CLUSTER EXPANSIONS AND PIROGOV SINAI THEORY FOR LONG RANGE SPIN SYSTEMS

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Abstract. We investigate the low temperature phases of lattice spin systems with interactions of Kac type, that is interactions that are weak but long range in such a way that the total interaction of one spin with all the others is of order unity. In particular we develop a systematic approach to convergent low temperature expansions in situations where interactions are weak but long range. This leads to a reformulation of the model in in terms of a generalized abstract Pirogov–Sinai model, that is a representation in terms of contours interacting through cluster fields. The main point of our approach is that all quantities in the contour representation satisfy estimates that are uniform in the range of the interaction and depend only on the overall interaction strength. The extension of the Pirogov–Sinai theory to such models developed in [Z3] allows then the investigation of the low-temperature phase diagram of these models.

1. Introduction

The theory of Pirogov and Sinai, introduced in their seminal paper in 1976 [PS], has become through the years the standard tool for the investigation of the low-temperature phases of classical spin systems. We mention the standard references [Z1,DZ,BKL1,BKL2,BS] and refer in particular to the lecture notes by one of us [Z2] for a good introduction. This theory covers a broad range of situations, including continuous spins [DZ]. However, in the spirits of its time, the theory has been developed primarily in view of *short range* interactions, with a focus of a predominance of the interactions between nearest neighbors. This does not mean that the existing theory is restricted to finite range models; it can easily accommodate infinite range but quickly decaying interactions. However, it always assumes that interactions between *close* neighbors is *strong*.

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On the other hand over the last years there has been a growing interest in a class of models introduced long ago by M. Kac et al. [KUH]. These models serve as interpolations between *short range* and *mean-field* models. Here spins interact via so called *Kac-potentials*, i.e. spins at distance r interact with a strength $J_\gamma(r) \equiv \gamma^d J(\gamma r)$, where $J(x)$ is some function of bounded support or rapid decay on a scale 1, and γ is a scaling parameter that is allowed to tend to zero. In the limit $\gamma \downarrow 0$ a number of simplifications occur, and in particular it becomes possible to compute the exact form of thermodynamic potentials which are seen to be related to those of mean field theory. More recently there has been an increasing interest in these models in the context of phase-coexistence [CMP,CP], interface dynamics [DOPT], mesoscopic profiles [COP], but also in disordered versions of Kac models [BGP1,BGP2,BGP3,Bo1,B1]. A basic ingredient in the understanding of these models, in particular in more than one dimension, is clearly the nature of the low temperature phase diagram.

It is natural to expect that the critical temperature in these models converges to that of the corresponding mean-field theories. However, the nature of the Kac-potentials is such that the basic assumptions of the Pirogov-Sinai theory in its standard form are not satisfied, since the interaction between any two spins will tend to zero as γ tends to zero! Thus all standard results do not apply in this model, and only very recently a version of the Peierls argument was developed that allows to give reasonable estimates on the critical temperature in the case of the Kac-Ising model [BZ,CP]. However, in these papers, as in the original work of Peierls [P], the exact spin flip symmetry of the Ising model was used in a crucial way.

To go beyond the symmetric situation and to develop the full Pirogov-Sinai theory, the crucial tool that has to be developed are convergent (low-temperature) cluster expansions. The fundamental importance of the concept of expansions in the Pirogov-Sinai theory is stressed in particular in the latest version of the theory that can be found in [HZ,Z2].

The main thrust of the present paper is therefore the development of such expansions. In this first paper we will not focus on more specific features of Kac models, like the possibility to prove the existence of first order transition up to the critical temperature (of a corresponding mean-field model, if $\gamma \to 0$). In particular, unlike in [BZ1], we will not use any block-spin techniques. Rather, our purpose here is to treat a rather broad class of predominantly "ferromagnetic" models and to develop convergent expansions for a range of temperatures that is *independent of the range of the interactions*, but depends only on the "total strength" of the dominant part of the attractive interaction (in a sense that will have to be made precise later). In our view, this will provide a very natural and desirable extension of the Pirogov-Sinai theory to a reasonable set of interactions and provides natural criteria for when to expect a phase transition. It should be observed that as far as the high-temperature phases are concerned, the class of interactions for which uniqueness properties can be proven is very general and qualitatively optimal. For low temperatures, so far, the existing results concern very restricted classes of interactions.

As a first exploration into this direction, however, we did not strive for full generality. Indeed, it will require more insight and reflection to formulate a general class of interactions for which our program should succeed. Thus we have restricted our attention to pair-interactions and finite state spaces. Some natural extensions could be accommodated without too much effort, but we feel that it is better to leave this for future publications while explaining our ideas in the simplest context that reveals the main features which we address. The same applies to the issue of pushing the estimates for the critical temperatures to the optimal values in Kac limits.

Let us mention the related results by Lebowitz, Mazel, and Presutti [LMP] that treat a particular Kac-type model in continuous space. While in principle this paper deals with rather similar problems, there are substantial differences that will not allow the direct application of their methods in our situation. It will be interesting to compare both approaches, also in other applications.

The remainder of this paper is organized as follows. In Section 2 we give a precise definition of the class of models we are studying and we formulate the main results. In Section 3 we explain how to define the crucial notion of *contours* and *regular regions*. Here we will encounter the main difference from standard Pirogov-Sinai theory: the configurations in regular regions will *not* simply look like ground states, but they will be characterized only by some carefully chosen nonlocal condition. The next two chapters will show that this definition was reasonable: In Section 4 we prove that contours satisfy the Peierls estimate, while in Section 5 we show how to perform high-temperature expansions in the "regular regions", i.e. outside of the contours. This requires to develop expansion techniques for spin ensembles satisfying non-local constraints. This will require a two-stage procedure: First we perform a high-temperature expansion and,*using* the constraints, partially resum it, mapping 'graphs' to 'trees'. Then we expand the contraints, which now have become unnecessary for ensuring convergence. In this way we arrive at a representation of our model in the form of an "abstract Pirogov Sinai model with additional cluster field". While this form of the abstract model differs slightly from the standard form of abstract Pirogov–Sinai models, it is not difficult to apply the machinery of Pirogov–Sinai theory to this setting and to obtain all the standard results of this theory. This is done in a separate paper [Z3].

2. THE MODELS

We now define the class of models we will treat. Let S be a finite set. The configuration space of our model will be $X \equiv S^{\mathbb{Z}^d}$. We will equip S with the discrete topology and the counting measure ρ. Correspondingly, X will be equipped with the product topology and the product measure $\rho^{\mathbb{Z}^d}$. For any finite subset $\Lambda \subset \mathbb{Z}^d$ we will denote write $X_\Lambda \equiv S^\Lambda$.

To define the interacting model, we introduce now a one-body potential $U : S \to \mathbb{R}$ and a two body interaction Φ, i.e. for any $i, j \in \mathbb{Z}^d$ we define a symmetric function $\Phi_{i,j} : S \times S \to \mathbb{R}$ with the following properties:

(i) For all $i, j, k \in \mathbb{Z}^d$, $\Phi_{i,j} = \Phi_{j,i} = \Phi_{i+k,j+k}$.

(ii) For any $i, j \in \mathbb{Z}^d$, and any $s, s' \in S$, $\Phi_{i,j}(s, s') = \Phi_{i,j}(s', s)$ and $\Phi_{i,i}(s, s) \equiv 0$.

The Hamiltonian of our model is for any finite volume $M \subset \mathbb{Z}^\nu$ and any boundary condition x_{M^c} given as follows (where we denote by $x = x_M \cup x_{M^c}$)

$$H_M(x) \equiv H_M(x_M|x_{M^c}) = \sum_{\{i,j\}\cap M\neq\emptyset} \Phi_{i,j}\{x_i, x_j\} + \sum_{i\in M} U(x_i). \tag{2.1}$$

Below, we will systematically consider the *periodic boundary conditions* i.e. the case when we have a finite, d-dimensional torus Λ. In that case the summation in the above equation is over all $\{i,j\} \subset \Lambda$ and we will write

$$H_\Lambda(x_\Lambda) = \sum_{\{i,j\}\subset\Lambda} \Phi_{i,j}\{x_i, x_j\} + \sum_{i\in\Lambda} U(x_i). \tag{2.2}$$

We will formulate specific assumptions on the interaction that will allow us to control the model via cluster expansion methods developed below.

Assumption 0 *(positivity of* Φ*)*: We assume that $\Phi_{i,j}\{x_i, x_j\} \geq 0$.

Note. Actually, this requirement can be greatly relaxed. We make this assumption only to get a simpler proof for the Peierls condition for the contours (defined below).The positivity of the interaction (together with some continuity - see below) assures the validity of such a condition, but it is of course not necessary. We keep the condition of positivity throughout this paper for simplicity. A more systematic investigation of the validity of the Peierls condition for long range, Kac type, models deserves a separate paper.

Our first assumption states that the interaction has finite range[1] R (where R *may* be a very large number).

Assumption 1 *(finite range R anf finite variance)*: Denote by

$$\Phi_i \equiv \max_{s,s'\in S\times S} |\Phi_{0,i}\{s, s'\}| \qquad \text{and} \qquad \Psi_i \equiv \min_{s,s'\in S\times S} |\Phi_{0,i}\{s, s'\}|\,.$$

We assume that there exists $0 < R < \infty$ such that $\Phi_i = 0$, if $|i| > R$. Moreover, we assume that there exists a finite positive constant $0 < D \leq 1$ such that for all $|i| \leq R$,

$$(\Phi_i \ \geq) \ \ \Psi_i \ \geq \ D\Phi_i. \tag{2.3}$$

Our second assumption assures a sufficiently attractive interaction: Again, we do not strive here for a maximal generality. We select now a subset $Q \subset \mathcal{S}$ containing all the "approximate minima" of U. Often (e.g. in our main example we have in mind here i.e. in the Kac Ising model with a rather small external field), this set of "reference colours" will be taken simply as $Q \equiv \mathcal{S}$).

Assumption 2 *(strong attraction)*: For $q \in Q$ and $s \in \mathcal{S}, s \neq q$ we define by

$$V_q(s) \ = \ U(s) - U(q) \ + \ \sum_{i\in\Lambda} \Phi_{0,i}\{s, q\} \quad \text{and} \quad V_q = \min_{s\in\mathcal{S}\setminus\{q\}} V_q(s). \tag{2.4}$$

We assume that there exists $\tau > 0$ such that for *each* $q \in Q$

$$\min_{q\in Q} V_q \equiv V \geq \tau. \tag{2.5}$$

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[1] Again this assumption is not strictly neessary, and as in the usual short-range situations, one may admitt an additional weak interaction that is rapidly decaying.

