Asymptotic of the integrated density of states of random acoustic operators with long range perturbations

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Abstract

In this paper we study the behavior of the integrated density of states of random acoustic operators of the form $A_\omega = -\nabla \frac{1}{\varrho_\omega} \nabla$. When the function ϱ_ω is considered as an Anderson type long range perturbation of some periodic function, then the behavior of the integrated density of states of A_ω in the vicinity of the internal spectral edges is given.

key words and phrases : spectral theory, random operators, integrated density of states, Lifshitz tails.

1 Introduction:

We consider A_ω, the self adjoint operator on $L^2(\mathbb{R}^d)$ formally defined by:

$$A_\omega = A(\varrho_\omega) = -\nabla \frac{1}{\varrho_\omega} \nabla = -\sum_{i=1}^{d} \partial_{x_i} \frac{1}{\varrho_\omega} \partial_{x_i}, \tag{1.1}$$

where ϱ_ω is a positive and bounded function.
A_ω is called the acoustic operator, see [4] for the physicals interpretations.

Let us start by defining the main object of our study, the integrated density of states. For this we consider Λ a cube of $\mathbb{R}^d$. We note by $A_{\omega,\Lambda}$ the restriction of A_ω to Λ with self adjoint boundary conditions. As A_ω is elliptic, the resolvent of $A_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $A_{\omega,\Lambda}$ is discrete and is made of isolated eigenvalues of finite multiplicity [26]. We define

$$N_\Lambda(E) = \frac{1}{\mathrm{vol}(\Lambda)} \cdot \#\{\text{eigenvalues of } \mathrm{A}_{\omega,\Lambda} \leq \mathrm{E}\}. \tag{1.2}$$

Here $\mathrm{vol}(\Lambda)$ is the volume of Λ in the Lebesgue sense and $\#E$ is the cardinal of E.
It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists almost surely and is independent of the boundary conditions. It is called the **integrated density of states** of A_ω (IDS for the short form). See [8].
The study of the integrated density of states and specially of its behavior is of interest for its relationship with physical interpretations. See [7]. The question we are interested in here deals with the behavior of N at the internal spectral edges of A_ω.

1.1 The IDS behavior

We give a brief history of the subject. In 1964, Lifshitz [19] argued that, for a Schrödinger operator of the form $H_\omega = -\Delta + V_\omega$, there exists $c_1, c_2 > 0$ such that $N(E)$ satisfies the asymptotic :

$$N(E) \simeq c_1 \exp(-c_2(E - E_0)^{-\alpha}), \quad E \to E_0. \tag{1.3}$$

Here E_0 is the bottom of the spectrum of H_ω and $\alpha > 0$. The behavior (1.3) is known as **Lifshitz tails** (for more details see part IV.9.A of [25]). Lifshitz predicted (1.3) also at fluctuating edges inside the spectrum.
The principal results known on Lifshitz tails are mainly shown for Schrödinger operators (for continuous and discrete cases). (See [7, 9, 13, 24, 27] and others). For an operator of type (1.1), see [20, 21].

1.2 The result

When ϱ_ω in (1.1) is obtained by a short range perturbation of Anderson type of some periodic function, it is proved [21] that the IDS of A_ω exhibits internal Lifshitz tails at the edges of the spectral gaps if and only if the IDS

of some periodic operator is non degenerate at the same edges. The essential goal of this work is to give the asymptotic of the IDS of the operator defined by (1.1) when ϱ_ω is given as a long range perturbation of some periodic function. Note that the main novelty of this paper compared to [21] is that the asymptotic is given without any assumption on the behavior of the IDS of the backgound operator. As we will see, in contrast to [21], in the present situation the kinetic energy i.e the Floquet eigenvalues of the background periodic operator does not play a rol or influence the IDS behavior. Indeed in this case the Lifshitz exponent (the α in equation (1.3)) does not depend on the certainly principal i.e on the kinetic energy. We refer to this situation as the classical regime. Note that here as in [21] we consider the case where the decreasing rate of the probability density at the edges of its support is 0. See **(H.3)**.
The proof of the result is based on the use of the technique of periodic approximations [13, 21] and is composed of two main parts, the upper and the lower bounds.
To present our result, let us consider the following plan :
In section 2, we define the model to be studied and specify various assumptions. We introduce a periodic reference operator A_{ω^+}. We state the principal theorem (Theorem 2.1) which gives the asymptotic of the IDS.
To prove Theorem 2.1, the technique of the periodic approximations [21, 13] enable to approximate the IDS of A with that of well chosen periodic operators. This will be done in sections 3. Section 4 is devoted to the proof of Theorem 2.1.
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2 The model

Let us start this section by giving the expression of ϱ_ω. We assume that ϱ_ω is a function which satisfies
(**H.0**)

$$\varrho_\omega = \varrho_0\Big(1 + \sum_{\gamma\in\Gamma} \omega_\gamma u_\gamma\Big),$$

where
(i) ϱ_0 is measurable real and $\mathbb{Z}^d$-periodic function i.e.

$$\varrho_0(x) = \varrho_0(x+\gamma), \ \forall x \in \mathbb{R}^d, \gamma \in \mathbb{Z}^d.$$

(ii) There exists constants $\varrho_{0,+} > \varrho_{0,-} > 0$ such that for almost all $x \in \mathbb{R}^d$, we have

$$0 < \varrho_{0,-} \leq \varrho_0(x) \leq \varrho_{0,+}. \tag{2.4}$$

(iii) For $\gamma \in \mathbb{Z}^d$, we set $u_\gamma(\cdot) = u(\cdot - \gamma)$. We suppose that u is a real function such that there exists $U_+ > 0$: such that for almost all $x \in \mathbb{R}^d$

$$0 \leq U(x) \equiv \sum_{\gamma \in \mathbb{Z}^d} u_\gamma(x) \leq U_+ < \infty. \tag{2.5}$$

(iv) $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d}$ is a family of non constant and positive, independent identically distributed random variables whose common probability measure is noted by $\mathbb{P}_{\omega_0}$. We note the probability space by $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathbb{P}_{\omega_0}$ is compactly supported.

Let $\mathcal{A}(\varrho_\omega)$ the quadratic form defined as follow: for $u \in H^1(\mathbb{R}^d) = \mathcal{D}(\mathcal{A}(\varrho_\omega))$

$$\mathcal{A}(\varrho_\omega)[u,u] = \int_{\mathbb{R}^d} \frac{1}{\varrho_\omega(x)} \nabla u(x) \overline{\nabla u(x)} dx.$$

$\mathcal{A}(\varrho_\omega)$ is a symmetrical, closed and positive quadratic form. A_ω given by (1.1) is defined to be the self adjoint operator associated to $\mathcal{A}(\varrho_\omega)$ [26].
Assumption **(H.0)** ensures that A_ω is a measurable family of self adjoint operators and ergodic [7, 25]. Indeed, if τ_γ refers to the translation by γ, then $(\tau_\gamma)_{\gamma \in \mathbb{Z}^d}$ is a group of unitary operators on $L^2(\mathbb{R}^d)$ and for $\gamma \in \mathbb{Z}^d$ we have

$$\tau_\gamma A_\omega \tau_{-\gamma} = A_{\tau_\gamma \omega}.$$

According to [7, 25] we know that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and Σ_{sc} closed and non random sets of $\mathbb{R}$ such that Σ is the spectrum of A_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of A_ω, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one.

