|e^{\kappa G_\gamma(z)} P_S\phi\|, \\
\|e^{\kappa G_\gamma(z)}(Id - P_S)\phi\| &\le \|(Id - P_S)e^{\kappa G_\gamma(z)}\phi\| + \frac{\kappa}{S}\|e^{\kappa G_\gamma(z)} P_S\phi\|,
\end{aligned}$$

which imply (1). We shall prove (2). We put

$$\psi_\gamma = e^{\kappa G_\gamma(z)} P_S\psi, \quad \gamma > 0,$$

and consider the family $\{\psi_\gamma\}_{\gamma>0}$. We shall first show that this family forms a Cauchy sequence. Using (4.23), we may estimate

$$\|\psi_{\gamma_1} - \psi_{\gamma_2}\| \le \|(e^{\kappa G_{\gamma_1}(z)} - e^{\kappa G_{\gamma_2}(z)})\psi\| + \frac{\kappa}{S}\{\|\psi_{\gamma_1} - \psi_{\gamma_2}\| + \|(F_{\gamma_1}(z) - F_{\gamma_2}(z))\psi_{\gamma_1}\|\}.$$

This is rewritten as

$$\|\psi_{\gamma_1} - \psi_{\gamma_2}\| \le \frac{S}{S-\kappa}\left\{\|(e^{\kappa G_{\gamma_1}(z)} - e^{\kappa G_{\gamma_2}(z)})\psi\| + \frac{\kappa}{S}\|(F_{\gamma_1}(z) - F_{\gamma_2}(z))\psi_{\gamma_1}\|\right\}. \tag{4.24}$$

We consider the second term. Noting that

$$i[(F_{\gamma_1}(z) - F_{\gamma_2}(z))e^{\kappa G_{\gamma_1}(z)}, P_S] = \frac{-i\kappa}{S} P_S \frac{d}{dz}\{(F_{\gamma_1}(z) - F_{\gamma_2}(z))e^{\kappa G_{\gamma_1}(z)}\} P_S$$

and $\|F'_{\gamma_1}(z) - F'_{\gamma_2}(z)\| \le C/\min(\gamma_1, \gamma_2)$, we have

$$\begin{aligned}
\|(F_{\gamma_1}(z) - F_{\gamma_2}(z))\psi_{\gamma_1}\| \le{}& \|(F_{\gamma_1}(z) - F_{\gamma_2}(z))e^{\kappa G_{\gamma_1}(z)}\psi\| \\
&+ \frac{C\kappa}{S\min(\gamma_1,\gamma_2)}\|e^{\kappa G_{\gamma_1}(z)} P_S\psi\| \\
&+ \frac{\kappa}{S}\|(F_{\gamma_1}(z) - F_{\gamma_2}(z))\psi_{\gamma_1}\|.
\end{aligned}$$

This is rewritten as

$$\begin{aligned}
\|(F_{\gamma_1}(z) - F_{\gamma_2}(z))\psi_{\gamma_1}\| \le{}& \frac{S}{S-\kappa}\|(F_{\gamma_1}(z) - F_{\gamma_2}(z))e^{\kappa G_{\gamma_1}(z)}\psi\| \\
&+ \frac{C\kappa}{(S-\kappa)\min(\gamma_1,\gamma_2)}\|e^{\kappa G_{\gamma_1}(z)} P_S\psi\| \\
\le{}& \frac{S}{S-\kappa}\|(F_{\gamma_1}(z) - F_{\gamma_2}(z))e^{\kappa G_{\gamma_1}(z)}\psi\| \\
&+ \frac{CS\kappa}{(S-\kappa)^2\min(\gamma_1,\gamma_2)}\|e^{\kappa G_{\gamma_1}(z)}\psi\|,
\end{aligned} \tag{4.25}$$

where we used (1). Noting that $\|e^{\kappa G_{\gamma_1}(z)}\psi\| \le \|e^{\kappa G_\infty(z)}\psi\| < \infty$ by assumption and that $F_{\gamma_1}(z) - F_{\gamma_2}(z)$ is supported in $\{z \mid z \ge (1-\delta)\min(\gamma_1,\gamma_2)\}$, where $0 < \delta < 1$ is as in the definition of functions F, we have by (4.25)

$$\begin{aligned}
&\|(F_{\gamma_1}(z) - F_{\gamma_2}(z))\psi_{\gamma_1}\| \\
\le&\frac{S}{S-\kappa}\sup\{e^{-\epsilon z_+} \mid z \ge (1-\delta)\min(\gamma_1,\gamma_2)\}\|e^{(\kappa G_\infty(z)+\epsilon z_+)}\psi\| \\
&+\frac{CS\kappa}{(S-\kappa)^2\min(\gamma_1,\gamma_2)}\|e^{\kappa G_\infty(z)}\psi\|. \qquad (4.26)
\end{aligned}$$

Also noticing that $e^{\kappa G_{\gamma_1}(z)} - e^{\kappa G_{\gamma_2}(z)}$ is supported in $\{z \mid z \ge (1-\delta)\min(\gamma_1,\gamma_2)\}$, where $0 < \delta < 1$ is as in the definition of functions F, we have

$$\|(e^{\kappa G_{\gamma_1}(z)} - e^{\kappa G_{\gamma_2}(z)})\psi\| \le C\sup\{e^{-\epsilon z_+} \mid z \ge (1-\delta)\min(\gamma_1,\gamma_2)\}\|e^{(\kappa+\epsilon)z_+}\psi\|.$$

Combining this with (4.24) and (4.26), we see that $\{\psi_\gamma\}_{\gamma>0}$ forms a Cauchy sequence. Thus the limit $\psi_\infty = \lim_{\gamma\to\infty}\psi_\gamma$ exists in $\mathcal{H}$. Therefore by virtue of the Lebesgue's dominated convergence theorem, we obtain the first statement of (2). Now in (4.24), putting for instance $\gamma_1 = 2$ and taking the limit as $\gamma_2 \to \infty$, we have

$$\|\psi_2 - \psi_\infty\| \le \frac{S}{S-\kappa}\left\{\|(e^{\kappa G_2(z)} - e^{\kappa G_\infty(z)}\psi\| + \frac{\kappa}{S}\|(F_2(z) - F(z \ge 1))\psi_2\|\right\}.$$

Since $S > 2\kappa$, we may estimate this as

$$\|\psi_2 - \psi_\infty\| \le 2\|(e^{\kappa G_2(z)} - e^{\kappa G_\infty(z)}\psi\| + \|(F_2(z) - F(z \ge 1))\psi_2\|. \qquad (4.27)$$

Putting $\phi = e^{(\kappa+\epsilon)z_+}\psi \in \mathcal{H}$, we have the following estimate by (4.27):

$$\|e^{\kappa G_\infty(z)}P_S e^{-(\kappa+\epsilon)z_+}\phi\| \le C\|\phi\|,$$

where $C > 0$ is independent of $S > 2\kappa$. This implies the second statement of (2).

