Drude weight in non solvable quantum spin chains

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Abstract

For a quantum spin chain or 1D fermionic system, we prove that the *Drude weight* D verifies the universal Luttinger liquid relation $v_s^2 = D/\kappa$, where κ is the *susceptibility* and v_s is the *Fermi velocity.* This result is proved by rigorous Renormalization Group methods and is true for any weakly interacting system, regardless its integrablity. This paper, combined with [1], completes the proof of the Luttinger liquid conjecture for such systems.

1 Introduction and main result

Quantum spin chains and one dimensional Fermi systems have been the subject of an intense theoretical investigation for decades, either for their remarkable properties or for the fact that they can be experimentally realized in systems like quantum spin chain models (KCuF3) [2] or carbon nanotubes [3]. We consider a quantum spin chain model with Hamiltonian

$$H = -\sum_{x=1}^{L-1}[S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2] - h\sum_{x=1}^{L} S_x^3 + \lambda \sum_{1\le x,y\le L} v(x-y) S_x^3 S_y^3 + U_L^1 \;, \tag{1}$$

where $S_x^\alpha = \sigma_x^\alpha/2$ for $i = 1, 2, \ldots, L$ and $\alpha = 1, 2, 3$, σ_x^α being the Pauli matrices, and U_L^1 is a boundary term; finally $v(x) = v(-x)$ and $|v(x)| \le Ce^{-\kappa|x|}$.

If $v(x-y) = \delta_{|x-y|,1}$ and $h = 0$, (1) is the hamiltonian of the XXZ spin chain in a zero magnetic field, which can be diagonalized by the Bethe ansatz [4]. No exact solution is known for more general interactions, but some particular models have been the subject of an extensive numerical analysis.

It is well known that the quantum spin model can be equivalently written in terms of fermionic anticommuting operators $a_x^\pm \equiv \prod_{y=1}^{x-1}(-\sigma_y^3)\sigma_x^\pm$; if $J_1 = J_2 = 1$, one gets

$$\begin{aligned} H = &-\sum_{x=1}^{L-1}\frac{1}{2}[a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] - h\sum_{x=1}^{L}(a_x^+ a_x^- - \frac{1}{2}) \\ &+ \lambda \sum_{1\le x,y\le L} v(x-y)(a_x^+ a_x^- - \frac{1}{2})(a_y^+ a_y^- - \frac{1}{2}) + U_L^2 \;, \end{aligned} \tag{2}$$

*Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica, I-00133, Roma where U_L^2 is the boundary term in the new variables. We choose it so that the fermionic Hamiltonian coincides with the Hamiltonian of a fermion system on the lattice with periodic boundary conditions.

If O_x is a local monomial in the S_x^α or $a_x^\pm$ operators, we call $O_{\mathbf{x}} = e^{Hx_0} O_x e^{-Hx_0}$ where $\mathbf{x} = (x_0, x)$ and x_0 is the "imaginary time"; moreover, if $A = O_{\mathbf{x}_1} \cdots O_{\mathbf{x}_n}$

$$< A >_{L,\beta} = \frac{\mathrm{Tr}[\mathrm{e}^{-\beta \mathrm{H}} \mathbf{T}(\mathrm{A})]}{\mathrm{Tr}[\mathrm{e}^{-\beta \mathrm{H}}]}\,, \tag{3}$$

$\mathbf{T}$ being the time order product, denotes its expectation in the grand canonical ensemble, while $< A >_{T;L,\beta}$ denotes the corresponding truncated expectation. We will use also the notation $< A >_T = \lim_{L,\beta\to\infty} < A >_{T;L,\beta}$.

The *response functions* measure the response of the system to an external probe. In particular, the spin conductivity properties of model (1) can be obtained in the model (2) from the *current-current* response function, whose Fourier transform is defined as

$$\widehat{G}^{0,2}_{J,J}(\mathbf{p}) = \lim_{\beta\to\infty} \lim_{L\to\infty} \int_{-\beta/2}^{\beta/2} dx_0 \sum_{x\in\Lambda} e^{i\mathbf{p}\mathbf{x}} \langle J_{\mathbf{x}} J_{\mathbf{0}} \rangle_{T;L,\beta}\,, \tag{4}$$

where $\mathbf{p} = (p_0, p)$, $p_0 = \frac{2\pi}{\beta} n, p = \frac{2\pi}{L} m$, $(n, m) \in \mathbb{Z}^2$, $-[L/2] \le m \le [(L-1)/2]$, $J_{\mathbf{x}} = e^{Ht} J_x e^{-Ht}$ and J_x is the paramagnetic part of the current