2.1 Reference operator

It is convenient to consider A_ω as a perturbation of some periodic operator A_{ω^+}. More precisely, for $\varrho_{\omega^+} = \varrho_0\Big(1+\omega^+ \sum_{\gamma\in\mathbb{Z}^d} u_\gamma\Big)$, where $\omega^+ = \sup(\mathrm{supp}\mathbb{P}_{\omega_0})$ we write :

$$A_\omega = A_{\omega^+} + \Delta A_\omega,$$

with

$$A_{\omega^+} = A(\varrho_{\omega^+}) \text{ and } \Delta \mathrm{A}_\omega = \mathrm{A}_\omega - \mathrm{A}_{\omega^+} = -\nabla \frac{\varrho_{\omega^+} - \varrho_\omega}{\varrho_{\omega^+}\varrho_\omega}\nabla \geq 0.$$

2.1.1 Some Floquet theory

Now we review some standard facts from the Floquet theory for periodic operators. Basic references for this material are [18, 26, 28].
As ϱ_{ω^+} is $\mathbb{Z}^d$-periodic, for any $\gamma \in \mathbb{Z}^d$, we have

$$\tau_\gamma A_{\omega^+} \tau_\gamma^* = \tau_\gamma A_{\omega^+} \tau_{-\gamma} = A_{\omega^+}.$$

Let $\mathbb{T}^* = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. We define $\mathcal{H}$ by

$$\mathcal{H} = \{u(x,\theta) \in L^2_{loc}(\mathbb{R}^d)\otimes L^2(\mathbb{T}^*); \forall (x,\theta,\gamma) \in \mathbb{R}^d\times\mathbb{T}^*\times\mathbb{Z}^d;\ u(x+\gamma,\theta) = e^{i\gamma\theta}u(x,\theta)\}.$$

There exists U a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$ such that A_{ω^+} admits the Floquet decomposition [18, 28]

$$UA_{\omega^+}U^* = \int_{\mathbb{T}^*}^{\oplus} A_{\omega^+}(\theta)d\theta.$$

Here $A_{\omega^+}(\theta)$ is the operator A_{ω^+} acting on $\mathcal{H}_\theta$, defined by

$$\mathcal{H}_\theta = \{u \in L^2_{loc}(\mathbb{R}^d); \forall\gamma \in \mathbb{Z}^d, u(x+\gamma) = e^{i\gamma\theta}u(x)\}.$$

As A_{ω^+} is elliptic, we know that, $A_{\omega^+}(\theta)$ has a compact resolvent; hence its spectrum is discrete [26]. We denote its eigenvalues, called Floquet eigenvalues of A_{ω^+}, by

$$E_0(\theta) \leq E_1(\theta) \leq \cdots \leq E_n(\theta) \leq \cdots.$$

The corresponding eigenfunctions are denoted by $(w(x,\cdot)_j)_{j\in\mathbb{N}}$. The functions $(\theta \to E_n(\theta))_{n\in\mathbb{N}}$ are Lipshitz-continuous, and we have

$$E_n(\theta) \to +\infty \text{ as } n \to +\infty \text{ uniformly in } \theta.$$

The spectrum $\sigma(A_{\omega^+})$ of A_{ω^+} is made of bands (i.e $\sigma(A_{\omega^+}) = \cup_{n\in\mathbb{N}} E_n(\mathbb{T}^*)$.)

2.1.2 Main assumptions

We assume that
(**H**.**1**)
There exists E_+ and $\delta > 0$ such that $\sigma(A_{\omega^+}) \cap [E_+, E_+ + \delta) = [E_+, E_+ + \delta)$ and $\sigma(A_{\omega^+}) \cap (E_+ - \delta, E_+] = \emptyset$.
To prove our result, we will need the following assumptions :
(**H**.**2**)
• There exists $\nu \in (d, d+2]$ and $0 \le g_- \le g_+$ two non vanishing functions, such that for any $\gamma \in \mathbb{Z}^d$ and almost every $x \in C_0 = \{x \in \mathbb{R}^d; \forall 1 \le j \le d; -\frac{1}{2} < x_j \le \frac{1}{2}\}$ one has

$$g_-(x) \le u(x+\gamma) \cdot (1+|\gamma|)^\nu \le g_+(x),$$

and for all $1 \le i \le d$,

$$g_-(x) \le |(\partial_{x_i} u)(x+\gamma)| \cdot (1+|\gamma|)^\nu \le g_+(x).$$

(**H**.**3**)
• $\limsup\limits_{\varepsilon \to 0^+} \dfrac{\log|\log \mathbb{P}_{\omega^0}([\omega^+ - \varepsilon, \omega^+])|}{|\log \varepsilon|} = 0.$
As, $\Delta A_\omega \ge 0$ and ω^+ is in the support of $\mathbb{P}_{\omega_0}$, Σ contains an interval of the form $[E_+, E_+ + a](a > 0)$ (see [10]).
As we are interested in the behavior of the IDS in the neighborhood of E_+, we require that E_+ remains always the edge of a gap for Σ, when the perturbation is turned on. More precisely, if for all $t \in [0,1]$, we define $A_{\omega,t} = A_{\omega^+} + t\Delta A_\omega$ and Σ_t is the almost sure spectrum of $A_{\omega,t}$, then one requires that the following assumption hold.
(**H**.**4**)
There exists $\delta' > 0$ such that for all $t \in [0,1]$, $\Sigma_t \cap [E_+ - \delta', E_+) = \emptyset$.

2.1.3 The main theorem

The main result of this work is:

Theorem 2.1 *Let A_ω the operator defined by (1.1). We assume that* **(H.1)** *,* **(H.2)** *and* **(H.3)** *hold. Then E_+ is continuity point for N and*

$$\lim_{\varepsilon \to 0, \varepsilon > 0} \frac{\log|\log\big(N(E_+ + \varepsilon) - N(E_+)\big)|}{\log \varepsilon} = -\frac{d}{\nu - d}. \tag{2.6}$$

Let us notice.

Remark 2.2 *-The result of Theorem 2.1 is stated for lower band edges. Under adequate assumptions the corresponding result can be proved for upper band edges.*
-As it has already been mentioned in remark 11 of [4], one can use the result of Theorem 2.1 to show either Anderson localization [4] or dynamical localization [2] under assumptions on the distribution of the random variables weaker than those required in these references. This was done in the Schrödinger case in [29] and for the divergence operator in [22].

