Some limiting properties of randomly forced 2D Navier–Stokes equations

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Abstract

We consider random perturbations of 2D Navier–Stokes equations. Under some natural conditions on random forces, we study asymptotic properties of solutions and stationary measures.

0 Introduction

We consider the 2D Navier–Stokes (NS) system perturbed by a random force:

$$\dot{u} - \nu\Delta u + (u, \nabla)u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0. \tag{0.1}$$

The space variable x belongs either to a bounded domain, and then the Dirichlet boundary condition is imposed, or to the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, and then we assume that $\int_{\mathbb{T}^2} u\,dx \equiv \int_{\mathbb{T}^2} \eta\,dx \equiv 0$. Denoting by H the corresponding L^2-space of divergence-free vector fields with the natural norm $|\cdot|$ and by Π the orthogonal projections to H, we write (0.1) as a random system in H:

$$\dot{u}(t) + \nu L u(t) + B(u(t), u(t)) = \eta(t). \tag{0.2}$$

Here $L = -\Pi\Delta$, $B(u, u) = \Pi(u, \nabla)u$, and $\eta(t) = \Pi\eta(t, \cdot)$ (see [CF88]). We denote by $\{e_j\}$ the Hilbert basis in H formed by the eigenfunctions of the operator L with eigenvalues $0 < \alpha_1 \le \alpha_2 \le \cdots$. Concerning the force η, we assume that it is smooth in x, whereas as a function of time t it is either a kick-process or a white noise. Thus, we consider the following two cases:

- The function η models random kicks with some period $\varepsilon > 0$:

$$\eta = \eta_\varepsilon(t, x) = \sqrt{\varepsilon} \sum_{k=-\infty}^{\infty} \eta_k(x)\delta(t - k\varepsilon), \quad \eta_k(x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x), \tag{0.3}$$

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where $\{b_j\}$ is a sequence of nonnegative constants such that $\sum_j b_j^2 < \infty$ and $\{\xi_{jk}\}$ is a family of bounded independent scalar random variables with k-independent distributions that satisfy certain regularity assumptions (see condition (H) in Section 3).

- The function $\eta(t)$ is the time derivative of an H-valued Wiener process:

$$\eta = \eta(t,x) = \frac{\partial}{\partial t}\zeta(t,x), \quad \zeta(t,x) = \sum_j b_j \beta_j(t) e_j(x), \qquad (0.4)$$

where $\{b_j\}$ is a sequence of constants as above and $\{\beta_j\}$ is a family of independent standard Brownian motions.

Under the assumption that

$$b_j \neq 0 \quad \text{for} \quad 1 \leq j \leq N, \qquad (0.5)$$

where $N = N_{\varepsilon\nu} \geq 0$ is a suitable integer, it was proved in [KS00] that the NS system (0.2), (0.3) has a unique stationary measure $\mu = \mu_{\varepsilon\nu}$ and that all solutions converge to μ in distribution:

$$\mathbb{E} f(u(t)) \to \int_H f(v)\mu(dv) \quad \text{as} \quad t \to \infty, \qquad (0.6)$$

where f is any bounded continuous functional on H. In subsequent publications, we have shown that the convergence is exponentially fast provided that the functional f is Lipschitz continuous; see [Kuk02] for discussions and references.

The NS system (0.2), (0.4), (0.5) was considered later by E, Mattingly, Sinai [EMS01] and Bricmont, Kupiainen, Lefevere [BKL02]. Under the additional restriction that the sum in (0.4) is finite, they proved that the system has a unique stationary measure and studied convergence of solutions to this measure; see [Kuk02] for discussions and references. In the work [KS02] we applied to the white-forced NS system (0.2), (0.4), (0.5) the techniques developed for studying the kicked equations. It was proved that convergence (0.6) is exponentially fast for any bounded Lipschitz functional f and any solution $u(t)$ of the NS system such that $\mathbb{E}|u(0)|^2 < \infty$.

In this work, we specify convergence (0.6) in the case of the white-forced equation and study relations between the stationary measures for the kick- and white-forced equations. The paper is organised as follows.

In Section 1, we establish a priori estimates for moments of Sobolev norms of solutions for the white-forced NS system under the periodic boundary conditions. In Section 2, we use these estimates to prove Theorem 2.1, which claims that convergence (0.6) holds for functionals f that are defined and Hölder continuous on a Sobolev space H^k (with an arbitrary $k \geq 0$) and have a polynomial growth at infinity. This generalisation is important since functionals relevant for statistical hydrodynamics are unbounded and usually are not continuous on the space H. For example, the correlation tensor gives rise to functionals $f(u) = u_i(x)u_j(y)$, where x and y are points in the space domain and $i, j = 1, 2$.

Section 3 is devoted to a comparison of solutions and stationary measures for kick- and white-forced equations. Namely, let $u_\varepsilon(t)$ and $u(t)$ be solutions of the NS system (0.2) that correspond to a deterministic initial function $u_0 \in H$ and right-hand sides (0.3) and (0.4), respectively. Assuming that

$$\mathbb{E}\xi_{jk} \equiv 0, \quad \mathbb{E}\xi_{jk}^2 \equiv 1, \tag{0.7}$$

we first show that $u_\varepsilon(t)$ converges to $u(t)$ in distribution. Furthermore, if all b_j are nonzero, then the kick-forced equation (0.2), (0.3) has a unique stationary measure μ_ε for any $\varepsilon > 0$. The main results of this paper is Theorem 3.1, according to which we have

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{as} \quad \varepsilon \to 0, \tag{0.8}$$

where μ is the unique stationary measure for the white-forced equation. Since $\mathcal{D}(u_\varepsilon(t)) \rightharpoonup \mu_\varepsilon$ as $t \to \infty$ and $\mathcal{D}(u_\varepsilon(t)) \rightharpoonup \mathcal{D}(u(t))$ as $\varepsilon \to 0$, convergence (0.8) implies that we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{D}\big(u_\varepsilon(t)\big) & \xrightarrow{t\to\infty} & \mu_\varepsilon \\
{\scriptstyle \varepsilon\to 0}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon\to 0} \\
\mathcal{D}\big(u(t)\big) & \xrightarrow{t\to\infty} & \mu
\end{array}$$

Convergence (0.8) manifests the fact that the white forced equations occupy a special place among randomly forced equations. The results proved and discussed in this paper suggests the following conjecture:

Conjecture. Let us set

$$\eta_\varepsilon(t,x) = \sqrt{\varepsilon}\sum_{j=1}^{\infty} b_j \xi_j(t/\varepsilon) e_j(x),$$

where $0 < \varepsilon \le 1$, $\{\xi_j\}$ is a sequence of i.i.d. stationary processes that are mixing (in an appropriate sense) and satisfy the conditions $\mathbb{E}\xi_j(t) \equiv 0$ and $\mathbb{E}\xi_j^2(t) \equiv 1$. Then the assertions below take place on condition that $b_j \neq 0$ for all $j \geq 1$:

(i) Equation (0.2) with $\eta = \eta_\varepsilon(t,x)$ has a unique stationary measure μ_ε.