$$J_x = \frac{1}{2i}[a^+_{x+1} a^-_x - a^+_x a^-_{x+1}]\,. \tag{5}$$

A crucial quantity in the study of the conductivity properties is played by the *Drude weight*, defined in the following way. Let us consider the function

$$\widehat{D}(\mathbf{p}) = -\Delta - \widehat{G}^{0,2}_{J,J}(\mathbf{p})\,, \tag{6}$$

where $\Delta =< \Delta_x >$ and

$$\Delta_x = -\frac{1}{2}[a^+_x a^-_{x+1} + a^+_{x+1} a^-_x] \tag{7}$$

is the diamagnetic part of the current, whose mean value $< \Delta_x >$ is indeed independent of x, hence it is equal to $< H_T > /L$, with $H_T = \sum_x \Delta_x$, the value of H for $h = \lambda = 0$. Then the Drude weight is given by

$$D = \lim_{p_0\to 0} \widehat{D}(p_0, 0)\,. \tag{8}$$

If one assumes analytic continuation in p_0 around $p_0 = 0$, one can compute the conductivity in the linear response approximation by the Kubo formula, see *e.g.* [5], that is

$$\sigma = \lim_{\omega\to 0} \lim_{\delta\to 0} \frac{\widehat{D}(-i\omega + \delta, 0)}{-i\omega + \delta}\,. \tag{9}$$

Therefore, a nonvanishing D indicates infinite conductivity.

Another important quantity is the *susceptibility*, which can be calculated, in the fermionic representation, in terms of the *density-density* response function $G^{0,2}_{\rho,\rho}(\mathbf{x}) = \langle \rho_{\mathbf{x}} \rho_{\mathbf{0}} \rangle_T$, $\rho_x = a^+_x a_x$, by the equation

$$\kappa = \lim_{p\to 0} \widehat{G}^{0,2}_{\rho,\rho}(0, p)\,, \tag{10}$$

where $\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p})$ is defined analogously to (4). Note that, in the fermionic representation, $\kappa = \kappa_c \rho^2$, where κ_c is the *fermionic compressibility* and ρ is the fermionic density, see *e.g.* (2.83) of [6].

The large distance behavior of the response functions is given (for coupling not too large) by power laws with non-universal exponents depending on all details of the Hamiltonian, like the form of the potential and the value of the magnetic field. Only for the interaction $v(x-y) = \delta_{|x-y|,1}$ a solution is known by Bethe ansatz [4], if $h = 0$; by using this explicit solution,κ and D can be computed. However, even in that case, only a single exponent can be calculated [7].

In [8, 9, 10] rigorous RG methods have been applied to spin chains or fermionic 1D systems, regardless their integrability; the outcome of such analysis is that several physical observables, and in particular the critical exponents, can be written as convergent series. The exponents are interaction-dependent but nevertheless verify universal model independent relations; if η is the exponent of the 2-point function (see *e.g.* (4) of [1]), $\bar{\nu}$ is the correlation length exponent (see *e.g.* (11) of [1]), X_+ is the density exponent (see *e.g.* (7-9) of [1]) and X_- is the Copper pair exponent (see *e.g.* (10) of [1]), it has been proved in [11, 1] that, for λ small enough,

$$X_+ = K\,, \quad X_- = K^{-1}\,, \quad \bar{\nu} = \frac{1}{1-K^{-1}}\,, \quad 2\eta = K + K^{-1} - 2\,, \tag{11}$$

where $K(\lambda)$ is an analytic function such that

$$K(\lambda) = 1 - \lambda \frac{\widehat{v}(0) - \widehat{v}(2p_F)}{\pi \sin p_F} + O(\lambda^2)\,, \tag{12}$$

with $\cos p_F = -h - \lambda$. Note that the exact relations (11) are *universal*, *i.e.* do not depend on the Hamiltonian details, for instance on the form of the interaction $v(x)$, contrary to the function $K(\lambda)$, see (12).

Universal relations connect also the critical exponents with the susceptibility κ; in [1] it was proved that

$$\kappa = \frac{K}{\pi v_s}\,. \tag{13}$$

In this paper we prove the following Theorem for the Drude weight.