We will choose Q in such a way that the one-body potential varies not too much over Q compared to the strength of the interaction[2],

$$Q \equiv \{q \in \mathcal{S} : U(q) \leq \min_{s\in\mathcal{S}} U(s) + \tau/4\}. \tag{2.6}$$

The third assumption expresses some smoothness of the interaction:

Assumption 3 *(continuity)*: For $x \in X$ set (compare with the quantity $V_q(x_i)$ for $U \equiv 0$)

$$V_x(i,q) = \sum_{j\neq i} \Phi_{i,j}\{q, x_j\}. \tag{2.7}$$

We assume that there exists a constant $C > 0$, such that for any vector $k \in \Lambda$, any $q \in Q$, and any $x \in X$,

$$|V_x(i+k,q) - V_x(i,q)| \leq \frac{C\tau\ |k|}{R}. \tag{2.8}$$

It is both natural and important that the continuity assumption is related to the fact that the individual interactions have to be rather small. In fact we have

LEMMA 2.1.*If Φ satisfies the assumptions above, then*

$$\max_{q,q'\in\mathcal{S}} \max_{j\in\mathbb{Z}^d} |\Phi_{i,j}\{q,q'\} \leq C/R \tag{2.9}$$

Proof. Note that by translation invariance, for any $i, k \in \mathbb{Z}^d$,

$$\begin{aligned} V_x(i,q) - V_x(i+k,q) &= \sum_{j\neq i} \Phi_{i,j}\{q,x_j\} - \sum_{j\neq i+k} \Phi_{i+k,j}\{q,x_j\} \\ &= \sum_{j\neq i} \left(\Phi_{i,j}\{q,x_j\} - \Phi_{i,j}\{q,x_{j+k}\}\right) \end{aligned} \tag{2.10}$$

Now choose the configuration x^ℓ to be

$$x_j^\ell = \begin{cases} q', & \text{if } j = \ell - nk, \text{ for some } n \in \mathbb{N}_0 \\ q, & \text{else} \end{cases} \tag{2.11}$$

Then by (2.10),

$$\begin{aligned} V_x(i,q) - V_x(i+k,q) &= \sum_{n=0}^{\infty} \left(\Phi_{i,\ell-nk}\{q,q'\} - \Phi_{i,\ell-(n+1)k}\{q,q'\}\right) \\ &= \Phi_{i,\ell}\{q,q'\} \end{aligned} \tag{2.12}$$

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[2] The precise choice is of course to a large extent arbitrary

where we used that the last sum is telescopic and that $(\Phi_{i,\ell-nk}\{q,q'\}$ tends to zero as n tends to infinity. Choosing k to be a unit vector, (2.12) together with the upper bound on the left-hand side given by (2.8) yields the statement of the lemma. $\diamondsuit$

Our aim is to develop the Pirogov-Sinai theory in this setting under suitable requirements on the values of the three constants D, τ, C, and the temperature T. Following the traditions of one of us (M.Z.), we will, however, always set

$$T = 1 \tag{2.13}$$

and thus incorporate the temperature in the interactions. We could just as well set $\tau = 1$ in Assumption 2 and express our conditions in terms of large enough $\beta \equiv 1/T$ only.

Note. Assumption 2 assures that the constant configurations with value $q \in Q$ give a strong energetic penalty against changing of a single spin. However, this energy is not enough to compensate the entropic loss occuring when fixing all spins in an R-neighborhood of one site, if R is large (and the temperature not too small). Thus the usual definition of Peierls contours (calling any spin part of a contour if at least one of its R-neighbors do not agree with him) would not allow us to obtain an R-independent constant in the Peierls condition. In other words, it is not reasonable to think of performing a low-temperature expansion around a single "ground state" configuration[3], but rather about some more relaxed, nontrivial measures that are in some specific way concentrated in some neighborhoods of these constant configurations. To make this precise will be somewhat technical, and requires the Assumption 3.

Basic examples.

1) In the classical Kac-Ising model, where $S = Q = \{-1, 1\}$, $\Phi_{i,j}(x,y) \equiv \frac{\beta}{2} J_\gamma(i-j)(x-y)^2$ and $J_\gamma(k) = \gamma^d J(\gamma k)$ with J e.g. being the indicator function of the unit cube, the assumptions above are satisfied with $R = \gamma^{-1}$, $D = 1$, $\tau = \beta/2$, $C = 2\beta d$.

2) Analogously, in the ferromagnetic Blume Capel model with positive interactions $\Phi_{i,j}\{x_i, x_j\}$, $|i-j| \leq R$ of the above Kac type and with potentials $U(s), s \in \mathcal{S} = \{-1, 0, 1\}$ the above assumptions are also satisfied at small enough temperatures, if we choose suitably the subset $Q \subset \mathcal{S}$: If all $U(s)$ are roughly the same and if all interactions $\Phi_{i,j}\{x_i, x_j\}$ are roughly of the same order (for any $x_i \neq x_j$) then it is natural to choose $Q = \mathcal{S}$.

3) However, interpreting sites occupied by 0 as "empty" and those occupied by ±1 as particles having a strong fugacity $(U(1) \doteq U(-1) \doteq 0); U(0) > U(\pm 1))$, if we have a strong Kac repulsion resp. attraction between the particles of different resp. same type and no interaction between the particles and the empty sites, then the only appropriate choice of the set of "reference colors" may be $Q = \{1, -1\}$ if $U(0) \gg U(\pm 1)$.

We now announce the central result on the low temperature phases for the class of models introduced above.

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[3] Except for *very* large τ, corresponding to very small temperatures of order R^{-d}.

THEOREM 2.2. *Take a class of model satisfying the assumptions* $0--3$ *above. Then there exists a constant* $\tau_c < \infty$*, depending on the constants* C *and* D *from Assumptions 1 and 3, but not on* R*, such that the following is true. There exist Lipshitz continuous functions (of* β*, resp.* Φ*, and* $U(q')$*)* h_q*,* $q \in Q$*, such that*

- *(i) If for some* $q^* \in Q$*,* $h_{q^*} > \max_{q \in Q \backslash q^*} h_q$*, then there exists a unique infinite volume Gibbs state* μ^{q^*}*.*
- *(ii) If there is a subset* $Q^* \subset Q$ *such that for all* $p, p' \in Q^*$*,* $h_p = h_{p'}$*, and for all other* $q \in Q \backslash Q^*$*,* $h_p > h_q$*, then there exist exactly one distinct extremal translation invariant Gibbs state for each value* $p \in Q^*$*.*
- *(iii) There is a finite positive constant* c *such that* $|h_q - U(q)| \leq ce^{-\tau/c}$*. The functions* h_q *are given in terms of rapidly converging series.*

Remark. Theorem 2.2 is the key statement of the Pirogov-Sinai theory. It implies in particular that in the $|Q|$-dimensional space of the parameters $U(q)$, the sets where the Gibbs measure is unique are open sets, and the sets where k Gibbs measures co-exists are closed, Lipshitz contnuous submanifolds of co-dimension $k-1$. Furthermore, all expectations of local functions with respect to the pure states mentionened in the theorem can be computed in terms of convergent expansions. Note on the other hand that when more than one Gibbs state co-exists, it is in general possible that there exist further, non-translation invariant extremal state.

Note. The statements of the theorem are the standard consequences of the Pirogov-Sinai theory once a model can be formulated in terms of what it called a "abstract Pirogov Sinai" model. In this paper we will show that this can be done in the sense of a slightly generalized sense compared to the standard formulation, namely where additional, quickly decaying, "cluster fields" produce a weak interaction between contours. The reformulation of Pirogov–Sinai theory in that class is done in a companion paper [Z3]. The proof of the uniqueness result (i) will be given in [BZ3].

The aim of this paper is to prepare the ground for the application of methods of [Z3], i.e. to reformulate our model such that its partition functions are expressed in a way to allow to apply the results of [Z3]. Emphasis is put on showing that all estimates in the resulting abstrat model are *uniform* in the range of the interaction, R.

Moreover, we devote a great care to the construction of cluster expansions. These expansions are absolutely crucial and give in fact much more information than just the condensed statement of the theorem. Rather, they allow, in principle and even in practice, to compute all physical quantities in terms of rapidly converging series. We emphasize this point also in view of applications to disordered models where renormalization group methode will rely heavily on the availability of convergent expansions.

3. q–POINTS, CONTOURS AND STARS. REFORMULATION OF THE HAMILTONIAN.

This section introduces the crucial notions of q-correct points and contours. At the same time we will reformulate the Hamiltonian in a way that is adapted to these new notions and that will be *used systematically* below.

3.1. The cleaning of a configuration.

The basic notion of the Pirogov-Sinai theory is that of a contour. Countours (of a configuration) are connected regions in space where the configuaration has locally a "high" energy density so that one would expect that at low temperatures the appearence of large contours should be "unlikely" with repect to the Gibbs measure. The rest of Λ is then occupied, for any given configuration, restricted ensembles of almost q configurations. The the energy of a configuration is expressed as a sum of the energies of contours (the "energy barrier" terms) and energies of the (almost constant) configurations around them.

The aim is to show that under suitable conditions the Gibbs measures is concentrated on configurations that are "mostly constant" and moreover such that the contours form only small and rare islands in such a – predominantly constant – configuration.

In the standard Pirogov Sinai theory developed for short range models, contours are connected components of the set of "incorrect" points. A point i is called q correct, for some $q \in Q$, if the configuration has value q anywhere at the distance $\leq R$ from i, and often the choice of $Q \subset S$ is made corresponding only to the minima of U.

The q–like configurations on the whole Λ are thus simply constant (or in a suitably more general setting periodic) configurations that correspond to the local grounds states of the Hamiltonian. Regions where the configurations are everywhere non-constant (when looking up to a distance R) are then included into *contours*. The important point is that q-like regions with different values of q must be separated by contours, so that if one can show that large contours are indead higly unlikely, one can establish the desired result that Gibbs measure are concentrated on essentilly constant configurations. A sufficient condition for this to hold is a so-called Peirls estimate that relates the energy of a contour to the geometric size of the volume of its support. In short range model like the Ising models such an estimate is readily proven with a proportinality constant (the "Peierls constant") of order of the strength of the nearest neighbor interaction.

However in the situation of Kac type models that we are interested in, a direct application of the above procedure would lead to estimates for the Peierls constant of the order of τR^{-d} and whence our estimate on the critical temperature would get arbitrarily bad as the range R of the interaction increases. The point here is that the interactions are locally too weak to enforce constant configurations as "most likely" ones, and that it is necessary in the the definition of the restricted ensembles (in the complements of contours)to take also the local entropy into account.

This will imply that the collection of the allowed configurations in the "restricted ensembles" will be much larger, characterized by some non-local condition to be "q-like" in an averaged sense. Contours will correspondingly have to be defined in a rather complicated way, as will be detailed below.

Thus, while the general philosophy of the Pirogov-Sinai theory will be unchanged, the details of the implementation, and in particular the expansion techniques used will have to be adopted to this more complicated situation.

The definition of the q-like configurations begins with a necessary criterion that establishes whether a configuration x could be considered q-like at a point i:

Definition of a q – point. *For $s, s' \in S$ and $q \in S$ and $i, j \in \Lambda$ denote by*

$$W^q_{i,j}\{s,s'\} = \Phi_{i,j}\{s,s'\} - \Phi_{i,j}\{s,q\} - \Phi_{i,j}\{q,s'\}. \tag{3.1}$$

Let $x \in X$ be some configuration. Then i is called a q – point of x and we write[4] $c_x(i) = q$, *if for any $s \neq q$*

$$\sum_j |W^q_{i,j}\{s,x_j\}| \le \delta\, V_q, \tag{3.2}$$

where $0 < \delta < 1$ will be chosen later. If for no value $q \in \mathcal{S}$, $c_x(i) = q$, or if $c_x(i) \notin Q$, then we call i an incorrect *point and write $c_x(i) = \bullet$.*

Remark. Notice that we do not assume that $x_i \neq q$; it is important to realize that the fact whether i will be called a q–point will be *independent* of the value of the spin at site i. It is determined by the fact that the spin configuration *outside* i, in an R-neighborhood of i is favours the color q at site i.