(ii) All solutions for (0.2) with deterministic initial data converge to μ_ε in distribution as $t \to \infty$.

(iii) We have $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$, where μ is the unique stationary measure for the white-forced equation.

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1 A priori estimates for higher Sobolev norms

1.1 Periodic boundary conditions

Let $H^s(\mathbb{T}^2)$ be the Sobolev space of order s on $\mathbb{T}^2$ endowed with the norm $\|u\|_s = |L^{s/2}u|$ and let $H^s = H^s(\mathbb{T}^2) \cap H$. We shall also use the notation $\|\cdot\| = \|\cdot\|_1$. In this subsection, we derive some estimates for the expectation $\mathbb{E}\|u\|_s^m$, where $m \geq 1$ and $s \geq 0$ are arbitrary integers and $u(t)$ is a solution of the NS system (0.2), (0.4) under the periodic boundary conditions.

Let us set $b_{\max} = \sup_j b_j$ and $B_k = \sum_j \alpha_j^k b_j^2$. In what follows, we shall always assume that

$$B_0 < \infty, \quad \mathbb{E}e^{\sigma\nu|u_0|^2} < \infty, \tag{1.1}$$

where u_0 is the initial function and $\sigma > 0$ is so small that $2\sigma b_{\max}^2 \leq \alpha_1$. Condition (1.1) ensures that the Cauchy problem has a unique solution $u(t)$ in an appropriate functional space, and the following inequality holds for a second exponential moment of $u(t)$ (see [BKL00, Lemma 1] and [Shi02, Remark 3.2]):

$$\mathbb{E}e^{\sigma\nu|u(t)|^2} \leq e^{-\sigma\nu t}\mathbb{E}e^{\sigma\nu|u_0|^2} + C, \quad t \geq 0, \tag{1.2}$$

where $C > 0$ depends only on σ and B_0.

For any $T > 0$ and any integer $k \geq 0$, we set

$$V_k(T) = \sup_{0\leq t\leq T}\Big(t^k\|u(t)\|_k^2 + \nu\int_0^t s^k\|u(s)\|_{k+1}^2 ds\Big). \tag{1.3}$$

In the theorem below, we confine ourselves to the case $\nu = 1$. The dependence of the constants on ν will be specified later.

Theorem 1.1. *Suppose that $B_k < \infty$ for an integer $k \geq 0$. Then for any $T > 0$ and any integer $p \geq 1$ there is a constant $C_{kp} = C_{kp}(\sigma, T) > 0$ such that*

$$\mathbb{E}V_k(T)^p \leq C_{kp}\mathbb{E}e^{\sigma|u_0|^2}. \tag{1.4}$$

Inequality (1.4) immediately implies a priori estimates for moments of the random variables

$$U_k(t_0, T) = \sup_{t_0\leq t\leq t_0+T}\Big(\|u(t)\|_k^2 + \nu\int_{t_0}^t \|u(s)\|_{k+1}^2 ds\Big). \tag{1.5}$$

Namely, we have the following corollary:

Corollary 1.2. *Under the conditions of Theorem 1.1, for any k, p, $t_0 \geq 1$, and $T > 0$ we have*

$$\mathbb{E}U_k(t_0, T)^p \leq C'_{kp}(\sigma, T)\left(e^{-\sigma t_0}\mathbb{E}e^{\sigma|u_0|^2} + 1\right), \tag{1.6}$$

where the positive constant $C'_{kp}(T)$ does not depend on t_0 and u_0.

To establish inequality (1.6), it suffices to apply an analogue of Theorem 1.1 for the interval $[t_0-1, t_0+T]$ and use (1.2) to estimate $\mathbb{E} e^{\sigma|u(t_0-1)|^2}$.

Proof of Theorem 1.1. For $k=0$, inequality (1.4) follows from Lemma 1.3 in [KS02]. We now assume that $k=m\ge 1$ and that for $k\le m-1$ inequality (1.4) is already established. Application of Itô's formula to the functional $f_m(t)=t^m\|u(t)\|_m^2$ results in

$$df_m(t)=mt^{m-1}\|u\|_m^2 dt-2t^m\left\|L^{\frac{m+1}{2}}u\right\|^2 dt-$$
$$-2t^m\left(L^{\frac{m+1}{2}}u, L^{\frac{m-1}{2}}B(u,u)\right)dt+B_m t^m dt+2t^m\left(L^{\frac{m}{2}}u, L^{\frac{m}{2}}d\zeta\right). \quad (1.7)$$

Integrating (1.7) with respect to t and using the inequality (see [KS01, Section 6.3])

$$\left|\left(L^{\frac{m+1}{2}}u, L^{\frac{m-1}{2}}B(u,u)\right)\right| \le c_m'\|u\|_{m+1}^{\frac{4m-1}{2m}}\|u\|^{\frac{m+1}{2m}}|u|^{\frac{1}{2}}$$
$$\le \tfrac{1}{4}\|u\|_{m+1}^2+c_m\|u\|^{2(m+1)}|u|^{2m},$$

we derive

$$t^m\|u(t)\|_m^2+\nu\int_0^t s^m\|u(s)\|_{m+1}^2 ds\le m\int_0^t s^{m-1}\|u\|_m^2 ds+$$
$$+\tfrac{B_m t^{m+1}}{m+1}+2c_m\int_0^t s^m\|u\|^{2(m+1)}|u|^{2m}ds+\xi_m(t),$$

where $c_m=0$ for $m=1$ and

$$\xi_m(t)=2\int_0^t s^m\left(L^{\frac{m}{2}}u, L^{\frac{m}{2}}d\zeta\right)-\frac{1}{2}\int_0^t s^m\|u\|_{m+1}^2 ds.$$

It follows that

$$V_m(T)\le mV_{m-1}(T)+\tfrac{B_m T^{m+1}}{m+1}+2c_m V_0(T)^{m+1}V_1(T)^m+\sup_{0\le t\le T}\xi_m(t). \quad (1.8)$$

We now repeat the argument used in the proof of Lemma 1.3 in [KS02] to show that, if $\gamma>0$ is sufficiently small, then

$$\mathbb{P}\Big(\sup_{0\le t\le T}\xi_m(t)\ge\rho\Big)\le e^{-\gamma\rho}, \quad \rho\ge 0. \quad (1.9)$$

Indeed, the quadratic variation of the integral $M_t := \int_0^t s^m(L^{\frac{m}{2}}u, L^{\frac{m}{2}}d\zeta)$ has the form

$$\langle M\rangle_t=\sum_j \alpha_j^{2m}b_j^2\int_0^t s^{2m}u_j^2(s)\,ds\le B_{m-1}\,T^m\int_0^t s^m\|u\|_{m+1}^2 ds$$
$$\le\frac{1}{4\gamma}\int_0^t s^m\|u\|_{m+1}^2 ds,$$

where we set $\gamma = (4T^m B_{m-1})^{-1}$. It follows that

$$\xi_m(t) \le 2M_t - 2\gamma\langle M\rangle_t.$$

Since $\exp(2\gamma M_t - 2\gamma^2\langle M\rangle_t)$ is a supermartingale whose mean value does not exceed 1, the required estimate (1.9) follows from the classical supermartingale inequality (see Theorem VI.T1 in [Mey66] or Theorem III.6.11 in [Kry95]).