Outline of the proof: To show Theorem 2.1, we use periodic approximations. This technique allows us to approximate exponentially the initial IDS by that of some periodic operators. Then we have just to control the behavior of the IDS of those periodic operators and take the limit. To do this the upper and lower bounds are proven separately.
Note that in [21] the upper bound is proved under the non degeneracy assumption of the IDS of the background operator A_{ω^+} which is relaxed here. The upper bound is proved by the use of probabilistic arguments and Markov inequalities [16].
The lower bound is proved by constructing a large enough number of orthogonal approximate eigenfunctions of $A_{\omega,\Lambda}$ associated with approximate eigenvalues in $[E_+ - \varepsilon, E_+ + \varepsilon]$. This will enable us to lower bound the number of the eigenvalues of $A_{\omega,\Lambda}$ in the interval $[E_+ - \varepsilon, E_+ + \varepsilon]$.
We end this section by remarks about our assumptions. Let us start with **(H.1)**. Figotin and Kuchment in [5] studied the existence of open spectral gaps in the spectrum of certain periodic acoustic operators for $d = 2$ and 3. In assumption **(H.1)** we asked that $E_+ > 0$ which excludes the spectral gap $(-\infty, 0)$. Lifshitz tails is likely to occur at the neighborhood of the fluctuating edges. See [25]. It should be noted that 0 is not a fluctuating edge of the spectrum. It belongs to the spectrum of A_ω independently of the choice of $\varrho_\omega(x)$.
If the support of $\mathbb{P}_{\omega_0}$ is connected, the assumption **(H.4)** can be replaced by:
(H.4.bis) There exists $\delta' > 0$ such that $\Sigma \cap [E_+ - \delta', E_+) = \emptyset$.
By adding a disorder parameter g in the equation which defines ϱ_ω i.e.

$$\varrho_\omega = \varrho_0\Big(1 + g \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma u_\gamma\Big),$$

we can choose g small enough so that the spectral gap in $\sigma(A_{\omega^+})$ will not be closed after the perturbation [4].

3 Approximation of the density of states

In this section we will approach the density of states of A_ω by the density of states of periodic approximations. In a neighborhood of E_+, we will control the behavior of the density of states of periodic approximations via the density of states of periodic approximations of the reference operators. We then compute the limit for the density of states of the reference operators and we obtain the sought for result.

3.1 The periodic approximations

Let $k \in \mathbb{N}^*$. We define the following periodic operator

$$A_{\omega,k} = -\nabla \frac{1}{\varrho_{\omega,k}} \nabla,$$

where the function $\varrho_{\omega,k}$ is defined by

$$\varrho_{\omega,k} = \varrho_0 \left(1 + \sum_{\gamma \in C_k \cap \mathbb{Z}^d} \omega_\gamma \sum_{\beta \in (2k+1)\mathbb{Z}^d} u_{\gamma+\beta} \right),$$

C_k is the cube

$$C_k = \{x \in \mathbb{R}^d; \forall 1 \leq j \leq d,\ -\frac{2k+1}{2} < x_j \leq \frac{2k+1}{2}\}.$$

$A_{\omega,k}$ is $(2k+1)\mathbb{Z}^d$-periodic and essentially self adjoint operator. Let $\mathbb{T}_k^* = (\mathbb{R}^d)/\frac{2\pi}{2k+1}\mathbb{Z}^d$. We define $N_{\omega,k}$ the IDS of $A_{\omega,k}$ by

$$N_{\omega,k}(E) = \frac{1}{(2\pi)^d} \sum_{n \in \mathbb{N}} \int_{\{\theta \in \mathbb{T}_k^*,\ E_{\omega,k,n}(\theta) \leq E\}} d\theta. \tag{3.7}$$

Let $dN_{\omega,k}$ the derivative of $N_{\omega,k}$ in the distribution sense. As $N_{\omega,k}$ is increasing, $dN_{\omega,k}$ is a positive measure; it is the density of states of $A_{\omega,k}$. We denote by dN the density of states of A_ω. For all $\varphi \in C_0^\infty(\mathbb{R}), dN_{\omega,k}$ verifies (see [13, 26])

$$\begin{aligned} \langle \varphi, dN_{\omega,k} \rangle &= \frac{1}{(2\pi)^d} \int_{\theta \in \mathbb{T}_k^*} \mathrm{tr}_{\mathcal{H}_\theta}\big(\varphi(A_{\omega,k,\theta})\big) d\theta, \\ &= \frac{1}{\mathrm{vol}(C_k)} \mathrm{tr}\Big(\chi_{C_k} \varphi(A_{\omega,k}) \chi_{C_k}\Big), \end{aligned} \tag{3.8}$$

where for $\Lambda \subset \mathbb{R}^d$, χ_Λ will design the characteristic function of Λ and $\mathrm{tr}(A)$ is the trace of A (we index by $\mathcal{H}_\theta$ if the trace is taken in $\mathcal{H}_\theta$). The proof of (3.8) is given in [13].

Theorem 3.1 *1) For any $\varphi \in C_0^\infty(\mathbb{R})$ and for almost all $\omega \in \Omega$ we have*

$$\lim_{k \to \infty} \langle \varphi, dN_{\omega,k} \rangle = \langle \varphi, dN \rangle.$$

2) For any $\lambda \in \mathbb{R}$ a continuity point for N, we have $\lim\limits_{k\to\infty} \mathbb{E}(N_{\omega,k}(\lambda)) = N(\lambda)$ almost surely .

Remark 3.2 *The result of Theorem 3.1 is close to that of Theorem 5.1 of [13]. The proof is also similar and is based on functional analysis. The unique difference in the proof comes from the control of the behavior of the resolvent. In [13], the perturbation was a potential; in our case, it is a differential operator of the same order as the background operator. The proof is given in [21].*

In what follow we prove that the IDS of A_ω is exponentially well approximated by the expectation of the IDS of the periodic operators $A_{\omega,k}$ when k is polynomial in ε^{-1}. More precisely we prove

Lemma 3.3 *For any $\eta_0 > 1$, there exists $\nu_0 > 0$ and $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$ and $k \geq k_1 = \varepsilon^{-\nu_0}$, we have*

$$\begin{aligned} &\mathbb{E}[N_{\omega,k}(E + \varepsilon/2) - N_{\omega,k}(E - \varepsilon/2)] - e^{\varepsilon^{-\eta_0}} \\ &\qquad \leq N(E+\varepsilon) - N(E) \\ &\qquad \leq \mathbb{E}[N_{\omega,k}(E + 2\varepsilon) - N_{\omega,k}(E - 2\varepsilon)] + e^{-\varepsilon^{-\eta_0}}. \end{aligned} \tag{3.9}$$

Proof. The last result is well known for operators with compact single sit potentials [14, 16, 17]. For this we need to define another operator. More precisely let f a function on $\mathbb{R}^d$, one define $f^\varepsilon(x) = f(x)\chi_{\{\varepsilon\cdot|x|\le 1\}}$, f^ε is compactly supported. We define the following random operator :

$$A_{\varepsilon,\omega} = -\nabla \cdot \frac{1}{\varrho_{\varepsilon,\omega}} \cdot \nabla,$$

where $\varrho_{\varepsilon,\omega}(\cdot)$ is the function given by

$$\varrho_{\varepsilon,\omega}(\cdot) = \varrho_0\big(1 + \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma u^\varepsilon(\cdot - \gamma)\big).$$

