Percolation in $\infty + 1$ dimensions at the uniqueness threshold

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Abstract: For independent density p site percolation on the (transitive non-amenable) graph $\mathbb{T}_b \times \mathbb{Z}$, where $\mathbb{T}_b$ is a homogeneous tree of degree $b+1$, and b is supposed to be large, it is shown that for $p = p_u = \inf\{p\colon \text{a.s. there is a unique infinite cluster}\}$ there are a.s. infinitely many infinite clusters. This contrasts with a recent result of Benjamini and Schramm, according to whom for transitive non-amenable planar graphs there is a.s. a unique infinite cluster at p_u.

1. Introduction and results

Percolation theory has for a long time been a central area of research in both Probability and Mathematical Physics. Two complementary reasons concur for this. In the words of Kesten in the preface to [Kes2]: " ... it is a source of fascinating problems of the best kind a mathematician can wish for: problems which are easy to state with a minimum of preparation, but whose solutions are (apparently) difficult and require new methods. At the same time many of the problems are of interest to or proposed by statistical physics and not dreamed up merely to demonstrate ingenuity."

Until recent years most of the interest on percolation concerned the case of the d-dimensional cubic lattice $\mathbb{Z}^d$ (and other graphs adapted to the geometry of the Euclidean space $\mathbb{R}^d$ in some periodic fashion; see [Kes2] and [Gri]). This state of affairs has been changing with the perception that at least from the mathematical viewpoint the proper setting for studying percolation problems is a more general graph. [Lyo2] studied percolation on trees and found a sharp relation between the value of the critical point and the branching number of the tree. The paper [GN] showed that phenomena which are excluded on $\mathbb{Z}^d$ can occur on other (decent) graphs, e.g., we can have the presence of a regime with infinitely many infinite clusters, and another one with a single infinite cluster. More recently a number of papers have appeared presenting a series of results and raising a number of natural and interesting questions concerning percolation on very general graphs. This includes the papers [BS1], [Lal1], [BB], [BLPS1], [BLPS2], [HP], [Sch2], [BS2], [HPS]. Moreover the study of percolation on graphs is part of a larger trend of studying processes which also include interacting particle systems and statistical mechanics models on such general graphs. In this connection the reader is invited to consult, for instance, [Lyo1], [NW], [Pem], [SeS], [Wu3], [SaS1], [SaS3], [ST], [JS], [BHW], and references therein. Good reasons exist for the study of such processes, even if one's ultimate interest lies in the processes on $\mathbb{Z}^d$. The study of the processes on more general graphs has given raise to new techniques and has helped in elucidating which mathematical features of $\mathbb{Z}^d$ are crucial for the behavior of a process there to be what it is. This study may also provide new insights on what to expect on $\mathbb{Z}^d$.

Before we can turn to the contribution in the present paper, we will review some of the relevant concepts and results pertaining to percolation on an infinite connected graph of locally bounded degree $G = (V, E)$. Here V is the (necessarily countable) set of vertices (also called sites) of G, and E is the set of edges (also called bonds) of G. Sites will be said to be neighbors if they belong to a common edge. A chain is a sequence $x_0, x_1, ..., x_n$, of distinct sites in which for each i, x_i is neighbor to x_{i+1}; the length of this chain is n, and x_0 and x_n are its end-points. The distance $d(y, z)$ between sites y and z is the minimal length of the chains which have y and z as end-points. The distance $d(y, S)$ between a site y and a set of sites S is the minimal distance between y and any site in S. The ball of center $x \in V$ and radius n is the set

$$B(x, n) = \{y \in V : d(x, y) \leq n\}.$$

The outer boundary of a set $S \subset V$ is the set

$$\partial S = \{y \notin S : d(y, S) = 1\}.$$

Given a pair of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we will let $G_1 \times G_2$ denote the graph which has as vertices the elements of the Cartesian product $V_1 \times V_2$ and an edge connects (x, u) to (y, v) iff either $x = y$ and $\{u, v\} \in E_2$ or else $\{x, y\} \in E_1$ and $u = v$.

An isomorphism between two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is a one-to-one mapping from V_1 onto V_2 which preserves the graph structure, i.e., such that the set of edges of G_2 can be obtained as the set of pairs of images of vertices of G_1 which form edges. An isomorphism between a graph G and itself is called an automorphism of G.

A graph is said to be transitive (or homogeneous) if for each pair x and y of its vertices there is an automorphism of the graph which maps x into y. Intuitively speaking, in a transitive graph all the vertices play exactly the same role.