Theorem 1.1 *If λ is small enough, the function $\widehat{D}(\mathbf{p})$ defined in (6) can be written, for $\mathbf{p}$ small but different from 0, in the form*

$$\widehat{D}(\mathbf{p}) = \frac{v_s}{\pi} K \frac{p_0^2}{p_0^2 + v_s^2 p^2} + H(\mathbf{p})\,, \tag{14}$$

where $H(\mathbf{p})$ is a continuous function, such that $|H(\mathbf{p})| \le C|\mathbf{p}|^{\vartheta}$, with $0 < \vartheta < 1$; therefore the Drude weight is given by

$$D = \frac{v_s K}{\pi} \tag{15}$$

and satisfies the identity

$$v_s^2 = D/\kappa\,. \tag{16}$$

The validity of the relations (11), (13),(15) and (16) is the content of the *Luttinger liquid conjecture* formulated in [12] (see also [13, 14]); given the Drude weight and the susceptibility, one can determine exactly all the exponents and the Fermi velocity. All these relations are true in the *Luttinger model*, describing interacting fermions with a relativistic linear dispersion relation and solved by bosonization [15]; the content of the Luttinger liquid conjecture is that they are true also in the model (1), even if the exponents are completely different. This is by no means obvious; the exponents, κ and D are non universal functions of the interaction, and surely depend on the dispersion relation and the details of the Hamiltonian. The validity of the conjecture was partially checked on the solvable XXZ chain; v_s, κ, D can be computed from the Bethe ansatz solution [12, 16] and the validity of the relation (16) (following from (15),(13)) is verified. Moreover, by using (11), the exponents can be exactly determined from the knowledge of κ and D; note that the value of $\bar\nu$ found in this way agrees with the one obtained in [7]. A number of arguments have been proposed along the years [17, 18, 19] in order to justify the validity of (11), (13) and (15), but they rely on unproved assumptions or approximations in non solvable cases.

The present paper completes the proof of the Luttinger liquid conjecture for quantum spin chain or 1D fermionic system with generic weak short range interaction. The proof relies on a number of technical results previously established in [8, 9, 10, 11, 1]; in particular the present paper extends and completes the analysis of [1], which we assume the reader familiar with.

The Drude weight in quantum spin chains has been the subject in recent years of an intense numerical investigation [20, 21, 22, 23, 24, 25], with the main objective of detecting a possible different behavior of conductivity at finite temperatures between the integrable and the non integrable cases; it has been conjectured that the Drude weight is non vanishing also at finite temperature in the integrable cases, while it is vanishing in non integrable systems, but the results are still controversial. Our methods for the calculation of the Drude weight at zero temperature can be applied either to solvable or non solvable systems, and we believe that an extension of these methods would allow us to understand also the properties of the Drude weight at non zero temperature.

2 Ward Identities

We shall proceed as in App. B of [1]. Let us consider the (imaginary time) conservation equation:

$$\frac{\partial\rho_{\mathbf{x}}}{\partial x_0}=e^{Hx_0}[H,\rho_x]e^{-Hx_0}=-i\partial^{(1)}_x J_{\mathbf{x}}\equiv -i[J_{x,x_0}-J_{x-1,x_0}]\ , \tag{17}$$

where we have used that $[H,\rho_x]=[H_T,\rho_x]$, H_T being the value of H for $h=\lambda=0$. This equation implies some exact identities involving various correlation functions, that play the role in the lattice models of the usual Ward Identities (WI) of continuous relativistic models. They are valid at any finite β and L, but we shall use them only in the limit $L=\beta=\infty$.

We shall call $G^2(\mathbf{x},\mathbf{y})=<a^-_{\mathbf{x}}a^+_{\mathbf{y}}>$ the (imaginary time) Green's function, while

$$G^{2,1}_\rho(\mathbf{x},\mathbf{y},\mathbf{z})=<\rho_{\mathbf{x}}a^-_{\mathbf{y}}a^+_{\mathbf{z}}>_T\ \text{ and }\ G^{2,1}_J(\mathbf{x},\mathbf{y},\mathbf{z})=<J_{\mathbf{x}}a^-_{\mathbf{y}}a^+_{\mathbf{z}}>_T$$

will be the vertex functions. By using (17) one gets the WI

$$\begin{aligned}\frac{\partial}{\partial x_0}G^{2,1}_\rho(\mathbf{x},\mathbf{y},\mathbf{z})&=-i\partial^{(1)}_x G^{2,1}_J(\mathbf{x},\mathbf{y},\mathbf{z})+\\ &+\delta(x_0-z_0)\delta_{x,z}G^2(\mathbf{y},\mathbf{x})-\delta(x_0-y_0)\delta_{x,y}G^2(\mathbf{x},\mathbf{z})\ ,\end{aligned} \tag{18}$$

where $\partial^{(1)}_x$ is the lattice derivative. In the same way a WI for the density-density correlations is derived. If we define

