STRATEGIES IN LOCALIZATION PROOFS FOR ONE-DIMENSIONAL RANDOM SCHRÖDINGER OPERATORS

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Dedicated to Jean-Michel Combes on the Occasion of his 60th Birthday

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ABSTRACT. Recent results on localization, both exponential and dynamical, for various models of one-dimensional, continuum, random Schrödinger operators are reviewed. This includes Anderson models with indefinite single site potentials, the Bernoulli-Anderson model, the Poisson model, and the random displacement model. Among the tools which are used to analyze these models are generalized spectral averaging techniques and results from inverse spectral and scattering theory. A discussion of open problems is included.

1. INTRODUCTION

The mathematical theory of random operators started in the late 60s and 70s and came to full bloom in the 1980s. All its basic notions were set on rigorous footing and, in particular, ground breaking results on localization of states were obtained for both one and multi-dimensional models. New discoveries were made in rapid succession, often in parallel by several groups of researchers. The state of the art at the end of this pioneering period is very well summarized in the monographs [7] and [26].

By the early 90s a state had been reached, where many of the remaining open problems had become quite hard and progress has become slower since then. This does not mean that no important open problems are left. In fact, the central problem of understanding the existence of extended states is probably as open now as it was in the 80s.

Even the seemingly better understood phenomenon of localization leaves a lot of open challenges. In particular, most of the initial results on localization were found for Anderson models with additional restrictions placed on the parameters of the model. A number of physically interesting classes of random operators could not be treated.

In this paper we review work by the author, partly in collaboration with D. Buschmann, D. Damanik, and R. Sims, which was aimed at getting a more complete understanding of mechanisms which lead to localization. We restrict our presentation to one-dimensional continuum models, where quite complete results

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are now available. Many of these results have not yet been established in higher dimension.

Our goal is to describe the mathematical differences in various ways of modeling randomness in a Schrödinger operator. After reviewing some of the "classical" strategies to prove localization for one-dimensional continuum Anderson-type models, we will discuss ideas and tools which allow to treat other models, such as the Poisson, random displacement, and Bernoulli-Anderson models. Among these tools are generalized spectral averaging methods and results from inverse spectral and scattering theory. For detailed proofs we refer to other sources, in particular [23, 31, 5, 32, 29, 9]. We also mention some work in progress. We do not attempt to include a complete bibliography of related works.

In Section 2 we start by introducing a list of different models of 1-d continuum random operators. We mainly work with random Schrödinger operators, but also discuss models related to the classical wave equation. While introducing the models we describe some of their technical properties, which have led to the necessity of new methods. Section 3 and its various subsections contain, in more or less chronological order, the results and methods which were developed to handle these models.

Finally, we comment on open problems in Section 4.

2. Models and their characteristic properties

In this section we introduce several models of one-dimensional continuum random operators. We will mainly work with Schrödinger operators, but at the end of the section also mention corresponding models for the propagation of classical waves in disordered media.

We will in particular comment on the presence or absence of the following two properties:

1. **Monotonicity:** Does the random operator depend monotonically in the sense of quadratic forms on the random parameters?
2. **Smoothness:** Are the random parameters smoothly distributed? For example, are their distributions absolutely continuous? Or, are they singular, in the extreme case discrete?

All the methods to be discussed in Section 3 below can be seen as ways to deal with the lack of monotonicity and/or smoothness in most of these models.

(i) The Standard Continuum Anderson Model: By this we denote the random operator

$$H_A(\omega) = -\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}} q_n(\omega) f(x-n) \tag{2.1}$$

in $L^2(\mathbb{R})$, where the single site potential $f \neq 0$ is *non-negative*, bounded and $\operatorname{supp} f \subset [0,1]$. The coupling constants q_n are independent, identically distributed random variables with an absolutely continuous distribution of bounded, compactly supported density.

The assumption $f \geq 0$ leads to a monotonic model: If ω and ω' are two random configurations with $q_n(\omega) \geq q_n(\omega')$ for all n, then $H_A(\omega) \geq H_A(\omega')$ in form sense. Also, the absolute continuity of the distribution of q_n yields smooth dependence of the model on the random parameters. In this paper we will continue to use the term "smooth" to denote absolute continuity of the distribution of random parameters (rather than differentiability properties).

The boundedness assumption on f and the assumptions on the density of q_n could be relaxed. In fact, most of the smoothness based methods discussed below only require a non-trivial absolutely continuous component for q_n. We will not dwell on these generalizations here. Problems related to relaxing the compact support of f are discussed in Section 4.

(ii) The Poisson Model: This is the operator in $L^2(\mathbb{R})$ given by

$$H_P(\omega) = -\frac{d^2}{dx^2} + \sum_i f(x - X_i(\omega)). \tag{2.2}$$

The $X_i(\omega)$ are the points of a Poisson process on $\mathbb{R}$ with constant density $\alpha > 0$. They can almost surely be measurably labeled by $i \in \mathbb{Z} \setminus \{0\}$ such that $\ldots < X_{-1} < 0 < X_1 < X_2 < \ldots$ and have the property that for every Borel set $B \subset \mathbb{R}$

$$\mathbb{P}(\#\{i : X_i \in B\} = n) = \frac{(\alpha|B|)^n}{n!} e^{-\alpha|B|},$$

where $|\cdot|$ is Lebesgue measure. If $B_1 \cap B_2 = \emptyset$, then the random variables $\#\{i : X_i \in B_1\}$ and $\#\{i : X_i \in B_2\}$ are independent. If $X_0 := 0$ (but *not* a Poisson point), then the random distances $p_n := X_{n+1} - X_n$, $n \in \mathbb{Z}$, are i.i.d. with distribution density $\alpha e^{-\alpha t}$.