A graph is said to be quasi-transitive if there is a finite set of vertices, V_0, with the property that each vertex of the graph can be mapped into one of the vertices of V_0 by an automorphism. Intuitively speaking, in a quasi-transitive graph there is a finite number of types of vertices and all the vertices of the same type play exactly the same role.

A graph is said to be amenable if there is a sequence of finite sets $S_n \subset V$, such that $\lim_{n\to\infty} |\partial S_n|/|S_n| = 0$.

An important class of transitive graphs is that of the Cayley graphs of finitely generated groups. Suppose that V is such a group and that g is a finite symmetric set of generators for it. Then the (right) Cayley graph of V for this set of generators is the graph $G = (V, E)$ which has $E = \{\{x, y\} : x, y \in V,\ y = xz \text{ for some } z \in g\}$. Cayley graphs are easily seen to be transitive. A number of important facts conjectured to hold for percolation on the full class of transitive graphs has been proved for Cayley graphs. This is so because for Cayley graphs a very powerful technique known as "mass transport technique" is available (see [BLPS1], [BLPS2], [HP]). This basic tool is also available for a larger class of graphs, called unimodular quasi-transitive graphs, but this class of graphs is known not to contain all the transitive graphs (see definition and discussion in [BLPS1]).

We will consider independent site percolation on graphs. Independent bond percolation is a particular case thereof, since the edges of a graph can be considered as the sites of another graph. (Note that in this procedure, if the original graph is quasi-transitive, then so is also the new graph.) Declare each site of a graph $G = (V, E)$ to be independently occupied with probability p and to be vacant otherwise. The corresponding probability measure will be denoted by $\mathbb{P}_p = \mathbb{P}$, and the corresponding expectation will be denoted by $\mathbb{E}_p = \mathbb{E}$. We will use the notation $A \longleftrightarrow B$ to denote the event that there is a fully occupied chain with one endpoint in $A \subset V$ and the other one in $B \subset V$. In this case we say that A is connected to B. The notation $A \overset{C}{\longleftrightarrow} B$ will denote the event that the mentioned chain is contained in the set C. The notation $A \longrightarrow \infty$ will denote the event that some site of the set $A \subset V$ is connected by occupied chains to infinitely many sites. As usual, we may replace singleton sets with their unique element, in order to simplify the notation. Also to simplify the notation, we may confuse a graph with its set of vertices.

The connected components of the graph which has as vertices the occupied sites and as edges the edges in E which connect any two of these vertices are called clusters. The random variable N will denote the number of infinite clusters. The following result was proved first in [HP] in the case of unimodular quasi-transitive graphs and then in [Sch2] with the assumption of unimodularity removed.

Theorem 1. *Suppose that G is a quasi-transitive graph. Then there are two critical points $0 < p_c \le p_u \le 1$ such that*

(1) *if $p < p_c$, then $\mathbb{P}_p(N = 0) = 1$,*
(2) *if $p_c < p < p_u$, then $\mathbb{P}_p(N = \infty) = 1$,*
(3) *if $p_u < p$, then $\mathbb{P}_p(N = 1) = 1$.*

The fact that for each value of p one of the three events in the alternatives above must have probability 1 is an older result from [NS], based on the 0-1 law for automorphism invariant events. The extra non-trivial information in the theorem is the statement that the uniqueness of the infinite cluster is a monotone property of p, i.e., once it holds for a certain value of p it will also hold for larger values of p. It is easy to see that without making any assumption on the graph this would be false.

There are, of course, cases in which p_c, p_u, or both are degenerate, i.e., take the value 1, or are identical to each other. The most remarkable result in this direction is that $p_c = p_u$ on $\mathbb{Z}^d$. This was first proved in [AKN], and later in a more elementary and also more general fashion in [BK]. The [BK] proof can actually be adapted to show the same result for all the amenable quasi-transitive graphs.

We will denote by $\mathbb{T}_b$ the homogeneous tree with degree $b+1$. The case $b = 1$ corresponds to the linear chain $\mathbb{Z}$, which has $p_c = p_u = 1$. When $b \geq 2$ we have from elementary branching process theory that $p_c = 1/b < 1 = p_u$.

The first example of a transitive graph for which it was shown that there is a non-degenerate interval of values of p on which $\mathbb{P}_p(N = \infty) = 1$ and another one on which $\mathbb{P}_p(N = 1) = 1$ – so that in the terminology above $p_c < p_u < 1$ – is the graph $\mathbb{T}_b \times \mathbb{Z}$, with b large enough. This was done in [GN] for bond percolation, but the same arguments apply also to site percolation. Note that the graph $\mathbb{T}_b \times \mathbb{Z}$ is not only transitive, but is also a Cayley graph. Further results of this nature appear in [BS1] and [Lal1].