Combining inequalities (1.8) and (1.9) and the induction hypothesis, we see that all the moments of $V_m(T)$ are finite. This completes the proof of the theorem. $\square$

We now study the dependence of the constant C'_{kp} in (1.6) on ν. To simplify the presentation, we shall only consider the case of a stationary solution.

Let us begin with the situation in which the right-hand side is analytic in x. More precisely, we assume that

$$\sum_{j=1}^{\infty} \exp(2\rho\sqrt{\alpha_j})b_j^2 < \infty, \tag{1.10}$$

where $\rho > 0$ is a constant not depending on ν.

Proposition 1.3. *Let $u(t,x)$ be a stationary solution of* (0.2), (0.3). *Then for any integers $k, m \ge 1$ and any $\varepsilon > 0$ and $T > 0$ there is a constant $C_{km} = C_{km}(\varepsilon, T) > 0$ such that*

$$\mathbb{E}\Bigl(\sup_{t_0\le t\le t_0+T} \bigl\|u(t)\bigr\|_k^m\Bigr) \le C_{km}\nu^{-2km+\frac{3m}{2}-\varepsilon}, \quad t_0 \ge 0. \tag{1.11}$$

We note that, for $k = 1$, a stronger assertion holds if we are interested in estimating the mean value at a fixed point t (see [Shi02, Theorem 2.2]): there is a constant $C > 0$ such that

$$\mathbb{E}\exp\bigl(\sigma\nu\|u(t,\cdot)\|^2\bigr) \le C, \quad t \ge 0. \tag{1.12}$$

In particular, since $y^m \le C_m\nu^{-\frac{m}{2}}e^{\sigma\nu y^2}$, for any $m \ge 1$ we have

$$\mathbb{E}\|u(t,\cdot)\|^m \le C_m\,\nu^{-\frac{m}{2}}, \quad t \ge 0.$$

Proof of Proposition 1.3. Since $u(t,x)$ is a stationary solution, it suffices to consider the case $t_0 = 0$. According to Theorems 2.1 and 2.2 in [Shi02], for any $\delta > 0$ and $T > 0$ there is a positive random variable r_ν such that for any integer $m \ge 1$ and an appropriate constant $K_m > 0$ we have

$$\mathbb{E}\Bigl(\sup_{0\le t\le T}\bigl\|\exp(r_\nu\nu^{2+\delta}L^{\frac12})u(t,\cdot)\bigr\|^m\Bigr) \le K_m\nu^{-\frac{m}{2}}, \quad \mathbb{E}r_\nu^{-m} \le K_m. \tag{1.13}$$

Let us fix an arbitrary integer $k \ge 1$. Direct verification shows that

$$\|u\|_k^2 = \sum_j \alpha_j^k|u_j|^2 \le C_k\nu^{-2(k-1)(2+\delta)}\,r_\nu^{-2(k-1)}\bigl\|\exp(r_\nu\nu^{2+\delta}L^{\frac12})u\bigr\|^2,$$

where $C_k > 0$ depends only on k. It follows that

$$\sup_{0\le t\le T} \|u\|_k^m \le C_k^{\frac{m}{2}}\, \nu^{-m(k-1)(2+\delta)}\, r_\nu^{-m(k-1)} \sup_{0\le t\le T} \bigl\|\exp(r_\nu \nu^{2+\delta} L^{\frac{1}{2}})u\bigr\|^m. \quad (1.14)$$

Combining (1.13) and (1.14) and choosing a sufficiently small $\delta = \delta_{km}(T) > 0$, we obtain (1.11). □

We conclude this subsection with a short discussion of the case when the right-hand side η is smooth, but not analytic¡ with respect to x.

Proposition 1.4. *Let $B_k < \infty$ for an integer $k \ge 0$ and and let $u(t,x)$ be a stationary solution. Then for any integer $m \ge 1$ and any $T > 0$ there is a constant $C_{km}(T) > 0$ not depending on ν and $u(t,x)$ such that*

$$\mathbb{E}\Bigl(\sup_{t_0\le t\le t_0+T} \bigl\|u(t)\bigr\|_k^m\Bigr) \le C_{km}(T)\, \nu^{-3km-\frac{m}{2}}, \quad t_0 \ge 0. \quad (1.15)$$

To establish inequality (1.15), it suffices to repeat the proof of Theorem 1.1, following the dependence of the constants on ν. We shall not dwell on it.

1.2 Dirichlet boundary condition

In this subsection, we briefly discuss the case in which Eq. (0.2), (0.3) is studied in a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary and is supplemented with the Dirichlet boundary condition.

As in the case of the periodic boundary conditions, we assume that (1.1) holds. This implies that the Cauchy problem has a unique solution. Moreover, inequality (1.2) remains valid for $2\sigma b_{\max}^2 \le \alpha_1$ (see [Shi02, Remark 3.2]). In particular, for any stationary solution $u(t,x)$ we have

$$\mathbb{E}\, e^{\sigma\nu|u(t)|^2} \le C, \quad t \ge 0, \quad (1.16)$$

whence it follows that

$$\mathbb{E}\bigl|u(t)\bigr|^m \le C_m \nu^{-\frac{m}{2}}, \quad t \ge 0, \quad m \ge 1. \quad (1.17)$$

2 Convergence to the stationary measure

In this section, we assume that $\nu = 1$ and that $B_{k+1} < \infty$ for some $k \ge 0$. We denote by $C_b(H)$ the space of bounded continuous functionals on H and by $\mathcal{P}(H)$ the set of all probability Borel measures on H. Let $P_t(u,\Gamma)$, $u \in H$, $\Gamma \in \mathcal{B}(H)$, be the transition function associated with the NS system (0.2) and let $\mathfrak{P}_t : C_b(H) \to C_b(H)$ and $\mathfrak{P}_t^* : \mathcal{P}(H) \to \mathcal{P}(H)$ be the corresponding Markov semigroups:

$$\mathfrak{P}_t f(u) = \int_H P_t(u,dv) f(v), \quad \mathfrak{P}_t^* \mu(\Gamma) = \int_H P_t(u,\Gamma)\mu(dv),$$

where $f \in C_b(H)$ and $\mu \in \mathcal{P}(H)$. A measure $\mu \in \mathcal{P}(H)$ is said to be *stationary* if $\mathfrak{P}_t^* \mu = \mu$ for all $t \geq 0$.