The periodic approximations of $A_{\varepsilon,\omega}$ it is defined analogously to $A_{\omega,k}$ and will be denoted by $A_{\varepsilon,\omega,k}$. Let N_ε (respectively $N_{\varepsilon,\omega,k}$) be the IDS of $A_{\varepsilon,\omega}$ (respectively of $A_{\varepsilon,\omega,k}$). Notice that from the decaying assumption **(H.2)** and the fact that the random variables are bounded uniformly in k and ω, we get that there exits $C > 0$ such that for any $\Psi = (-1-\Delta)^{-1}\varphi$, where $\varphi \in L^2(\mathbb{R}^d)$

$$0 \le \langle A_\omega \Psi, \Psi\rangle - \langle A_{\varepsilon,\omega}\Psi, \Psi\rangle \le C \cdot \varepsilon^{\nu-d} \cdot \|\Psi\|^2 .$$

The same inequality holds for the periodic approximations. This yields that, uniformly in k and ω and locally uniformly in the energy E we have

$$N_\varepsilon(E - C \cdot \varepsilon^{\nu-d}) \le N(E) \le N_\varepsilon(E + C \cdot \varepsilon^{\nu-d}), \tag{3.10}$$

and

$$N_{\varepsilon,\omega,k}(E - C \cdot \varepsilon^{\nu-d}) \le N_{\omega,k}(E) \le N_{\varepsilon,\omega,k}(E + C \cdot \varepsilon^{\nu-d}). \tag{3.11}$$

This tells us that the IDS of A_ω is well approximated by that of $A_{\varepsilon,\omega}$. But for the last operator the singles sites potentials are compactly supported and so many techniques and results are available [16]. We need the following lemma

Lemma 3.4 *[16] We assume that the single sit potential is supported in a ball of radius R_ε. Let I, a relatively compact open interval in $\mathbb{R}$. For any $\beta \in (0, 1)$, there exists $C > 1$ and $\rho > 0$ such that , for any $\varphi \in C_0^\infty(I)$, for $n \in \mathbb{N}^*$ and $k > R_\varepsilon$, we have*

$$\begin{aligned}|\mathbb{E}(\langle\varphi, dN_{\omega,k}\rangle) - \langle\varphi, dN\rangle| \le \\ C \cdot |k - R_\varepsilon|^{-(1-\beta)k} \cdot n^n \cdot \sup_{x\in\mathbb{R}, 0\le j\le n+\rho} \Big|(|x| + 1)^{\rho+n}\varphi^{(j)}(x)\Big|.\end{aligned} \tag{3.12}$$

Remark 3.5 *This lemma is proven in [16] for the Schrödinger case. It is still true for our case. The proof is based on the Helffer Sjöstrand formula and the resolvent equation with the exponential decay of the resolvent kernels (the Combes-Thomas argument).*

Let φ be a Gevrey class function, of Gevrey exponent $\alpha > 1$ (see [6]) such that φ is supported in $[-1,2]$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $[0,1]$. For $0 < \varepsilon < 1$, we set

$$\varphi_{E_+,\varepsilon}(\cdot) = \varphi\Big(\frac{\cdot - E_+}{\varepsilon}\Big).$$

From Lemma 3.4 and properties of Gevrey class functions, we deduce that there exists $C > 1$ such that for all $k > R_\varepsilon$, $n \geq 1$ and $0 < \varepsilon < 1$ we have

$$\Big|\mathbb{E}(\langle \varphi_{E_+,\varepsilon}, dN_{\omega,k}\rangle) - \langle \varphi_{E_+,\varepsilon}, dN\rangle\Big| \\ \leq \varepsilon^{-n-\rho}(n+\rho)^{2\alpha(n+\rho)}(k-R_\varepsilon)^{-(1-\beta)n}.$$

We pick $n \leq (k-R_\varepsilon)^{(1-\beta)/4\alpha} - \rho$. For $k - R_\varepsilon$ large, we get that, there exists $C > 1$ such that for $k > R_\varepsilon + 2$, $n > \rho$ and $0 < \varepsilon < 1$, we have

$$\Big|\mathbb{E}(\langle \varphi_{E_+,\varepsilon}, dN_{\omega,k}\rangle) - \langle \varphi_{E_+,\varepsilon}, dN\rangle\Big| \leq \Big(\varepsilon^{-1}(k-R_\varepsilon)^{(1-\beta)/4}\Big)^{(k-R_\varepsilon)^{(1-\beta)/4\alpha}}.$$

As $R_\varepsilon \sim \varepsilon^{1/(d-\nu)}$, for $\eta_0 > 1$ such that $\alpha\cdot\eta_0 > 1$ and $k = k_1 = \varepsilon^{-\nu_0} > \varepsilon^{-\eta_0 4\alpha/(1-\beta)} + R_\varepsilon$, (it suffice to take $\nu_0 > \sup(\eta_0 4\alpha/(1-\beta), \frac{1}{d-\nu})$) we get that there exist $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, we have

$$\Big|\mathbb{E}(\langle \varphi_{E_+,\varepsilon}, dN_{\omega,k_\varepsilon}\rangle) - \langle \varphi_{E_+,\varepsilon}, dN\rangle\Big| \leq e^{-\varepsilon^{-\eta_0}}. \tag{3.13}$$

As dN_ε and $dN_{\varepsilon,\omega,k}$ are positive measures and by the definition of φ, we have

$$\mathbb{E}(N_{\varepsilon,\omega,k}(E+\varepsilon) - N_{\varepsilon,\omega,k}(E-\varepsilon)) \leq \mathbb{E}(\langle \varphi_{E,\varepsilon}, dN_{\varepsilon,\omega,k}\rangle) \leq \mathbb{E}(N_{\varepsilon,\omega,k}(E+2\varepsilon) - N_{\varepsilon,\omega,k}(E-2\varepsilon))$$

and

$$N_\varepsilon(E+\varepsilon) - N_\varepsilon(E-\varepsilon) \leq \langle \varphi_{E,\varepsilon}, dN\rangle \leq N_\varepsilon(E+2\varepsilon) - N_\varepsilon(E-2\varepsilon).$$

This, gives (3.9), for $A_{\varepsilon,\omega}$. To get (3.9) for A_ω, it suffices to pick $\varepsilon = \varepsilon^{1/\nu-d}$ and take into account (3.10), (3.11) and (3.13).$\square$
Now we have the necessary tools to prove the main theorem.

4 The proof of Theorem 2.1

Notice that the first part of the theorem i.e the continuity of the IDS at E_+ is a consequence of the continuity if the IDS of the periodic operator. See [21].
As we mentioned in the introduction the proof of the main result is composed of two classical parts, the upper and the lower bounds. We start by the proof of the upper bound, then we turn to the lower bound.