It is very natural to ask what happens at the critical points p_c and p_u. A standard conjecture is that for all the quasi-transitive graphs for which $p_c < 1$, $P_{p_c}(N = 0) = 1$. This is known to be true for a class of planar graphs which includes bond and site percolation on $\mathbb{Z}^2$ (this was proved in one case in [Kes1] and generalized in [Rus]). And it is also known to hold for another class of graphs which includes bond and site percolation on $\mathbb{Z}^d$ with large d (this was proved in [HS] – in that paper the so called triangle condition was verified; a proof that this condition implies absence of percolation at p_c can be found in the union of the papers [AN] and [New] or in the paper [BA]). More recently it has been shown in [BLPS1] and [BLPS2] that for bond or site percolation on any unimodular quasi-transitive non-amenable graph this conjecture is also true.

Regarding now the behavior at p_u, [BS2] proved that for planar non-amenable transitive graphs $\mathbb{P}_{p_u}(N = 1) = 1$. Here we will prove that in contrast

Theorem 2. *For the graph $\mathbb{T}_b \times \mathbb{Z}$ with $b \geq 2$,*

$$\mathbb{P}_{p_u}(N = 1) = 0.$$

So if b is such that $p_c < p_u$, then

$$\mathbb{P}_{p_u}(N = \infty) = 1.$$

Therefore we learn that for transitive graphs (or even Cayley graphs) with $p_c < p_u < 1$ both behaviors are possible at p_u: we can have $\mathbb{P}_{p_u}(N = 1) = 1$ or $\mathbb{P}_{p_u}(N = \infty) = 1$. It would be interesting to know for which graphs each alternative holds. The technique in this paper can be generalized to a small extent, so that if H is a quasi-transitive graph, then in the statement of Theorem 2, we can have the graph $\mathbb{T}_b \times H$ in place of the graph $\mathbb{T}_b \times \mathbb{Z}$. The proof can also be adapted to bond percolation on the same graphs. The main point of this paper is, nevertheless, to simply provide an example which shows that a certain type of behavior is possible. For this reason, and since the generalizations would require more involved notation, we restrict ourselves to only proving Theorem 2 as it is stated above. Note that Question 5 in [BS1] is now answered by Theorem 1, the work of [BS2] and Theorem 2.

The condition that b be large for $p_c < p_u$ to happen can be improved. Using techniques from [Sta] and from the present paper one can show that $p_c < p_u$ for all $b \geq 2$, for bond or site percolation. This proof will be omitted from this paper since it is quite technical and moreover it is a very special result; it is conjectured that for all the non-amenable quasi-transitive graphs one should have $p_c < p_u$. Similarly, using also techniques from [Sch1], one can prove that the so called triangle condition holds below p_u when $b \geq 2$. This improves a result of [Wu1], and implies that several critical exponents (at p_c) take their mean-field value when $b \geq 2$. Again, the proof is omitted from this paper for the same reason explained above.

One of the ingredients of our proof of Theorem 2 will be a characterization of p_u in terms of connectivities between large balls. This result, from [Sch2], states the following.

Theorem 3. *For every quasi-transitive graph*

$$p_u = \inf\left\{p\colon \lim_{n\to\infty} \inf_{x,y\in V} \mathbb{P}_p(B(x,n) \longleftrightarrow B(y,n)) = 1\right\}. \tag{1.1}$$

Another ingredient of the proof will be the behavior of a function $\alpha(p)$, defined below, and which is simply an inverse correlation length. To define it, we need to present some standard arguments next.

We will think of $\mathbb{Z}$ as a subgraph of $\mathbb{T}_b$, so that it will make sense to talk about the site $n \in \mathbb{T}_b$, and let $n \to \infty$.

From the Harris' inequality we have $\mathbb{P}_p((0,0) \longleftrightarrow (m+n,0)) \geq$

$\mathbb{P}_p((0,0) \longleftrightarrow (m,0))\mathbb{P}_p((0,0) \longleftrightarrow (n,0))$, and therefore

$$\lim_{n\to\infty} (\mathbb{P}_p((0,0) \longleftrightarrow (n,0)))^{1/n} = \sup_{n\geq 1}\{(\mathbb{P}_p((0,0) \longleftrightarrow (n,0)))^{1/n}\} = \alpha(p). \tag{1.2}$$

Note that

$$\mathbb{P}_p((0,0) \longleftrightarrow (n,0)) \leq (\alpha(p))^n, \tag{1.3}$$

for $n \geq 0$. Use now (1.2) to write

$$\alpha(p) = \sup_{n\geq 1} \sup_{k\geq 1} \left(\mathbb{P}_p\left((0,0) \overset{B((0,0),k)}{\longleftrightarrow} (n,0)\right)\right)^{1/n},$$

and observe that here we are taking the supremum of a continuous function of p. Therefore $\alpha(\cdot)$ is lower-semi-continuous and since it is clearly non-decreasing, it is also left-continuous.