We denote by $u(t; u_0)$ a solution for (0.2) equal to a random variable u_0 at $t = 0$. In [KS02], we prove that the NS equation has a unique stationary measure μ and

$$\left| \mathbb{E} f\big(u(t;u_0)\big) - \int_H f(v)\mu(dv) \right| \leq C\left(1 + \mathbb{E}|u_0|^2\right) e^{-\alpha\gamma t}, \quad t \geq 0, \tag{2.1}$$

for any function $f \in C_b(H)$ satisfying the conditions

$$\sup_{u\in H} \big|f(u)\big| \leq 1, \quad \sup_{u \neq v} \frac{|f(u) - f(v)|}{|u - v|^\alpha} \leq 1.$$

Here α is any number such that $0 < \alpha \leq 1$, and the constants $C \geq 1$ and $\gamma > 0$ are independent of α. Inequality (2.1) implies that

$$\mathcal{D}\big(u(t;u_0)\big) \rightharpoonup \mu \quad \text{as} \quad t \to \infty, \quad \text{for all} \quad u_0 \in H. \tag{2.2}$$

Below we use the results from the previous section to show that estimates similar to (2.1) hold for a much larger class of functionals f.

For $\alpha \in (0,1]$ and integers $p \geq 1$ and $l \geq 0$, we denote by $\mathcal{O}_l^p(\alpha)$ the set of continuous functionals g on H^l such that

$$\big|g(u)\big| \leq 1 + \|u\|_l^p, \tag{2.3}$$

$$\big|g(u) - g(v)\big| \leq \|u - v\|_l^\alpha \big(1 + \|u\|_l^{p-1} + \|v\|_l^{p-1}\big), \tag{2.4}$$

where $u, v \in H^l$. Let us take any constant $R \geq 1$ and consider two solutions $\tilde{u}_1(t)$ and $\tilde{u}_2(t)$ that correspond to initial functions u_{10} and u_{20} satisfying the inequality $\mathbb{E} e^{\sigma |u_{i0}|^2} \leq R < \infty$ for some $\sigma > 0$. It is proved in [KS02] (see there relation (2.39)) that there are weak solutions u_1 and u_2 for (0.2) that are distributed as $\tilde{u}_1$ and $\tilde{u}_2$, respectively, and satisfy the inequality

$$\begin{aligned} \mathbb{P}\left\{|u_1(t) - u_2(t)| \leq C_1 e^{-\gamma t}\right\} &\geq 1 - C_1\big(1 + \mathbb{E}|u_{10}|^2 + E\,|u_{20}|^2\big)e^{-\gamma t} \\ &\geq 1 - C_2 R\, e^{-\gamma t}. \end{aligned} \tag{2.5}$$

Denoting by $\Omega_1(t)$ the event in the left-hand side of (2.5), we have

$$\mathbb{P}\big(\Omega_1(t)^c\big) \leq C_2 R\, e^{-\gamma t}, \tag{2.6}$$

$$|u_1(t) - u_2(t)| \leq C_1 e^{-\gamma t}, \quad \omega \in \Omega_1(t). \tag{2.7}$$

Let us fix any $p \geq 1$, $k \geq 0$, and $\alpha \in (0,1]$ and take a functional $g \in \mathcal{O}_k^p(\alpha)$. Assuming that $B_{k+1} < \infty$, we conclude from Corollary 1.2 that

$$\mathbb{E}\,\big\| u_i(t)\big\|_r^m \leq K(r,m,\sigma)R, \quad t \geq 1, \tag{2.8}$$

where $i = 1, 2$, $0 \leq r \leq k+1$, $m \geq 1$, and $K(r,m,\sigma) > 0$ is a constant not depending on solutions. Using the interpolation inequality

$$\big\|u\big\|_k \leq \big|u\big|^{\frac{1}{k+1}} \big\|u\big\|_{k+1}^{\frac{k}{k+1}},$$

we derive from (2.4) and (2.7) that

$$\begin{aligned}\big|g(u_1(t)) - g(u_2(t))\big| &\le \big\|u_1(t) - u_2(t)\big\|_k^{\alpha}\big(1 + \|u_1(t)\|_k^{p-1} + \|u_2(t)\|_k^{p-1}\big) \\ &\le C_3\, e^{-\gamma_k \alpha t}\big(1 + \|u_1(t)\|_{k+1}^p + \|u_2(t)\|_{k+1}^p\big), \qquad (2.9)\end{aligned}$$

where $\omega \in \Omega_1(t)$ and $\gamma_k = \frac{\gamma}{k+1}$. It follows from (2.9) and (2.8) that

$$\mathbb{E}\Big(I_{\Omega_1(t)}\big|g(u_1(t)) - g(u_2(t))\big|\Big) \le C_4 K(k+1,p,\sigma) R\, e^{-\gamma_k \alpha t}, \quad t \ge 0. \qquad (2.10)$$

Furthermore, in view of (2.3) and (2.6), we have

$$\begin{aligned}\mathbb{E}\Big(I_{\Omega_1(t)^c}\big|g(u_1(t)) - g(u_2(t))\big|\Big) &\le \mathbb{E}\Big(I_{\Omega_1(t)^c}\big(2 + \|u_1(t)\|_k^p + \|u_2(t)\|_k^p\big)\Big) \\ &\le \mathbb{P}\big(I_{\Omega_1(t)^c}\big)^{\frac{1}{2}}\Big(2 + \big(\mathbb{E}\|u_1(t)\|_k^{2p}\big)^{\frac{1}{2}} + \big(\mathbb{E}\|u_1(t)\|_k^{2p}\big)^{\frac{1}{2}}\Big) \\ &\le C_5 K(k,2p,\sigma)^{\frac{1}{2}} R\, e^{-\gamma t/2}. \qquad (2.11)\end{aligned}$$

Combining (2.10) and (2.11), we obtain

$$\big|\mathbb{E}g(u_1(t)) - \mathbb{E}g(u_2(t))\big| \le C(k,p,\sigma)\, R\, e^{-\gamma_k' \alpha t}, \quad t \ge 1, \qquad (2.12)$$

where $g \in \mathcal{O}_k^p(\alpha)$. In particular, assuming that $u_2(t)$ is a stationary solution, we arrive at the following result:

Theorem 2.1. *Let $B_{k+1} < \infty$ for some integer $k \ge 0$, let μ be the stationary measure for the NS system (0.2), (0.3) with the periodic boundary conditions and let $u(t) = u(t; u_0)$ be a solution for (0.2) with initial function u_0 such that $R := \mathbb{E}e^{\sigma |u_0|^2} < \infty$, where $\sigma > 0$ is a constant. Then for any $p \ge 1$, $\alpha \in (0,1]$, and $g \in \mathcal{O}_k^p(\alpha)$ we have*

$$\big|\mathbb{E}g\big(u(t)\big) - (\mu, g)\big| \le C(k,p,\sigma)\, R\, e^{-\gamma' t}, \quad t \ge 0, \qquad (2.13)$$

where $\gamma' > 0$ depends on α and k.