4.1 The upper bound

From Lemma 3.3 and for $\eta_0 > 1/(\nu - d)$ and $k \sim \varepsilon^{-\delta}$ such that $\delta > \nu_0$ the proof of the upper bound is reduced to prove that

$$\limsup_{\varepsilon \to 0^+} \frac{\log|\log(\mathbb{E}(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)))|}{\log \varepsilon} \leq -\frac{d}{\nu - d}. \qquad (4.14)$$

Lemma 4.1 *Let $k \sim \varepsilon^{-\rho}$ with $\rho > 1/(\nu - d)$. Define the event,*

$$\mathbf{E}_{\varepsilon,\omega} = \{\omega; \Delta A_{\omega,k} \geq -\varepsilon \Delta = -\varepsilon \sum_{i=1}^{d} \partial_{x_i}^2\}.$$

Then $\mathbf{E}_{\varepsilon,\omega}$ has a probability at least $1 - \mathbb{P}_\varepsilon$ where $\mathbb{P}_\varepsilon$ satisfies

$$\limsup_{\varepsilon \to 0^+} \frac{\log|\log(\mathbb{P}_\varepsilon)|}{\log \varepsilon} \leq -\frac{d}{\nu - d}. \qquad (4.15)$$

Proof. Using the assumption **(H.2)**, we get that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
\langle \Delta A_{\omega,k}\varphi, \varphi\rangle &= \sum_{\gamma \in \mathbb{Z}^d; 1 \leq i \leq d} \omega_{\widetilde{\gamma}} \langle u(\cdot - \gamma)\partial_{x_i}\varphi, \partial_{x_i}\varphi\rangle \ \text{ where } \ \widetilde{\gamma} = \gamma \mod (2k+1)\mathbb{Z}^d \\
&\geq g_- \sum_{\gamma, \alpha \in \mathbb{Z}^d;} \omega_{\widetilde{\gamma}} (1 + |\alpha - \gamma|)^{-\nu} |\nabla \varphi|^2 \\
&= \sum_{\alpha \in \mathbb{Z}^d} \Big(\sum_{\gamma \in \mathbb{Z}^d} \omega_{\widetilde{\gamma}} g_-(x - \gamma - \alpha)(1 + |\alpha - \gamma|)^{-\nu} \Big) |\nabla \varphi|^2 \\
&= \sum_{\alpha \in \mathbb{Z}^d} A_\alpha(\omega) g_-(x - \alpha) |\nabla \varphi|^2.
\end{aligned}$$

$$(4.17)$$

Here $A_\alpha(\omega) = \sum_{\gamma \in \mathbb{Z}^d} \omega_{\tilde{\gamma}}(1 + |\alpha - \gamma|)^{-\nu}$.
Notice that ; $\omega_{\tilde{\gamma}}$ is $(2k+1)\mathbb{Z}^d$ periodic so is $A_\alpha(\omega)$. We set $\mathbb{Z}^d_{2k+1} = \{\alpha \in \mathbb{Z}^d; |\alpha| < k\}$. Then we have

$$\begin{aligned} \mathbb{P}(\{\Delta A_{\omega,k} \geq -\varepsilon\Delta\}) &\geq \mathbb{P}(\{\forall \alpha \in \mathbb{Z}^d_{2k+1}; A_\alpha \geq \varepsilon\}) \\ &\geq 1 - \sum_{\alpha \in \mathbb{Z}^d_{2k+1}} \mathbb{P}(\{A_\alpha(\omega) \leq \varepsilon\}). \end{aligned}$$

As the random variables are i.i.d, we have

$$\mathbb{P}(\{\Delta A_{\omega,k} \geq -\varepsilon\Delta\}) \geq 1 - (2k+1)^d \mathbb{P}(\{A_0(\omega) \leq \varepsilon\}). \tag{4.18}$$

To estimate $\mathbb{P}(\{A_0(\omega) \leq \varepsilon\})$ it suffices to follow the same computation done in [16] and based on the Markov's inequality, and the Taylor expansion of e^{-x} to get that

$$\mathbb{P}(\{A_0(\omega) \leq \varepsilon\}) \leq e^{-\frac{1}{C}\varepsilon^{-\frac{d}{\nu-d}}}. \tag{4.19}$$

This complete the proof of Lemma 4.1. $\square$

Lemma 4.2 *There exists $C > 0$ and $\varepsilon_0 > 0$ (uniform in k and ω) such that, if $0 < \varepsilon < \varepsilon_0$ and ω satisfies $\Delta A_{\omega,k} \geq -C\varepsilon\Delta$, for $k \in \mathbb{N}$, one has*

$$N_{\omega,k}(E_+) = N_{\omega,k}(E_+ + \varepsilon).$$

Let us use this lemma to finish the proof of the upper bound. Lemma 4.2 says that if $\Delta A_{\omega,k} \geq -C\varepsilon\Delta$, then the spectrum of $A_{\omega,k}$ does not intersect $(E_+, E_+ + \varepsilon)$ for ε small . This lemma will be proved in the end of this section.
Now estimate (4.14) is an immediate consequence of Lemma 4.1 and 4.2. Indeed, picking C as in Lemma 4.2; one computes

$$\begin{aligned} \mathbb{E}\Big(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)\Big) &= \mathbb{E}\Big([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega; \Delta A_{\omega,k} \geq -C\varepsilon\Delta\}}\Big) \\ &+ \mathbb{E}\Big([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega; \Delta A_{\omega,k} < -C\varepsilon\Delta\}}\Big) \\ &= \mathbb{E}\Big([N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)]\mathbf{1}_{\{\omega; \Delta A_{\omega,k} < -C\varepsilon\Delta\}}\Big) \\ &\leq C_0 \mathbb{P}(\{\omega; \Delta A_{\omega,k} < -C\varepsilon\Delta\}) \\ &= C_0(1 - \mathbb{P}(\mathbf{E}_{C\cdot\varepsilon,\omega})) = C_0 \mathbb{P}_{C\cdot\varepsilon}. \end{aligned}$$

Here, we have used the fact that $N_{\omega,k}$ is bounded locally uniformly in energy, uniformly in ω, k by C_0. Tanking (4.15) into account we end the proof of (4.14) and consequently the proof of the upper bound. $\square$

The proof of Lemma 4.2. Let us take $\Delta A_{\omega,k} > -C\varepsilon\Delta$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\langle A_{\omega,k}\varphi, \varphi\rangle > E_+$, then

$$\begin{aligned}\langle A_{\omega,k}\varphi,\varphi\rangle &= \langle A_{\omega^+,k}\varphi,\varphi\rangle + \langle \Delta A_{\omega,k}\varphi,\varphi\rangle \\ &\geq \langle A_{\omega^+,k}\varphi,\varphi\rangle + C\varepsilon|\nabla\varphi|^2 \\ &> \langle A_{\omega^+,k}\varphi,\varphi\rangle + C\varepsilon\varrho_{0,-}E_+. \end{aligned} \tag{4.20}$$

But by assumption **(H.1)**, below the energy E_+ there is a gap in the spectrum of A_{ω^+} of length at least $\delta > 0$; taking point (ii) of **(H.0)** into account, we get that for $\varepsilon < \varepsilon_0 = \frac{\delta}{E_+ C\varrho_{0,-}}$; $A_{\omega,k}$ has no spectrum in $(E_+, E_+ + C\varepsilon E_{+0,-})$. So the proof of Lemma 4.2 is ended if we choose $C = \frac{1}{E_+\varrho_{0,-}} > 0$. $\square$

4.2 The lower bound

The lower bound is proven in the same way as in [21] and consists on proving the following theorem.