Notice that W^q is defined in such a way that it vanishes if one of its arguments equals q. In the sequal it will be much more convenient to work with this function rather than with the original interaction $\Phi_{i,j}\{x_i,x_j\}$ when investigating regions where the predominant color is q.

For a given choice of δ, once all q-points of a configuration have been determined, it is useful to associate to any $x \in X(\Lambda) = S^\Lambda$ the "cleaned", configuration $\bar{x}(x)$ by putting

$$\bar{x}_i(x) = \begin{cases} q, & \text{if}\, c_x(i) = q \in Q \\ x_i, & \text{if}\, c_x(i) = \bullet \end{cases} \tag{3.3}$$

Let us denote by $S^\bullet$ the extended spin set $S^\bullet = S \cup \{\bullet\}$.

Notation. Denote by $\Lambda_q = \Lambda^\delta_q = \Lambda^\delta_q(x)$ the collection of all q – points of x. Denote by

$$\Lambda^\bullet(x) = \Lambda \setminus \cup_{q \in S} \Lambda_q. \tag{3.4}$$

"Stars" of a configuration. This name will be used for spins x_i in q-points i of x_Λ having a value $x_i \neq q$. In the reformulation of the Hamiltonian we will give below, it will become evident that to each star there will be associated with a large "fugacity" $V_q(x_i) + U(x_i)$ term which will help to suppress their appearance in the q–correct region.

The first thing we need to clarify is whether the value of q is *uniquely determined* in the definition of a q-point above. To prove this, we will rewrite the assumption 1 (see (2.3)) in the following way, with new constants D_q and $D_q{}'$:

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[4] We will show later that this definition makes sense for suitable choices of δ that will guarantee that the map c_x is single valued.

Assumption 1'. There are constants $D_q, D'_{q'}$ such that[5] for any $s \neq q, s' \neq q$

$$D_q \max_s |\Phi_{i,j}\{s,q\}| \;\le\; |W^q_{i,j}\{s,s'\}| \;\le\; D'_q \min_{s\neq q} |\Phi_{i,j}\{s,q\}|. \tag{3.5}$$

Note. The assumptions (2.3) and (3.5) are easily shown to be valid e.g. in the important special case when $\Phi_{i,j}\{x_i,x_j\}$ is decoupled like

$$\Phi_{i,j}\{x_i,x_j\} = \phi_{i-j}\ \Phi\{x_i,x_j\} \tag{3.6}$$

where Φ is a finite interaction on $S \times S$ and ϕ_i is some numerical, Kac "kernel", like that mentioned above for the Kac Ising model. (Then, obviously, $D_+ = D_- = 2$.) Notice also that $W^q_{i,j}\{x_i,x_j\} \le 2\max|\Phi_{i,j}\{s,s'\}|$.

Uniqueness of the value $\bar{x}_i(x)$ is implied by the following two lemmata:

LEMMA 3.1. *If $i \in \Lambda_q$ and $s \neq q$ then*

$$\sum_{j:x_j=q} \Phi_{i,j}\{s,q\} > V_q(1-\frac{\delta}{D_q}). \tag{3.7}$$

Proof. The l.h.s of the relation (3.7) is obviously bigger, by (3.2) and (3.5), than

$$V_q - \sum_{j:x_j\neq q} \Phi_{i,j}\{s,q\} > V_q - \frac{1}{D_q}\sum_{j:x_j\neq q} W_{i,j}\{s,q\} \ge V_q\ (1-\frac{\delta}{D_q}). \tag{3.8}$$

LEMMA 3.2. *If $i \in \Lambda_{\tilde q}$, $\tilde q \neq q$ then for any $s \neq q$ we have the bound*

$$\sum_{j:x_j=q} \Phi_{i,j}\{s,q\} < \frac{\delta}{D_q}\ V_q. \tag{3.9}$$

Proof. The l.h.s of the relation (3.9) is obviously smaller, by (3.2) and (3.5), than

$$\sum_{j:x_j=q} \Phi_{i,j}\{s,q\} < \frac{1}{D_{\tilde q}}\sum_{j:x_j=q} W^{\tilde q}_{i,j}\{x_j,s\} \le \frac{1}{D_{\tilde q}}\sum_{j:x_j\neq\tilde q} W^{\tilde q}_{i,j}\{x_j,s\} < \frac{V_q\,\delta}{D_{\widetilde{q}}}. \tag{3.10}$$

COROLLARY 3.3.

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[5] In the following, the constants D, D' will be always taken from (3.1) rather then from (2.3). Actually here (and only here!) it would be more convenient to have $\Phi_{i,j}\{x_i,x_j\}$ in the definition above. Notice that we do not exclude $s = s'$ in Assumption 1'.

(i) If $2\delta < D \equiv \min D_q$ *then for any* $q \neq \widetilde{q}$ *we have* $\Lambda_q^\delta \cap \Lambda_{\widetilde{q}}^\delta = \emptyset$.

(ii) If in addition $\frac{2\delta}{D} \min_q V_q < \min_q V_q - \frac{Ck}{R}$*, then, for any* $q \neq \tilde{q}$, $\operatorname{dist}(\Lambda_q^\delta, \Lambda_{\tilde{q}}^\delta) > k$.

Proof. The first statement follows immediately from (3.7) and (3.10) (used for the value $x_i = q$), the second uses in addition the continuity relation (2.5) in order to estimate the change of the l.h.s of (3.9) when evaluated at a point j at distance k from i. $\diamondsuit$

The reader may notice that that with $\Phi_{i,j}\{x_i, x_j\}$ instead of $W_{i,j}^q\{x_i, x_j\}$ in (3.2) the above argument would be even simpler, using no constants D_q. However, the definition of correctedness based on $W_{i,j}^q\{x_i, x_j\}$ will be crucial later.

Agreement on the choice of δ. In the following we will choose δ such that part (i) of Corollary 3, (ii) holds, with $k = R$.

3.2. Contours of a configuration x_Λ.

For the given value of δ, we have now achieved a decomposition of $\mathbb{Z}^d$ into a union of disjoints sets $\Lambda_q^\delta(x)$, $q \in \mathcal{S}$ and the remainder which we will denote by $B_\delta(x)$. Naively one might think that the connected components of B_δ should represent the contours of x. However, it will be necessary to enlarge this set somewhat in order to reduce the interaction of contours with their surroundings. To do so we introduce are led to introduce another, *smaller* threshold value $\tilde{\delta} < \delta$.

We start with the set $B_\delta(x)$ that will form the "cores" of the contours. For any set $M \subset \Lambda$ define $M^R \equiv \{i \in \Lambda : \operatorname{dist}(i, M) \leq R\}$.Then set, for $\tilde{\delta} < \delta$,

$$\widetilde{B}(x) \equiv \widetilde{B}_{\delta,\tilde{\delta}}(x) \equiv \arg\min\left\{M \supset (B_\delta(x))^R : B_{\tilde{\delta}}(x_M \cup \bar{x}_{M^c}) = \emptyset\right\}. \tag{3.11}$$

The crucial point is that the set $\widetilde{B}$ has an outer layer of points that are $\tilde{\delta}$-correct while none of its points is a distance farther than R away from a $\tilde{\delta}$-correct point. This construction will look at first glance rather surprising. In particular, the reader may wonder what the role of δ and $\tilde{\delta}$ in the construction is. This will only become clear once the expansions in the complements of this set are discussed.

Now we can finally give the definition of contours.

Definition of a contour. *Let* $\underline{\mathbf{\Gamma}}$ *be a connected component of the set* $\widetilde{B}(x)$*. Then the pair* $\mathbf{\Gamma} = (\underline{\mathbf{\Gamma}}, x_{\underline{\mathbf{\Gamma}}} \cup \bar{x}_{\underline{\mathbf{\Gamma}}^c})$ *is called a contour of* x.

The set $\underline{\mathbf{\Gamma}}$ is usually called the *support* of the contour $\mathbf{\Gamma}$. It will be useful to distinguish between the *core* $\underline{\mathbf{\Gamma}}^\bullet \equiv \underline{\mathbf{\Gamma}} \cap B_\delta(x)$, the *extended core* $(\underline{\mathbf{\Gamma}})^{\bullet R} \equiv \underline{\mathbf{\Gamma}} \cap B_\delta^R(x)$, and the *belts* of $\mathbf{\Gamma}$ of color q, $\widetilde{\underline{\mathbf{\Gamma}}}^q \equiv (\underline{\mathbf{\Gamma}} \backslash \underline{\mathbf{\Gamma}}^\bullet) \cap \Lambda^q(x)$. The restriction of $\mathbf{\Gamma}$ to $\widetilde{\underline{\mathbf{\Gamma}}}^q$ resp. to $\underline{\mathbf{\Gamma}}^\bullet$ will be denoted as $\widetilde{\mathbf{\Gamma}}^q$ resp. $\mathbf{\Gamma}^\bullet$.[6]

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[6] The notation $\widetilde{\mathbf{\Gamma}}^q$ (the use of the tilde sign is logical, because the choice of $\tilde{\delta}$ affects $\widetilde{\mathbf{\Gamma}}^q$ substantially) should not be confused with another notation $\mathbf{\Gamma} = \mathbf{\Gamma}^q$ marking contours $\mathbf{\Gamma}$ having the external colour q.

Finally, it will be useful to introduce the q-regular regions (the decomposition, according to "colours", of the set $(\cup\underline{\mathbf{\Gamma}})^c$)

$$\widetilde{\Lambda}^q \equiv \widetilde{\Lambda}_q(x) \equiv \Lambda_q(x)\backslash\widetilde{B}(x) \tag{3.12}$$

whose complement will be occasionally denoted also by

$$\widetilde{\Lambda}^\bullet = \Lambda \setminus \cup_q \widetilde{\Lambda}^q \quad (= \widetilde{B}(x)). \tag{3.13}$$

3.3. Reformulation of the Hamiltonian.

Now that we have defined contours, some simple algebraic manipulations provide a very useful reformuation of the Hamiltonian that allow to distinguish betwen contributions that are to be interpreted as energies of contours, energies of restricted ensembles, and interactions terms, respectively. For simplicity we formulate this statement, as always in this article, for a finite system on a torus Λ.