The results from [KS02] given in (2.1) and (2.5) apply to solutions of the NS equations both under the Dirichlet and periodic boundary conditions. Since our proof of Theorem 2.1 uses estimates (2.8) for $r = k+1 \ge 1$, it applies for the periodic case only. Still, if we take $k = 0$, then inequality (2.8) with $r = 0$ remains valid (see (1.2) and Subsection 1.2), while (2.8) with $r = 1$ was used only in the derivation of inequality (2.10). The latter is now implied by (2.6), (2.7), and (2.8) with $r = 0$. After (2.8) and (2.10) are established, the proof of inequality (2.12) goes as before. Therefore, we have the following result:

Proposition 2.2. *If $k = 0$, then the assertion of Theorem 2.1 holds true for solutions of the NS system under Dirichlet's boundary condition.*

In fact, the arguments used to prove (2.13) with $k = 0$ apply to functionals $g \in C_b(H)$ such that

$$\big|g(u) - g(v)\big| \le |u - v|\,\big(e^{\varepsilon |u|^2} + e^{\varepsilon |u|^2}\big),$$

where $u, v \in H$, and $\varepsilon > 0$ is a sufficiently small constant.

3 Kick-forced equations

3.1 Statement of the results

In this section, we consider the NS system (0.2) under the periodic boundary conditions, perturbed by short short-period random kick-force η_ε defined in (0.3):

$$\dot u_\varepsilon + \nu L u_\varepsilon + B(u_\varepsilon, u_\varepsilon) = \eta_\varepsilon(t,x). \tag{3.1}$$

We assume that $0 < \varepsilon \le 1$, the constants $b_j \ge 0$ are the same as in (0.4), and the distributions $\mathcal{D}(\xi_{jk})$ of the independent random variables ξ_{jk} satisfy (0.7) and the following condition:

(H) $\mathcal{D}(\xi_{jk}) = p_j(s)\,ds$, where p_j's are functions of bounded total variation such that $\operatorname{supp} p_j \subset [-1,1]$ and $\int_{-\delta}^{\delta} p_j(s)\,ds > 0$ for all $j \ge 1$ and $\delta > 0$;

To simplify notations, in what follows it is assumed that $\nu = 1$. We normalise solutions of (3.1) to be continuous from the right and set $u_\varepsilon^k = u_\varepsilon(k\varepsilon)$. Denoting by S_ε the operator of time ε shift along trajectories of the free NS system, we see that

$$u_\varepsilon^k = S_\varepsilon\left(u_\varepsilon^{k-1}\right) + \eta_k. \tag{3.2}$$

Since $|S_\varepsilon(u)| \le e^{-\varepsilon}|u|$ and $\mathbb{E}\eta_k = 0$ due to (0.7), it follows from the independence of η_k and u_ε^k that

$$\mathbb{E}\left|u_\varepsilon^k\right|^2 = \mathbb{E}\left|S_\varepsilon(u_\varepsilon^{k-1})\right|^2 + \mathbb{E}|\eta_k|^2 \le e^{-2\varepsilon}\mathbb{E}\left|u_\varepsilon^{k-1}\right|^2 + \varepsilon B_0,$$

where we used the relation

$$\mathbb{E}|\eta_k|^2 = \varepsilon\sum_j b_j^2\,\mathbb{E}\xi_{jk}^2 = \varepsilon\sum_j b_j^2 = \varepsilon B_0.$$

In particular, if $u_\varepsilon(0) = 0$, then

$$\mathbb{E}\left|u_\varepsilon^k\right|^2 \le \frac{\varepsilon B_0}{1 - e^{-2\varepsilon}} \le \frac{B_0}{2} \quad \text{for each} \quad k \ge 0.$$

For $0 \le \tau < \varepsilon$, we have $|u_\varepsilon(k\varepsilon + \tau)| = |S_\tau(u_\varepsilon^k)| \le |u_\varepsilon^k|$. Therefore, if $u_\varepsilon(0) = 0$, then

$$\mathbb{E}\left|u_\varepsilon(t)\right|^2 \le \tfrac{1}{2}B_0 \quad \text{for all} \quad t \ge 0. \tag{3.3}$$

Since (3.1) is the NS system under the periodic boundary conditions, we also have $\|S_\varepsilon(u)\| \le e^{-\varepsilon}\|u\|$. Repeating the above arguments, we get

$$\mathbb{E}\left\|u_\varepsilon(t)\right\|^2 \le \tfrac{1}{2}B_1 \quad \text{for all} \quad t \ge 0, \tag{3.4}$$

if $u_\varepsilon(0) = 0$.

For $\varepsilon \in (0,1]$, let us denote by $\zeta_\varepsilon(t) \in H$ the piecewise constant process

$$\zeta_\varepsilon(t) = \int_{0+}^{t+0} \eta_\varepsilon(s)\, ds, \quad t \geq 0, \tag{3.5}$$

(note that $\zeta_\varepsilon(t)$ vanishes at zero and is continuous from the right). We denote by $\tilde{\zeta}_\varepsilon(t)$ the process equal to $\zeta_\varepsilon(t)$ for $t \in \varepsilon\mathbb{Z}_+$ and extended to $\mathbb{R}_+$ by the linear interpolation. Let us fix any $T > 0$ and restrict these processes to the time interval $[0,T]$. Then, by the Donsker theorem (see [Bil99]),

$$\mathcal{D}(\tilde{\zeta}_\varepsilon) \rightharpoonup \mathcal{D}(\zeta) \quad \text{as} \quad \varepsilon \to 0 \tag{3.6}$$

in the space of measures on $C([0,T]; H^1)$. As we show below in Corollary 3.3, this convergence implies that

$$\mathcal{D}\big(u_\varepsilon(t,u_0)\big) \rightharpoonup \mathcal{D}\big(u(t,u_0)\big) \quad \text{as} \quad \varepsilon \to 0 \tag{3.7}$$

for each $t \in [0,T]$ and each $u_0 \in H$, where $u_\varepsilon(t;u_0)$ (accordingly, $u(t;u_0)$) stands for a solution for (3.2) (accordingly, for (0.2)) equal to u_0 at $t=0$.