Theorem 4.3 *Let A_ω, the operator defined by (1.1). We assume that* **(H.1)**, *and* **(H.2)** *hold. Then, we have*

$$\liminf_{\varepsilon\to 0^+} \frac{\log\left|\log\Big(N(E_+ + \varepsilon) - N(E_+)\Big)\right|}{\log\varepsilon} \geq -\frac{d}{\nu - d}. \tag{4.21}$$

Proof. We will sketch the proof, for more details see [21]. As by assumption, there is a spectral gap below E_+ of length at least $\delta' > 0$. Thus, for $\varepsilon < \delta'$ we have

$$N(E_+ + \varepsilon) - N(E_+) = N(E_+ + \varepsilon) - N(E_+ - \varepsilon).$$

To prove Theorem 4.3, we will lower bound $N(E_+ + \varepsilon) - N(E_+ - \varepsilon)$. So, for N large, we will show that A_{ω,Λ_N} (A_{ω,Λ_N} is A_ω restricted to Λ_N with Dirichlet boundary conditions) has a large number of eigenvalues in $[E_+ - \varepsilon, E_+ + \varepsilon]$ with large a probability. For this we will construct a family of approximate eigenvectors associated to approximate eigenvalues of A_{ω,Λ_N} in $[E_+ - \varepsilon, E_+ + \varepsilon]$. These functions will be constructed from an eigenvector of A_0 associated to E_+. Locating this eigenvector in θ, one obtains an approximate eigenfunction of A_{ω,Λ_N}. Notice that the main difference point with the proof given in [21]

appears on the choice of the box where we locate this eigenvector in θ. Then we Locate this eigenfunction in x in several disjointed places, we get several eigenfunctions two by two orthogonal.
In order to simplify the notations, in what follows we assume that $\theta^0 = 0$ is a point where $E_1(\theta)$ reaches E_+. From [13, 21] there exists $C > 1$, V a neighborhood of 0 and $f : \theta \in V \to f(\cdot, \theta)$ a real analytic function such that, $||f(\cdot, \theta)||_{L^2(C_0)} = 1$ and

$$||(A_0(\theta) - E_+)f(\cdot, \theta)||_{L^2(C_0)} \leq C|\theta|^2. \tag{4.22}$$

Let $0 < \xi < 1$ be a small constant. Let $\chi \in C_0^\infty(\mathbb{R})$ such that it is positif, supported in $[\frac{\xi}{2}, \xi]$ and $\displaystyle\int_{[\frac{\xi}{2},\xi]} \chi(t)^2 dt = 2$.
For $\varepsilon > 0$, we define

$$\mathcal{W}_\varepsilon(\theta) = \varepsilon^{-d/4} \prod_{j=1}^{d} \chi(\varepsilon^{-\frac{1}{2}}\theta_j) \in L^2(\mathbb{T}^*)$$

and

$$\mathcal{W}_\varepsilon^f(\cdot, \theta) = \mathcal{W}_\varepsilon(\theta) \cdot f(\cdot, \theta).$$

Now we estimate

$$\begin{aligned} ||(A_0 - E_+)\mathcal{W}_\varepsilon^f||_{\mathcal{H}}^2 &= \frac{1}{\mathrm{vol}(\mathbb{T}^*)} \int_{\mathbb{T}^*} ||(A_{\omega^+}(\theta) - E_+)f(\cdot, \theta)||_{L^2(C_0)}^2 |\mathcal{W}_\varepsilon(\theta)|^2 d\theta \\ &\leq C^2 \int_{\mathbb{T}^*} |\theta|^4 |\mathcal{W}_\varepsilon(\theta)|^2 d\theta \\ &\leq C^2 \varepsilon^2 \int_{[\frac{\xi}{2},\xi]^d} |\theta|^4 \prod_{j=1}^{d} \chi^2(\theta_j) d\theta \\ &\leq \frac{\varepsilon^2}{8}, \text{ if } \xi \text{ is small enough.} \end{aligned} \tag{4.23}$$

For $\beta \in \mathbb{Z}^d$, we define

$$\mathcal{W}_{\varepsilon,\beta}^f(\cdot, \theta) = e^{-i\beta\cdot\theta}\mathcal{W}_\varepsilon^f(\cdot, \theta) \ \text{ and } \ \mathcal{W}_{\alpha,\varepsilon,\beta,\zeta}^f(\cdot, \theta) = e^{-i\beta\cdot\theta}(\Pi_{\Lambda_\alpha(\zeta)}\mathcal{W}_\varepsilon^f)(\cdot, \theta),$$

where $\Lambda_\alpha(\zeta)$ is the cube defined by

$$\Lambda_\alpha(\zeta) = \{\gamma \in \mathbb{Z}^d;\ \text{for } 1 \leq j \leq d;\ |\gamma_j| \leq \zeta^{-(\frac{1}{\nu-d}+\alpha)}\},$$

and $\Pi_{\Lambda_\alpha(\zeta)}$ is the orthogonal projection on $\Lambda_\alpha(\zeta)$.
We set

$$\mathcal{U}^f_{\varepsilon,\beta}(x) = \int_{\mathbb{T}^*} \mathcal{W}^f_{\varepsilon,\beta}(x,\theta)d\theta \text{ and } \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x) = \int_{\mathbb{T}^*} \mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta}(x,\theta)d\theta.$$

For N large and well chosen β and $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$, $\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}$ will be an approximate eigenfunction of A_{ω,Λ_N} associated with an approximate eigenvalue in the interval $[E_+ - \varepsilon, E_+ + \varepsilon]$.
We show initially that $||\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}||_{L^2(\mathbb{R}^d)} > C > 0$. Note that

$$\begin{aligned}(\mathrm{vol}(\mathbb{T}^*))||\mathcal{U}^f_{\varepsilon,\beta}||^2_{L^2(\mathbb{R}^d)} &= ||\mathcal{W}^f_{\varepsilon,\beta}||^2_{\mathcal{H}} \\ &= \int_{\mathbb{T}^*} ||f(\cdot,\theta)||^2_{L^2(C_0)}|\mathcal{W}_\varepsilon(\theta)|^2 d\theta \geq 2^d.\end{aligned}$$

As in [13] using the non-stationary phase we see that $\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}$ and $\mathcal{U}^f_{\varepsilon,\beta}$ are close to each others. More precisely, for any $n \in \mathbb{N}$ and $\beta \in \mathbb{Z}^d$, there exists $C_n > 0$ such that

$$(\mathrm{vol}(\mathbb{T}^*)) \cdot ||\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta} - \mathcal{U}^f_{\varepsilon,\beta}||_{L^2(\mathbb{R})} = ||\mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta} - \mathcal{W}^f_{\varepsilon,\beta}||_{\mathcal{H}} \leq C_n \varepsilon^{-n/2}\zeta^{n(\frac{1}{\nu-d}+\alpha)}. \tag{4.24}$$