$$G^{0,2}_{\rho,\rho}(\mathbf{x},\mathbf{y})=<\rho_{\mathbf{x}}\rho_{\mathbf{y}}>_T,\ G^{0,2}_{\rho,J}(\mathbf{x},\mathbf{y})=<\rho_{\mathbf{x}}J_{\mathbf{y}}>_T,\ G^{0,2}_{J,J}(\mathbf{x},\mathbf{y})=<J_{\mathbf{x}}J_{\mathbf{y}}>_T,$$

we get

$$\frac{\partial}{\partial x_0} G^{0,2}_{\rho,\rho}(\mathbf{x},\mathbf{y}) = -\partial^{(1)}_x G^{0,2}_{J,\rho}(\mathbf{x},\mathbf{y}) + \delta(x_0-y_0) < [\rho_{(x,x_0)}\,,\quad \rho_{(y,x_0)}] >,$$
$$\frac{\partial}{\partial x_0} G^{0,2}_{\rho,J}(\mathbf{x},\mathbf{y}) = -i\partial^{(1)}_x G^{0,2}_{J,J}(\mathbf{x},\mathbf{y}) + \delta(x_0-y_0) < [\rho_{(x,x_0)}, J_{(y,x_0)}] > \;. \tag{19}$$

Noting that $[\rho_{(x,x_0)},\rho_{(y,x_0)}]=0$, while

$$[\rho_{(x,x_0)}, J_{(y,x_0)}] = -i\delta_{x,y}\Delta_{(x,x_0)} + i\delta_{x-1,y}\Delta_{(y,x_0)}, \tag{20}$$

we get, using that $<\Delta_{\mathbf{x}}>=<\Delta_x>=\Delta$,

$$-ip_0\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p}) - i(1-e^{-ip})\widehat{G}^{0,2}_{J,\rho}(\mathbf{p}) = 0\,,$$
$$-ip_0\widehat{G}^{0,2}_{\rho,J}(\mathbf{p}) - i(1-e^{-ip})\widehat{G}^{0,2}_{J,J}(\mathbf{p}) = i(1-e^{-ip})\Delta\;. \tag{21}$$

Hence, by using the definition (6), the WI (21) and the fact that $\widehat{G}^{0,2}_{\rho,J}(\mathbf{p})=\widehat{G}^{0,2}_{J,\rho}(-\mathbf{p})$, we get

$$p_0^2\,\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p}) - 4\sin^2(p/2\,)\widehat{D}(\mathbf{p}) = 0\;. \tag{22}$$

The above equation holds quite generally for fermionic lattice systems. If $\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p})$ and $\widehat{D}(\mathbf{p})$ were continuous in $\mathbf{p}=0$, it would imply that both κ and D are vanishing. In the case we are considering, we will see in the next section that $\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p})$ and $\widehat{D}(\mathbf{p})$ are bounded but not continuous in $\mathbf{p}=0$, which is sufficient to prove only that:

$$\widehat{G}^{0,2}_{\rho,\rho}(p_0,0)=0\,,\quad \widehat{D}_\beta(0,p)=0\;. \tag{23}$$

3 Renormalization Group anaysis

It is well known that the correlations of the quantum spin chain can be derived by the following Grassmann integral, see §2.1 of [8]:

$$e^{\mathcal{W}_{L,\beta,M}(A,J,\phi)} = \int P(d\psi)e^{-\mathcal{V}(\psi)+B(A,J,\psi)+\int d\mathbf{x}[\phi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}+\psi^-_{\mathbf{x}}\phi^+_{\mathbf{x}}]}\,, \tag{24}$$

where $\cos p_F=-\lambda-h-\nu$, $v_s=v_F(1+\delta)$, $\psi^\pm_{\mathbf{x}}$ and $\phi^\pm_{\mathbf{x}}$ are Grassmann variables, $\int d\mathbf{x}$ is a shortcut for $\sum_x\int_{-\beta/2}^{\beta/2}dx_0$, $P(d\psi)$ is a Grassmann Gaussian measure in the field variables $\psi^\pm_{\mathbf{x}}$ with covariance (the free propagator) given by

$$g_M(\mathbf{x}-\mathbf{y}) = \frac{1}{\beta L}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\frac{\chi(\gamma^{-M}k_0)e^{i\delta_M k_0}e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0+(v_s/v_F)(\cos p_F-\cos k)}\,, \tag{25}$$