PROPOSITION 3.4. *Using the notations (3.1) we have the relation*

$$H_\Lambda(x_\Lambda) = H_{\widetilde{B}}(x_{\widetilde{B}} \cup \bar{x}_{(\widetilde{B})^c}) + \sum_q \left(H_q^{\widetilde{\Lambda}_q}(x_\Lambda) + U(q)|\widetilde{\Lambda}_q| \right) \tag{3.14}$$

where Δ denotes the symmetric difference of sets and $q = q\{i,j\}$ is such that $\{i,j\} \cap \widetilde{\Lambda}_q \neq \emptyset$. The funtions H_q^M, $M \subset \mathbb{Z}^d$, are defined by

$$H_q^M(x) = \sum_{i \in M} (V_q(x_i) + U(x_i) - U(q)) + \sum_{\{i,j\} \cap M \neq \emptyset} W_{i,j}^q\{x_i, x_j\}. \tag{3.15}$$

Proof. The proof is simple bookkeeping. Recall definitions (2.4) and (3.1) and notice that the cleaned configuration $\bar{x}_\Lambda$ has a constant value q in each $\widetilde{\Lambda}_q$. Notice the arithmetical rearrangements we made for those $\Phi_{i,j}\{x_i, x_j\}$ with $\{i,j\} \cap \widetilde{B} = \emptyset$ and also with $\{i,j\} \cap \widetilde{B} = \{j\}$. Namely, in the latter case the third term from the r.h.s of the expression

$$\Phi_{i,j}\{x_i, x_j\} = W_{i,j}^q\{x_i, x_j\} + \Phi_{i,j}\{q, x_j\} + \Phi_{i,j}\{x_i, q\} \quad i \in \widetilde{B}, \;\; j \in \widetilde{L}^q$$

still contributes to $V_q(x_i)$, while the second term already contributes to $H(x_{\widetilde{B}} | \bar{x}_{(\widetilde{B})^c})$. ♢

4. THE PEIERLS ESTIMATES OF THE ENERGY OF CONTOURS

The representation of the Hamiltonian given in Proposition 3.4 allows to identify the first term as the energy of the contours. Indeed, the value of this term depends entirely on the configurations on $\widetilde{B}$, and can be expressed as a sum of terms depending only on the individual contours (since the color of the cleaned configuration outside of $\underline{\mathbf{\Gamma}}$ can be read off from the colors on its boundary). In this sense,

$$H_{\widetilde{B}}(x_{\widetilde{B}} \cup \bar{x}_{(\widetilde{B})^c}) = \sum_{\mathbf{\Gamma}} (E(\mathbf{\Gamma}) - U|\underline{\mathbf{\Gamma}}|) \tag{4.1}$$

where for any single contour $\mathbf{\Gamma}$ we have, in analogy to (4.1),

$$E(\mathbf{\Gamma}) = H_{\underline{\mathbf{\Gamma}}}(x_{\underline{\mathbf{\Gamma}}} \cup \bar{x}_{\underline{\mathbf{\Gamma}}^c}) - U|\underline{\mathbf{\Gamma}}| \tag{4.2}$$

where $U = \min_{q \in \mathcal{S}} U(q)$ is the minimum of the potential and could be normalized to zero, if desired. In this section we will prove a Peierl's estimate for the contour energies, i.e. we will prove:

THEOREM 4.1. *Under the assumptions 0 – 3, we have for any contour $\mathbf{\Gamma}$ the bound*

$$E(\mathbf{\Gamma}) \geq \tau^{\bullet}\, |\underline{\mathbf{\Gamma}}| \tag{4.3}$$

where $\tau^{\bullet}$ is given as

$$\tau^{\bullet} = C_d\, \tilde{\delta}\, V \left(\frac{\tilde{\delta}}{CD'} \right)^d \tag{4.4}$$

with $D' = \max D'_q$, the constant C being taken from Assumption 3 and with C_d is a numerical constant depending only on the dimension.

Proof. Note that by definition of the contours, there can be no point i within $\underline{\mathbf{\Gamma}}$ whose R-neighborhood is free of points j that are $\tilde{\delta}$-incorrect with respect to the configuration $x_{\underline{\mathbf{\Gamma}}} \cup \bar{x}_{\underline{\mathbf{\Gamma}}^c}$. For, otherwise, the R-neighborhood of i could be removed from $\underline{\mathbf{\Gamma}}$ and it would still be true that all point in $(\underline{\mathbf{\Gamma}} \backslash \{i\}^R)^R$ were $\tilde{\delta}$ correct with respect to $(x_{\underline{\mathbf{\Gamma}} \backslash \{i\}^R} \cup \bar{x}_{(\underline{\mathbf{\Gamma}}^c \cup \{i\}^R)})$ (by the positivity of the interaction). Therefore we can find a R-connected set $\widetilde{G} \subset \underline{\mathbf{\Gamma}}$ such that all points $i \in \widetilde{G}$ are $\tilde{\delta}$-incorrect points of the configuration $x_{\underline{\mathbf{\Gamma}}} \cup \bar{x}_{\underline{\mathbf{\Gamma}}^c}$.

LEMMA 4.2. *Set $R^{\circ} = \frac{\tilde{\delta}}{D'C} R$. Let $\widetilde{G} \subset \underline{\mathbf{\Gamma}}$ be a set of $\tilde{\delta}$-incorrect points of $x_{\underline{\mathbf{\Gamma}}} \cup \bar{x}_{(\underline{\mathbf{\Gamma}}^c}$ and assume for simplicity that for any $i \neq j \in \widetilde{G}$, $|i - j| > R^{\circ}$. Then*

$$E(\mathbf{\Gamma}) \geq \frac{1}{2} \sum_{K=1}^{R^{\circ}} (\frac{\tilde{\delta} V}{D'} - \frac{CK\tau}{R}) |\widetilde{G}^{(K)}| \tag{4.5}$$

where $\widetilde{G}^{(K)}$ denotes the set of points i that have distance K from the set $\widetilde{G}$.

Proof. By definition,

$$E(\mathbf{\Gamma}) = \sum_{i \in \underline{\mathbf{\Gamma}}} V_{x_{\mathbf{\Gamma}} \cup \bar{x}_{\underline{\mathbf{\Gamma}}^c}}(i, x_i) + \sum_{i \in \underline{\mathbf{\Gamma}}} (U(x_i) - U) \tag{4.6}$$

where $V_x(i,q)$ was defined in (2.7). Let us assume that all points i in $\widetilde{G}$ are actually incorrect w.r.t. all colors in $\mathcal{S}$. Then it is enough to bound the right hand side of (4.6) from below by

$$\geq \sum_{i\in\underline{\mathbf{\Gamma}}} V_{x_{\mathbf{\Gamma}}\cup\bar{x}_{\underline{\mathbf{\Gamma}}^c}}(i,x_i) \tag{4.7}$$

Using the upper bound of Assumption 1', together with the definition of $\widetilde{G}$ one sees readily that for all $i\in\widetilde{G}$ for any $q\in\mathcal{S}$,

$$V_{x_{\mathbf{\Gamma}}\cup\bar{x}_{\underline{\mathbf{\Gamma}}^c}}(i,q) \geq \frac{V\delta}{D'} \tag{4.8}$$

Note that this estimate uses in a crucial way the fact that correctness of a point does not depend on the values of the spin at this point.

Next note that by the continuity Assumption 3,

$$V_{x_{\mathbf{\Gamma}}\cup\bar{x}_{\underline{\mathbf{\Gamma}}^c}}(i,x_i) \geq V_{x_{\mathbf{\Gamma}}\cup\bar{x}_{\underline{\mathbf{\Gamma}}^c}}(i+k,x_i) - \frac{C|k|\tau}{R} \tag{4.9}$$

But for any point i at distance $K\leq R^\circ$ from $\widetilde{G}$ one can find a vector k of length K such that $i+k\in\widetilde{G}$, and therefore for such points,

$$V_{x_{\mathbf{\Gamma}}\cup\bar{x}_{\underline{\mathbf{\Gamma}}^c}}(i,x_i) \geq V_{x_{\mathbf{\Gamma}}\cup\bar{x}_{\underline{\mathbf{\Gamma}}^c}}(i+k,x_i) - \frac{CK\tau}{R} \geq \frac{\delta V}{D'} - \frac{CK\tau}{R} \tag{4.10}$$

From here the lemma follows immediately under the above assumption. Now, if some of the points i are incorrect because $c_i(x)\in\mathcal{S}\backslash Q$, then this implies that there is a large fraction of points in the R-neighborhood of these points that have $U(x_i)\geq U+\tau/4$. But this implies an even stronger excess energy as was obtained above. $\diamondsuit$

We are now ready to conclude the proof of the theorem. Note first that Lemma 4.2 implies by a simple geometric consideration (see [BZ1]), that

$$E(\mathbf{\Gamma}) \geq C_d\frac{\tilde{\delta}V}{D'}\left(\frac{\tilde{\delta}}{CD'}\right)^d|\underline{\mathbf{\Gamma}}| \tag{4.11}$$

with $C_d>0$ depending only on the dimension. This gives the claimed estimate of the Theorem. $\diamondsuit$

5. Cluster expansion of restricted "low density" ensembles.

In the previous sections we defined the notion of contours and established a Peierls estimate for the contour energies. In terms of these objects we can now express the partition function on the torus Λ as follows

$$Z_\Lambda = \sum_{\mathbf{\Gamma}_1,\dots,\mathbf{\Gamma}_n} e^{-\sum_{i=1}^n E(\mathbf{\Gamma}_i)-U|\underline{\mathbf{\Gamma}}_i|} \prod_{q\in\mathcal{S}} Z_{\widetilde{\Lambda}_q}(\mathbf{\Gamma}_1,\dots,\mathbf{\Gamma}_n)\ \ e^{U(q)|\widetilde{\Lambda}_q|} \tag{5.1}$$

where the sum is over all families of compatible contours (that is to say the components $\underline{\mathbf{\Gamma}}_i$ are mutually disconnected and the colors on the components of the boundaries of the nested arrangement of contours match), the sets $\widetilde{\Lambda}_q$ make the decomposition

$$\cup_q \widetilde{\Lambda}_q = \Lambda \setminus (\cup_i \underline{\mathbf{\Gamma}}_i) \quad (\equiv (\cup_i \underline{\mathbf{\Gamma}}_i)^c)$$

of the complement of the union of the supports of the contours $\mathbf{\Gamma}_i$ of colour q, and the *restricted partition function* $Z_{\widetilde{\Lambda}_q}(\mathbf{\Gamma}_1, \dots, \mathbf{\Gamma}_n) = Z^{\delta}_{\widetilde{\Lambda}_q}(\mathbf{\Gamma}_1, \dots, \mathbf{\Gamma}_n)$ in the volume $\widetilde{\Lambda}_q$ equals to

$$Z^{\delta}_{\widetilde{\Lambda}_q}(\mathbf{\Gamma}_1, \dots, \mathbf{\Gamma}_n) = \sum_{x_{\widetilde{\Lambda}_q} \in \mathcal{S}^{\widetilde{\Lambda}_q}} \prod_{i \in \widetilde{\Lambda}_q} \mathbb{1}^{\delta}_i(x) \exp(-\sum_{i \in \widetilde{\Lambda}_q} (V_q(i) + U(x_i) - U(q)) - \sum_{\{i,j\} \cap \widetilde{\Lambda}_q \neq \emptyset} W^q_{i,j}(x_i, x_j)) \tag{5.2}$$

where for notational simplicity we set $x = \cup_q x_{\widetilde{\Lambda}_q} \cup x_{\underline{\mathbf{\Gamma}}_1} \cup \dots \cup x_{\underline{\mathbf{\Gamma}}_n}$ (the configuration that equals $x_{\widetilde{\Lambda}_q}$ within $\widetilde{\Lambda}_q$ and the one imposed by the fixed contours on their support) and $\mathbb{1}^{\delta}_i(x)$ is the indicator of the event that i is a δ correct point of x.[7]

In the present chapter we will expand the restricted partition functions. Our approach will be based on the following observations. The energy of a configuration on $\widetilde{\Lambda}_q$ is expressed in such a way that the pair interactions are non-zero only between spins x_i for which $x_i \neq q$. Moreover, any such spin x_i has a "potential" or "activity" term $V_q(x_i) + U(x_i) - U > 0$.

It is thus reasonable to think of the configuration as a set of (colored) stars interacting through a pair interaction. What is important is that the constraint of δ-correctness imposes a constraint on the density of these stars. Under this constraint, we will see that the pair interaction is effectively weak (a fact that would *fail* in the absence of such a constraint) and can be dealt with by high-temperature expansion techniques. It should be noted that the stars interact with the boundary condition also only by pair interactions between stars, and that by construction, the boundary layer of contours (of thickness R) carries configurations that are even $\tilde{\delta}$-correct if "looked from the point of view of the cleaned configuration outside" and thus have an even lower density of stars then otherwise allowed in $\widetilde{\Lambda}_q$.