The random dynamical system (3.2) defines a Markov chain in the space H. Let $P_{\varepsilon t}(u,\cdot)$, $t \in \varepsilon\mathbb{Z}_+$, be its transition function. In [KS00], we prove that if condition (0.5) is satisfied for a sufficiently large integer $N = N_\varepsilon \geq 1$, then the Markov chain has a unique stationary measure μ_ε and[1]

$$P_{\varepsilon t}(u,\cdot) \rightharpoonup \mu_\varepsilon \quad \text{as} \quad t \to \infty. \tag{3.8}$$

In particular, due to this convergence with $u=0$ and (3.3), for any $M>0$ we have $\int_H \big(|u|^2 \vee M\big)\, \mu_\varepsilon(du) \leq \frac{1}{2}B_0$. Applying the Levi theorem, we find that

$$\int_H |u|^2\, \mu_\varepsilon(du) \leq \tfrac{1}{2}B_0 \quad \text{for} \quad 0 < \varepsilon \leq 1. \tag{3.9}$$

If $B_1 < \infty$, then due to similar arguments we have

$$\int_H \|u\|^2\, \mu_\varepsilon(du) \leq \tfrac{1}{2}B_1 \quad \text{for} \quad 0 < \varepsilon \leq 1. \tag{3.10}$$

By (3.10), the family $\{\mu_\varepsilon, \varepsilon \in (0,1]\}$ regarded as measures on H is tight. In fact, it converges to a limit as $\varepsilon \to 0$:

Theorem 3.1. *Let $B_1 < \infty$, $b_j \neq 0$ for all $j \geq 1$, and μ be the stationary measure for (0.2). Then $\mu_\varepsilon \rightharpoonup \mu$ as $\varepsilon \to 0$.*

3.2 Proof of Theorem 3.1

Since the family of measures $\{\mu_\varepsilon\}$ is compact, to prove the theorem it suffices to verify the following property: if a sequence $\{\varepsilon_n\}$ is such that $\varepsilon_n \to 0$ and

[1] In [KS00], we prove the convergence (3.8) under some unessential additional restrictions which were removed in our subsequence publications, see in [Kuk02].

$\mu_{\varepsilon_n} \rightharpoonup \lambda \in \mathcal{P}(H)$, then $\lambda = \mu$. Below we fix a sequence $\{\varepsilon_n\}$ as above and study the limiting measure λ.

Let us consider the processes ζ_ε and $\tilde{\zeta}_\varepsilon$ (see (3.5)) and choose any $T > 0$. Due to Skorokhod's representation theorem (see [Bil99]), there exist random processes $\{\hat{\zeta}_{\varepsilon_n}(t), 0 \le t \le T\}$ and $\{\hat{\zeta}(t), 0 \le t \le T\}$ defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{D}(\hat{\zeta}_{\varepsilon_n}) = \mathcal{D}(\tilde{\zeta}_{\varepsilon_n})$, $\mathcal{D}(\hat{\zeta}) = \mathcal{D}(\zeta)$, and

$$\hat{\zeta}_{\varepsilon_n}(\cdot) \underset{n\to\infty}{\longrightarrow} \hat{\zeta}(\cdot) \quad \text{in} \quad C([0,T]; H^1) \quad \text{a.s.} \tag{3.11}$$

We define processes $\zeta'_{\varepsilon_n}(t) \in H^1$ using the following relation:

$$\zeta'_{\varepsilon_n}(t) = \hat{\zeta}_{\varepsilon_n}(t) \quad \text{if} \quad k\varepsilon_n \le t < (k+1)\varepsilon_n. \tag{3.12}$$

Clearly, $\mathcal{D}(\zeta'_{\varepsilon_n}) = \mathcal{D}(\zeta_{\varepsilon_n})$.

Setting $d_n = \max_{0\le t\le T} \|\hat{\zeta}_{\varepsilon_n}(t) - \hat{\zeta}(t)\|$, we have

$$d_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{a.s.}$$

Since $B_1 < \infty$, then $\hat{\zeta}$ is a Wiener process in H^1. For any $K > 0$, let us consider the event

$$\Omega_K = \big\{\omega \in \Omega : \|\hat{\zeta}(t_1) - \hat{\zeta}(t_2)\| \le K|t_1 - t_2|^{1/3} \text{ for } 0 \le t_1, t_2 \le T\big\}. \tag{3.13}$$

Then $\mathbb{P}(\Omega_K) \to 1$ as $K \to \infty$. Due to (3.12), we have

$$\|\zeta'_{\varepsilon_n}(t) - \hat{\zeta}(t)\| \le d_n(\omega) + K\varepsilon_n^{1/3} \quad \text{for} \quad \omega \in \Omega_K. \tag{3.14}$$

For any $\varepsilon \in \{\varepsilon_1, \varepsilon_2, \dots\}$, let us denote

$$\theta_\varepsilon = \max\{t \in \varepsilon\mathbb{Z} : t \le T\}. \tag{3.15}$$

Then $\mathfrak{P}^*_{\varepsilon\theta_\varepsilon}\mu_\varepsilon = \mu_\varepsilon$. We rewrite this relation as follows:

$$\mu_\varepsilon = \big(\mathfrak{P}^*_{\varepsilon\theta_\varepsilon} - \mathfrak{P}^*_T\big)\mu_\varepsilon + \mathfrak{P}^*_T(\mu_\varepsilon - \lambda) + \mathfrak{P}^*_T\lambda. \tag{3.16}$$

We claim that (3.16) implies the equality

$$\mathfrak{P}^*_T\lambda = \lambda. \tag{3.17}$$

If (3.17) is proved, then $\lambda = \mu$. Indeed, let $u(t)$ be a solution for (0.2) such that $\mathcal{D}(u(0)) = \lambda$. Then $\mathcal{D}(u(kT)) = \lambda$ for $k \in \mathbb{Z}_+$. Since $\mu_{\varepsilon_n} \rightharpoonup \lambda$, inequality (3.9) implies that the measure λ has a finite second moment. So (2.1) applies and $\mathcal{D}(u(kT)) \rightharpoonup \mu$. Hence, $\mu = \lambda$.

To prove (3.17), it suffices to check that

$$(\mathfrak{P}^*_T\lambda, f) = (\lambda, f) \quad \text{for any} \quad f \in \mathcal{X}, \tag{3.18}$$

where $\mathcal{X} \subset C_b(H)$ is formed by all functions f which extend to Lipschitz functions $f\colon H^{-3} \to \mathbb{R}$ satisfying the conditions $|f| \le 1$ and $\mathrm{Lip}(f) \le 1$. Indeed, the set $\mathcal{X}$ contains all bounded Lipschitz cylindrical[2] functions on H (multiplied by a suitable constant), and therefore (3.18) implies that all cylindrical projections of the two measures coincide. So (3.17) follows.

To prove (3.18), we integrate f against the signed measure in the left- and right-hand sides of (3.16) and next send $\varepsilon = \varepsilon_n$ to zero. Then:

(i) The term (μ_{ε_n}, f) converges to (λ, f);

(ii) since the map $\mathfrak{P}_T^* : \mathcal{P}(H) \to \mathcal{P}(H)$ is continuous with respect to the weak convergence, we have $(\mathfrak{P}_T^*(\mu_{\varepsilon_n} - \lambda), f) \to 0$.

Hence, to prove (3.18), it remains to check that

(iii) $(\mathfrak{P}_{\varepsilon\theta_\varepsilon}^* \mu_\varepsilon, f) \to 0$ as $\varepsilon = \varepsilon_n \to 0$.