The technically most involved part of the proof of Theorem 2 is the proof of the following proposition, which is deferred to Section 2.

Proposition 1. *If $\alpha(p) > 1/\sqrt{b}$ then*

$$\lim_{n\to\infty} \inf_{x,y\in V} \mathbb{P}_p(B(x,n) \longleftrightarrow B(y,n)) = 1. \tag{1.4}$$

From this proposition and (1.1) we learn that for $p < p_u$ we must have $\alpha(p) \leq 1/\sqrt{b}$. By the left-continuity of $\alpha(\cdot)$ we obtain then

$$\alpha(p_u) \leq 1/\sqrt{b}.$$

But when $P_p(N=1) = 1$ we obtain from Harris' inequality and transitivity of the graph that, for all $x, y \in V$,

$$\mathbb{P}_p(x \longleftrightarrow y) \geq \mathbb{P}_p(x \longrightarrow \infty, y \longrightarrow \infty) \geq \mathbb{P}_p(x \longrightarrow \infty)\mathbb{P}_p(y \longrightarrow \infty) > 0,$$

uniformly in x, y, so that in particular $\alpha(p) = 1$. This concludes the proof of Theorem 2 based on Proposition 1.

The proof of Proposition 1 is the subject of Section 2; a heuristic reasoning that gives meaning to the value $1/\sqrt{b}$ in Proposition 1 will also be provided there. Section 3 will contain some remarks about the origin of the approach in this paper and its connection with the study of the contact process on homogeneous trees.

This section is ended with the following insight, which is brought up by Theorem 2. As mentioned before, the standard conjecture about the behavior of independent percolation on transitive graphs at p_c, when $0 < p_c < 1$, is that one should have $\mathbb{P}_{p_c}(N = 0) = 1$. This conjecture seems to be motivated mostly by the fact that there is now a number of cases in which it has been proved (see above) and there is no known counterexample. In contrast we know now that we can have $p_c < p_u < 1$ with each one of the possible behaviors at p_u: $\mathbb{P}_{p_u}(N = 1) = 1$ and $\mathbb{P}_{p_u}(N = \infty) = 1$. Couldn't it then also be the case that at p_c we do not always have the same behavior, but that for some transitive graphs we have $0 < p_c < 1$ and $\mathbb{P}_{p_c}(N \neq 0) = 1$?

2. Proof of Proposition 1

We will use the following notation for subsets of $\mathbb{Z}$. For $j \leq k$, $[j,k] = \{j, ..., k\}$; for $j + 1 \leq k$, $]j,k] = \{j + 1, ..., k\}$, $[j,k[= \{j, ..., k - 1\}$; for $j + 1 \leq k - 1$, $]j,k[= \{j + 1, ..., k - 1\}$.

Regarding some subgraphs of $\mathbb{T}_b$, we will use the following notation. The tree obtained by removing all the sites which can only reach the site 0 through a chain which contains the site -1 is denoted $\mathbb{T}_b^+$. Given a site $x \neq 0$, we denote by $\mathbb{T}_b(x)$ the tree obtained by removing the sites which can reach the site 0 through a chain which does not contain x. Note that each graph $\mathbb{T}_b(x)$ is isomorphic to $\mathbb{T}_b^+$, with x taking the role of 0.

Given a subgraph H of $\mathbb{T}_b$, the notation $S_H(n)$ will denote the set of sites which in $\mathbb{T}_b$ are at distance n from 0 and are also sites of H.

Similarly to the considerations related to the definition of $\alpha(p)$ we have also some further standard facts. For each k the following limit exists.

$$\lim_{n\to\infty} \left(\mathbb{P}\left((0,0) \overset{\mathbb{T}_b \times [-k,k]}{\longleftrightarrow} (n,0)\right)\right)^{1/n}$$
$$= \sup_{n\geq 1} \left\{\left(\mathbb{P}\left((0,0) \overset{\mathbb{T}_b \times [-k,k]}{\longleftrightarrow} (n,0)\right)\right)^{1/n}\right\} = \alpha_k(p),$$

and the bound

$$\mathbb{P}\left((0,0) \overset{\mathbb{T}_b \times [-k,k]}{\longleftrightarrow} (n,0)\right) \leq (\alpha_k(p))^n, \tag{2.1}$$

holds for all $n \geq 0$.