The only obstacle we will encounter is the presence of the density constraint. This constraint in crucial on the one hand since it ensures that the interaction is weak and that thus a high-temperature expansion may converge. On the other hand, this constraint imposes a non-local interaction between the high-temperature polymers, which is somewhat unconventional. The way we will deal with this is that first, under the presence of the constraint, we will perform a partial resummation of the original high-temperature expansion based on the expansion of the parentheses in (5.4). This yields as usual a sum over certain graphs. We then associate to each graph a spanning tree, and re-sum over all graphs corresponding to the same tree. Taking advantage of the presence of the density constraint one can show that the resulting activities of the trees are sufficiently small to ensure convergence of the Mayer expansion of this tree ensemble even if there were no density constraints (i.e. in this ensemble

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[7] Emphasise that while $\tilde{\delta}$ played a crucial role in the definition of contours the definition of the ensemble over which we sum in (5.2) is actually taken does not depend on $\tilde{\delta}$ once the collection of contours $\{\mathbf{\Gamma}_i\}$ is fixed,

the low density constraint would be automatically respected with overwhelming probability). At this point the constraint is thus a pure nuisance, and we would like to get rid of it. To do so we actually treat it as a perturbation and again expand it in the "F.K. way". This will produce a new class of objects (high density graphs) which we will call “galaxies” and which in many ways share the properties of the original contours.

After these preparations the partition function is reformulated in the form of a standard polymer model and conventional techniques can be used to deal with it.

To simplify notation from the somewhat gruesome looking (5.2), let M be any set and let x_{M^c} be a boundary condition of the type that can arise in our situation, i.e. such that $M \cap B_{\tilde\delta}(q_M \cup x_{M^c}) = \emptyset$. Using Proposition 3.4, we can write the partition function of interest is (up to an overall factor $\exp(U(q)|M|)$)

$$\begin{aligned} Z^q_M(x_{M^c}) &= \sum_{x_M \in \mathcal{S}^M} 1\!\!\mathrm{I}_{M \subset B^c_{\tilde\delta}(x)} \exp(-H^q_M(x)) \\ &= \sum_{x_M \in \mathcal{S}^M} 1\!\!\mathrm{I}_{M \subset B^c_{\tilde\delta}(x)} \prod_{i \in M} e^{-(V_q(x_i)+U(x_i)-U(q))} \prod_{\{i,j\} \cap M \neq \emptyset} e^{-W^q_{i,j}(x_i,x_j)}. \end{aligned} \tag{5.3}$$

where $B^c_{\tilde\delta}(x) \equiv \Lambda \setminus B_\delta(x)$ and it is understood that $x = x_M \cup q_{M^c}$.

5.1. Construction of the polymer representation in the volume M.

We begin the program outlined above with the first step, the high-temperature expansion of the interaction between stars.

For $i \neq j$ write the term $e^{-W^q_{i,j}\{x_i,x_j\}}$ as

$$\exp(-W^q_{i,j}\{x_i,x_j\}) = 1 + w^q_{i,j}\{x_i,x_j\} \equiv 1 + w_b \qquad \text{where} \qquad b = \{x_i, x_j\}. \tag{5.4}$$

We also write

$$v(x_i) = \exp(-U(x_i) + U(q) - V_q(x_i)). \tag{5.5}$$

Note that if $x_i = q$, then $w^q_{i,j}\{x_i,x_j\} = 0$ and $v(x_i) = 1$. Recall that we assumed in Assumption 2 and 3 ((2.6) and (2.5)) that we have both the bound for the “oscillation of U”as well as a lower bound for $V_q(x_i)$. We may summarize these two bounds by assuming that for any $x_i \neq q$

$$V_q(x_i) + U(x_i) - U(q) > 3/4 \; V_q(x_i) \tag{5.6}$$

Already at this point it is reasonable to formulate our later requirements on the fugacities $V_q(x_i)$ – which will have to “beat” both the repulsive energies of the bond terms $W^q_{i,j}\{x_i,x_j\}$ (whose sum over j may substantially lower $V_q(x_i)$) as well as the entropy.

In order to simplify the notation let us introduce the modified potential

$$\widetilde{V}_q(x_i) = (1/2 - \delta) \; V_q(x_i) + U(x_i) - U(q) > 0. \tag{5.7}$$

LEMMA 5.1. *There exists $\tau^* > 0$ such that for each i and each $x_i \in S$*

$$\sum_j \sum_{x_j \in S} w^q_{i,j}\{x_i, x_j\}\; e^{-\widetilde{V}_q(x_i)} \;\le\; e^{-\tau^*}. \tag{5.8}$$

Moreover, for large enough τ, $\tau^* > \tau/3$.

Proof. By Lemma 2.1 we have that $|w^q_{i,j}\{x_i, x_j\}| \le W^q_{i,j}(x_i, x_j)e^{2C/R}$. Then, taking into account Assumption 1', we see that

$$\sum_j \sum_{x_j \in S} w^q_{i,j}\{x_i, x_j\}\; e^{-\widetilde{V}_q(x_i)} \le\; e^{-\widetilde{V}_q(x_i)} \frac{\tau e^{2C/R}}{D'} \le \frac{\tau e^{2C/R}}{D'} e^{-\tau(1/2-\delta)} \equiv e^{-\tau^*} \tag{5.9}$$

From this the lemma is obvious. $\diamondsuit$

In the following we keep in mind that the reference color is q and we do not make this explicit in notations anymore in this paragraph. Let us associate with a configuration $x = x_M \cup x_{M^c}$ the set of stars

$$A(x) \equiv \{i \in M \cup M^R \,:\, x_i \ne q\} \tag{5.10}$$

Then

$$\exp\left(-H^q_M(x)\right) = \prod_{i \in A(x) \cap M} v(x_i) \prod_{b \subset A(x), b \cap M \ne \emptyset} (1 + w_b) \tag{5.11}$$

In a standard way the product over bonds can be expanded as a sum over (unoriented, simple) graphs G on $A(x)$, whose connected components may be also single points. We denote by $\underline{G}$ the set of vertices of G. The graphs G that occur have the properties that

(i) $A(x) \supset \underline{G} \supset M \cap A(x)$.

(ii) Every bond and (in case of $|\underline{G}| = 1$) every connected component of G contains at least one point of M. Denote the set of such graphs $\mathcal{G}(M, x)$, $x = x_M \cup x_{M^c}$. Then

$$\sum_{G \in \mathcal{G}(M,x)} \prod_{i \in \underline{G}} v(x_i) \prod_{b \in G} w_b = \prod_{i \in A(x) \cap M} v(x_i) \prod_{b \subset A(x), b \cap M \ne \emptyset} (1 + w_b). \tag{5.12}$$

Inserting this expression into the formula (5.2) for the partition function, one observes that the summation over the configurations x_M and the graphs $G \in \mathcal{G}(M, x)$ can be interchanged: Consider the class set $\mathcal{G}_q(M, x_{M^c})$ of ("colored by q") graphs that are of the same type as those specified above, except that the vertex set $\{x_i \ne q\}$ within M is now arbitrary and while each point outside of M carries the prescribed vertex x_i given by $(x_{M^c})_i$. For $G_q \in \mathcal{G}(M, x_{M^c})$ we set

$$w_G = \prod_{i \in \underline{G}} v_i(x_i) \prod_{b = \{x_i, x_j\} \in G} w_b \qquad \text{where} \qquad x_b = w^q_{i,j}\{x_i, x_j\}. \tag{5.13}$$

Then the partition function (5.3) can be written, using (5.4), (5.5) and (5.13) as

$$Z^q_M(x_{M^c}) = \sum_{G\in\mathcal{G}(M,x_{M^c})} \mathbb{1}_{B_\delta(x)\subset M^c}\ w_G = \sum_{G\in\mathcal{G}(M,x_{M^c})} w_G \prod_{i\in M} \chi_\delta(i) \tag{5.14}$$

where $\chi_\delta(i)$ is an indicator of the event that i a δ correct q-point of $x_\Lambda = x_{M^c} \cup x_M$ and $x_M \equiv x_M(G)$ is defined as a configuration on M having the value q outside the vertices of G and the value x_i in the vertices $x_i \in \operatorname{supp} G$.

5.2. From graphs to forests (and trees).

Consider some mapping, defined in a translational invariant way, denoted by $\{G \to T_G\}$ which assigns to any graph G a *forest* $T_G \subset G$ such that $\underline{T}_G = \underline{G}$, all connected components of T_G are *trees*[8], and, for all points $i \in \underline{G}$, the values of the spins (vertices of G resp. T_G) are the same: $x_i(T_G) = x_i(G)$. It is important to note that by this mapping the spin configuration associated to G is entirely determined by T_G. In particular, a graph G occurs in the partition sum $Z^q_M(x_{M^c})$ if and only if T_G occurs.

For any forest T, denote by $\widetilde{w}_T$ the quantity

$$\widetilde{w}_T = \sum_{G: T_G = T} w_G. \tag{5.15}$$

Note that if the forest $T = \{t_1, \dots, t_n\}$ where t_l are connected trees, then

$$\widetilde{w}_T = \prod_{l=1}^{n} \widetilde{w}(t_l) \tag{5.16}$$

LEMMA 5.2.*For any forest T which is δ correct*[9] *we have the following bound:*

$$|\widetilde{w}_T| \ \le \prod_{b\in T} |w^q_{i,j}\{x_i, x_j\}| \prod_{i\in\underline{T}} e^{-\widetilde{V}_q(x_i)}. \tag{5.17}$$

Proof. Recall that since only such forests can contribute in the partition function for which all points i are δ-correct, we have that for any $i \in \underline{T}$,

$$\sum_{j:\{i,j\}\in G} |W^q_{i,j}(x_i, x_j)| \le \delta V_q(x_i) \tag{5.18}$$

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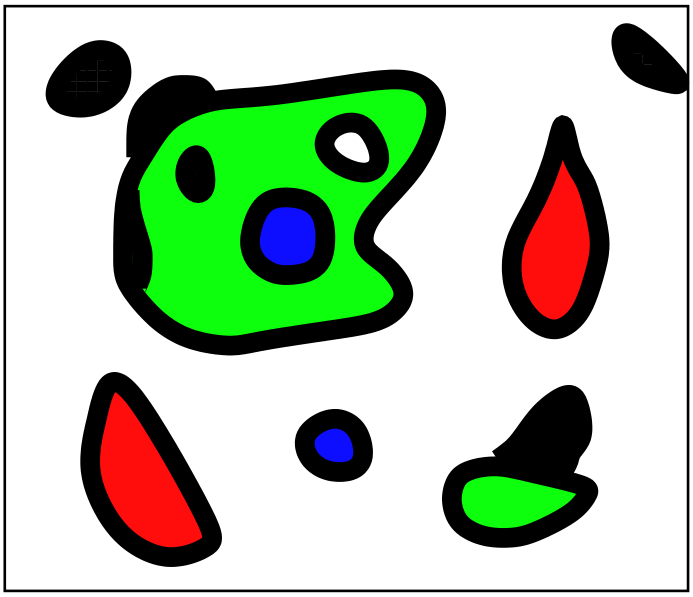
[8] We consider a single point also as a tree.

[9] In other words, which occurs as a subgraph of some graph index in the sum (5.3),(5.4) over the restricted ensembles of graphs in the partition functions Z^q_M.