Let us consider Eq. (0.2) with a deterministic right-hand side:

$$\dot{u} + \nu L u + B(u,u) = \partial_t \zeta(t,x), \quad \zeta \in C([0,T]; H^1), \quad \zeta(0) = 0, \tag{3.19}$$

$$u(0) = u_0. \tag{3.20}$$

Substituting $u = \zeta + v$ into (3.19), we get for v the equation

$$\dot{v} + Lv + B(v+\zeta, v+\zeta) = -L\zeta, \quad v(0) = u_0. \tag{3.21}$$

Due to the basic properties of the 2D NS system (see [CF88]), the problem (3.21) has a unique solution $v \in C([0,T]; H) \cap L^2([0,T]; H^1)$. We call the corresponding function $u = v + \zeta$ a *weak solution* for (3.19), (3.20) and denote it by $u = U(\zeta) = U(\zeta, u_0)$. It is known (see [Kry95]) that solutions for (0.2) and (3.1) "can be treated path-wise," that is,

$$u(t; u_0) = U(\zeta; u_0)(t), \quad u_\varepsilon(t; u_0) = U(\zeta_\varepsilon; u_0)(t). \tag{3.22}$$

To verify (iii), we need the following lemma, which is proved at the end of this subsection:

Lemma 3.2. *Let $|u_0| \le R$, let $\max_{0\le t\le 1} \|\zeta_i(t)\| \le N$ for $i = 1,2$, and let $\max_{0\le t\le 1} \|\zeta_1(t) - \zeta_2(t)\| \le \varepsilon < 1$. We set $u_i = U(\zeta_i; u_0)$. Then for $0 \le t \le T$ we have*

$$|u_1(t)|, |u_2(t)| \le F_1(R,N), \tag{3.23}$$

$$|u_1(t) - u_2(t)| \le \varepsilon F_2(R,N), \tag{3.24}$$

where $F_1 \ge 1$ and $F_2 \ge 1$ are fixed continuous functions.

Corollary 3.3. *Convergence* (3.7) *holds for any u_0 and any $t \in [0,T]$.*

[2]A function f on H is called *cylindrical* if $f(u)$ depends only on finitely many Fourier coefficients of u.

Proof. It is sufficient to verify (3.7) for any sequence $\varepsilon_n \to 0$. Due to (3.14) and (3.24) with $N = KT^{1/3}$,

$$\big|U(\zeta'_{\varepsilon_n})(t) - U(\zeta)(t)\big| \le \big(d_n(\omega) + K\varepsilon_n^{1/3}\big)\, F_2(|u_0|, KT^{1/3}) \quad \text{for} \quad \omega \in \Omega_K,$$

where $d_n \to 0$ a.s. Hence, $U(\zeta'_{\varepsilon_n})(t) \to U(\hat\zeta)(t)$ a.s. on the set Ω_K for every K. Since $\mathbb{P}(\Omega_K) \to 1$ as $K \to \infty$, we have $U(\zeta'_{\varepsilon_n})(t) \to U(\zeta)(t)$ a.s. It remains to note that the processes ζ'_{ε_n} and $\hat\zeta$ have the same distributions as ζ_{ε_n} and ζ, and therefore (3.7) follows from (3.22). □

Corollary 3.4. *Let the assumptions of Lemma 3.2 be satisfied, let $0 \le t_1 \le t_2 \le t_1 + \delta \le T$, and let $\|\zeta_1(t_1) - \zeta_1(t_2)\| = C_\delta$. Then*

$$\big\|u_1(t_1) - u_1(t_2)\big\|_{-3} \le C_\delta + \delta C F_1^2(R, N). \tag{3.25}$$

Proof. Since u_1 is a weak solution for (3.19), (3.20) with $\zeta = \zeta_1$, we have

$$u_1(t_2) - u_1(t_1) = \big(\zeta_1(t_2) - \zeta_1(t_1)\big) - \int_{t_1}^{t_2} \big(Lu_1(s) + B(u_1(s), u_1(s))\big)\, ds.$$

It follows from (3.23) that $\|Lu_1\|_{-2} \le F_1$ and $\|B(u_1, u_1)\|_{-3} \le C_1 F_1^2$. Hence, the H^{-3}-norm of the integral is bounded by $\delta C F_1^2$, and (3.25) follows. □

The transition function $P_{\varepsilon t}(v, \Gamma)$ was defined for $t \in \varepsilon\mathbb{Z}_+$. We now extend it to $t \in \mathbb{R}_+$ by the relation

$$P_{\varepsilon t}(v, \Gamma) = \mathbb{P}\big\{u_\varepsilon(t; v) \in \Gamma\big\}.$$

Then for any $f \in \mathcal{X}$ we have

$$\big((\mathfrak{P}^*_{\varepsilon\theta_\varepsilon} - \mathfrak{P}^*_{\varepsilon T})\mu_\varepsilon, f\big) = \int_H \mathbb{E}\big(f(u_\varepsilon(\theta_\varepsilon; v) - f(u_\varepsilon(1; v))\big)\mu_\varepsilon(dv).$$

To estimate the right-hand side of this relation, we replace $u_\varepsilon(\theta_\varepsilon; v)$ and $u_\varepsilon(T; v)$ by $U(\zeta_\varepsilon; v)(\theta_\varepsilon) =: u'_\varepsilon(\theta_\varepsilon)$ and $U(\zeta_\varepsilon; v)(T) =: u'_\varepsilon(T)$, respectively. Since $f \in \mathcal{X}$, we conclude that the right-hand side is bounded by

$$2\mu_\varepsilon\big\{|v| > R\big\} + \int_{|v| \le R} \mathbb{E}\big(\|u'_\varepsilon(\theta_\varepsilon) - u'_\varepsilon(T)\|_{-3} \wedge 2\big)\, \mu_\varepsilon(dv). \tag{3.26}$$

By (3.9), the first term is majorised by $R^{-2}B_0$. To estimate the integral, let us note that for $\varepsilon = \varepsilon_n$ and $\omega \in \Omega_K$ (see (3.13)), due to (3.14), we have

$$\begin{aligned}\|\zeta'_\varepsilon(\theta_\varepsilon) - \zeta'_\varepsilon(T)\| &\le \|\zeta'_\varepsilon(\theta_\varepsilon) - \hat\zeta(\theta_\varepsilon)\| + \|\hat\zeta(\theta_\varepsilon) - \hat\zeta(T)\| + \|\hat\zeta(T) - \zeta'_\varepsilon(T)\| \\ &\le 2d_n(\omega) + 3K\varepsilon^{1/3}.\end{aligned}$$

Applying Corollary 3.4, we find that

$$\|u'_\varepsilon(\theta_\varepsilon) - u'_\varepsilon(T)\| \le \big(2d_n(\omega) + 3K\varepsilon^{1/3}\big) + C\,\varepsilon_n F_1(R, N) \quad \text{for} \quad \omega \in \Omega_K,$$

where $N = KT^{1/3}$. Therefore,

$$(3.26) \le R^{-2}B_0 + 2\mathbb{E}\,d_n + 3K\varepsilon_n^{1/3} + C\,\varepsilon_n F_1(R,N) + 2\big(1 - \mathbb{P}(\Omega_K)\big).$$