Thinking of the p_n rather than the X_i as the defining parameters, we therefore have *smoothness* of the model in the parameters. However, the model is *not monotonic* in the p_n (or the X_i): Change of the parameters makes the single site bumps move left or right rather than up or down. A sign restriction on f doesn't change this and thus we only assume that $f \neq 0$ is bounded and compactly supported.

An additional feature of the Poisson model is that dense clusters of single sites as well as large gaps between sites occur. Technically, this is sometimes helpful and a nuisance at other times.

(iii) The Displacement Model: In several ways intermediate between H_A and H_P is the displacement model

$$H_D(\omega) = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} f(x - n - d_n(\omega)). \tag{2.3}$$

The random displacements d_n are i.i.d. random variables, which can be thought of as modeling temperature fluctuations in a crystal. Thus it is reasonable to assume that they have an a.c. distribution with bounded density supported in $[-d_{\max}, d_{\max}]$, i.e. that the model is *smooth* in the d_n. The single site potential f is assumed to be non-zero, bounded and supported in $[-s, s]$ with $s + d_{\max} \leq 1/2$, which guarantees non-overlapping adjacent sites.

H_D is *non-monotonic* in the displacements d_n, similar to the Poisson model. It differs from H_P by the more crystalline single site structure. Neither large gaps nor dense clusters occur.

(iv) Indefinite Anderson: Monotonicity of the Anderson model H_A is due to the way in which the random parameters enter, but also to the sign-definiteness of the single site potential f. If we drop the assumption $f \geq 0$ (or, essentially equivalent,

$f \leq 0$), but keep all other assumptions from (i), one gets a non-monotonic version of the Anderson model. This is due to possible sign changes of f.

To distinguish this model from the monotonic random operator $H_A(\omega)$, we will denote it by $H_{IA}(\omega)$, the indefinite Anderson model. We don't change the assumptions on q_n and thus still have smoothness.

(v) Singular Random Parameters: Not just for the sake of generality, but also for physical reasons it is desirable to understand the Anderson model H_A if the distributions of the q_n are singular. An extreme case is the Bernoulli-Anderson model where the q_n take only two values, i.e. $\mathbb{P}(q_n = a) = p \in (0,1)$ and $\mathbb{P}(q_n = b) = 1-p$. Looking at this and other discrete distributions is motivated by thinking of the q_n as the nuclear charge at site n in an alloy consisting of finitely many components.

We will write $H_{SA}(\omega)$ (singular Anderson) when referring to the Anderson model with general, possibly singular, distributions of the i.i.d. random variables q_n, i.e. it will only be assumed that the support of their distribution contains at least two points. For simplicity, we will still assume that the support is bounded.

Thus H_{SA} is in general singular in its random parameters, i.e. lacks smoothness. Requiring a definite sign for f would still leave us with monotonicity. But it will turn out that due to the lack of smoothness we will not be able to exploit monotonicity any more. One way to understand this is to look at the restrictions of $H_A(\omega)$ to a fixed finite interval. If $f \geq 0$ and for a configuration ω the restricted operator has an eigenvalue at E_0, then by choosing a configuration ω' with slightly larger couplings $q_n(\omega')$, the eigenvalue can be made to move away from E_0. If the q_n have discrete distribution, then the nearest possible larger values of $q_n(\omega')$ might cause a lower lying eigenvalue to move up to E_0, which diminishes the technical advantage of monotonicity. Therefore our assumptions on f for H_{SA} will be the same as for H_{IA}.

Below, we will occasionally discuss specific properties of the Bernoulli-Anderson model, denoting it by H_{BA}.

We may also introduce a singular version of the displacement model H_D by dropping the absolute continuity assumption in (iii) and only requiring that the support of the distribution of the d_n contains at least two points. This will be referred to as H_{SD}.

(vi) Anderson with periodic background: A generalized Anderson model is obtained if in addition to the random potential a periodic background potential V_{per} is introduced:

$$H_{P+A}(\omega) = -\frac{d^2}{dx^2} + V_{\text{per}}(x) + \sum_{n\in\mathbb{Z}} q_n(\omega) f(x-n). \tag{2.4}$$

Here V_{per} is bounded with $V_{\text{per}}(x+1) = V_{\text{per}}(x)$, while q_n and f satisfy the general assumptions from (v).

Studying this model is partly motivated by applications, where it can be used to model a crystal with random impurities. There is also a mathematical reason for introducing H_{P+A}: Starting from a singular Anderson model $H_{SA} = -d^2/dx^2 + \sum q_n f(x-n)$, we may pick any number $q^{(0)}$ in the support of the distribution of

q_n, define $V_{\rm per}(x) = \sum q^{(0)} f(x-n)$ and $q_n^{(0)} = q_n - q^{(0)}$ to get $H_{SA} = -d^2/dx^2 + V_{\rm per}(x) + \sum q_n^{(0)} f(x-n)$.