Next observe that since for any $z \in \mathbb{R}$, $1 + |e^z - 1| \le e^{|z|}$

$$\begin{aligned}\widetilde{w}_T \le \sum_{G:T_G=T} |w_G| &\le \prod_{i\in\underline{T}} v_i(x_i) \prod_{b\in T} |w^q_{i,j}\{x_i,x_j\}| \prod_{i,j\in\underline{T}} (1+|w^q_{i,j}\{x_i,x_j\}|) \\ &\le \prod_{i\in\underline{T}} v_i(x_i) \prod_{b\in T} |w^q_{i,j}\{x_i,x_j\}| \prod_{i,j\in\underline{T}} e^{|Wij|} \\ &\le \prod_{i\in\underline{T}} v_i(x_i) \prod_{b\in T} |w^q_{i,j}\{x_i,x_j\}| \prod_{i\in\underline{T}} e^{\delta V_q(x_i)}\end{aligned} \tag{5.19}$$

Using (5.6), this yields the Lemma.



A collection of contours in a model with 4 colors
Contours are depicted as black regions

5.3. A new representation of the partition function (5.3).

We can now represent the partition function (5.3)as

$$Z^q_M(x_{M^c}) = \sum_{T\in\mathcal{G}(M,x_{M^c})} \mathbb{1}_{B_\delta(x_M\cup x_{M^c})\subset M^c}\, \widetilde{w}_T = \sum_{T\in\mathcal{G}(M,x_{M^c})} \widetilde{w}_T \prod_{i\in M} \chi_\delta(i). \tag{5.20}$$

The point is that due to Lemma 5.1 for large enough τ the estimates (5.17) above on the tree activities $\widetilde{w}_T$ suffice to guarantee the convergence of the Mayer expansion for the logarithm of the partition function, *now even in the absence of the density constraints*. Thus, at this point, the constraint has become a nuisance rather than a help, and we would rather get rid of it.

The problem is that the test functions $\chi(i)$ in (5.20) act on the whole configuration x_Λ thus imposing multi-body compatibility relation between components of the forest T. Writing

$$\prod_{i\in M}\chi(i)=\prod_{i\in M}(1-(1-\chi(i)))=\sum_{M_1\subset M}(-1)^{|M_1|}\prod_{i\in M_1}(1-\chi(i)) \tag{5.21}$$

we replace the ensemble of forests $T=\cup_j t_j$ obeying the "low density" tests $\chi(i)$ by an ensemble of all forests; the constraint being now reflected by a creation of artificial new polymer formed by conglomerates of those i where the low density test of G is *violated*. Note that nothing prevents us to maintain the fact that we sum only over forests made of trees satisfying individually the low density constraint, and in particular the estimate (5.17). We will denote the subset of forests in $\mathcal{G}(M,x_{M^c})$ whose elements t_i satisfy the bounds (5.17) by $\mathcal{T}(M,x_{M^c})$.

Now we can write the partition function above as

$$\begin{aligned} Z^q_M(x_{M^c}) &= \sum_{T\in\mathcal{T}(M,x_{M^c})}\prod_{i\in M}\big(1-(1-\chi_\delta(i))\big)\widetilde{w}_T \\ &= \sum_{T\subset\mathcal{T}(M,x_{M^c})}\sum_{M_1\subset M}\prod_{i\in M_1}(-\chi^c_\delta(i))\ \widetilde{w}_T \\ &\equiv \sum_{T\subset\mathcal{T}(M,x_{M^c})}\sum_{M_1\subset M}\widetilde{w}(M_1,T) \end{aligned} \tag{5.22}$$

where we introduce the provisional notation

$$\widetilde{w}(M_1,T)=\prod_{i\in M_1}(-\chi^c_\delta(i))\ \widetilde{w}_T. \tag{5.23}$$

Since the indicator functions χ_δ depend only on the forest in an R-neighborhood of i, it is natural to lump all objects that intersect connected components of the R-neighborhoods of the set M_1 together and to treat them as *single objects*; the connected components of the agglomerates obtained in this way will be called *galaxies.*

We will now make this notion precise.

5.4. Constellations and galaxies in a volume M.

We note that while the definitions below will be given for any volume $M\subset\Lambda$ and any boundary condition x_{M^c}, they will be used only in the situation when M is a component of the union of all contours of some x_Λ, i.e. in the situation when the boundary condition x_{M^c} is such that any point $i\in M$ is a $\tilde{\delta}$–correct point of the configuration $x_{M^c}\cup q_M$.

Let $\mathcal{P}$ denote the set

$$\mathcal{P}\equiv\mathcal{P}(M_1,T)=M_1^R\cup\underline{T}. \tag{5.24}$$

We say that that a subset P of $\mathcal{P}$ is R,T–connected, if any two points in P can be joined by a path made *either* of bonds of T *or* nearest neighbor bonds on the set $M_1^R\cup\partial^R M$. An R,T–connected component P of $\mathcal{P}$ together with the restriction T_P of T to P will be called

(i) A *constellation*, t , if $P \cap M_1^R = \emptyset$, and

(ii) a *pre-galaxy* P, more precisely M_1 pre-galaxy if $P \cap M_1^R \neq \emptyset$.

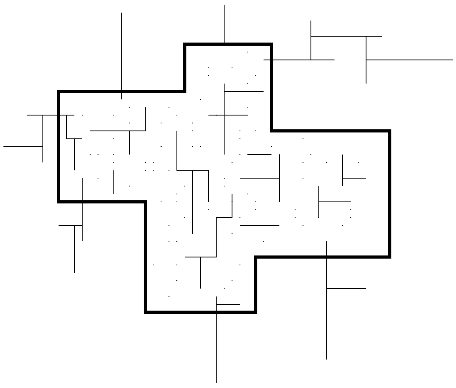
Constellations and pre-galaxies will be treated quite differently. Indeed, the constraint-free constellations are ready for being Mayer-expanded. This will be done later.

On the other hand, the pre-galaxies will be re-summed to form "galaxies" by summing over all possible arrangements of the sets $M_1 \subset P$ giving the same M_1^R and also over the "body" of the forest T_P. We will sum neither over the vertices $x_i \notin M_1^R$ nor over the bonds $\{x_i, x_j\}, \{i,j\} \setminus M_1^R \neq \emptyset$ of the trees of the pre-galaxy.

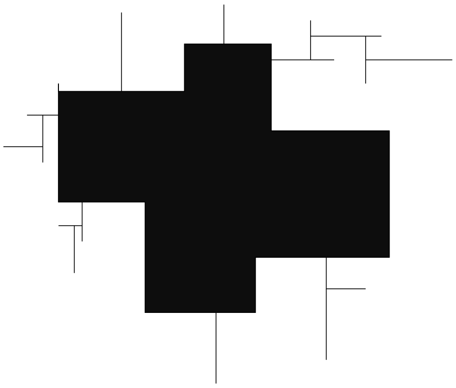
The remaining trees of such a pre-galaxy will be called the *legs* of the pre-galaxy.[10] On the other hand the set M_1^R resp. $P \cap M_1^R$ (the latter is the complement of the union of the "outer" points of the legs of the pre-galaxy) will be called the "body" resp. the "skeleton" of a given pre-galaxy $(P, T_P.M_1)$ and denoted by P° resp. $P^\bullet$.

A *galaxy* P^* is then defined as a pair $P^* = (P^\circ, \{L_i\})$ where the set P° (the "body"of the galaxy P^*) is a body of some pre-galaxy $(P, T_P.M_1)$ and the "legs" L_i of P^* are some trees intersecting P° such that $\{L_i\}$ is the collection of all legs of a suitable pre-galaxy (P, T_P, M_1) above.[11]

We may say that constellation is a galaxy with an empty body. In the following we will reserve the name of a galaxy only for the objects having a *nonempty* body.



A pre - galaxy



The corresponding galaxy

5.5. Restricted ensemble as a gas of constellations and galaxies.

For any M_1- pre-galaxy $Q = (P, T_P)$ define the quantity (5.23) now denoted as

$$\widetilde{w}_Q = \widetilde{w}(M_1, P, T_P) = \prod_{i \in M_1} (-\chi_\delta^c(i)) \; \widetilde{w}_T. \tag{5.25}$$

---

[10] Both the constellations and the "legs" of a galaxy are connected trees.

[11] The legs have at least one link $\{x_i, x_j\}$ intersecting the core P^0 of the galaxy - but not belonging to it.

Given a galaxy P^* denote by $\mathcal{P}(P^*)$ the collection of all possible pre-galaxies corresponding to the given P^* (for a suitable M_1). Define the weight of a galaxy P^*

$$\widetilde{w}(P^*) = \sum_{Q \in \mathcal{P}(P^*)} \widetilde{w}_Q. \tag{5.26}$$

Using these notation we can write the expression (5.22) as follows

$$Z_M^q(x_{M^c}) = \sum_{\{P_k^*, t_l\}_c} \prod_k \widetilde{w}(P_k^*) \prod_l \widetilde{w}(t_l) \tag{5.27}$$

where the summation is over all collections $\{P_k^*, t_l\}_c$ of mutually disconnected galaxies and constellations in a given volume M, under a boundary condition x_{M^c}. Note that by construction, the weights $\widetilde{m}(P^*)$ may depend on x_{M^c}, however only if $\underline{P}^* \cap M^c \neq \emptyset$.

5.6. Peierls bounds for pre-galaxies.

We will show that due to the high density constraints in M_1^R, the 'weight" ("activity") of a galaxy is exponentially small. A technical problem arises here from points in the vicinity of M^c, since there the δ-incorrectness of a point may be provoked partly also by the presence of stars outside M which do not contribute an activity factor of the galaxy. To avoid this problem we had defined contours with the R-belt of points (forming now the R belt of M in M^c) that are $\tilde{\delta}$ correct if looked from outside M^c (and if the configuration outside the contours i.e. in M is replaced by the cleaned configuration). This will make sure that the presence of nearby contours cannot be mainly responsible for δ-*in*correctness inside M. In other words, the protection belts we imposed around contours make sure that contours cannot trigger the nucleation of galaxies in their vicinity.

LEMMA 5.3.*Let the boundary condition x_{M^c} be such that each point $i \in M$ is a $\tilde{\delta}$ - correct point of the configuration $x_{M^c} \cup q_M$. Let P^* be a galaxy of a configuration $x_\Lambda = x_M \cup x_{M^c}$ with a body P^0 (not necessarily $P^0 \subset M$). Assume that $\tilde{\delta}$ was chosen such that $\tilde{\delta} < \delta$. Then we have the bound, using the same constants D' and C_q as in (4.3)*

$$\sum_{i \in M \cap P^0} V_q(x_i) \geq \tau_q^0 \, |P^0| \qquad \textit{where} \qquad \tau_q^0 = C_d V_q \left(\frac{\delta - \tilde{\delta}}{CD'_q} \right)^d . \tag{5.28}$$

Note. Notice that we have the constants D'_q and V_q instead of D' and V here, and that we have P^0 on the right hand side and not $M \cap P^0$ (even if $P^0 \cap M^c \neq \emptyset$). It should be also noted that for small δ the quantity in (5.28) is proportional to the energy of the core of a"contour" P^0 and the constant $\tau^c irc$ is essentially the same as $\tau^\bullet$ in the proof of Peierls condition (4.3) (if we consider there the special case of a contour $\mathbf{\Gamma}$ having an empty core $\mathbf{\Gamma}^\bullet = \emptyset$).