where $\chi(t)$ is a smooth compact support function equal to 0 if $|t|\geq\gamma>1$ and equal to 1 for $|t|<1$, $\mathbf{k}=(k,k_0)$, $\mathbf{k}\cdot\mathbf{x}=k_0x_0+kx$, $\mathcal{D}_{L,\beta}\equiv\mathcal{D}_L\times\mathcal{D}_\beta$, $\mathcal{D}_L\equiv\{k=2\pi n/L, n\in\mathbb{Z}, -[L/2]\leq n\leq[(L-1)/2]\}$, $\mathcal{D}_\beta\equiv\{k_0=2(n+1/2)\pi/\beta, n\in Z\}$ and

$$\mathcal{V}(\psi)=\lambda\int d\mathbf{x}d\mathbf{y}\widetilde{v}(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{x}}\psi^+_{\mathbf{y}}\psi^-_{\mathbf{y}}\psi^-_{\mathbf{x}}+\nu\int d\mathbf{x}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} -$$
$$-\delta\int d\mathbf{x}[\cos p_F\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} - (\psi^+_{\mathbf{x}+\mathbf{e_1}}\psi^-_{\mathbf{x}}+\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}+\mathbf{e_1}})/2]\,,$$

with $\mathbf{e_1} = (0,1)$, $\widetilde{v}(\mathbf{x}-\mathbf{y}) = \delta(x_0-y_0)v(x-y)$. Moreover

$$\begin{aligned} B(A,J,\psi) = \int d\mathbf{x} \Big\{ & \psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}} A_0(\mathbf{x}) + \frac{1}{2i}[\psi^+_{\mathbf{x}+\mathbf{e}_1}\psi^-_{\mathbf{x}} - \psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}+\mathbf{e}_1}]A_1(\mathbf{x}) - \\ & - \frac{J}{2}[\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}+\mathbf{e}_1} + \psi^+_{\mathbf{x}+\varepsilon_1}\psi^-_{\mathbf{x}}]) \Big\} . \end{aligned} \tag{26}$$

Note that, due to the presence of the ultraviolet cut-off γ^M, the Grassmann integral has a finite number of degree of freedom, hence it is well defined. The constant $\delta_M = \beta/\sqrt{M}$ is introduced in order to take correctly into account the discontinuity of the free propagator $g(\mathbf{x})$ at $\mathbf{x} = 0$, where it has to be defined as $\lim_{x_0\to 0^-} g(0,x_0)$; in fact our definition guarantees that $\lim_{M\to\infty} g_M(\mathbf{x}) = g(\mathbf{x})$ for $\mathbf{x} \neq 0$, while $\lim_{M\to\infty} g_M(0,0) = g(0,0^-)$. The density and current correlations can be written in terms of functional derivatives of (24)

$$\begin{aligned} G^{0,2}_{\rho,\rho}(\mathbf{x},\mathbf{y}) &= \lim_{\beta\to\infty}\lim_{L\to\infty}\lim_{M\to\infty} \frac{\delta^2}{\delta A_0(\mathbf{x})\delta A_0(\mathbf{y})} W_{L,\beta,M}(A,0,0)\big|_{A=0} , \\ G^{0,2}_{\rho,J}(\mathbf{x},\mathbf{y}) &= \lim_{\beta\to\infty}\lim_{L\to\infty}\lim_{M\to\infty} \frac{\delta^2}{\delta A_0(\mathbf{x})\delta A_1(\mathbf{y})} W_{L,\beta,M}(A,0,0)\big|_{A=0} , \\ G^{0,2}_{J,J}(\mathbf{x},\mathbf{y}) &= \lim_{\beta\to\infty}\lim_{L\to\infty}\lim_{M\to\infty} \frac{\delta^2}{\delta A_1(\mathbf{x})\delta A_1(\mathbf{y})} W_{L,\beta,M}(A,0,0)\big|_{A=0} , \\ \Delta &= \lim_{\beta\to\infty}\lim_{L\to\infty}\lim_{M\to\infty} \frac{1}{\beta L}\frac{\delta}{\delta J} W_{L,\beta,M}(0,J,0)\big|_{J=0} . \end{aligned} \tag{27}$$

In [8, 9, 10] a multiscale integration procedure combined with Ward Identities allows us to write the above correlations in terms of a convergent expansion; the counterterms ν,δ are chosen so that p_F is the Fermi momentum and v_s is the Fermi velocity. By using Theorem 3.12 of [8], one can easily prove that Δ is a finite constant. However, the bounds obtained from the muliscale analysis for $G^{0,2}_{\rho,\rho}(\mathbf{x},\mathbf{y})$ and $G^{0,2}_{J,J}(\mathbf{x},\mathbf{y})$ are not sufficient to prove that their Fourier transforms are bounded around $\mathbf{p} = (0,0)$. In fact, by using theorem (1.5) of [8], we see that their non-oscillating part behaves for large $|\mathbf{x}-\mathbf{y}|$ as $|\mathbf{x}-\mathbf{y}|^{-2}$, so that logarithmic divergences in the Fourier transform cannot be excluded.