At the cost of the new $V_{\rm per}$ we have reached the advantage that the new random couplings $q_n^{(0)}$ have 0 in the support of their distribution. To some extent this will allow to treat H_{SA} as a perturbation of $-d^2/dx^2 + V_{\rm per}$. In particular, we will outline in Section 3 how ideas from scattering theory at periodic background can be exploited.

Introducing $V_{\rm per}$ also allows to view certain displacement models as Anderson models: If the d_n in H_{SD} take only two values $d^{(0)}$ and $d^{(1)}$ ("Bernoulli displacement model"), then

$$H_{SD} = -\frac{d^2}{dx^2} + \tilde{V}_{\rm per}(x) + \sum_n \tilde{q}_n \tilde{f}(x-n),$$

where $\tilde{V}_{\rm per}(x) = \sum_n f(x-n-d^{(0)})$, $\tilde{f}(x) = f(x-d^{(1)}) - f(x-d^{(0)})$, and $\tilde{q}_n = 0$ if $d_n = d^{(0)}$ and $\tilde{q}_n = 1$ if $d_n = d^{(1)}$. This will allow results and methods developed for Anderson models to carry over to displacement models.

(vii) Models for Random Wave Equations: All the models introduced in (i) to (vi) are random Schrödinger operators, i.e. quantum mechanical models to study electron waves in disordered media. Random operators can also be used to model the propagation of classical waves (electro-magnetic, acoustic, etc.) in disordered media. This is the main motivation behind studying models of the type

$$H^W(\omega) = -\frac{d}{dx} a(x,\omega) \frac{d}{dx}, \tag{2.5}$$

a random operator derived from the one-dimensional wave equation. The random coefficient $a(x,\omega)$ describes properties of an inhomogeneous medium, such as the reciprocals of the mass density, compressibility, dielectricity, etc., depending on the type of waves to be studied.

Specific choices of $a(x,\omega)$ allow us to introduce randomness of Anderson, Poisson, or displacement type. The corresponding operators will be denoted by H_A^W, H_P^W, H_D^W. For example, we get H_A^W by choosing

$$\frac{1}{a_A(x,\omega)} = 1 + \sum_{n\in\mathbb{Z}} q_n(\omega) f(x-n), \tag{2.6}$$

with assumptions on q_n and f as in (i), but modified to guarantee that the right hand side of (2.6) remains strictly positive. A sign-change in f again leads to a non-monotonic model H_{IA}^W, and singular models arise as H_{SA}^W and H_{SD}^W. Note that the use of $1/a(x,\omega)$ in (2.6) and its analogues has physical reasons. It is $1/a$, not a, which corresponds to physical quantities.

3. Strategies in localization proofs

3.1. **Basic consequences of ergodicity and Kotani theory.** All the Schrödinger-type models from above almost surely define self-adjoint operators in $L^2(\mathbb{R})$ in the sense that they are essentially self-adjoint on $C_0^\infty(\mathbb{R})$. The only model where this is slightly non-trivial is H_P due to the existence of dense clusters in the Poisson potential. But it can be shown that the latter is almost surely logarithmically bounded in x, e.g. [14], guaranteeing essential self-adjointness.

The classical wave models H^W from (2.5) are defined as the self-adjoint operators corresponding to the non-negative form $\int a(x,\omega)u'(x)\overline{v'(x)}\,dx$, $u, v \in C_0^\infty(\mathbb{R})$. Note that this does not require smoothness of $a(x,\omega)$ and therefore $C_0^\infty(\mathbb{R})$ may not be in the operator domain of H^W. In particular, $a(x,\omega)$ may be a step function.

All random operators $H(\omega)$ from Chapter 2 are ergodic (sometimes called metrically transitive or covariant), i.e. there is an ergodic dynamical system $\{T_y\}$ on the underlying probability space ($y \in \mathbb{R}$ for the Poisson model, $y \in \mathbb{Z}$ for the others) such that

$$U_y H(\omega) U_{-y} = H(T_y\omega) \qquad \text{for all } y \text{ and } \omega, \tag{3.1}$$

where $(U_y f)(x) = f(x-y)$. This implies that $H(\omega)$ has non-random spectrum and spectral types, i.e. there exist deterministic sets Σ, Σ_{ac}, Σ_{sc}, and Σ_{pp}, such that almost surely

$$\sigma(H(\omega)) = \Sigma, \quad \sigma_{ac/sc/pp}(H(\omega)) = \Sigma_{ac/sc/pp}, \tag{3.2}$$

where σ_{ac}, σ_{sc}, and σ_{pp} denote the absolutely continuous, singular continuous, and point spectrum (closure of the set of eigenvalues), respectively. Since our models are one-dimensional, it also holds that each fixed $E \in \mathbb{R}$ is almost surely not an eigenvalue of $H(\omega)$. For these results and other facts from the general theory of ergodic operators, see the monographs [7] and [26].

The deepest general results on one-dimensional ergodic operators are due to Kotani, e.g. [21, 22]. To cover all our models, we also need the extensions of Kotani theory to $\mathbb{Z}$-ergodic operators by Kirsch [17] and to operators of type H^W by Minami [24, 25].