Finally we prove (3). This may be immediately seen by combining (4.19) with (2) and the assumption of (3). □

§5. Propagation estimates for the Floquet Hamiltonians

In this section, we shall derive some propagation estimates for the Floquet Hamiltonian K. We shall follow the idea of ourselves [A2], in which we derived some propagation estimates of time-pointwise type for N-body Stark Hamiltonians. But, since the relative boundedness of some operators with respect to the Hamiltonian may not hold, we have to take care of this point.

First we shall mention the Mourre estimate [M] for the Hamiltonian K. By virtue of Lemma 3.7 and Theorem 4.4, we may prove the following Mourre estimate in the same way as in [HMS1].

Theorem 5.1. (The Mourre Estimate) *(1) Let $0 < \nu < |E| < \nu'$. Then one can take $\epsilon > 0$ so small uniformly in $\lambda \in \boldsymbol{R}$ that*

$$\eta_\epsilon(K-\lambda) i[K, A] \eta_\epsilon(K-\lambda) \geq \nu \eta_\epsilon(K-\lambda)^2, \tag{5.1}$$

$$\eta_\epsilon(K-\lambda) i[K, -A] \eta_\epsilon(K-\lambda) \geq -\nu' \eta_\epsilon(K-\lambda)^2 \tag{5.2}$$

hold.
(2) The spectrum of K is purely absolutely continuous.

Proof. We prove the inequality (5.1) only. The inequality (5.2) may be proved similarly, and the statement (2) is a consequence of the estimate (5.1).

Computing the commutator $i[K, A]$, we have

$$i[K, A] = |E| - \sum_{\alpha \not\subset c} \omega^\alpha \cdot \nabla_{x^\alpha} V_\alpha(x^\alpha + c^\alpha(t)).$$

Since $\alpha \not\subset c$, we see that $\sigma_{pp}(K^a) = \emptyset$ for all a such that $\alpha \subset a \subset a_{\max}$, by virtue of Theorem 4.4. For each term $\omega^\alpha \cdot \nabla_{x^\alpha} V_\alpha(x^\alpha + c^\alpha(t))$ in the sum, inserting $1 = F(|x^\alpha| \leq R) + (1 - F(|x^\alpha| \leq R))$ and taking $R > 0$ sufficiently large, we have

$$\left\| \sum_{\alpha \not\subset c} \omega^\alpha \cdot \nabla_{x^\alpha} V_\alpha(x^\alpha + c^\alpha(t))(1 - F(|x^\alpha| \leq R)) \right\| \leq \frac{|E| - \nu}{2},$$

where we noticed that $c(t)$ is bounded in t. We fix this R. On the other hand, by virtue of Lemma 3.7, we obtain

$$\begin{aligned} &\eta_\epsilon(K-\lambda) \left(-\sum_{\alpha \not\subset c} \omega^\alpha \cdot \nabla_{x^\alpha} V_\alpha(x^\alpha + c^\alpha(t)) F(|x^\alpha| \leq R) \right) \eta_\epsilon(K-\lambda) \\ &\geq -\frac{|E| - \nu}{2} \eta_\epsilon(K-\lambda)^2, \end{aligned}$$

by taking $\epsilon > 0$ sufficiently small. We should note that we may take ϵ independent of $\lambda \in \boldsymbol{R}$ by virtue of Lemma 3.7. These two estimates obtained above imply that the inequality (5.1) holds. $\square$

Now we introduce the notion of the Heisenberg derivative. We denote the Heisenberg derivative of $\Phi(\sigma)$ by

$$\mathbf{D}(\Phi(\sigma)) = \frac{\partial \Phi}{\partial \sigma} + i[K, \Phi(\sigma)].$$

The following proposition may be proved by the abstract theory of Skibsted [Sk].

Proposition 5.2. (The Maximal Acceleration Bound) *Let $f \in C_0^\infty(\boldsymbol{R})$ and $s \geq l \geq 0$. Then there exists $M > 0$ such that the following estimate holds as $\sigma \to \infty$:*

$$\left\| \langle p \rangle^l F\left(\frac{\langle p \rangle}{\sigma} \geq M\right) e^{-i\sigma K} f(K) \langle p \rangle^{-s} \right\| = O(\sigma^{l-s}). \tag{5.3}$$

Proof. We have only to follow the proof of Theorem 3.4 of [Sk], and check the conditions which should be satisfied. In this case, we should note that, for example, $[K, \langle p \rangle]$ is bounded. We omit the detail of the proof (see [A2] and [Sk]). $\square$

The following lemma may follow from Theorem 5.1 (1) and the abstract theory of Skibsted [Sk].

Lemma 5.3. *Let $f \in C_0^\infty(\mathbf{R})$ and $s > 0$. Let ν and ν' be as in Theorem 5.1. Then the following estimate holds as $\sigma \to \infty$:*

$$\left\| F\left(\frac{A}{\sigma} \leq \nu\right) e^{-i\sigma K} f(K) \langle A \rangle^{-s} \right\| = O(\sigma^{-s}), \tag{5.4}$$

$$\left\| F\left(\frac{A}{\sigma} \geq \nu'\right) e^{-i\sigma K} f(K) \langle A \rangle^{-s} \right\| = O(\sigma^{-s}). \tag{5.5}$$

Proof. The lemma can be proved as in the proof of Lemma 5.5 of [A1] (see also Example 1 in section 3 of [Sk]). $\square$

The following proposition means the minimal acceleration bound of the evolution of the Floquet Hamiltonian K, which played an important role for the scattering theory for N-body Stark Hamiltonians (see e.g. [AT1,AT2] and [HMS2]).

Proposition 5.4. (The Minimal Acceleration Bound) *Let $f \in C_0^\infty(\boldsymbol{R})$, and $0 < \nu < |E|$. Then the following estimate holds as $\sigma \to \infty$:*

$$\left\| F\left(\frac{z}{\sigma^2} \leq \frac{\nu}{2}\right) e^{-i\sigma K} f(K) \langle p \rangle^{-1} \langle D_t \rangle^{-1} \right\| = O(\sigma^{-1/2}). \tag{5.6}$$

Proof. To prove this, we follow the idea of Møller [Mø], which was inspired by [HMS2]. We put $\zeta(\sigma) = F(z/\sigma^2 \leq \nu/2) e^{-i\sigma K} f(K) \langle p \rangle^{-1} \langle D_t \rangle^{-1} \psi$ for $\psi \in \mathcal{H}$. We compute

$$\begin{aligned} \| |p| \zeta(\sigma) \|^2 &= (2(K + |E|z - V(x + c(t)) - D_t)\zeta(\sigma), \zeta(\sigma)) \\ &\leq |E|\nu\sigma^2 \|\zeta(\sigma)\|^2 + 2\|D_t \zeta(\sigma)\| \|\zeta(\sigma)\| + C\|\psi\|^2. \end{aligned}$$

Here we should note that, by virtue of Proposition 5.2,

$$\left\| \left[K, F\left(\frac{z}{\sigma^2} \leq \frac{\nu}{2}\right)\right] e^{-i\sigma K} f(K) \langle p \rangle^{-1} \langle D_t \rangle^{-1} \psi \right\| = O(\sigma^{-1}) \|\psi\|.$$