So, for $\zeta = \varepsilon$ small enough, we get

$$||\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}||_{L^2(\mathbb{R}^d)} \geq 1.$$

Now we have to look to the conditions for which we have

$$\Big|\Big|\Big(A_\omega - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 \leq \varepsilon^2.$$

Note that

$$\begin{aligned}\Big|\Big|\Big(A_{\omega,\Lambda_N} - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 &\leq \Big|\Big|\Big(A_\omega - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 \\ &\leq 2\Big|\Big|\Big(A_{\omega^+} - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 + 2\Big|\Big|\Delta A_\omega \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2.\end{aligned} \tag{4.25}$$

Equations (4.23) and (4.24) give the bound on the first member of (4.25), it just remains to us to control the second term. For $\omega^+_\gamma = \omega^+ - \omega_\gamma$, we have

$$\begin{aligned}\Big|\Big|\Big(\Delta A_\omega\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2 \leq 2\Big|\Big| \sum_{\gamma\in\mathbb{Z}^d, 1\leq i\leq d} \omega^+_\gamma (\partial_{x_i}u)(\cdot - \gamma)\cdot\Big(\partial_{x_i}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big)\Big|\Big|^2 + \\ 2\Big|\Big| \sum_{\gamma\in\mathbb{Z}^d, 1\leq i\leq d} \omega^+_\gamma u(\cdot-\gamma)\partial^2_{x_ix_i}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2.\end{aligned} \tag{4.26}$$

To estimate (4.26), one needs the following lemmas, proven in [21].

Lemma 4.4 *There exists $K>0$, such that for any $1\le i,j\le d$ we have*

$$\Big|\Big|\sum_{\gamma\in\mathbb{Z}^d,1\le i\le d}\omega_\gamma^+(\partial_{x_i}u)(\cdot-\gamma)\cdot\Big(\partial_{x_i}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big)\Big|\Big|^2\le\varepsilon^4+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)}\omega_\gamma^+\Big)^2. \tag{4.27}$$

Lemma 4.5 *There exists $K>0$, such that we have*

$$\Big|\Big|\sum_{\gamma\in\mathbb{Z}^d,1\le i\le d}\omega_\gamma^+u(\cdot-\gamma)\partial^2_{x_i}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big|\Big|^2\le\varepsilon^4+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)}\omega_\gamma^+\Big)^2. \tag{4.28}$$

Now, combining (4.27), (4.28) and taking (4.26) into account we get that there exists $K>0$ such that

$$\Big|\Big|(\Delta A_\omega)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon}\Big|\Big|^2\le\varepsilon^3+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon);1\le i\le d}\omega_\gamma^+\Big)^2. \tag{4.29}$$

By (4.23),(4.24) and (4.29), it follows that :

$$\Big|\Big|\Big(A_\omega-E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon}\Big|\Big|^2\le\frac{\varepsilon^2}{2}+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon);1\le i\le d}\omega_\gamma^+\Big)^2. \tag{4.30}$$

Now, for N large, we may divide Λ_N into $N(\varepsilon)$ disjoints cubes of size $2\Lambda_\alpha(\varepsilon)$. One has

$$N(\varepsilon)\simeq\frac{(2N)^d}{\varepsilon^{-d(\frac{1}{\nu-d}+\alpha)}}; \tag{4.31}$$

and

$$\cup_{j=1}^{N(\varepsilon)}(\beta_j+\Lambda_\alpha(\varepsilon))\subset\Lambda_N \text{ and for } j\neq j'(\beta_j+2\Lambda_\alpha(\varepsilon))\cap(\beta_{j'}+2\Lambda_\alpha(\varepsilon))=\emptyset.$$

This implies that for $j\neq j'$, $\mathcal{U}^f_{\alpha,\varepsilon,\beta_j,\varepsilon}$ and $\mathcal{U}^f_{\alpha,\varepsilon,\beta_{j'},\varepsilon}$ are orthogonal.
We denote the counting function of the eigenvalues of A_{ω,Λ_N} below E by $\Theta_{\Lambda_N}(E)$, then

$$\mathbb{E}\Big(\Theta_{\Lambda_N}(E+\varepsilon)-\Theta_{\Lambda_N}(E-\varepsilon)\Big)=\mathbb{E}\Big(\#\Big\{\text{eigenvalues of }\Pi_\mathrm{N}\mathrm{A}_\omega\Pi_\mathrm{N}\text{ in }[\mathrm{E}_+-\varepsilon,\mathrm{E}_++\varepsilon]\Big\}\Big)$$

$$\ge\mathbb{E}\Big(\#\Big\{1\le j\le N(\varepsilon);||(A_\omega-E_+)\mathcal{U}^f_{\alpha,\varepsilon,\beta_j,\varepsilon}||_{L^2(\mathbb{R}^d)}\le\varepsilon\Big\}\Big)\ge\mathbb{E}\Big(\sum_{j=1}^{N(\varepsilon)}B_j(\omega)\Big), \tag{4.32}$$

where

$$B_j(\omega) = \begin{cases} 1 & \text{if } \mathrm{K}\left(\varepsilon^{\mathrm{s}\alpha(\nu-\mathrm{d})} \cdot \varepsilon + \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon);} \omega_\gamma^+\right)^2 \leq \frac{\varepsilon^2}{2}. \\ 0 & \text{if not.} \end{cases}$$

The $(B_j)_{1\leq j\leq N(\varepsilon)}$ are independent, identically distributed, Bernoulli random variables. So equations (4.31) and (4.32), imply that there exists $C > 0$, such that one has

$$\begin{aligned} &N_{\Lambda_N}(E+\varepsilon) - N_{\Lambda_N}(E-\varepsilon) = \\ &\quad \frac{1}{((2N+1))^d}\mathbb{E}\Big(\#\Big\{\text{eigenvalues of } \Pi_\mathrm{N}\mathrm{A}_\omega\Pi_\mathrm{N} \text{ in } [\mathrm{E}_+-\varepsilon, \mathrm{E}_++\varepsilon]\Big\}\Big) \\ &\qquad\qquad \geq \frac{N(\varepsilon)}{(2N+1)^d}\mathbb{P}(B_1=1) \geq \frac{1}{C}\varepsilon^{d(1/(\nu-d)+\alpha}\mathbb{P}(B_1=1). \end{aligned}$$

Hence, taking the limit $N\to\infty$, we get that, for $\varepsilon>0$ small, we obtain

$$N(E_++\varepsilon) - N(E_+-\varepsilon) \geq \frac{1}{C}\varepsilon^d\mathbb{P}(B_1=1). \tag{4.33}$$