Assuming in each case that the single site potential is compactly supported makes all our models non-deterministic in Kotani's sense. By one of the results of Kotani theory this implies that the Lyapunov exponent $\gamma(E)$ is strictly positive for almost every energy E. Recall that the Lyapunov exponent is defined by

$$\gamma(E) = \lim_{x\to\pm\infty} \frac{1}{|x|}\mathbb{E}(\log \|T(x,E,\omega)\|), \tag{3.3}$$

where $T(x,E,\omega)$ is the transfer matrix from 0 to x of $H(\omega)$ at E, and $\mathbb{E}$ is the expectation with respect to ω.

By another general result this is equivalent to the fact that $\Sigma_{ac} = \emptyset$, i.e. that $H(\omega)$ almost surely has no a.c. spectrum.

In order to get results which go beyond these general facts, it seems to be necessary to use more specific assumptions. In particular, all the work on localization properties has been done for specific models.

We will follow the convention that *localization* of a random operator $H(\omega)$ means almost surely pure point spectrum, i.e. that $\Sigma_{ac} = \Sigma_{sc} = \emptyset$. *Exponential localization* means in addition that all eigenfunctions of $H(\omega)$ decay exponentially, again almost surely in ω.

In Section 3 we will discuss the various methods which enter the proof of

Theorem 1. *All the random Schrödinger operators defined in Section 2(i) to 2(vi) are exponentially localized.*

Under suitable assumptions this also holds for the operators of type H^W from Section 2(vii), see the remarks in Sections 3.5 and 3.6.

More recently, and closer to what is relevant in physics, attention has been shifted to proving *dynamical localization* of random operators. We will use the following definition, which is one of the stronger versions to be found in the literature: $H(\omega)$ is said to satisfy *dynamical localization* on a subset $I \subset \mathbb{R}$ if

$$\mathbb{E}\left\{\sup_{t>0} \||X|^p e^{-itH(\omega)} P_I(H(\omega))\chi_K\|\right\} < \infty \tag{3.4}$$

for every compact set K in $\mathbb{R}$ and every $p > 0$. Here P_I is the spectral projection onto I and X the operator of multiplication by the variable in $L^2(\mathbb{R})$. We will discuss dynamical localization for the models from Section 2 in Section 3.6.

All self-adjoint Hamiltonians H considered in this paper are Sturm-Liouville operators in the limit point case at $+\infty$ and $-\infty$. Thus their spectral type is characterized by their Weyl-Titchmarsh spectral measure ρ_H, i.e. the trace measure corresponding to the standard 2×2-matrix-valued spectral measure for H [8]. The absolutely continuous, singular continuous and point spectra of H are the corresponding parts of the Lebesgue decomposition of ρ_H.

3.2. **Spectral Averaging in Monotonic Models.** To see how a proof of localization for smooth, monotonic models can be completed, let us first focus on the Anderson model H_A under the assumptions of Section 2(i). We essentially outline the argument in [23], where more details can be found.

The quantity $|x|^{-1} \log \|T(x, E, \omega)\|$ converges to $\gamma(E)$ not only in expectation but also almost surely as $x \to \pm\infty$. Fubini's and Osceledec's Theorems yield that for almost every ω there exists $S(\omega) \subset \mathbb{R}$ with $|\mathbb{R} \setminus S(\omega)| = 0$ such that for every $E \in S(\omega)$ solutions u_+ and u_- of $H(\omega)u = Eu$ exist which decay exponentially at $+\infty$ and $-\infty$, respectively. All other solutions increase exponentially at $\pm\infty$.

Let $\rho_\omega = \rho_{H_A(\omega)}$ be the Weyl-Titchmarsh spectral measures of H_A. It follows from either Shnol's Theorem on the polynomial boundedness of generalized eigenfunctions or from Subordinacy Theory (e.g. [5, Sect.2]) that ρ_ω restricted to $S(\omega)$ is a pure point measure.

The set $S(\omega)$ is independent of the value of $q_0(\omega)$, since a change of q_0 only yields a local change in the differential equation, not effecting the asymptotics of solutions at $\pm\infty$. This is crucial since it allows to use the following result on *spectral averaging*, which is essentially Lemma 2.2 of [23] (for a detailed proof in the stated form see also [30, Sect.5.3]). It can be seen as a version of "Kotani's trick".

Theorem 2. *Let f be as in Section 2(i), V a real-valued potential such that $H_\lambda = -d^2/dx^2 + V + \lambda f$ is in limit point case at $\pm\infty$, ρ_λ the Weyl-Titchmarsh spectral measure of H_λ, and $-\infty < \lambda_1 < \lambda_2 < \infty$. Then the measure μ defined for Borel sets B by*

$$\mu(B) = \int_{\lambda_1}^{\lambda_2} \rho_\lambda(B)\, d\lambda \tag{3.5}$$

is absolutely continuous on $\mathbb{R}$.

The proof of this result crucially uses $f \geq 0$, i.e. monotonicity of the model. Smoothness comes in as follows: We apply Theorem 2 to the case where $\lambda = q_0$ and $V(x) = \sum_{n\neq 0} q_n f(x-n)$. Since $|\mathbb{R}\setminus S(\omega)| = 0$, it follows for the q_0-averaged measure μ that $\mu(\mathbb{R}\setminus S(\omega)) = 0$. This implies that $\rho_\omega(\mathbb{R}\setminus S(\omega)) = 0$ for Lebesgue-a.e. value of q_0 (let $[\lambda_1, \lambda_2]$ exhaust $\mathbb{R}$). Using that $q_0(\omega)$ has absolutely continuous distribution we get that $\rho_\omega(\mathbb{R} \setminus S(\omega)) = 0$ for a.e. $q_0(\omega)$ and thus, using independence, for a.e. ω.