Because of $\|D_t \zeta(\sigma)\| = O(\sigma)\|\psi\|$, which may be seen by the fact that the Heisenberg derivative of D_t is $O(1)$, we obtain

$$\| |p| \zeta(\sigma) \|^2 \leq |E|\nu\sigma^2 \|\zeta(\sigma)\|^2 + C'\sigma\|\psi\|^2. \tag{5.7}$$

On the other hand, we may estimate $\|A\zeta(\sigma)\|^2$ as follows: for $\nu_1 > 0$ such that $|E|\nu < (\nu_1)^2 < |E|^2$,

$$\|A\zeta(\sigma)\|^2 \geq (\nu_1)^2\sigma^2 \left\| F\left(\frac{A^2}{\sigma^2} \geq (\nu_1)^2\right)\zeta(\sigma)\right\|^2$$
$$\geq (\nu_1)^2\sigma^2\|\zeta(\sigma)\|^2 - (\nu_1)^2\sigma^2 \left\|\bar{F}\left(\frac{A^2}{\sigma^2} \leq (\nu_1)^2\right)\zeta(\sigma)\right\|^2. \tag{5.8}$$

Here we took $F(u \geq (\nu_1)^2)$ and $\bar{F}(u \leq (\nu_1)^2)$ as $1 = F(u \geq (\nu_1)^2)^2 + \bar{F}(u \leq (\nu_1)^2)^2$. Combining (5.7) with (5.8) and using (5.4), we have

$$\{(\nu_1)^2 - |E|\nu\}\sigma^2\|\zeta(\sigma)\|^2 \leq (\nu_1)^2\sigma^2 \left\|\bar{F}\left(\frac{A^2}{\sigma^2} \leq (\nu_1)^2\right)\zeta(\sigma)\right\|^2 + C'\sigma\|\psi\|^2$$
$$\leq C''\sigma\|\psi\|^2, \tag{5.9}$$

where we should notice that

$$\left\|\left[\bar{F}\left(\frac{A^2}{\sigma^2} \leq (\nu_1)^2\right), F\left(\frac{z}{\sigma^2} \leq \frac{\nu}{2}\right)\right]\right\| = O(\sigma^{-3}).$$

Noting that $(\nu_1)^2 - |E|\nu > 0$, we obtain the proposition by (5.9). □

The following proposition is related to the estimate (5.2).

Proposition 5.5. *Let $f \in C_0^\infty(\boldsymbol{R})$, and $\nu' > |E|$. Then the following estimate holds as $\sigma \to \infty$:*

$$\left\|F\left(\frac{z}{\sigma^2} \geq \frac{\nu'}{2}\right) e^{-i\sigma K} f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1/2}). \tag{5.10}$$

Proof. First we note that for $g \in C_0^\infty(\boldsymbol{R})$

$$\langle z\rangle^{1/2} g(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}$$

is bounded on $\mathcal{H}$. We put

$$\xi(\sigma) = F\left(\frac{z}{\sigma^2} \geq M'\right) F\left(\frac{|p|}{\sigma} \leq M\right) e^{-i\sigma K} f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\psi$$

for $\psi \in \mathcal{H}$. Here M, $M' > 0$ which will be chosen so large later. Now we shall compute

$$\|\sqrt{|E|z}\xi(\sigma)\|^2 = \left(\left(-K + \frac{1}{2}p^2 + V(x + c(t)) + D_t\right)\xi(\sigma), \xi(\sigma)\right)$$
$$\leq \frac{M^2}{2}\sigma^2\|\xi(\sigma)\|^2 + C\sigma\|\psi\|^2 \tag{5.11}$$

as in the proof of Proposition 5.4. Here, by virtue of Proposition 5.2, by taking $M > 0$ sufficiently large,

$$\left\|F\left(\frac{|p|}{\sigma} \geq M\right) e^{-i\sigma K} f(K)\langle p\rangle^{-1}\right\| = O(\sigma^{-1}) \tag{5.12}$$

holds. We fix such an M. Since clearly

$$\|\sqrt{|E|z}\xi(\sigma)\|^2 \geq M'|E|\sigma^2\|\xi(\sigma)\|^2,$$

by taking M' so large that $M'|E| > M^2/2$, we have

$$\left(M'|E| - \frac{M^2}{2}\right)\sigma^2\|\xi(\sigma)\|^2 \leq C'\sigma\|\psi\|^2,$$

i.e.

$$\left\|F\left(\frac{z}{\sigma^2} \geq M'\right)F\left(\frac{|p|}{\sigma} \leq M\right)e^{-i\sigma K}f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1/2}).$$

Noticing the choice of M, this implies

$$\left\|F\left(\frac{z}{\sigma^2} \geq M'\right)e^{-i\sigma K}f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1/2}), \tag{5.13}$$

by virtue of (5.12).

Next we put

$$\zeta(\sigma) = F\left(\frac{\nu'}{2} \leq \frac{z}{\sigma^2} \leq M'\right)F\left(\left|\frac{A}{\sigma} - |E|\right| \leq \epsilon'\right)e^{-i\sigma K}f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\psi,$$

where M' is taken as above, and $F'(\nu'/2 \leq u \leq M') \geq 0$ for $\nu'/2 \leq u \leq \nu'/2 + \delta$. We compute

$$\left\|\sqrt{z - \frac{|E|}{2}\sigma^2}\zeta(\sigma)\right\|^2 = \left(\left(z - \frac{|E|}{2}\sigma^2\right)\zeta(\sigma), \zeta(\sigma)\right).$$

We note that the Heisenberg derivatives of $z - |E|\sigma^2/2$, $F(z/\sigma^2 \geq \nu'/2)$ and $F(|A/\sigma - |E|| \leq \epsilon')$ are as follows:

$$\mathbf{D}\left(z - \frac{|E|}{2}\sigma^2\right) = A - |E|\sigma, \tag{5.14}$$

$$\begin{aligned}&\mathbf{D}\left(F\left(\frac{z}{\sigma^2} \geq \frac{\nu'}{2}\right)\right)\\ =&F'\left(\frac{z}{\sigma^2} \geq \frac{\nu'}{2}\right)\left\{-2\sigma^{-3}\left(z - \frac{|E|}{2}\sigma^2\right) + \sigma^{-2}(A - |E|\sigma)\right\} + O(\sigma^{-4}),\end{aligned} \tag{5.15}$$

$$\mathbf{D}\left(F\left(\left|\frac{A}{\sigma} - |E|\right| \leq \epsilon'\right)\right) = O(\sigma^{-1})F\left(\left|\frac{A}{\sigma} - |E|\right| \geq \epsilon''\right) + O(\sigma^{-\infty}), \tag{5.16}$$

for some $0 < \epsilon'' < \epsilon'$. Here we used the almost analytic extension method [HeSj] and pseudodifferential calculus to obtain the last estimate. Noting that $F'(\nu'/2 \leq u \leq M') \geq 0$ for