The following lemma is also standard.

Lemma 1.

$$\lim_{k\to\infty} \alpha_k(p) = \alpha(p).$$

Proof. It is clear that for all k, $\alpha_k(p) \leq \alpha(p)$ and that $\alpha_k(p)$ increases with k

to some limit bounded above by $\alpha(p)$. On the other hand, by (1.2) and (2.1), for any small $\epsilon > 0$ there is an n such that

$$(\alpha(p)-\epsilon))^n \le \mathbb{P}((0,0) \longleftrightarrow (n,0)) = \lim_{k\to\infty} \mathbb{P}\left((0,0) \overset{\mathbb{T}_b\times[-k,k]}{\longleftrightarrow} (n,0)\right)$$
$$\le \lim_{k\to\infty} (\alpha_k(p))^n = \left(\lim_{k\to\infty} \alpha_k(p)\right)^n.$$

Therefore $\lim_{k\to\infty} \alpha_k(p) \ge \alpha(p) - \epsilon$. Since ϵ is arbitrary, we have $\lim_{k\to\infty} \alpha_k(p) \ge \alpha(p)$, finishing the proof. ■

For $x \in \mathbb{T}_b$, let now E_x be the event that all the sites in $\{x\} \times [0,s[$ are occupied, and define

$$Y_{n,s} = \mathbb{P}\left(E_0 \cap E_n \cap \left\{(0,0) \overset{\mathbb{T}_b^+\times[0,s[}{\longleftrightarrow} (n,s-1)\right\}\right)$$

Lemma 2. *There exists a sequence $(s(n))_{n\ge1}$, $s(n) \ge 1$, such that*

$$\liminf_{n\to\infty} \left(Y_{n,s(n)}\right)^{1/n} \ge \alpha(p).$$

Proof. By Harris' inequality, when s is even,

$$Y_{n,s} \ge p^{2s}\, \mathbb{P}\left((0,s/2) \overset{\mathbb{T}_b\times[0,s[}{\longleftrightarrow} (n,s/2)\right).$$

So that the lemma follows at once from Lemma 1 and the homogeneity of the graph $\mathbb{T}_b \times \mathbb{Z}$. ■

The statement and the proof of the next lemma will be easier to understand with the following picture in mind. Suppose that for some $C_1, C_2 > 0$ the probability of connecting $(0,0)$ to $(C_1 i, C_2 i)$ is of order $\ge \alpha^{C_1 i}$, when i is large. Then the probability of connecting $(0,0)$ to $(0, 2C_2 i)$ by a path which goes through $(C_1 i, C_2 i)$ (or any other fixed site in $S_{\mathbb{T}_b}(C_1 i) \times \{C_2 i\}$) is of order $\ge \alpha^{2C_1 i}$. There are $\sim b^{C_1 i}$ vertices in $\mathbb{T}_b$ at distance $C_1 i$ from 0. So, summing over the sites of $S_{\mathbb{T}_b}(C_1 i) \times \{C_2 i\}$, we get an heuristic lower bound $(b\alpha^2)^{C_1 i}$ for the order of magnitude of the probability of $\{(0,0) \longleftrightarrow (0, 2C_2 i)\}$. Of course, this reasoning only makes sense when $b\alpha^2 << 1$, but it also indicates what should happen when α approaches $1/\sqrt{b}$ and becomes larger than this value. Technically the structure of $\mathbb{T}_b$ allows us to find enough "independence" to show that the heuristic prediction, contained in the next lemma, is correct. (In the proof of this lemma, the role of C_1 will be played by the quantity nl and the role of C_2 will be played by the quantity sl.)

Lemma 3. *If $\alpha(p) > 1/\sqrt{b}$ then*

$$\inf_{z\in\mathbb{Z}} \mathbb{P}\left((0,0) \stackrel{\mathbb{T}_b^+\times\mathbb{Z}}{\longleftrightarrow} (0,z)\right) > 0. \tag{2.2}$$