Thus, knowing that the restriction of ρ_ω to $S(\omega)$ is pure point, we conclude that ρ_ω almost surely is a pure point measure, which completes the proof of localization for H_A. In fact, the proof shows that ρ_ω almost surely is supported on the set where the generalized eigenfunctions decay exponentially at the rate of the Lyapunov exponent, so that actually exponential localization is proven.

Much of the above strategy also works for the other smooth models from Section 2. The only difficulty is that it is not immediate how to find a replacement for Theorem 2 in non-monotonic models. One exception is the Poisson model H_P at positive energies: Independence of the random distances d_n allows to adopt the above strategy for H_A and to fix all d_n for $n \neq 0$ and average spectral measures over d_0. Essentially, this leads to the following one-parameter model (e.g. [31]):

Let $W_1, W_2 \in L^\infty_{\text{loc}}(\mathbb{R})$ be real-valued, $|W_i(x)| = O(x^2)$ as $|x| \to \infty$, $i = 1, 2$, and $\operatorname{supp} W_1 \subset [0, \infty)$, $\operatorname{supp} W_2 \subset (-\infty, 0]$. For $a > 0$ let $V_a(x) = W_1(x-a) + W_2(x+a)$ and ρ_a the Weyl-Titchmarsh spectral measure of $H_a = -d^2/dx^2 + V_a(x)$.

Theorem 3. *For fixed $a_2 > a_1 > 0$ and arbitrary Borel sets $B \subset \mathbb{R}$ define*

$$\mu(B) = \int_{a_1}^{a_2} \rho_a(B)\, da.$$

Then the measure μ is absolutely continuous on $(0, \infty)$.

This result allows to deduce exponential localization of H_P at positive energies. The proof of Theorem 3 in [31] is based on a monotonicity property of the family H_a. In [5, Sect.1] it is explained how this can be understood in terms of growing phase space volume at positive energy as a increases. Technically, monotonicity enters the proof of Theorem 3 in the form of monotonic dependence on a of Prüfer phases for solutions of $H_a u = Eu$, $E > 0$, see [31].

The measure μ in Theorem 3 is not necessarily absolutely continuous at negative energies. In fact, it is easy to construct an example such that H_a has the same eigenvalue $E_0 < 0$ for all a, see [5, Sect.1]. Thus μ also gets a pure point component. Also, Theorem 3 does not apply to the situation of the displacement model, since for H_D the displacements, not the distances, of the single sites are independent.

3.3. **Two-Parameter Averaging and Inverse Spectral Theory.** Averaging the Weyl-Titchmarsh spectral measures of a random operator over one of its random parameters does not necessarily lead to an a.c. measure for the non-monotonic models from Section 2. It turns out that simultaneously averaging over two parameters can be used to overcome this problem in many situations. We will describe this here for the displacement model H_D, but similar ideas also yield a proof of exponential localization for the Poisson model H_P at all, including negative, energies.

The following two-parameter family of operators arises when fixing all but two of the d_n in H_D: Let $0 < L < 1$, $q \neq 0$ bounded and real-valued with $\operatorname{supp} q \subset [0, L]$, and $a, b \in (0, 1-L)$. Let

$$V_{a,b}(x) = \begin{cases} q(x-a), & a \le x \le a+L, \\ q(x-1-b), & 1+b \le x \le 1+b+L, \\ 0, & \text{elsewhere in } [0,2], \\ V(x), & x \notin [0,2], \end{cases}$$

where V is a fixed potential such that $-d^2/dx^2 + V$ is limit point at $+\infty$ and $-\infty$. By $\rho_{a,b}$ we denote the Weyl-Titchmarsh spectral measures of the operators $-d^2/dx^2 + V_{a,b}$.

Theorem 4. *There is a discrete subset $M \subset \mathbb{R}$ such that for $E_0 \in \mathbb{R} \setminus M$ the following holds: For every $a_0 \in (0, 1-L)$ and for every $b_0 \in (0, 1-L) \setminus M(a_0, E_0)$, where $M(a_0, E_0)$ is finite, there exist $\varepsilon > 0$ and $\delta > 0$ such that the measure defined by*

$$\rho(B) = \int_{a_0-\delta}^{a_0+\delta} \int_{b_0-\delta}^{b_0+\delta} \rho_{a,b}(B)\, db\, da$$

is absolutely continuous on $(E_0 - \varepsilon, E_0 + \varepsilon)$.

This version of two-parameter spectral averaging still allows to complete a proof of exponential localization for H_D along the lines of the general arguments from Section 3.2, see [5] for details and a proof of Theorem 4. This uses that the finite set which is excluded for the values of b_0 has zero probability since the d_n have a.c. distribution. Also, the discrete set of energies M where averaging does not apply can not carry any continuous spectrum. In fact, M almost surely carries no spectrum.