$\nu'/2 \le u \le \nu'/2+\delta$, and using Lemma 5.3 and (5.14)–(5.16), $d(\|\sqrt{z-|E|\sigma^2/2}\zeta(\sigma)\|^2)/d\sigma$ can be estimated as follows:

$$\frac{d}{d\sigma}\left(\left\|\sqrt{z-\frac{|E|}{2}\sigma^2}\zeta(\sigma)\right\|^2\right) \le O(1)\|\psi\|^2 + C\epsilon'\sigma\|\zeta(\sigma)\|^2. \tag{5.17}$$

Hence we have

$$\begin{aligned}\frac{\nu'-|E|}{2}\sigma^2\|\zeta(\sigma)\|^2 \le & \left\|\sqrt{z-\frac{|E|}{2}\sigma^2}\zeta(\sigma)\right\|^2 \\ & \le O(\sigma)\|\psi\|^2 + \int_1^\sigma C\epsilon'\tau\|\zeta(\tau)\|^2 d\tau,\end{aligned} \tag{5.18}$$

by integrating (5.17) in σ. Taking $\epsilon' > 0$ so small that $C\epsilon' < (\nu' - |E|)/2$, we obtain

$$\sigma^2\|\zeta(\sigma)\|^2 \le O(\sigma)\|\psi\|^2, \tag{5.19}$$

by virtue of Gronwall's lemma. Combining Lemma 5.3 with (5.13) and (5.19), this implies the proposition. $\square$

The following theorem is one of the main results in this section.

Theorem 5.6. *Let $f \in C_0^\infty(\boldsymbol{R})$. Then the following estimate holds as $\sigma \to \infty$:*

$$\left\|F\left(\left|\frac{p}{\sigma}-E\right| \ge \epsilon\right)e^{-i\sigma K}f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1/2}). \tag{5.20}$$

Proof. We write $\zeta(\sigma) = F(\epsilon \le |p/\sigma - E| \le M)e^{-i\sigma K}f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\psi$ for $\psi \in \mathcal{H}$, where $M > 0$ is so large that

$$\left\|F\left(\left|\frac{p}{\sigma}-E\right| \ge M\right)e^{-i\sigma K}f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1}) \tag{5.21}$$

holds, by virtue of Proposition 5.2. We compute

$$\begin{aligned}&\||p-E\sigma|\zeta(\sigma)\|^2 \\ =&(\zeta(\sigma),(p-E\sigma)^2\zeta(\sigma)) \\ =&(\zeta(\sigma),\{2(K-V(x+c(t))-D_t)+2|E|z-2|E|A\sigma+|E|^2\sigma^2\}\zeta(\sigma)) \\ =&\left(\zeta(\sigma),\left[2(K-V(x+c(t))-D_t)+2|E|\left\{\left(z-\frac{|E|}{2}\sigma^2\right)-\sigma(A-|E|\sigma)\right\}\right]\zeta(\sigma)\right).\end{aligned} \tag{5.22}$$

Here we should note that Lemma 5.3, Propositions 5.4 and 5.5 imply that for any $\epsilon' > 0$,

$$\left\|F\left(\left|\frac{A}{\sigma}-|E|\right| \ge \epsilon'\right)e^{-i\sigma K}f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1/2}), \tag{5.23}$$

$$\left\|F\left(\left|\frac{z}{\sigma^2}-\frac{|E|}{2}\right| \ge \epsilon'\right)e^{-i\sigma K}f(K)\langle z\rangle^{-1/2}\langle p\rangle^{-1}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1/2}). \tag{5.24}$$

Using these estimates, and using the almost analytic extension method [HeSj] to control the commutators, we obtain the following estimate:

$$\epsilon^2\sigma^2\|\zeta(\sigma)\|^2 \leq C\sigma\|\psi\|^2 + C'\epsilon'\sigma^2\|\zeta(\sigma)\|^2, \tag{5.25}$$

where we should note (5.11) holds. Hence, taking $\epsilon' > 0$ so small that $C'\epsilon' < \epsilon^2$, we have

$$\|\zeta(\sigma)\| \leq C\sigma^{-1/2}\|\psi\|,$$

by virtue of Gronwall's lemma. Combining this estimate with Proposition 5.2, we obtain the theorem. □

The following proposition will be needed in order to guarantee the maximal acceleration bound with respect to the position x, not the velocity p.

Proposition 5.7. *Let $f \in C_0^\infty(\boldsymbol{R})$. Then there exists $M > 0$ such that the following estimate holds as $\sigma \to \infty$:*

$$\left\| F\left(\frac{\langle x_\perp\rangle}{\sigma^2} \geq M\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2} \right\| = O(\sigma^{-1}). \tag{5.26}$$

Here $x_\perp = x - z\omega$.

Proof. First we put

$$A_\perp = \frac{1}{2}(x_\perp \cdot p_\perp + p_\perp \cdot x_\perp),$$

where $p_\perp = -i\nabla_\perp = p - A\omega$, and consider

$$\xi(\sigma) = F\left(\frac{A_\perp}{\sigma^3} \geq M'\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2}\psi$$

for $\psi \in \mathcal{H}$, where $M' > 0$ will be taken very large later, and $F'(u \geq M') \geq 0$. We should note that

$$\langle x\rangle g(K)\langle x\rangle^{-1}\langle p\rangle^{-1}$$

is bounded on $\mathcal{H}$. We shall compute the Heisenberg derivatives of $A_\perp$ and $F(A_\perp/\sigma^3 \geq M')$:

$$\begin{aligned}
&\mathbf{D}(A_\perp) = {p_\perp}^2 - x_\perp \cdot \nabla_\perp V(x + c(t)),\\
&\mathbf{D}\left(F\left(\frac{A_\perp}{\sigma^3} \geq M'\right)\right)\\
=& -\frac{3}{\sigma^4}A_\perp F'\left(\frac{A_\perp}{\sigma^3} \geq M'\right) + \frac{1}{\sigma^3}F'\left(\frac{A_\perp}{\sigma^3} \geq M'\right)({p_\perp}^2 - x_\perp \cdot \nabla_\perp V(x + c(t)))\\
&+ O(\sigma^{-6})\langle p_\perp\rangle^2.
\end{aligned}$$

Noting that $F'(u \geq M') \geq 0$ has a compact support, and that, by virtue of the assumption on the potentials, $x_\perp \cdot \nabla_\perp V(x + c(t))$ is bounded, $d(A_\perp\xi(\sigma), \xi(\sigma))/d\sigma$ can be estimated as follows:

$$\frac{d}{d\sigma}(A_\perp\xi(\sigma), \xi(\sigma)) \leq C\sigma^2\|\xi(\sigma)\|^2 + C'\|\psi\|^2, \tag{5.27}$$

where in order to deal with the terms related to ${p_\perp}^2$, we used Proposition 5.2. Hence we have

$$\begin{aligned} M'\sigma^3\|\xi(\sigma)\|^2 &\leq (A_\perp\xi(\sigma),\xi(\sigma)) \\ &\leq O(\sigma)\|\psi\|^2 + \int_1^\sigma C\tau^2\|\xi(\tau)\|^2\,d\tau, \end{aligned} \tag{5.28}$$

by integrating (5.27) in σ. Taking $M' > 0$ so large that $C < M'$, we obtain

$$\sigma^3\|\xi(\sigma)\|^2 \leq O(\sigma)\|\psi\|^2,$$

by virtue of Gronwall's lemma. This implies

$$\left\| F\left(\frac{A_\perp}{\sigma^3} \geq M'\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2} \right\| = O(\sigma^{-1}). \tag{5.29}$$

Moreover, by using (5.29), we have

$$\frac{d}{d\sigma}(A_\perp\xi(\sigma),\xi(\sigma)) \leq C''\|\psi\|^2,$$

i.e. there exists $C > 0$ such that

$$(A_\perp\xi(\sigma),\xi(\sigma)) \leq C\sigma\|\psi\|^2 \tag{5.30}$$

for $\sigma \geq 1$.