Proof. Suppose that $\alpha(p) > 1/\sqrt{b}$. Thanks to Lemma 2, we can take n and $s \geq 1$ such that

$$(Y_{n,s})^{1/n} = a > 1/\sqrt{b}. \tag{2.3}$$

We will show that if, for a proper choice of a positive integer l, we set

$$r_i = \mathbb{P}\left((0,0) \stackrel{\mathbb{T}_b^+\times[0,2ils]}{\longleftrightarrow} (0,2ils)\right), \qquad i = 0,1,2,...$$

then

$$\inf_{i=0,1,2,...} r_i > 0. \tag{2.4}$$

This clearly suffices for our purposes, since then, by Harris' inequality and symmetry,

$$\inf_{z\in\mathbb{Z}} \mathbb{P}\left((0,0) \stackrel{\mathbb{T}_b^+\times\mathbb{Z}}{\longleftrightarrow} (0,z)\right) \geq p^{2ls} \inf_{i=0,1,2,...} r_i > 0.$$

For $j = 1,2,...$ set

$$\bar{Z}_j = \left|\left\{x \in S_{\mathbb{T}_b^+}(jn) : (0,0) \stackrel{\mathbb{T}_b^+\times[0,js[}{\longleftrightarrow} (x, js-1)\right\}\right|.$$

Clearly $\bar{Z}_j \geq Z_j$, where $(Z_j)_{j=0,1,2,...}$ is a branching process with offspring distribution identical to the distribution of $\bar{Z}_1$. This distribution is concentrated on $\{0,1,...,b^n\}$ and has mean $\mu_n \geq b^n Y_{n,s} = (ba)^n$. Since the offspring distribution has a finite support and in particular a finite second moment, it follows from standard branching-process theory (see, e.g., example 4.3 in Sect. 4.4, p. 254 of [Dur2]) that for some random variable X with mean $\mathbb{E}(X) = 1$

$$\frac{Z_j}{(\mu_n)^j} \to X \qquad \text{a.s.} \qquad \text{as } j \to \infty.$$

In particular, there is $\epsilon > 0$ such that

$$\mathbb{P}\left(\bar{Z}_l \geq \frac{(ba)^{nl}}{2}\right) \geq \mathbb{P}\left(Z_l \geq \frac{(ba)^{nl}}{2}\right) \geq \mathbb{P}\left(Z_l \geq \frac{(\mu_n)^l}{2}\right) \geq \epsilon, \tag{2.5}$$

for all large enough l.

Choose now l large enough for (2.5) to hold, and also so that

$$\left(1 - \frac{\epsilon}{2} a^{nl}\right)^{\frac{(ba)^{nl}}{2}} \leq \frac{1}{2}. \tag{2.6}$$

This last requirement can be fulfilled because

$$\limsup_{l\to\infty} \left(1 - \frac{\epsilon}{2} a^{nl}\right)^{\frac{(ba)^{nl}}{2}} \leq \limsup_{l\to\infty} \ \exp\left(-\frac{\epsilon}{4}(ba^2)^{nl}\right) = 0,$$

since $ba^2 > 1$ by (2.3).

Next we will show inductively in i that

$$r_i \geq \frac{\epsilon}{2} a^{nl}, \tag{2.7}$$

verifying therefore the validity of (2.4).

For $i = 0$ inequality (2.7) is clearly true (since $r_0 = 1$ and $\epsilon, a \leq 1$), and we will show now that if it is true for i it is true for $i+1$. From the spherical symmetry of $\mathbb{T}_b$ we have

$$\begin{aligned} r_{i+1} \geq \mathbb{P}&\left((0,0) \overset{\mathbb{T}_b^+ \times [0,(2i+1)ls]}{\longleftrightarrow} S_{\mathbb{T}_b^+}(nl) \times \{(2i+1)ls\}\right) \\ &\cdot \mathbb{P}\left((nl,(2i+1)ls+1) \overset{\mathbb{T}_b^+ \times [(2i+1)ls+1, 2(i+1)ls]}{\longleftrightarrow} (0, 2(i+1)ls)\right). \end{aligned} \tag{2.8}$$

We will now look separately at each one of the factors in the right hand side of (2.8). First, noting that the graphs $\mathbb{T}_b(x) \times [ls,(2i+1)ls]$, $x \in S_{\mathbb{T}_b^+}(nl)$ are disjoint and isomorphic to $\mathbb{T}_b^+ \times [0, 2ils]$, we can use independence to write

$$\begin{aligned} \mathbb{P}&\left((0,0) \overset{\mathbb{T}_b^+ \times [0,(2i+1)ls]}{\longleftrightarrow} S_{\mathbb{T}_b^+}(nl) \times \{(2i+1)ls\}\right) \\ &\geq \mathbb{P}\left(\bar{Z}_l \geq \frac{(ba)^{nl}}{2}\right) \cdot \left(1 - (1 - r_i)^{\frac{(ba)^{nl}}{2}}\right). \end{aligned}$$