The crucial aspect of the work in [5] is an explicit description of M which allows to prove its discreteness. For this let q be as above and $u_0(\cdot, \theta, E, q)$ the solution of the initial value problem $-u'' + qu = Eu$, $u(0) = \sin\theta$, $u'(0) = \cos\theta$. Then $M = M_+ \cup \{0\} \cup M_-$, where

$$M_+ := \{E > 0 : u_0'(D, \theta, E, q)^2 + E u_0(D, \theta, E, q)^2 = \cos^2\theta + E\sin^2\theta \text{ for all } \theta \in \mathbb{R}\}, \tag{3.6}$$

$$M_- := \{E < 0 : \text{There is a solution of } -u'' + qu = Eu \text{ such that } \frac{u'(0)}{u(0)}, \frac{u'(D)}{u(D)} \in \{-\sqrt{|E|}, \sqrt{|E|}\}\}. \tag{3.7}$$

Discreteness of M_+ in $[0, \infty)$ follows by analytic continuation (in E) from discreteness of M_- in $(-\infty, 0]$. The latter can be seen to follow from a classical result of Borg in inverse spectral theory: Let Q be the potential on $\mathbb{R}$ found by L-periodic extension of q. If M_- were not discrete, then one can show that $\sigma(-d^2/dx^2 + Q) = [0, \infty)$, see [5]. Thus we can use

Theorem 5 ([3])**.** *If Q is periodic and $\sigma(-d^2/dx^2 + Q) = [0, \infty)$, then $Q = 0$.*

Since we have assumed $q \neq 0$, we conclude that M_- is discrete.

3.4. **The Indefinite Anderson Model.** It turns out that two-parameter spectral averaging and a simple fact from inverse spectral theory can also be used to prove exponential localization for the indefinite Anderson model H_{IA}. We sketch the two main ingredients from the proof in [32], which in fact allows the presence of a periodic background V_{per} as in (2.4) in H_{IA}, but needs an a.c. component in the distribution of q_n.

Fixing all but two of the q_n we arrive at a family

$$H_{\lambda,\mu} = -d^2/dx^2 + V + \lambda W_1 + \lambda W_2,$$

where V is a fixed background potential such that we have limit point case at $\pm\infty$, and $W_1 \neq 0$ and $W_2 \neq 0$ are integrable potentials of disjoint, compact supports, but not necessarily of fixed sign. For their spectral measures $\rho_{\lambda,\mu}$ we have the following [32, Section 2]

Theorem 6. *Let $E_0 \in \mathbb{R}$ be fixed. Then there is a discrete set $M \subset \mathbb{R}$ such that for every $\lambda_0 \in \mathbb{R} \setminus M$ there is a discrete set $N(\lambda_0) \subset \mathbb{R}$ with the following property: If $\mu_0 \in \mathbb{R} \setminus N(\lambda_0)$ then there are $\varepsilon > 0$ and $\delta > 0$ such that the Borel measure μ defined by*

$$\mu(B) = \int_{\lambda_0-\delta}^{\lambda_0+\delta} \int_{\mu_0-\delta}^{\mu_0+\delta} \rho_{\lambda,\mu}(B)\, d\mu\, d\lambda$$

is absolutely continuous on $(E_0 - \varepsilon, E_0 + \varepsilon)$.

Note that in Theorem 6, as opposed to Theorem 4, there is no exceptional set of energies. Spectral averaging is possible in a neighborhood of any given energy E_0.

But it is still crucial for the proof of localization that the exceptional sets M and $N(\lambda_0)$ for the parameters λ_0 and μ_0 are discrete. This is derived in [32] from the following inverse spectral result:

Theorem 7. *Let q and w be integrable real potentials on a finite interval $[a, b]$ and let T_λ be a self-adjoint operator on $L^2(a, b)$ defined by $-d^2/dx^2 + q + \lambda w$ and arbitrary, λ-independent, boundary conditions at a and b. If E_0 is an eigenvalue of T_λ for every λ, then $w = 0$.*

This follows quite easily from the variational principle and the fact that no eigenvalue crossings are allowed in dimension one.

3.5. **Tools from Scattering Theory.** It is possible to replace Borg's result from inverse spectral theory by tools from inverse scattering theory in proofs of localization. This is closer to physics since it supports the following heuristics: If scattering at each single site is non-trivial and a global potential is constructed by randomly dispersing these sites, then states should be localized.

A way to characterize "non-trivial scattering at single sites" is by using reflection and transmission coefficients. For their definition let f be compactly supported in $[-1/2, 1/2]$, $k \in \mathbb{C} \setminus \{0\}$, and $u(\cdot, k)$ the Jost solution of $-u'' + fu = k^2 u$, i.e. the solution which satisfies

$$u(x,k) = \begin{cases} e^{ikx} & \text{for } x \le -1/2, \\ a(k)e^{ikx} + b(k)e^{-ikx} & \text{for } x \ge 1/2. \end{cases} \tag{3.8}$$

This characterizes $a(k)$ and $b(k)$ uniquely, which are related to the reflection and transmission coefficients from physics by $r(k) = b(k)/a(k)$ and $t(k) = 1/a(k)$. The coefficients $a(k)$ and $b(k)$ are analytic in k and for real k satisfy $|a(k)|^2 - |b(k)|^2 = 1$. One has

Theorem 8. *If $-u'' + fu = k^2 u$ is reflectionless, i.e. if $b(k) \equiv 0$, then $f \equiv 0$.*

This follows from inverse scattering theory, e.g. [12], where all reflectionless potentials (solitons) have been characterized, and none of them are compactly supported. One can also deduce Theorem 8 from Borg's Theorem since it is seen quite easily that $b(k) \equiv 0$ implies that the potential found by periodically extending f has spectrum $[0, \infty)$.