Next we put

$$\zeta(\sigma) = F\left(\frac{\langle x_\perp\rangle}{\sigma^2} \geq M\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2}\psi$$

for $\psi \in \mathcal{H}$, where $M > 0$ will be taken very large later. We shall compute the Heisenberg derivatives of $\langle x_\perp\rangle^2$ and $F(\langle x_\perp\rangle/\sigma^2 \geq M)$:

$$\mathbf{D}(\langle x_\perp\rangle^2) = 2A_\perp,$$

$$\mathbf{D}\left(F\left(\frac{\langle x_\perp\rangle}{\sigma^2} \geq M\right)\right) = \frac{1}{2\sigma^2}\left(\frac{x_\perp}{\langle x_\perp\rangle}\cdot p_\perp + p_\perp\cdot\frac{x_\perp}{\langle x_\perp\rangle}\right) F'\left(\frac{\langle x_\perp\rangle}{\sigma^2} \geq M\right) + O(\sigma^{-4}).$$

Then $d(\langle x_\perp\rangle^2\zeta(\sigma),\zeta(\sigma))/d\sigma$ can be estimated as follows:

$$\frac{d}{d\sigma}(\langle x_\perp\rangle^2\zeta(\sigma),\zeta(\sigma)) \leq 2M'\sigma^3\|\zeta(\sigma)\|^2 + C'\sigma\|\psi\|^2,$$

where in order to deal with the terms related to $p_\perp$ and $A_\perp$, we used Proposition 5.2 and (5.30). Hence we have

$$\begin{aligned} M^2\sigma^4\|\zeta(\sigma)\|^2 &\leq (\langle x_\perp\rangle^2\zeta(\sigma),\zeta(\sigma)) \\ &\leq O(\sigma^2)\|\psi\|^2 + \int_1^\sigma 2M'\tau^3\|\zeta(\tau)\|^2\,d\tau. \end{aligned} \tag{5.31}$$

Taking $M > 0$ so large that $2M' \leq M^2$, we obtain

$$\sigma^4\|\zeta(\sigma)\|^2 \leq O(\sigma^2)\|\psi\|^2,$$

by virtue of Gronwall's lemma. This implies the proposition. □

The following theorem is one of the main results in this section. This implies one of the important propagation properties of the evolution of the Floquet Hamiltonian K.

Theorem 5.8. *Let $f \in C_0^\infty(\boldsymbol{R})$ and $\epsilon > 0$. Then the following estimate holds as $\sigma \to \infty$:*

$$\left\| F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \geq \epsilon\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1/2}). \tag{5.32}$$

Proof. We should note that by virtue of Propositions 5.4, 5.5 and 5.7, there exists $M > 0$ such that the following estimate holds:

$$\left\| F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \geq M\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2}\langle D_t\rangle^{-1}\right\| = O(\sigma^{-1/2}). \tag{5.33}$$

Here we write

$$\zeta(\sigma) = F\left(\epsilon \leq \left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \leq M\right) F\left(\left|\frac{p}{\sigma} - E\right| \leq \epsilon'\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2}\langle D_t\rangle^{-1}\psi$$

with $\psi \in \mathcal{H}$, where $F'(\epsilon \leq s \leq M) \geq 0$ for $\epsilon \leq s \leq \epsilon + \delta$. We compute

$$\left(\left(x - \frac{E}{2}\sigma^2\right)^2 \zeta(\sigma), \zeta(\sigma)\right).$$

We note that the Heisenberg derivatives of $x - E\sigma^2/2$, $F(\epsilon \leq |x/\sigma^2 - E/2| \leq M)$ and $F(|p/\sigma - E| \leq \epsilon')$ are as follows:

$$\begin{aligned}
&\mathbf{D}\left(x - \frac{E}{2}\sigma^2\right) = p - E\sigma,\\
&\mathbf{D}\left(F\left(\epsilon \leq \left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \leq M\right)\right)\\
=&F'\left(\epsilon \leq \left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \leq M\right)\left|\frac{x}{\sigma^2} - \frac{E}{2}\right|^{-1}\\
&\times\left(\frac{x}{\sigma^2} - \frac{E}{2}\right)\cdot\left\{-2\sigma^{-3}\left(x - \frac{E}{2}\sigma^2\right) + \sigma^{-2}(p - E\sigma)\right\} + O(\sigma^{-4}),\\
&\mathbf{D}\left(F\left(\left|\frac{p}{\sigma} - E\right| \leq \epsilon'\right)\right) = O(\sigma^{-1})F\left(\left|\frac{p}{\sigma} - E\right| \geq \epsilon''\right) + O(\sigma^{-\infty}),
\end{aligned}$$

for some $0 < \epsilon'' < \epsilon'$. Here we used the almost analytic extension method [HeSj] and pseudodifferential calculus to obtain the last result. Noting that $F'(\epsilon \leq s \leq M) \geq 0$ for $\epsilon \leq s \leq \epsilon + \delta$, and using Theorem 5.6 and the above formulas, $d((x - E\sigma^2/2)^2\zeta(\sigma), \zeta(\sigma))/d\sigma$ can be estimated as follows:

$$\frac{d}{d\sigma}\left(\left(x - \frac{E}{2}\sigma^2\right)^2 \zeta(\sigma), \zeta(\sigma)\right) \leq O(\sigma^2)\|\psi\|^2 + C\epsilon'\sigma^3\|\zeta(\sigma)\|^2.$$

Hence we have

$$\begin{aligned}
\epsilon^2\sigma^4\|\zeta(\sigma)\|^2 \leq& \left(\left(x - \frac{E}{2}\sigma^2\right)^2 \zeta(\sigma), \zeta(\sigma)\right)\\
\leq& O(\sigma^3)\|\psi\|^2 + \int_1^\sigma C\epsilon'\tau^3\|\zeta(\tau)\|^2 d\tau.
\end{aligned} \tag{5.34}$$

Taking $\epsilon' > 0$ so small that $C\epsilon' \le \epsilon^2$, by virtue of Gronwall's lemma, we obtain

$$\sigma^4 \|\zeta(\sigma)\|^2 \le O(\sigma^3)\|\psi\|^2.$$

Combining this with (5.33), this implies the theorem. □

In order to prove Theorem 2.3, we will need more propagation properties of the evolution of the Floquet Hamiltonian K.