Sending to ∞ first R and K and then n, we get

$$\big((\mathfrak{P}^*_{\varepsilon\theta_\varepsilon} - \mathfrak{P}^*_{\varepsilon T})\mu_\varepsilon, f\big) \to 0 \quad \text{as} \quad \varepsilon = \varepsilon_n \to 0.$$

Hence, to prove (iii), it remains to check that

$$I_\varepsilon := \big((\mathfrak{P}^*_{\varepsilon T} - \mathfrak{P}^*_T)\mu_\varepsilon, f\big) \to 0 \quad \text{as} \quad \varepsilon = \varepsilon_n \to 0, \tag{3.27}$$

for any $f \in \mathcal{X}$. To prove the convergence, we write $|I_\varepsilon|$ as

$$|I_\varepsilon| = \Big|\int_H \mathbb{E}\big(f(u'_\varepsilon(T)) - f(\hat{u}(T))\big)\mu_\varepsilon(dv)\Big|,$$

where $u'_\varepsilon(T) = U(\zeta'_\varepsilon; v)(T)$ and $\hat{u}(T) = U(\hat{\zeta}; v)(T)$. Then

$$|I_\varepsilon| \le 2\mu_\varepsilon\big\{|v| > R\big\} + \int_{|v|\le R} \mathbb{E}\big(\|u'_\varepsilon(T) - \hat{u}(T)\|_{-3} \wedge 2\big)\,\mu_\varepsilon(dv).$$

The first term on the right-hand side is bounded by $R^{-2}B_0$. To estimate the second term, we note that, by Lemma 3.2 and inequality (3.14), for $|v| \le R$ we have

$$\big|u'_{\varepsilon_n}(T) - \hat{u}(T)\big| \le \big(d_n + K\varepsilon_n^{1/3}\big)\,F_2(R,N), \quad \omega \in \Omega_K.$$

Therefore,

$$|I_{\varepsilon_n}| \le R^{-2}B_0 + 2\big(1 - \mathbb{P}(\Omega_K)\big) + \big(\mathbb{E}\,d_n + K\varepsilon_n^{1/3}\big)\,F_2(R,N),$$

and, for the same reason as above, $|I_{\varepsilon_n}| \to 0$ as $n \to \infty$.

Thus, convergence (iii) is established, and Theorem 3.1 is proved.

Proof of Lemma 3.2. To prove (3.23), we write $u_i = \zeta_i + v_i$, $i = 1,2$, and note that both v_1 and v_2 satisfy Eqs. (3.21), where $\|\zeta\| \le N$. Multiplying the equation by v in H, we get

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \|v\|^2 + \big(B(v+\zeta,\zeta),v\big) \le \|\zeta\|\,\|v\|.$$

Since $\|\zeta\| \le N$, we have $(B(v+\zeta,\zeta),v)| \le CN^2\|v\| + CN|v|\,\|v\|$, and therefore

$$\frac{d}{dt}|v|^2 + \|v\|^2 \le C_1\big(N^2 + N^4 + N^2|v|^2\big). \tag{3.28}$$

This implies the inequality

$$|v(t)|^2 \le e^{C_1N^2t}\big(R^2 + C_2tN^4\big), \quad 0 \le t \le T, \tag{3.29}$$

which proves (3.23). Besides, integrating (3.28) from 0 to 1 and using (3.29), we obtain

$$\int_0^T \|v\|^2 dt \le F(R,N), \tag{3.30}$$

where F is a continuous function.

We now set $v = v_1 - v_2$ and $\zeta = \zeta_1 - \zeta_2$. Subtracting the equation for v_2 from the equation for v_1 and multiplying the result by v in H, we derive

$$\frac{1}{2}\frac{d}{dt}|v|^2 + \|v\|^2 + \big(B(u_1,u_1) - B(u_2,u_2), v\big) = (Lv, \zeta).$$

Since $\|\zeta\| \le \varepsilon$, then evoking the usual estimates for the nonlinearity (see [CF88]), we get

$$\big|\big(B(u_1,u_1) - B(u_2,u_2), v\big)\big| \le C\,\varepsilon\,\|v\|\big(\|u_1\| + \|u_2\|\big) + C\,\|v\|\,|v|\,\|u_2\|.$$

It follows that

$$\frac{d}{dt}|v|^2 + \|v\|^2 \le C_1\big(\varepsilon^2(\|u_1\| + \|u_2\|)^2 + |v|^2\|u_2\|^2 + \varepsilon^2\big).$$

Since $v(0) = 0$, application of the Gronwall inequality results in

$$|v(t)|^2 \le C_1\varepsilon^2 \int_0^t \exp\Big(C_2 \int_s^t \|u_2\|^2 d\theta\Big)\,\big(1 + (\|u_1\| + \|u_2\|)^2\big)\,ds.$$

Using (3.30) and (3.23), we obtain (3.24). $\square$

References

[Bil99] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, 1999.

[BKL00] J. Bricmont, A. Kupiainen, and R. Lefevere, *Probabilistic estimates for the two-dimensional stochastic Navier–Stokes equations*, J. Statist. Phys. **100** (2000), 743–756.

[BKL02] J. Bricmont, A. Kupiainen, and R. Lefevere, *Exponential mixing of the 2D stochastic Navier-Stokes dynamics*, Comm. Math. Phys. **230** (2002), no. 1, 87–132.

[CF88] P. Constantin and C. Foias, *Navier-Stokes Equations*, University of Chicago Press, Chicago, 1988.

[EMS01] W. E, J. C. Mattingly, and Ya. G. Sinai, *Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation*, Comm. Math. Phys. **224** (2001), 83–106.

[Kry95] N. V. Krylov, *Introduction to the Theory of Diffusion Processes*, AMS Translations of Mathematical Monographs, vol. 142, Providence, RI, 1995.

[KS00] S. B. Kuksin and A. Shirikyan, *Stochastic dissipative PDE's and Gibbs measures*, Comm. Math. Phys. **213** (2000), 291–330.

[KS01] S. B. Kuksin and A. Shirikyan, *Ergodicity for the randomly forced 2D Navier–Stokes equations*, Math. Phys. Anal. Geom. **4** (2001), 147–195.

[KS02] S. B. Kuksin and A. Shirikyan, *Coupling approach to white-forced nonlinear PDE's*, J. Math. Pures Appl. **81** (2002), 567–602.

[Kuk02] S. B. Kuksin, *Ergodic theorems for 2D statistical hydrodynamics*, Rev. Math. Phys. **14** (2002), no. 6, 585–600.

[Mey66] P.-A. Meyer, *Probability and Potentials*, Balisdell, Waltham-Toronto-London, 1966.

[Shi02] A. Shirikyan, *Analyticity of solutions for randomly forced two-dimensional Navier–Stokes equations*, Russian Math. Surveys **57** (2002), no. 4, 785–799.