A proof of exponential localization for the displacement model H_D can be based on Theorem 8 by observing that the sets M_+ and M_- in (3.6) and (3.7) can be characterized as follows:

$$M_+ = \{k^2 > 0 :\ k > 0,\ b(k) = 0\}, \tag{3.9}$$

$$M_- = \{-\alpha^2 : \alpha > 0,\ b(i\alpha)b(-i\alpha)a(i\alpha)a(-i\alpha) = 0\}, \tag{3.10}$$

see [29]. For $f \neq 0$ one has that $b(k)$ and $a(k)$ are analytic and don't vanish identically. Thus they have discrete roots, yielding discreteness of M_+ and M_-. Therefore Theorem 4 applies and the proof of localization is concluded as before.

For the Schrödinger-type model H_D, the above ideas are mainly a variant of the previous proof, but they also can be applied to prove exponential localization for H_D^W, the displacement model for classical waves [29]. The main difficulty in extending the proof to this model is to find an analogue of Theorem 8. For this let a function $f \in L^1$ be supported in $[-1/2, 1/2]$ and such that $1 + f > 0$. For the single site equation

$$-((1+f)^{-1}u')' = k^2 u \tag{3.11}$$

one can define $a(k)$ and $b(k)$ as before by (3.8). One indeed has [29]

Theorem 9. *If $-((1+f)^{-1}u')' = k^2u$ is reflectionless, then $f \equiv 0$.*

If f is twice differentiable, then (3.11) can be transformed into a Schrödinger equation by a Liouville-Green transformation, which can be used to reduce Theorem 9 to Theorem 8. For non-smooth f the result is deeper. It was proved in [29] by using various facts about Herglotz function representations and the following result on the asymptotics of m-functions:

Theorem 10. *Let $m(z,x)$ be the Weyl-Titchmarsh m-function of the operator $-\frac{d}{dx}p\frac{d}{dx}$ on (x,∞) with Dirichlet-boundary condition at x, where $p > 0$, $1/p \in L^1_{loc}(\mathbb{R})$ and $p - 1$ is compactly supported. If x is a Lebesgue-point for $1/p$, then*

$$m(-\alpha^2, x) = -\alpha p(x)^{1/2} + o(\alpha)$$

for real $\alpha \to +\infty$.

An elementary proof of this fact is given in [29]. Much more general results which apply to our situation are contained in [1], see also [28].

3.6. **Singular Random Parameters.** We have demonstrated that the method of spectral averaging can be made to work in a number of non-monotonic models. But it is not possible to extend this method to models with singularly distributed random parameters, in particular the singular Anderson model H_{SA}. In an attempt to prove localization this dooms much of the basic strategy described in Sections 3.1 and 3.2 to failure. For example, Kotani's theory guarantees positivity of $\gamma(E)$ only for a.e. $E \in \mathbb{R}$. The set where $\gamma(E) = 0$ might be uncountable and therefore, without spectral averaging at hand, support singular continuous spectrum with positive probability.

Thus it becomes important to have stronger results on positivity of Lyapunov exponents. Since we assume that $\operatorname{supp} f \subset [-1/2, 1/2]$ the transfer matrices for $H_{SA}(\omega)$ from $n - 1/2$ to $n + 1/2$ are i.i.d. unimodular random matrices. This allows us to use Fürstenberg's Theorem, see e.g. [4], as a powerful tool to prove positivity of Lyapunov exponents. This has frequently been exploited for discrete models, compare [26, Sect.14.A]. In particular it shows that $\gamma(E) > 0$ for *all* $E \in \mathbb{R}$ in the discrete one-dimensional Bernoulli-Anderson model, which has served as a starting point in a proof of localization for this model in [6]. Based on Fürstenberg's Theorem one can show the following result for the continuum case:

Theorem 11. *Let H_{P+A} be the singular Anderson model with periodic background as introduced in Sections 2(v) and (vi). Then there exists a discrete set M such that $\gamma(E) > 0$ for every $E \in \mathbb{R} \setminus M$.*

In fact, the set M of critical energies where $\gamma(E)$ may vanish, can be characterized explicitly in terms of reflection and transmission coefficients at the single site potentials. While this is done in [9] in the context of scattering at periodic background, we only state the result here for the special case $V_{\text{per}} = 0$, where we can work with the reflection and transmission coefficients as defined in (3.8). It is also sufficient to focus on the Bernoulli case, since by the nature of Fürstenberg's Theorem the critical set becomes smaller as the support of q_n increases.

Theorem 12. *Let $H_{BA}(\omega) = -d^2/dx^2 + \sum_n q_n(\omega)f(x-n)$ with $f \neq 0$, supp $f \subset [-1/2, 1/2]$ and i.i.d. random variables q_n taking only the values 0 and 1. Then the critical set M in Theorem 11 can be chosen as*

$$M = M_+ \cup M_- \cup \{(n\pi/2)^2 : n \in \mathbb{Z}\},$$

where M_+ and M_- are the sets from (3.9) and (3.10).

Thus discreteness of M follows from the results in Section 3.5. Theorem 12 follows from the more general results in [9]. A more direct proof of this special case is provided in [10].