Proposition 5.9. *Let* $0 < \epsilon < \min_{\alpha \not\subset c} |E^\alpha|/2$. *Set*

$$\zeta(\sigma) = F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) F\left(\left|\frac{p}{\sigma} - E\right| \le \epsilon\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2}\langle D_t\rangle^{-1}\psi$$

for $\psi \in \mathcal{H}$. *Then the following estimate holds as* $\sigma \to \infty$*:*

$$\left|\left(\zeta(\sigma), \left(z - A\sigma + \frac{|E|}{2}\sigma^2\right)\zeta(\sigma)\right)\right| = O(\sigma)\|\psi\|^2. \tag{5.35}$$

Proof. The Heisenberg derivative of $z - A\sigma + |E|\sigma^2/2$ is as follows:

$$\mathbf{D}\left(z - A\sigma + \frac{|E|}{2}\sigma^2\right) = \sigma\omega \cdot \nabla V(x + c(t)). \tag{5.36}$$

Hence, by assumption, we see that $\mathbf{D}(z - A\sigma + |E|\sigma^2/2) = O(\sigma^{-\rho})$ on $\operatorname{supp} F(|x/\sigma^2 - E/2| \le \epsilon)$, with $\rho > 1$. We note that the Heisenberg derivative of $F(|x/\sigma^2 - E/2| \le \epsilon)$ is computed as

$$\begin{aligned}
&\mathbf{D}\left(F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right)\right) \\
=&F'\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right)\left|\frac{x}{\sigma^2} - \frac{E}{2}\right|^{-1} \\
&\times \left(\frac{x}{\sigma^2} - \frac{E}{2}\right) \cdot \left\{-2\sigma^{-3}\left(x - \frac{E}{2}\sigma^2\right) + \sigma^{-2}(p - E\sigma)\right\} + O(\sigma^{-4}),
\end{aligned}$$

where we should note that for some $0 < \epsilon' < \epsilon$

$$F'\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) = F'\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \ge \epsilon'\right)$$

holds. Noting these facts, by straightforward computation, we have

$$\left|\left(\zeta(\sigma), \left(z - A\sigma + \frac{|E|}{2}\sigma^2\right)\zeta(\sigma)\right)\right| = O(\sigma)\|\psi\|^2.$$

Here we used Theorems 5.6 and 5.8. This implies the proposition. □

The following theorem, which will be needed in the proof of Theorem 2.3, may be proved similarly to [A2], by virtue of Proposition 5.9.

Theorem 5.10. *Let* $0 < \epsilon < \min_{\alpha \not\subset c} |E^\alpha|/2$. *Put*

$$Z(\sigma) = F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) F\left(\left|\frac{p}{\sigma} - E\right| \le \epsilon\right) e^{-i\sigma K} f(K)\langle x\rangle^{-1}\langle p\rangle^{-2}\langle D_t\rangle^{-1}.$$

Then we have as $\sigma \to \infty$

$$\||p - E\sigma|Z(\sigma)\| = O(\sigma^{1/2}), \tag{5.37}$$

$$\left\|\left|x - \frac{E}{2}\sigma^2\right| Z(\sigma)\right\| = O(\sigma^{3/2}). \tag{5.38}$$

Proof. Now we prove (5.37). We have by virtue of Proposition 5.9

$$\begin{aligned}\||p - E\sigma|\zeta(\sigma)\|^2 =& \left(\zeta(\sigma), 2(K - V(x + c(t)) - D_t) + 2|E|\left(z - A\sigma + \frac{|E|}{2}\sigma^2\right)\zeta(\sigma)\right) \\ \le& O(\sigma)\|\psi\|^2,\end{aligned}$$

where we should also note that the Heisenberg derivative of D_t is $O(\sigma^{-(1+\rho)})$ on the support of $F(|x/\sigma^2 - E/2| \le \epsilon)$. This implies that (5.37) holds.

Next we shall prove (5.38). Noting the Heisenberg derivative of $x - E\sigma^2/2$ is $p - E\sigma$ and using (5.37), we estimate $\|d(e^{i\sigma K}(x - E\sigma^2/2)Z(\sigma)/d\sigma\|$ as follows:

$$\left\|\frac{d}{d\sigma}\left(e^{i\sigma K}\left(x - \frac{E}{2}\sigma^2\right) Z(\sigma)\right)\right\| \le O(\sigma^{1/2}).$$

By integration in σ, we have (5.38). □

§6. Proof of the asymptotic completeness

In this section, we shall prove Theorems 1.1 and 1.2.

First we prove Theorem 2.3.

Proof of Theorem 2.3. We have only to prove the existence of the adjoint of $\mathcal{W}_c^\pm$, that is,

$$\text{s - }\lim_{\sigma \to \pm\infty} e^{i\sigma K_c} e^{-i\sigma K}, \tag{6.1}$$

because we may prove the existence of the wave operators $\hat{W}_c^\pm$ similarly, and this fact implies the unitarity of $\hat{W}_c^\pm$ by a standard argument in the scattering theory. We shall consider the case $\sigma \to \infty$. Since the set $\mathcal{D} = \{\psi \in \mathcal{H} | f(K)\psi = \psi$ for some $f \in C_0^\infty(\boldsymbol{R})$ and $\langle D_t\rangle\langle p\rangle^2\langle x\rangle\psi \in \mathcal{H}\}$ is dense in $\mathcal{H}$, it suffices to show the existence of the limit

$$\lim_{\sigma \to \infty} e^{i\sigma K_c} e^{-i\sigma K}\psi \tag{6.2}$$

for $\psi \in \mathcal{D}$. By virtue of Theorem 5.8, we see that

$$\lim_{\sigma \to \infty} e^{i\sigma K_c}\left\{1 - F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right)\right\} e^{-i\sigma K}\psi = 0, \tag{6.3}$$

where we take $\epsilon > 0$ as $\epsilon < \min_{\alpha\not\subset c} |E^\alpha|/2$. Moreover by Theorem 5.6, we have

$$\lim_{\sigma\to\infty} e^{i\sigma K_c} F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) \left\{1 - F\left(\left|\frac{p}{\sigma} - E\right| \le \epsilon\right)\right\} e^{-i\sigma K}\psi = 0. \tag{6.4}$$

Thus we have only to show the existence of the limit

$$\lim_{\sigma\to\infty} e^{i\sigma K_c} F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) F\left(\left|\frac{p}{\sigma} - E\right| \le \epsilon\right) e^{-i\sigma K}\psi. \tag{6.5}$$