Proof. The idea is very similar to that of the proof of Peierls condition. By definition,

$$\sum_{i \in \underline{P}^0} V_q(x_i) = \sum_{i \in \underline{P}^0, j \in \Lambda, j \neq i} \Phi_{\{i,j\}}\{x_i, q\}. \tag{5.29}$$

Using the symmetry of Φ with respect to the interchange of the two arguments and interchanging the order of summation we get that

$$\sum_{i\in P^0\cap M} V_q(x_i) = \sum_{i\in P^0\cap M, j\in\Lambda, j\neq i} \Phi_{\{j,i\}}\{q,x_i\}. \tag{5.30}$$

But if $x_i = q$, $\Phi_{ij}(x_i,q) = 0 = W^q_{ij}(x_i,x_j)$, while if $x_i \neq q$, by Assumption 1', $\Phi_{i,j}(q,x_i) = \Phi_{i,j}(x_i,q) \geq \frac{1}{D'_q} W^q_{i,j}(x_i,s_j)$ for any choice of $s_j \neq q$. It is important to note here that the definition of a galaxy depends on a boundary condition x_{M^c}, but x_M will be mostly responsible for a possible δ incorrectness of any $i \in M$! Namely if i is not a $\delta-$ correct point of x_Λ then i is $(\delta-\tilde{\delta})$- incorrect point of the configuration $x_M \cap q_{M^c}$. So we have the lower bound

$$\sum_{i\in\Lambda, i\neq j} \Phi_{i.j}\{q,x_i\} \geq \frac{1}{D'_q}\sum_{i\in\Lambda} |W^q_{j,i}(s_j,x_i)| \geq \frac{(\delta-\widetilde{\delta})V_q}{D'_q}. \tag{5.31}$$

Since δ-incorrect points of the galaxy P^* are guaranteed only in the possibly very sparse subset M_1 of P^0, we will have to use the continuity Assumption 3, just as in the proof of the Peierls condition. Let us fix a point $j \in M_1 \cap \underline{P}^*$, and consider, for $|k| \leq R^*_q \equiv \frac{(\delta-\tilde{\delta})}{D'_q C}$ (it is defined analogously as in Lemma 4.2, the contribution from a point $j+k$. We have the lower bound

$$\sum_{i\in\Lambda} \Phi_{j+k,i}(q,x_i) \geq \sum_{i\in\Lambda} \Phi_{j,i}\{q,x_i\} - \frac{C|k|}{R}. \tag{5.32}$$

for it. As in the derivation of (4.11), it follows now that the sum over the R^* neighborhood of such a $(\delta-\tilde{\delta})$ incorrect point j of $x_M \cup q_{M^c}$ is not smaller than

$$\sum_k \left(\sum_{i\in\Lambda} \Phi_{j,i}\{q,x_i\} - \frac{C|k|}{R} \right). \tag{5.33}$$

Summing over the union $J \subset M$ of all such incorrect $j \in M$ and noticing that $J^R \supset P^0$ we finally have

$$\sum_{j\in\Lambda}\sum_{i\in P^0\cap M} \Phi_{j,i}\{q,x_i\} \geq V_q\,|P^0|\,C_d \left(\frac{\delta-\tilde{\delta}}{D'_q C}\right)^d. \tag{5.34}$$

This proves the lemma.$\diamondsuit$

5.7. Summing over pre-galaxies. Peierls bounds for galaxies.

Based on the previous lemma, we can now estimate rather easily also the activities of galaxies, by fixing the legs and summing over all possible skeletons of a given galaxy. One should remark that inside the body of a galaxy, a slightly different type of estimateof the bond weights forming the skeleton of this body) will be used than for the legs.

Let us denote by L the union of all legs of the galaxy P^*. Recall that P^0 denotes the body of P^* and that the weights $\widetilde{w}(P^*)$ is given by the sum (5.26),(5.25) where $\widetilde{w}_T$ satisfies the bound (5.17).

Recall that by re-summing over graphs G corresponding to the same trees T_G we were left in the situation where the tree weights can be imagined (5.17) as products of bond weights $w^q_{i,j}\{x_i,x_j\}$ and "remaining" vertex weights $e^{-\widetilde{V}_q(x_i)}$. Let us decompose the remaining fugacity on the vertices of trees as follows

$$\widetilde{V}_q(x_i) = V^\circ_q(x_i) + V^\bullet_q(x_i) \tag{5.35}$$

where we make the somewhat non-optimal choice $V^\circ_q(x_i) = V^\bullet_q(x_i) = \frac{1}{2}\widetilde{V}_q(x_i)$.

$$\sum_j w^q_{i,j}\{x_i,x_j\}\ e^{-V^\bullet_q(x_j)} \leq\ \zeta \tag{5.36}$$

with $\zeta \leq e^{-\tau/4}$ for τ large enough, which is easy to see from (5.9).

In the sequel it will turn out convenient to forget that the summations are only over forests, and to retain only the bound (5.36).

For the bonds outside the body of a galaxy we will not need any decomposition (5.35); in these cases we incorporate the whole remaining fugacity $\widetilde{V}_q(x_i)$ into $w^q_{i,j}\{x_i,x_j\}$; this case deserves a new notation in which the bound (5.17) will be written in the following form:

$$|\widetilde{w}_T| \leq \prod_b |\widetilde{w}_b| \tag{5.37}$$

where the newly defined bond activities $\widetilde{w}_b, b = \{x_i,x_j\}$ (smaller than that of (5.36))

$$\widetilde{w}_b =\ w_b\ e^{-\widetilde{V}_q(x_i)} \tag{5.38}$$

satisfy a bound, with ε small enough of the order $\varepsilon = e^{-K\tau}$ where $K = K(\delta)$

$$\sum_{b\ni x_i} \widetilde{w}_b \leq \varepsilon. \tag{5.39}$$

Then we have the following result.

$$\tau^\circ_q = C_d\, V^\circ_q\ \left(\frac{\delta-\tilde{\delta}}{CD'_q}\right)^d. \tag{5.40}$$

PROPOSITION 5.4. *The galaxy activities can be estimated as follows (L denotes the union of all legs of* P^**)*

$$|\widetilde{w}(P^*)|\ \leq\ e^{-\tau^*_q|P^0|}\prod_{b\in L}|\widetilde{w}_b| \tag{5.41}$$

where $\widetilde{w}_b$ *are bond weights from (5.38) and the constant* τ^*_q *satisfies the bound*

$$e^{-\tau^*_q} = 4\ \left(e^{-\tau^\circ_q+\zeta}|S|\right)^\alpha \tag{5.42}$$

where (see (5.28)) $\alpha = \; C_d \; \left(\frac{\delta-\tilde{\delta}}{CD'_q}\right)^d$.

Note. Let us emphasise that the estimate (5.37), used below for the constellations, is now just a special case of the estimate (5.41), for an empty body $P^0 = \emptyset$.

Proof. We noted already that our new formulation (5.37) with (5.39) of the bounds for tree weights will allow to forget the restriction that our summation was over trees rather than general graphs. Let us consider a fixed set $\underline{P}^\circ$ (the body of P^0) together with a fixed set (skeleton) $P^\bullet = Y$ of cardinality n of stars x_i. We want first to perform the sum over all forests (more generally, graphs) having Y as their support and then to sum over all possible Y. Denote by

$$\rho(Y) \equiv \sum_{T:\underline{T}=Y} \widetilde{w}_T. \tag{5.43}$$

To estimate $\rho(Y)$, we note that from each point $i \in Y$, links may emanate to points at distance less than R. However, the sum of weights of these links satisfies (5.36) (if the $V_q^\circ(x_i)$ part of the fugacity is "assigned to vertices only") and so

$$|\rho(Y)| \leq e^{-V_q^\circ |Y|} \, e^{\zeta |Y|} \qquad \text{where} \qquad V_q^\circ = \min\nolimits_{s \neq q} V_q^\circ(s). \tag{5.44}$$

The exponential factor $e^{\zeta|Y|} = 1 + \zeta + \frac{\zeta^2}{2} \ldots$ (compare (5.36)) appears here for similar reasons as in (5.19)); to estimate (when taking product over all $x_i \in Y$) the contribution of all possible graphs (not only trees) on Y. We actually use slightly less precise bounds here than in (5.19) and the relative smallness of ζ w.r. to V_q° is important.

To finish our estimate the weight of galaxies, it remains to sum over all possible choices of the set Y, of stars, and over the sets M_1 such that $M_1^R = \underline{P}^0$. In the latter sum we are generous and bound it by $2^{|\underline{P}^0|}$, even though this can clearly be improved. In the sum over the sets Y we must of course retain the fact, established in Lemma 5.3 that the number of stars it proportional to the volume of $\underline{P}^0$. Thus , with $\alpha = C_d(\frac{\delta-\tilde{\delta})}{CD'_q})^d$ (see (5.28)) and with $n^0 = \alpha|P^0|$ we have the bound[12]

$$|\widetilde{w}(P^*)| \leq 2^{|P^0|} \sum_{Y \subset P^0 : |Y| \geq n^0} \rho(Y) \prod_{b \in L} |\widetilde{w}_b| \leq 2^{|P^0|} \sum_{n \geq n^0} \binom{|P^0|}{n} (|S|-1)^n e^{n(\zeta - V_q^\circ)} \prod_{b \in L} |\widetilde{w}_b|.$$

Using a simple bound $\sum_{n \geq k} \binom{N}{n} x^n \leq 2^N x^k$, $x \ll 1$ this gives the final estimate

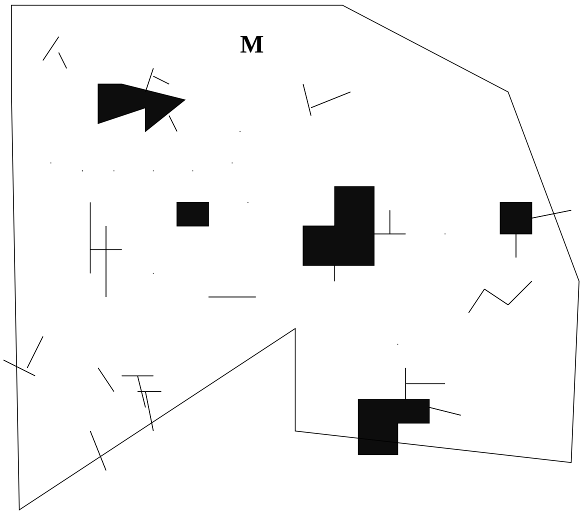
$$|\widetilde{w}(P^*)| \;\leq\; 4^{|P^0|} \left(e^{(\zeta - V_q^\circ)}(|S|-1)\right)^{\alpha|P^0|} \prod_{b \in L} |\widetilde{w}_b| \tag{5.45}$$

and this proves (5.41). $\diamondsuit$

---

[12] For a large continuity constant C in Assumption 3 this gives a rather poor estimate for τ_q^*, thus requiring correspondingly large value of τ.

At this stage we have reformulated the restricted partition function in terms of a polymer model where polymers are either constellations or galaxies, that are mutually disjoint, not interacting except for the volume exclusion, and whose activities satisfy exponential bounds that will be seen to ensure the convergence of the Mayer expansions, as we will explain shortly. The following picture depicts a typical configuration of such objects.