In the general case of Theorem 11, i.e. in the presence of V_{per}, one needs to redefine a and b as scattering coefficients relative to the periodic background, and can then again describe the critical set M in terms of their zeros. In particular, it can again be shown that their zeros are discrete, e.g. [9].

While the discrete set M can not support singular continuous spectrum, the argument from Section 3.2 can still not be pushed through to prove localization. By the way in which Fubini enters the argument at the beginning of Section 3.2, for fixed ω we still get the existence of exponentially decaying solutions only for a.e. E. Instead, one can use Theorem 11 as a starting point for a proof of localization following the arguments for the discrete case in [6]. We refer to [9] for details and only mention that the following facts can be established: Hölder continuity of the Lyapunov exponent and the integrated density of states, a Wegner-type estimate, and a so-called initial length scale estimate for the decay of resolvent kernels on finite intervals. It then follows from well established facts, based on multiscale analysis methods, that H_{P+A} is exponentially localized. In fact, one gets dynamical localization:

Theorem 13. *The singular Anderson model H_{P+A} with periodic background satisfies dynamical localization in the sense of (3.4) for compact intervals $I \subset \mathbb{R} \setminus M$, where M is the discrete critical set from Theorem 11.*

Critical energies with $\gamma(E) = 0$ do indeed exist in Bernoulli-Anderson models. For example, in the situation of Theorem 12 let $E = k^2 > 0$ be an energy such that $b(k) = 0$. It is then easily seen from (3.8) that all solutions of $H_{BA}u = Eu$ are globally bounded, implying $\gamma(E) = 0$. For the special case $f = \lambda\chi_{[-1/2,1/2]}$ this happens if $E = (n\pi)^2 + \lambda$, $n \in \mathbb{N}$. Critical energies also appear in the so-called dimer model, a generalized discrete Anderson model, where dynamical localization away from the critical energies has been shown in [2].

The restriction in Theorem 13 to intervals $I \subset \mathbb{R} \setminus M$ is crucial. Numerical results in the physics literature, e.g. [13], indicate that the dimer and related discrete models have positive diffusion exponents if wave packets contain critical energies in their energy support. This will be rigorously established for discrete as well as continuous models in [16]. This is quite remarkable since it is the first result which establishes at least subdiffusive wave transport for some types of one-dimensional random operators.

Finally, we note that the techniques which led to Theorem 13 can be applied to several other models: Results for singular displacement models H_{SD} and classical wave models H^W are contained in [28]. Generalized discrete Anderson models, which include the dimer model, will be considered in [11].

3.7. **Some Open Problems.** We conclude by mentioning some open problems. Among the many open problems in the field of random operators and their applications, we restrict our remarks to problems which arise directly in the context of results presented in this paper.

(i) In Section 3.6 we described methods which allow to prove localization for one-dimensional random operators with singularly distributed random parameters, in particular the Bernoulli-Anderson model. The proof uses multiscale analysis and therefore the entire collection of tools which enter into this method, which was originally developed mainly for the purpose of proving localization for multi-dimensional Anderson models. After establishing positivity of the Lyapunov exponent in Theorem 11, it seems to be somewhat inappropriate that one has to embark into this sophisticated machinery rather than being able to complete the proof within arguments from the one-dimensional theory. A more direct proof of localization for one-dimensional Bernoulli-Anderson models would be desirable.

Some ideas for this could come from an existing second method of proof of localization for the one-dimensional discrete Bernoulli-Anderson model which was given in [27]. However, the application of this method to continuum models seems to yield considerable difficulties.

(ii) Throughout this paper we have assumed that the single site potentials have compact support, most of the time even that they do not overlap. For many of the techniques which were used this is quite crucial, for example for most of the results on spectral averaging, or the results based on Fürstenberg's Theorem. While most of the results on localization should hold for non-compactly supported single sites under suitable decay assumptions, the proofs do not easily extend to this situation. Technically, the main reason for the difficulties is that long range correlations are introduced into the random potential. There are results which show localization near the bottom of the spectrum or, more generally, near band edges, for multi-dimensional Anderson models with non-compactly supported single site potentials, e.g. [18]. While these results include dimension one, they do not give localization at arbitrary energy in $d = 1$. It is quite surprising that there does not seem to be a proof of localization at all energies for Anderson models H_A from (2.1) where f is in a class of not necessarily compactly supported potentials.

The scattering theoretic arguments used in Section 3.5 indicate that some non-trivial difficulties may arise. If f is not compactly supported, then the structure of its reflection coefficient $b(k)$ may be much more complicated. f may be a soliton ($b(k) \equiv 0$) or, probably even worse, b may vanish on large sets without vanishing identically. This may happen in situations where f has subexponential decay and thus b is not real-analytic. Another difficulty is the non-linear interaction of scattering-coefficients at multiple overlapping sites. One has to ask if the sets of critical energies which have appeared in Sections 3.3, 3.5 and 3.6 could become significantly larger and give rise to continuous spectrum. We feel that this question is wide open and do not want to predict the answer.

(iii) While we have restricted this paper to the one-dimensional case, it should at least be mentioned that many of the results discussed here have not yet been extended to higher dimension. There are results for the multi-dimensional displacement model [19] and for the indefinite Anderson model [20, 15]. But there are no results yet on localization for the multi-dimensional Poisson model. And localization for the multi-dimensional Bernoulli-Anderson model currently seems to be quite far out of reach, even in the discrete case.

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