We compute

$$\begin{aligned}
&\frac{d}{d\sigma}\left(e^{i\sigma K_c} F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) F\left(\left|\frac{p}{\sigma} - E\right| \le \epsilon\right) e^{-i\sigma K}\psi\right)\\
=&e^{i\sigma K_c}\left\{\left(-\frac{2x}{\sigma^3} + \frac{p}{\sigma^2}\right)\cdot\left(\frac{x}{\sigma^2} - \frac{E}{2}\right)\left|\frac{x}{\sigma^2} - \frac{E}{2}\right|^{-1}\right.\\
&F'\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) F\left(\left|\frac{p}{\sigma} - E\right| \le \epsilon\right) + O(\sigma^{-1}) F\left(\left|\frac{p}{\sigma} - E\right| \ge \epsilon\right)\\
&\left.-F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) I_c(x + c(t)) F\left(\left|\frac{p}{\sigma} - E\right| \le \epsilon\right) + O(\sigma^{-4})\right\} e^{-i\sigma K}\psi,
\end{aligned} \tag{6.6}$$

where we used $[V(x + c(t)), F(|p/\sigma - E| \le \epsilon)] = O(\sigma^{-1})F(|p/\sigma - E| \ge \epsilon) + O(\sigma^{-\infty})$. Noting that $-2x/\sigma^3 + p/\sigma^2 = -2(x - E\sigma^2/2)/\sigma^3 + (p - E\sigma)/\sigma^2$, the property of ψ and

$$\left\|F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) I_c(x + c(t))\right\| = O(\sigma^{-\rho})$$

with $\rho > 1$, by virtue of Theorems 5.6 and 5.10, we have

$$\left\|\frac{d}{d\sigma}\left(e^{i\sigma K_c} F\left(\left|\frac{x}{\sigma^2} - \frac{E}{2}\right| \le \epsilon\right) F\left(\left|\frac{p}{\sigma} - E\right| \le \epsilon\right) e^{-i\sigma K}\psi\right)\right\| = O(\sigma^{-\min(3/2,\rho)})\|\psi\|,$$

which implies the existence of the desired limit by Cook's method. Thus the theorem is proved. □

Next we shall prove Theorem 2.2, which implies Theorem 2.1 as mentioned in §2. By the argument of §2, we have only to prove the existence of the wave operators $\tilde{W}_c^\pm(s)$, $s \in \boldsymbol{R}$ in (2.9). In order to prove their existence, we need some propagation properties of the evolution of the N-body Stark Hamiltonian H_c^S. Here we refer to the results in [A2], because one can compare the results obtained in §5 with them. One should also refer to [AT1,AT2] and [HMS2] about the propagation properties of $e^{-itH_c^S}$. We omit the proof of the following theorem (see [A2]).

Theorem 6.1. *Suppose that $(V)_{c,S}$ or $(V)_{c,L}$, and $(V)_{\bar{c}}$ are fulfilled. Let $f \in C_0^\infty(\boldsymbol{R})$.*
(1) Let $\epsilon > 0$ and $u > u' > 0$. Then the following estimates hold as $t \to \infty$:

$$\left\| F\left(\left|\frac{p}{t} - E\right| \geq \epsilon\right) e^{-itH_c^S} f(H_c^S)\langle x\rangle^{-u/2} \right\|_{B(L^2(X))} = O(t^{-u'}), \tag{6.7}$$

$$\left\| F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \geq \epsilon\right) e^{-itH_c^S} f(H_c^S)\langle x\rangle^{-u/2} \right\|_{B(L^2(X))} = O(t^{-u'}). \tag{6.8}$$

(2) Let $0 < \epsilon < \min_{\alpha\not\subset c}|E^\alpha|/2$. Then the following estimates hold as $t \to \infty$:

$$\left\| |p - Et| F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \leq \epsilon\right) e^{-itH_c^S} f(H_c^S)\langle x\rangle^{-u/2} \right\|_{B(L^2(X))} = O(1), \quad u > 1, \tag{6.9}$$

$$\left\| \left|x - \frac{E}{2}t^2\right| F\left(\left|\frac{x}{t^2} - \frac{E}{2}\right| \leq \epsilon\right) e^{-itH_c^S} f(H_c^S)\langle x\rangle^{-u/2} \right\|_{B(L^2(X))} = O(t), \quad u > 1. \tag{6.10}$$

Proof of Theorem 2.2. We prove the existence of $\tilde{W}_c^+(s)$ only. The existence of $\tilde{W}_c^-(s)$ may be proved similarly.

Since the set $\mathcal{D} = \{\psi \in L^2(X) \mid f(H_c^S)\psi = \psi$ for some $f \in C_0^\infty(\boldsymbol{R})$ and $\langle x\rangle^{u/2}\psi \in L^2(X)$ for some $u > 1\}$ is dense in $L^2(X)$, it suffices to show the existence of the limit

$$\lim_{t\to\infty} \tilde{U}(t,s)^* e^{-i(t-s)H_c^S}\psi \tag{6.11}$$

for $\psi \in \mathcal{D}$. By virtue of (6.8), we see that

$$\lim_{t\to\infty} \tilde{U}(t,s)^* \left\{1 - F\left(\left|\frac{x}{(t-s)^2} - \frac{E}{2}\right| \leq \epsilon\right)\right\} e^{-i(t-s)H_c^S}\psi = 0. \tag{6.12}$$

Thus we have only to show the existence of the limit

$$\lim_{t\to\infty} \tilde{U}(t,s)^* F\left(\left|\frac{x}{(t-s)^2} - \frac{E}{2}\right| \leq \epsilon\right) e^{-i(t-s)H_c^S})\psi, \tag{6.13}$$

where we take ϵ as $0 < \epsilon < \min_{\alpha\not\subset c}|E^\alpha|/2$. By virtue of (2.5), noting that $I_c(x + c(t))F(|x/(t-s)^2 - E/2| \leq \epsilon) = O((t-s)^{-\rho})$ with $\rho > 1$, we compute

$$\begin{aligned} &\frac{d}{dt}\left(\tilde{U}(t,s)^* F\left(\left|\frac{x}{(t-s)^2} - \frac{E}{2}\right| \leq \epsilon\right) e^{-i(t-s)H_c^S}\psi\right) \\ =&\tilde{U}(t,s)^* \left\{\left(-\frac{2x}{(t-s)^3} + \frac{p}{(t-s)^2}\right)\cdot\left(\frac{x}{(t-s)^2} - \frac{E}{2}\right)\left|\frac{x}{(t-s)^2} - \frac{E}{2}\right|^{-1}\right. \\ &\left.\times F'\left(\left|\frac{x}{(t-s)^2} - \frac{E}{2}\right| \leq \epsilon\right) + O((t-s)^{-\min(4,\rho)})\right\} e^{-i(t-s)H_c^S}\psi. \end{aligned} \tag{6.14}$$

Noting that $\psi = f(H_c^S)\psi$ for some $f \in C_0^\infty(\boldsymbol{R})$, $\langle x\rangle^{u/2}\psi \in L^2(X)$ for some $u > 1$ and that $-2x/(t-s)^3 + p/(t-s)^2 = -2(x - E(t-s)^2/2)/(t-s)^3 + (p - E(t-s))/(t-s)^2$, by virtue of (6.9) and (6.10), we have

$$\begin{aligned}&\left\| \frac{d}{dt}\left(\tilde{U}(t,s)^* F\left(\left|\frac{x}{(t-s)^2} - \frac{E}{2}\right| \le \epsilon\right) e^{-i(t-s)H_c^S}\psi\right)\right\|_{L^2(X)} \\ =&O(t^{-\min(2,\rho)})\|\langle x\rangle^{u/2}\psi\|_{L^2(X)},\end{aligned}$$

which implies the existence of (6.13) by Cook's method, because of $\rho > 1$. Thus, combining this with the argument in §2, the proof of Theorem 2.2 is completed. □