Using also symmetries of the graph $\mathbb{T}_b \times \mathbb{Z}$ and independence on disjoint subsets, we have for the second factor in the right hand side of (2.8),

$$\mathbb{P}\left((nl,(2i+1)ls+1) \overset{\mathbb{T}_b^+ \times [(2i+1)ls+1,2(i+1)ls]}{\longleftrightarrow} (0,2(i+1)ls)\right)$$
$$= \mathbb{P}\left((nl,0) \overset{\mathbb{T}_b^+ \times [0,ls[}{\longleftrightarrow} (0,ls-1)\right) \geq (Y_{n,s})^l.$$

Combining the last three displays and using (2.5), the induction hypothesis (2.7), (2.6) and (2.3) we obtain

$$r_{i+1} \geq \mathbb{P}\left(\bar{Z}_l \geq \frac{(ba)^{nl}}{2}\right) \cdot \left(1-(1-r_i)^{\frac{(ba)^{nl}}{2}}\right) \cdot (Y_{n,s})^l \geq \frac{\epsilon}{2} a^{nl}.$$

This completes the proof of (2.7) and of Lemma 3. ■

Proof of Proposition 1. The infimum in (1.4) can be taken over $x=(0,0)$, $y=(z,t)$, with $z \in \mathbb{T}_b^+$. We suppose below that x and y are of this form.

Note that Lemma 3 implies

$$\mathbb{P}\left((0,0) \overset{\mathbb{T}_b^+ \times \mathbb{Z}}{\longleftrightarrow} (0,j) \text{ for infinitely many } j \geq 0\right) > 0. \tag{2.9}$$

Let $A_1(n)$ be the event that some $v \in S_{\mathbb{T}_b \setminus \mathbb{T}_b^+}(n)$ is connected inside of $\mathbb{T}_b(v) \times \mathbb{Z}$ to infinitely many sites in $\{v\} \times \mathbb{Z}_+$. Noting that the b^{n-1} graphs $\mathbb{T}_b(v) \times \mathbb{Z}$, $v \in S_{\mathbb{T}_b \setminus \mathbb{T}_b^+}(n)$ are disjoint and isomorphic to $\mathbb{T}_b^+ \times \mathbb{Z}$ we obtain from (2.9)

$$\lim_{n\to\infty} \mathbb{P}(A_1(n)) = 1. \tag{2.10}$$

In order to define the next event of interest, we let $|z|$ be the distance in the graph $\mathbb{T}_b$ between the root and the site z, and we set

$$\mathbb{T}_b(z,n) = \bigcup_{u \in S_{\mathbb{T}_b(z)}(|z|+n)} \mathbb{T}_b(u),$$

Note that this is a disjoint union of copies of $\mathbb{T}_b^+$. Now let $A_2(n,y)$ be the event that some $v \in S_{\mathbb{T}_b \setminus \mathbb{T}_b^+}(n)$ is connected inside of $(\mathbb{T}_b(z,n))^c \times \mathbb{Z}$ to all sites in $\partial \mathbb{T}_b(z,n) \times \{j\}$ for some $j \in \mathbb{Z}_+$. It is clear that for all n,

$$\mathbb{P}(A_2(n,y) \mid A_1(n)) = 1. \tag{2.11}$$

Since the graph $\mathbb{T}_b(z,n) \times \mathbb{Z}$ is a disjoint union of b^n copies of $\mathbb{T}_b^+ \times \mathbb{Z}$, we obtain using Lemma 3 again and independence on disjoint subsets of the graph $\mathbb{T}_b \times \mathbb{Z}$ that

$$\lim_{n\to\infty} \mathbb{P}(B(x,n) \longleftrightarrow B(y,n) \mid A_1(n) \cap A_2(n,y)) = 1, \tag{2.12}$$

uniformly in $y = (z,t)$.

Combining (2.10), (2.11) and (2.12), we obtain (1.4). ■

3. Relations with work on the contact process

In this section we will explain how the work in the present paper is related to work on the contact process. The contact process is one of the fundamental examples of an interacting particle system and can be described as follows. At each time $t \geq 0$ each site of a graph $G = (V, E)$ can be healthy (state 0) or infected by a disease (state 1). Infected sites recover at rate 1, independently of anything else, while each healthy site is infected at a rate which is the product of a constant $\lambda > 0$ by the number of its neighboring sites which are infected. The constant λ is called the infection parameter and plays a role similar to that of p in percolation. In what follows we will generally suppose that the reader is familiar with this model, otherwise we refer him/her to [Lig1] or [Dur1] for introductions to it.