In order to compute the Fourier transform of the current-current correlation we will follow the same strategy used in [1] for the density-density correlation. We introduce a continuous model with linear dispersion relation regularized by a non local fixed interaction, together with ultraviolet γ^N and an infrared γ^l momentum cut-offs. The model is expressed in terms of the following Grassmann integral:

$$\begin{aligned} e^{\mathcal{W}_N(J,\widetilde{J},\phi)} &= \int P_Z(d\psi) e^{-\mathcal{V}^{(N)}(\sqrt{Z}\psi + \sum_{\omega=\pm}\int d\mathbf{x}[Z^{(3)}J_{\mathbf{x}} + \omega\,\widetilde{Z}^{(3)}\widetilde{J}_{\mathbf{x}}]\rho_{\mathbf{x},\omega}} \cdot \\ &\cdot e^{Z\sum_{\omega=\pm}\int d\mathbf{x}[\psi^+_{\mathbf{x},\omega}\phi^-_{\mathbf{x},\omega} + \phi^+_{\mathbf{x},\omega}\psi]} , \end{aligned} \tag{28}$$

where $\rho_{\mathbf{x},\omega} = \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}$, $\mathbf{x} \in \widetilde{\Lambda}$ and $\widetilde{\Lambda}$ is a square lattice of side L, whose size is of order γ^{-l}, say $\gamma^{-l}/2 \leq L \leq \gamma^{-l}$; $P_Z(d\psi^{[l,N]})$ is the fermionic measure with propagator

$$\frac{1}{Z} g_{th,\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{Z}\frac{1}{L^2}\sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_N(\mathbf{k})}{-ik_0 + \omega c k} , \tag{29}$$

where Z and c are two parameters, to be fixed later, and $\chi_{l,N}(\mathbf{k})$ is a cut-off function depending on a small positive parameter ε, nonvanishing for all $\mathbf{k}$ and reducing, as $\varepsilon \to 0$, to a compact support function equal to 1 for $\gamma^l \leq |\mathbf{k}| \leq \gamma^{N+1}$ and vanishing for $|\mathbf{k}| \leq \gamma^{l-1}$

or $|\mathbf{k}| \geq \gamma^{N+1}$ (its precise definition can be found in (21) of [9]); moreover, the interaction is

$$\mathcal{V}^{(N)}(\psi) = \frac{\lambda_\infty}{2} \sum_\omega \int d\mathbf{x} \int d\mathbf{y} v_0(\mathbf{x}-\mathbf{y}) \psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} \psi^+_{\mathbf{y},-\omega} \psi^-_{\mathbf{y},-\omega} \;, \tag{30}$$

where $v_0(\mathbf{x}-\mathbf{y})$ is a rotational invariant potential, of the form

$$v_0(\mathbf{x}-\mathbf{y}) = \frac{1}{L^2} \sum_{\mathbf{p}} \widehat{v}_0(\mathbf{p}) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \;, \tag{31}$$

with $|\widehat{v}_0(\mathbf{p})| \leq C e^{-\mu |\mathbf{p}|}$, for some constants C, μ, and $\widehat{v}_0(0) = 1$. We define

$$\begin{aligned}
G^{2,1}_{th,\rho;\omega}(\mathbf{x},\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty} \lim_{a^{-1},L\to\infty} \frac{\partial}{\partial J_{\mathbf{x}}} \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega} \partial \phi^-_{\mathbf{z},\omega}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;, \\
G^{2,1}_{th,J;\omega}(\mathbf{x},\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty} \lim_{a^{-1},L\to\infty} \frac{\partial}{\partial \widetilde{J}_{\mathbf{x}}} \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega} \partial \phi^-_{\mathbf{z},\omega}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;, \\
G^{2}_{th;\omega}(\mathbf{y},\mathbf{z}) &= \lim_{-l,N\to\infty} \lim_{a^{-1},L\to\infty} \frac{\partial^2}{\partial \phi^+_{\mathbf{y},\omega} \partial \phi^-_{\mathbf{z},\omega}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;, \\
G^{0,2}_{th,\rho,\rho}(\mathbf{x},\mathbf{y}) &= \lim_{-l,N\to\infty} \lim_{a^{-1},L\to\infty} \frac{\partial^2}{\partial J_{\mathbf{x}} \partial J_{\mathbf{y}}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;, \\
G^{0,2}_{th,J,J}(\mathbf{x},\mathbf{y}) &= \lim_{-l,N\to\infty} \lim_{a^{-1},L\to\infty} \frac{\partial^2}{\partial \widetilde{J}_{\mathbf{x}} \partial \widetilde{J}_{\mathbf{y}}} \mathcal{W}_{l,N}(J,\widetilde{J},\phi)|_{J=\widetilde{J}=\phi=0} \;.
\end{aligned} \tag{32}$$