We have just obtained Theorem 2.1 as well as Theorem 2.2. Now we shall prove Theorems 1.1 and 1.2. Since their proofs are similar to each other, we prove Theorem 1.2 only. First we need the following theorem proved by Dereziński [D] (see also [DG] and [Z]), which is concerned with the asymptotic completeness for the subsystem Hamiltonian H^c. We omit the proof. We should note that H^c is not a many body Stark Hamiltonian but an usual many body Schrödinger operator. Before mentioning its statement, we introduce some notations. Suppose $a \subset c$. We define the cluster Hamiltonian

$$H_a^c = -\frac{1}{2}\Delta^c + V^a$$

on $L^2(X^c)$ and put

$$U_{a,D}^c(t,s) = e^{-i(t-s)H_a^c} e^{-i\int_s^t I_a^c(p_a u)\,du},$$

which is acting on $L^2(X^c)$, where we noted the definition of I_a^c (see §1). We denote the orthogonal complement of X^a in X^c with respect to the metric $\cdot$ by X_a^c. Then we have $X^c = X^a \oplus X_a^c$ and see that $L^2(X^c)$ is decomposed into the tensor product $L^2(X^a) \otimes L^2(X_a^c)$. Thus the cluster Hamiltonian H_a^c is decomposed into

$$H_a^c = H^a \otimes Id + Id \otimes T_a^c \text{ on } L^2(X^c) = L^2(X^a) \otimes L^2(X_a^c),$$

where $T_a^c = -\Delta_a^c/2$ and Δ_a^c is the Laplace-Beltrami operator on X_a^c. It follows from this that

$$U_{a,D}^c(t,s) = e^{-i(t-s)H^a} \otimes (e^{-i(t-s)T_a^c} e^{-i\int_s^t I_a^c(p_a u)\,du}) \text{ on } L^2(X^c) = L^2(X^a) \otimes L^2(X_a^c). \tag{6.15}$$

Theorem 6.2. *Assume that $(V)_{c,L}$ and $(V)_{\bar{c}}$ are fulfilled. Then the modified wave operators*

$$\Omega_a^{c,\pm}(s) = \mathrm{s}\text{ - }\lim_{t\to\pm\infty} e^{i(t-s)H^c} U_{a,D}^c(t,s)(P^a \otimes Id)$$

acting on $L^2(X^c)$, $s \in \boldsymbol{R}$, exist for all $a \subset c$, and are asymptotically complete

$$L^2(X^c) = \sum_{a\subset c} \oplus Ran\, \Omega_a^{c,\pm}(s).$$

Proof of Theorem 1.2. We first prove the existence of the modified wave operators $W_{a,D}^{\pm}(s)$, $s \in \boldsymbol{R}$, in (1.4). Since we have seen the existence of $\hat{W}_c^{\pm}(s)$ in (2.7) by virtue of Theorem 2.1, by the chain rule, we have only to show the existence of the strong limits

$$\operatorname{s-}\lim_{t\to\pm\infty} U_c(t,s)^* U_{a,D}(t,s)(P^a \otimes Id) \tag{6.16}$$

for $a \subset c$ and $s \in \boldsymbol{R}$. By the definition of $T_a(t)$, we see that

$$T_a(t) = T_a^c \otimes Id + Id \otimes T_c(t) \text{ on } L^2(X_a) = L^2(X_a^c) \otimes L^2(X_c).$$

Thus $\bar{U}_a(t,s)$ in (1.1) is decomposed into

$$\bar{U}_a(t,s) = e^{-i(t-s)T_a^c} \otimes \bar{U}_c(t,s).$$

Combining this with (6.15), (6.16) is rewritten as

$$\begin{aligned}\operatorname{s-}\lim_{t\to\pm\infty} U_c(t,s)^* U_{a,D}(t,s)(P^a \otimes Id) &= \operatorname{s-}\lim_{t\to\pm\infty} e^{i(t-s)H^c} U_{a,D}^c(t,s)(P^a \otimes Id) \otimes Id \\ &= \Omega_a^{c,\pm}(s) \otimes Id\end{aligned}$$

on $L^2(X) = L^2(X^c) \otimes L^2(X_c)$. The existence of the right-hand side is guaranteed by Theorem 6.2. Thus the existence of the modified wave operators $W_{a,D}^{\pm}(s)$ is proved. The closedness and mutual orthogonality of their ranges can be easily seen. Finally we prove the asymptotic completeness. By Theorem 2.1, for any $\psi \in L^2(X)$, there exists $\psi_c^{\pm} \in L^2(X)$ such that

$$U(t,s)\psi = U_c(t,s)\psi_c^{\pm} + o(1), \quad t \to \pm\infty. \tag{6.17}$$

In fact, we have $\psi_c^{\pm} = \hat{W}_c^{\pm}(s)^*\psi$. On the other hand, $\psi_c^{\pm} \in L^2(X)$ is decomposed into

$$\psi_c^{\pm} = \sum_{j:\text{finite}} \psi_j^{c,\pm} \otimes \psi_c^{j,\pm} + O(\epsilon),$$

with $\psi_j^{c,\pm} \in L^2(X^c)$ and $\psi_c^{j,\pm} \in L^2(X_c)$. Then by virtue of Theorem 6.2, we have, by (6.17), as $t \to \pm\infty$

$$\begin{aligned}U(t,s)\psi &= \sum_{j:\text{finite}} e^{-i(t-s)H^c}\psi_j^{c,\pm} \otimes \bar{U}_c(t,s)\psi_c^{j,\pm} + O(\epsilon) + o(1) \\ &= \sum_{j:\text{finite}} \sum_{a\subset c} e^{-i(t-s)H^c}\Omega_a^{c,\pm}(s)\phi_{a,j}^{c,\pm} \otimes \bar{U}_c(t,s)\psi_c^{j,\pm} + O(\epsilon) + o(1) \\ &= \sum_{j:\text{finite}} \sum_{a\subset c} U_{a,D}^c(t,s)\phi_{a,j}^{c,\pm} \otimes \bar{U}_c(t,s)\psi_c^{j,\pm} + O(\epsilon) + o(1)\end{aligned}$$

for some $\phi_{a,j}^{c,\pm}$, whose existence is guaranteed by Theorem 6.2. This implies

$$\left\| \psi - \sum_{j:\text{finite}} \sum_{a\subset c} W_{a,D}^{\pm}(s)(\phi_{a,j}^{c,\pm} \otimes \psi_c^{j,\pm}) \right\|_{L^2(X)} = O(\epsilon).$$

Because $\epsilon > 0$ is arbitrary and $\sum_{a \subset c} \oplus Ran\, W^{\pm}_{a,D}(s)$ is closed, we see

$$\psi \in \sum_{a \subset c} \oplus Ran\, W^{\pm}_{a,D}(s).$$

This implies the asymptotic completeness. The proof of Theorem 1.2 is completed. □

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