That close relations exist between percolation and the contact process is not a surprise, since the graphical construction of the contact process shows that this model is identical to a percolation process which is partially oriented and continuous. This percolation process takes place on $V \times \mathbb{R}_+$, which can be seen as the space-time setting for the model. Most of the relations between work on percolation and work on the contact process have been in the direction of percolation results and techniques leading to developments in our understanding of the contact process. For instance [DG] introduced a rescaling scheme for the one-dimensional contact process and obtained a series of exponential estimates for this model inspired by the rescaling scheme for percolation introduced in [Rus]. One of the most fundamental papers ever written about the contact process, the paper [BG], built on a dynamic rescaling technique devised for percolation in [BGN]. What is particularly interesting about the percolation–contact-process relation in the present paper is that here the relation is inverted, with contact process techniques feeding back to percolation theory.

The relatively long digression provided below seems to be worthwhile for two reasons. For readers who are interested in following the developments in percolation and contact process theory it will explain and emphasize some important connections. Readers who study the papers referred to below will certainly find the technical work in Section 2 above less mysterious. The other reason is to advertise once more this important point: the study of statistical mechanics models, percolation and interacting particle systems are interrelated in many ways and progress in each one of them may lead to progress in the others, sometimes in unexpected ways.

The paper [GN] seems to have been one of the sources of motivation for the paper [Pem], where the study of the contact process on homogeneous trees was started. The reason for this connection should be clear from the nature of the space-time setting for this contact processes. While [GN] showed that percolation on $\mathbb{T}_b \times \mathbb{Z}$ (with b large) has three phases, with $N = 0, \infty, 1$, respectively, [Pem] showed that the contact process on $\mathbb{T}_b$ (with b large) has three phases also. In the first one the process (actually the infection) dies out, i.e., if started from a finite set of infected sites, eventually the infection will disappear. In the second phase there is survival (of the infection) in the global sense, meaning that the system does not die out, but there is no local survival (of the infection) as defined below. In the third phase there is local survival (of the infection), meaning that if the process is started from a finite set of infected sites, then with positive probability every site will be infected at arbitrarily large times. The values of λ at the boundary between the first regime and the second is denoted λ_1, and the one between the second and third regimes is denoted λ_2.

It is worth pointing out that while percolation on $\mathbb{T}_b$ is basically a trivial model, since it amounts to a simple branching process, other processes like the Ising model on $\mathbb{T}_b$ have also an interesting theory (see Chapter 12 of [Geo]).

[Pem] was followed by a large number of papers dedicated to the study of the contact process on $\mathbb{T}_b$ and some related processes. In chronological order we have: [MS], [MSZ], [DS], [Wu2], [Zha], [Lig2], [Sta], [Lig3], [LS], [SaS2], [Lal2], and [Sch1]. We will explain now how the work in some of these papers led to the work presented here. [Zha] proved the following two results: (a) for $\lambda > \lambda_2$ complete convergence holds (roughly speaking, if the process is started from any set of infected sites and it survives, then we have convergence in distribution to a unique non-trivial invariant probability distribution); (b) at $\lambda = \lambda_2$ the process does not survive locally. [Lig3] introduced an analogue of the quantity $\alpha(p)$ in the current paper (let's call it $\alpha_{\mathrm{cp}}(\lambda)$); he obtained several of its properties and conjectured that for $\lambda \leq \lambda_2$, $\alpha_{\mathrm{cp}}(\lambda) \leq 1/\sqrt{b}$. This conjecture was proved by [LS], who also stressed, as pointed out by Liggett, that this provides a simplified proof of the result (b) of [Zha]. In parallel to these developments, and motivated by the work on the contact process on $\mathbb{T}_b$, [SaS1] and [SaS3] studied the contact process on fairly arbitrary graphs. It was shown in the first of these papers that for the contact process on a quasi-transitive graph the threshold for complete convergence (with survival) can be characterized in a way which is analogous to the characterization of p_u in (1.1) (see Theorem 2 parts (b) and (i) in [SaS1]). A combination of this result from [SaS1] with ideas from [Zha] and [LS] led to the paper [SaS2], in which a simplified proof of the result (a) of [Zha] was presented. The relation (1.1) used here was derived in [Sch2], motivated also by the result of [SaS1] quoted above. Finally here we are building on the techniques of [SaS2], when we prove (1.4) and use (1.1) in order to prove Theorem 2. A guiding principle which helped us follow the path just described was the heuristic idea that complete convergence is for the contact process what uniqueness of the infinite cluster is for percolation.

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