The existence of the $N \to \infty$ limit has been proved in [26] and in §3 of [11], extending the method used in [27] for the analysis of the Yukawa model in two dimensions; the existence of the limit $l \to -\infty$ has been proved in [8, 9, 10].

The model (28) is a sort of *effective model* for the lattice fermionic model (2); it is indeed well known that a non relativistic gas of fermions in one dimension admits an effective description in terms of massless Dirac fermions in $d = 1+1$ dimension. We can make precise this idea via the following lemma, whose proof is an immediate extension of the proof given in §3 of [1] for the density-density correlation.

Lemma 3.1 *Given λ small enough, there exist constants Z, $Z^{(3)}$, $\widetilde{Z}^{(3)}$, λ_∞, depending analytically on λ, such that $Z = 1 + O(\lambda^2)$, $Z^{(3)} = 1 + O(\lambda)$, $\widetilde{Z}^{(3)} = v_F + O(\lambda)$, $\lambda_\infty = \lambda + O(\lambda^2)$ and, if $c = v_s$ and $|\mathbf{p}| \leq \kappa \leq 1$,*

$$\begin{aligned}
\widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p}) &= \widehat{G}^{0,2}_{th,\rho,\rho}(\mathbf{p}) + A_{\rho,\rho}(\mathbf{p}) \;, \\
\widehat{G}^{0,2}_{J,J}(\mathbf{p}) &= \widehat{G}^{0,2}_{th,J,J}(\mathbf{p}) + A_{J,J}(\mathbf{p}) + \Delta \;,
\end{aligned} \tag{33}$$

with $A_{\rho,\rho}(\mathbf{p}), A_{J,J}(\mathbf{p})$ Lipschitz continuous in $\mathbf{p}$. Moreover, if we put $\mathbf{p}^\omega_F = (0, \omega p_F)$ and we suppose that $0 < \kappa \leq |\mathbf{p}|, |\mathbf{k}'|, |\mathbf{k}'-\mathbf{p}| \leq 2\kappa$, $0 < \vartheta < 1$, then

$$\begin{aligned}
\widehat{G}^{2,1}_{\rho}(\mathbf{k}'+\mathbf{p}^\omega_F, \mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F) &= \widehat{G}^{2,1}_{th,\rho;\omega}(\mathbf{k}', \mathbf{k}'+\mathbf{p})[1 + O(\kappa^\vartheta)] \;, \\
\widehat{G}^{2,1}_{J}(\mathbf{k}'+\mathbf{p}^\omega_F, \mathbf{k}'+\mathbf{p}+\mathbf{p}^\omega_F) &= \widehat{G}^{2,1}_{th,J;\omega}(\mathbf{k}', \mathbf{k}'+\mathbf{p})[1 + O(\kappa^\vartheta)] \;, \\
\widehat{G}^{2}(\mathbf{k}'+\mathbf{p}^\omega_F) &= \widehat{G}^{2}_{th,\omega}(\mathbf{k}')[1 + O(\kappa^\vartheta)] \;.
\end{aligned} \tag{34}$$

This lemma says that the vertex functions of the two models are essentially coinciding close to the Fermi momenta, if the bare parameters are chosen properly, while the response functions differ by a continuous function. Note also the the bare parameters of the model

(28) are expressed by convergent expansions depending on all model details, but the WI imply that they are not independent parameters, as we will see shortly.

The main reason behind the introduction of the model (28) is that, while the model (1) is invariant only under the phase transformation $\psi^\pm_{\mathbf{x}} \to e^{\pm i\alpha}\psi^\pm_{\mathbf{x}}$, the model (28) is invariant under *two* phase transformations, the total $\psi^\pm_{\mathbf{x},\omega} \to e^{\pm i\alpha}\psi^\pm_{\mathbf{x},\omega}$ and the chiral $\psi^\pm_{\mathbf{x},\omega} \to e^{\pm\omega i\alpha}\psi^\pm_{\mathbf{x},\omega}$. This implies that the Fourier transforms of the response functions can be completely determined from the WI, see app. A of [1]; if $D_\omega(\mathbf{p}) = -ip_0 + \omega c p$, we get:

$$\begin{aligned} \widehat{G}^{0,2}_{th,J,J} &= \frac{-1}{4\pi c Z^2}\frac{(\widetilde{Z}^{(3)})^2}{1-\tau^2}\left[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} + 2\tau\right] + O(\mathbf{p})\,, \\ \widehat{G}^{0,2}_{th,\rho,\rho} &= \frac{-1}{4\pi c Z^2}\frac{(Z^{(3)})^2}{1-\tau^2}\left[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} - 2\tau\right] + O(\mathbf{p})\,, \end{aligned} \tag{35}$$

where $\tau = \frac{\lambda_\infty}{4\pi c}$. Therefore, from (35) and (33), since $c = v_s$,

$$\begin{aligned} \widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p}) &= \frac{-1}{4\pi v_s Z^2}\frac{(Z^{(3)})^2}{1-\tau^2}\left[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} + 2\tau\right] + A_{\rho,\rho}(0) + R_\rho(\mathbf{p}), \\ \widehat{D}(\mathbf{p}) &= \frac{-1}{4\pi v_s Z^2}\frac{(\widetilde{Z}^{(3)})^2}{1-\tau^2}\left[\frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} + \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} - 2\tau\right] + A_{J,J}(0) + \Delta \\ &\qquad + R_j(\mathbf{p})\,, \end{aligned} \tag{36}$$

with $|R_\rho(\mathbf{p})|, |R_j(\mathbf{p})| \le C|\mathbf{p}|^\vartheta$, $0 < \vartheta < 1$. The constants $A_{\rho,\rho}(0)$, $A_{J,J}(0)$ and Δ are expressed by convergent expansions, but their values can be determined from the WI for the model (1); indeed, by (35), $\widehat{G}^{0,2}_{th,j,J}$ and $\widehat{G}^{0,2}_{th,\rho,\rho}$ are not continuous in $\mathbf{p} = 0$, but they are bounded, so that (23) holds; this condition fixes the values of $A_{J,J}(0) + \Delta$ and $A_{\rho,\rho}(0)$ so that

$$\begin{aligned} \widehat{G}^{0,2}_{\rho,\rho}(\mathbf{p}) &= \frac{1}{\pi v_s Z^2}\frac{Z^{(3)})^2}{1-\tau^2}\frac{v_s^2 p^2}{p_0^2 + v_s^2 p^2} + R_\rho(\mathbf{p})\,, \\ \widehat{D}(\mathbf{p}) &= \frac{1}{\pi v_s Z^2}\frac{(\widetilde{Z}^{(3)})^2}{1-\tau^2}\frac{p_0^2}{p_0^2 + v_s^2 p^2} + R_j(\mathbf{p})\,. \end{aligned} \tag{37}$$

Moreover, the vertex functions verify the following WI, see (35) of [1]:

$$\begin{aligned} &-ip_0\frac{Z}{Z^{(3)}}\widehat{G}^{2,1}_{th,\rho;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p}) + \omega p\; v_s\frac{Z}{\widetilde{Z}^{(3)}}\widehat{G}^{2,1}_{th,J;\omega}(\mathbf{k},\mathbf{k}+\mathbf{p}) = \\ &\quad = \frac{1}{1-\tau}[\widehat{G}^2_{th;\omega}(\mathbf{k}) - \widehat{G}^2_{th;\omega}(\mathbf{k}+\mathbf{p})]\,; \end{aligned} \tag{38}$$

hence, by using (34) and by comparing (38) with the WI (18), we get that the bare parameters are not independent, but verify the relations:

$$\frac{Z^{(3)}}{(1-\tau)Z} = 1\,, \quad v_s\frac{Z^{(3)}}{\widetilde{Z}^{(3)}} = 1\,, \tag{39}$$

implying that

$$\begin{aligned} \widehat{\Omega}_{\rho\rho}(\mathbf{p}) &= \frac{K}{\pi v_s}\frac{v_s^2 p^2}{p_0^2 + v_s^2 p^2} + R_\rho(\mathbf{p})\,, \\ \widehat{D}(\mathbf{p}) &= \frac{v_s}{\pi}K\frac{p_0^2}{p_0^2 + v_s^2 p^2} + R_j(\mathbf{p})\,, \end{aligned} \tag{40}$$

with $K = \frac{1-\tau}{1+\tau}$. Eq. (52) of [8] shows that K is indeed the critical index X_+, see (11); hence, by using (8) and (10), we get the relations (15) and (13), which immediately imply (16), so that Theorem 1.1 is proved.

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