A set M with galaxies and constellations

5.8. Expression of the "restricted ensemble" through a polymer model..

Let us recall the form in which the partition function in a volume M (a connected component of the union of supports of all contours of x_Λ) may now be re-written:

$$Z_M^q(x_{M^c}) = \sum_{\{P_1^*,\dots,P_k^*,t_1,\dots,t_l\}_c} \prod_{i=1}^{n} \widetilde{w}(P_i^*) \prod_{j=1}^{m} \widetilde{w}_{t_j} \tag{5.46}$$

where the sum is over all compatible collections[13] of galaxies and constellations in M. Compatibility of a collection $\{P_i^*\} \cup \{t_j\}$ "in M" means that

$$\underline{P}_i^* \cap M \neq \emptyset,\ \underline{t}_j \cap M \neq \emptyset \qquad \text{and} \tag{5.47}$$

$$\underline{P}_i^* \cap (\underline{P}_1^* \dots \underline{P}_k^* \cup \underline{t}_1 \cdots \cup \underline{t}_l) = \emptyset,\ \ \underline{t}_j^* \cap (\underline{P}_1 \dots \underline{P}_k \cup \underline{t}_1 \cdots \cup \underline{t}_l) = \emptyset. \tag{5.48}$$

The following Corollary summarizes what we have done until now, in the reformulation of our original spin model.

COROLLARY 5.5.*For any torus Λ we have the expression*

$$Z(\Lambda) = \sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m,P_1^*,\dots,P_k^*,t_1,\dots,t_l\}_c} \prod_{i=1}^{m} e^{-E(\mathbf{\Gamma}_i)} \prod_q \exp(-U(q)\Lambda_q) \prod_{i=1}^{k} \widetilde{w}_{\mathbf{\Gamma}}(P_i^*) \prod_{j=1}^{l} \widetilde{w}_{t_j} \tag{5.49}$$

---

[13] The symbol $\{P_i\}_c$ will be used throughout to denote a *compatible* collection of polymers P_i.

where Λ_q *is the set of all* $q-$ *correct points of the system* $\{\mathbf{\Gamma}_1, \dots \mathbf{\Gamma}_n\}$ *of contours and the sum is taken over all compatible collections as defined by (5.47), (5.48). We normalized the potential such that* $\min U(q) = 0$*. The weights* $\widetilde{w}_{\mathbf{\Gamma}}(P_i^*)$ *and* $\widetilde{w}_t$ *satisfy the bounds stated in Lemma 5.2 and Proposition 5.4, respectively, and* $E(\mathbf{\Gamma})$ *satisfy the Peierls condition (4.3). Moreover, the weights of galaxies* $\widetilde{w}_{\mathbf{\Gamma}}(P^*) \equiv \tilde{w}(P^*)$ *depend on the collection of contours* $\mathbf{\Gamma} \equiv \{\mathbf{\Gamma}_1, \dots \mathbf{\Gamma}_n\}$ *only through the value of* q *if their support lies in* $\widetilde{\Lambda}_q$*, except if* $P*$ *intersects the support of* $\mathbf{\Gamma}$*, when it also may depend on the spin configuration on the R-boundary of* $\underline{\mathbf{\Gamma}}$*.*

5.9. Mayer expansion of the polymers.

The main step now is to perform a Mayer expansion for the sum over the constellations and galaxies. To simplify the language, we will call galaxies and constellations indiscriminately *polymers* and denote them by P. The purpose of the Mayer expansion is to write the sum over these polymers in the presence of a fixed collection of contours as an exponential of a sum of new objects, called 'clusters'. This sum over clusters can then be interpreted as a 'free energy' (depending on the colour of the set in which the polymers used to live), plus a correction to the contour energies. In this form the partition function is very similar to that of what is called an 'abstract Pirogov-Sinai model', the only difference being that the ground state energies of the different colors are replaced by free energy functions and that there are some non-local interactions between contours due to the interactions of clusters with the contours. In [Z3] it is shown that the Pirogov-Sinai theory can be developed without problems in this general context.

Performing the Mayer expansion as indicated above is actually a very standard exercise and it is well-known that this expansions converges under the conditions we have for the polymer activities, provided τ is large enough. The possibly simplest way of seeing this is by using a version of the Kotecký-Preiss-Dobrushin [KP,Do] method as given in [BZ2] (see also [NOZ]). In fact all we need is part of the Theorem 2.2 of that article, that we rephrased slightly for our purposes.

Let $\mathcal{P}$ be a collection of polymers, and let ι be a binary relation on $\mathcal{P}$ (called 'incompatibility'), and let c denote its logical converse (i.e. $P \, c \, P'$ unless $P \, \iota \, P$). The relation $\subset$ induces a natural graph structure on any set made of elements of $\mathcal{P}$. We will always assume that $P \, \iota \, P$, for all $P \in \mathcal{P}$.

Say that a set of elements of $\mathcal{P}$ (with multiplicity) is compatible, if the corresponding graph has no edges, and call it a *cluster* if its graph is connected. Let $\mathcal{C}(\mathcal{P})$ denote the set of clusters (all connected graphs on $\mathcal{P}^{\mathbb{N}}$), and $\mathcal{F}(\mathcal{P})$ denote the set of all compatible subsets of $\mathcal{P}$.

Let $w : \mathcal{P} \to \mathbb{R}$ be some function on $\mathcal{P}$.

THEOREM 5.6. [BZ2]*Assume that for some function* $a : \mathcal{P} \to \mathbb{R}^+$*, for some* $0 < \delta < 1$*, for all* $P \in \mathcal{P}$

$$|w(P)| e^{a(P)} \leq \delta \tag{5.50}$$

and for any $Q \in \mathcal{P}$,

$$\sum_{P \in \mathcal{P} \setminus Q,\, P \,\iota\, Q} |w(P)| e^{a(P)+d(P)} \leq \frac{a(Q)}{L(\delta)} \tag{5.51}$$

where $L(\delta) = \frac{-\ln(1-\delta)}{\delta}$ *and where* $d : \mathcal{P} \to \mathbb{R}^+$ *is another function of* P. *Then*

$$\ln \sum_{F \in \mathcal{F}(\mathcal{P})} \prod_{P \in F} w(P) = \sum_{C \in \mathcal{C}(\mathcal{P})} \widehat{w}(C) \tag{5.52}$$

where $\widehat{w}(C)$ *satisfies for all* $Q \in \mathcal{P}$ *the bound*

$$\sum_{C \in \mathcal{C}(\mathcal{P}),\, C \,\iota\, Q} |\widehat{w}(C)| e^{d(C)} \leq a(Q) \tag{5.53}$$

where $d(C) = \sum_{P \in C} d(P)$ *measures the decay of* $\widehat{w}(C)$.

We will use this theorem with $\mathcal{P}$ be the collection of all trees and galaxies that can exist in the presence of a given configuration of contours $\mathbf{\Gamma}$, and with $w(P)$ the corresponding activities $\widetilde{w}_t$ and $\widetilde{w}_{\mathbf{\Gamma}}(P^*)$ depending on whether P is a tree or a galaxy. As the function $a(P)$ we will simply use $a|t|$ resp. $b|\underline{P}^*|$ with suitable constants a and b. Since activities are decaying exponentially in these same quantities with a rate controllable by τ, and the number of these objects of given size is at most exponentially increasing with a rate that is independent of both τ and R, it is an elementary exercise that the hypothesis of Theorem 5.6 are satisfied if τ is large enough.

Next we need to understand a little more about the geometric structure of the clusters obtained by computing the logarithm of the sum over all polymers existing in the presence of $\mathbf{\Gamma}$. Recall the the only possible polymers are whose whose support is in the regions $\widetilde{\Lambda}_q^R$. Therefore it is geometrically impossible that polymers whose support intersects different connected components of the complement of $\underline{\mathbf{\Gamma}}$ are incompatible, implying that the sum over all clusters can be split into a sum over contributions from clusters whose support intersects different connected components of the complement of $\underline{\mathbf{\Gamma}}$. Next we observe that if a cluster does not intersect the support of $\mathbf{\Gamma}$, then its weight is independent of the the contours $\mathbf{\Gamma}$ except that it depends on the color of the region it is supported in. Since in all other respects cluster weights are translationally invariant, we can introduce translation invariant cluster weights $w^q(C)$ which are simply equal to the weight of a translate of the cluster C that has support in $\widetilde{\Lambda}_q$ and does not touch any contour. We may also consider the ensemble $\mathcal{F}^q$ of all polymers that may exist in an infinite volume restricted ensemble of color q and the corresponding set of all cluster $\mathcal{C}^q$. Let us also define

$$f_q \equiv U(q) - \sum_{C \in \mathcal{C}^q,\, \underline{C} \ni 0} \frac{\widehat{w}^q(C)}{|\underline{C}|} \tag{5.54}$$

Then Theorem 5.6 allows us to express the partition function as follows.

THEOREM 5.7. *There exist $\tau_0 < \infty$, independent of R such that the following holds if $\tau > \tau_0$:*

$$
\begin{aligned}
Z_\Lambda &= \sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m\}} \prod_{i=1}^m e^{-E(\mathbf{\Gamma}_i)} \exp\left(-\sum_{q\in Q} U(q)|\Lambda_q| + \sum_{C\in\mathcal{C}(\mathbf{\Gamma})} \widehat{w}_{\mathbf{\Gamma}}(C)) \right) \\
&= \sum_{\{\mathbf{\Gamma}_1,\dots\mathbf{\Gamma}_m\}} \prod_{i=1}^m e^{-E(\mathbf{\Gamma}_i)} \exp\left(-\sum_{q\in Q} f_q|\Lambda_q| + \sum_{C\in\widetilde{\mathcal{C}}:\underline{C}\cap\underline{\mathbf{\Gamma}}\neq\emptyset} \widetilde{w}(C) \right)
\end{aligned}
\tag{5.55}
$$

where $\mathbf{\Gamma} \equiv \{gb_1, \dots, \mathbf{\Gamma}?n\}$, $\mathcal{C} \equiv \mathcal{C}(\mathbf{\Gamma})$ is the collection of clusters constructed by Theorem 5.6 and in the second formula we use the resummation (5.54). Then the sum is only over clusters intersecting $\underline{\mathbf{\Gamma}}$. The values $\widetilde{w}(C)$ are then defined, for any $C \in \widetilde{\mathcal{C}}$ such that $\underline{C} \cap \underline{\mathbf{\Gamma}} \neq \emptyset$ as

$$
\widetilde{w}(C) = \widehat{w}_{\mathbf{\Gamma}}(C) - \sum_q \widehat{w}^q(C) \frac{|\underline{C} \setminus \underline{\mathbf{\Gamma}}|}{|\underline{C}|} \tag{5.56}
$$

where we set $\widehat{w}_{\mathbf{\Gamma}} = 0$ if $C \notin \mathcal{C}(\mathbf{\Gamma})$.

The quantities f_q are analytic functions of the parameters of the Hamiltonian, and the cluster weights $\widetilde{w}(C)$ satisfy an estimate

$$
\sum_{C:\underline{C}\ni i} |\widetilde{w}(C)| e^{c\tau|\underline{C}|} \leq e^{-c'\tau} \tag{5.57}
$$

for $c, c' > 0$ independent of R.

Proof. This theorem is a straightforward consequence of Corollary 5.5 and Theorem 5.6, as explained above. ♢

With Theorem 5.7 we have achieved the goal of this paper: we have formulated the partition function to our models in the form of an abstract Pirogov–Sinai model in the sense of [Z3], with bounds on the Peierls constant and the cluster weights that do not depend on the range of the interaction R but only on the overall strength expressed by τ. Theorem 2.2 now follows from the general results of [Z3].

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