It just remains to estimate $\mathbb{P}(B_1=1)$. If for $1\leq j\leq N(\varepsilon)$, and any $\gamma\in\beta_j+\Lambda_\alpha(\varepsilon)$;, one has $\omega_\gamma \leq \dfrac{\varepsilon}{2K}$, then for ε rather small

$$K\Big(\varepsilon^{\alpha(\nu-d)}\cdot\varepsilon + \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon);}\omega_\gamma^+\Big)^2 \leq \varepsilon^2\cdot K\Big(\varepsilon^{\alpha(\nu-d)} + \frac{1}{2K}\Big)^2 \leq \frac{\varepsilon^2}{2}.$$

As the random variables are independent identically distributed, one has the estimate

$$\mathbb{P}(B_j=1) \geq \Big(\mathbb{P}(\omega_0 \leq \frac{\varepsilon}{2K})\Big)^{d^2\#\Lambda_\alpha(\varepsilon)}.$$

So, taking the double logarithm of (4.33), using assumption **(H.3)** and the fact that $\#\Lambda_\alpha(\varepsilon) = \varepsilon^{-d(\frac{1}{\nu-d}+\alpha)}$, we get that

$$\lim_{\varepsilon\to0^+}\frac{\log\Big|\log\Big(N(E_++\varepsilon)-N(E_+)\Big)\Big|}{\log\varepsilon} \geq -\frac{d}{\nu-d} - d\alpha. \tag{4.34}$$

The equation (4.34) is true for any $\alpha>0$, by taking α small we end the proof of Theorem 4.3. □

The proof of Lemmas 4.4 and 4.5. we start by the proof of Lemma 4.4. It is similar to that given in [21]. we start by noticing that

$$\Big\|\sum_{\gamma\in\mathbb{Z}^d,}\Big(\omega^+_\gamma\partial_{x_i}u(\cdot-\gamma)\cdot\Big(\partial_{x_j}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}\Big)\Big\|^2\le\varepsilon^5+\int_{\mathbb{R}^d}\Big(\sum_{\gamma\in\mathbb{Z}^d}\omega^+_\gamma\partial_{x_i}u(\cdot-\gamma)\Big)^2\Big|\mathcal{U}^{\partial_{x_j}f}_{\varepsilon,\beta}(x)\Big|^2dx. \tag{4.35}$$

We used an estimate like (4.24) to replace $\mathcal{U}^{\partial_{x_i}f}_{\alpha,\varepsilon,\beta,\varepsilon}$ with $\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\beta}$.
We set

$$S_{\beta,\varepsilon}=\int_{\mathbb{R}^d}\Big(\sum_{\gamma\in\mathbb{Z}^d}\omega^+_\gamma\partial_{x_i}u(\cdot-\gamma)\Big)^2\Big|\mathcal{U}^{\partial_{x_j}f}_{\varepsilon,\beta}(x)\Big|^2dx.$$

By the use of **(H.2)**, we get that

$$S_{\beta,\varepsilon}\le\sum_{\eta\in\mathbb{Z}^d}\Big(\sum_{\gamma\in\mathbb{Z}^d}(1+|\eta-\gamma|)^{-\nu}\omega^+_\gamma\Big)^2\cdot\int_{C_0}g^2_+(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta-\beta}(x)\Big|^2dx. \tag{4.36}$$

There exists $B>0$ such that

$$\sum_{\eta\notin\Lambda_\alpha(\varepsilon)}(1+|\eta|)^{-\nu}\le B\cdot\varepsilon^{\alpha(\nu-d)}\cdot\varepsilon.$$

As the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d;}$ are bounded, we get that there exists $B'>0$ such that

$$S_{\beta,\varepsilon}\le\sum_{\eta\in\mathbb{Z}^d}\Big(B'\varepsilon^{\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\eta+\Lambda_\alpha(\varepsilon)}(1+|\eta-\gamma|)^{-\nu}\omega^+_\gamma\Big)^2\cdot\int_{C_0}g^2_+(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta-\beta}(x)\Big|^2dx.$$

One splits the sum on $\eta\in\mathbb{Z}^d$ in two parts according to whether η belongs to $\beta+\Lambda_\alpha(\varepsilon)$ or not.

- For the sum on $\eta\notin\beta+\Lambda_\alpha(\varepsilon)$, there exists $C>0$ such that

$$\begin{aligned}\sum_{\eta\notin\beta+\Lambda_\alpha(\varepsilon)}\Big(C\varepsilon^{\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\eta+\Lambda_\alpha(\varepsilon)}(1+|\eta-\gamma|)^{-\nu}\omega^+_\gamma\Big)^2\cdot\Big(\int_{C_0}g^2_+(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta-\beta}(x)\Big|^2dx\Big)\\ \le C\sum_{\eta\notin\Lambda_\alpha(\varepsilon)}\int_{C_0}g^2_+(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta}(x)\Big|^2dx.\end{aligned} \tag{4.37}$$

By the use of the non stationary phase and following the same computation done in [13] one proves that, for n entire, there exists $C_n>0$ such that

$$\sum_{\eta\notin\Lambda_\alpha(\varepsilon)}\Big(\int_{C_0}g^2_+(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta}(x)\Big|^2dx\Big)\le C_n\varepsilon^{\alpha n}. \tag{4.38}$$

• For the sum on η in $\beta+\Lambda_\alpha(\varepsilon)$, one gets

$$\sum_{\eta\in\beta+\Lambda_\alpha(\varepsilon)} \Big(C\varepsilon^{\alpha(\nu-d)}\cdot\varepsilon + \sum_{\gamma\in\Lambda_\alpha(\varepsilon)} (1+|\gamma|)^{-\nu}\omega^+_{\gamma+\eta}\Big)^2 \cdot \int_{C_0} g_+^2(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta}(x)\Big|^2 dx$$

$$\leq \Big(\varepsilon^{\alpha(\nu-d)}\cdot\varepsilon + \Big[\sum_{\gamma\in\Lambda_\alpha(\varepsilon)} (1+|\gamma|)^{-\nu}\Big] \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon)} \omega^+_\gamma\Big)^2 \cdot \sum_{\eta\in\Lambda_\alpha(\varepsilon)} \int_{C_0} g_+^2(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta}(x)\Big|^2 dx. \quad (4.39)$$

There exists $K>0$ such that

$$\sum_{\eta\in\Lambda_\alpha(\varepsilon)} \int_{C_0} g_+^2(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta}(x)\Big|^2 dx = \sum_{\eta\in\Lambda_\alpha(\varepsilon)} \int_{C_0} g_+^2(x)\Big|\int_{\mathbb{T}^*} e^{i\eta\cdot\theta}\mathcal{W}_\varepsilon(\theta)(\partial_{x_i}f)(x,\theta)d\theta\Big|^2 dx \leq K. \quad (4.40)$$

So we use (4.35), (4.36), (4.37), (4.38), (4.39) and (4.40) to end the proof of Lemma 4.4. □
The proof of Lemma 4.5 is similar to that of Lemma 4.5. Following the same computation done for the proof of (4.27), we prove (4.28). □

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