Nonperturbative absolutely continuous spectrum for 1D quasiperiodic operators.

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1 Introduction.

In this paper we study quasiperiodic operators $\tilde{H}_{\omega,\lambda,\theta}$, acting on $\ell^2(\mathbb{Z})$ and given by

$$(\tilde{H}_{\omega,\lambda,\theta}\Psi)(n) = \Psi(n+1) + \Psi(n-1) + \frac{1}{\lambda} f(\omega n + \theta)\Psi(n), \tag{1.1}$$

where $f(\theta)$ is an analytic 1−periodic function, in the regime of small coupling (large λ).

Write f as $f(x) = \sum a_k e^{2\pi i k x}$ with $a_{-k} = \overline{a}_k$ and $|a_k| \leq Ce^{-\mu k}$, $\mu > 0$. Without loss we assume $a_0 = 0$. We do not incorporate C into λ as we want to track the dependence of λ not only on μ but also on finer properties of f. We will assume ω is Diophantine: there exist $c(\omega) > 0$ and $1 < r(\omega) < \infty$ such that

$$\|j\omega/2\| > \frac{c(\omega)}{|j|^{r(\omega)}} \tag{1.2}$$

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for all $j \neq 0$, where $\|\cdot\|$ denotes distance to the integers.

Our main result is the following

THEOREM 1 *For any real analytic f there exists $\lambda_0(f)$ such that for $\lambda > \lambda_0$, any Diophantine ω and a.e. $\theta \in \mathbb{R}$ operator $\tilde{H}_{\omega,\lambda,\theta}$ has purely absolutely continuous spectrum.*

Remarks:

- By a theorem of Last-Simon [17], ac spectrum is constant as a function of θ, so this also implies the existence of an ac component in the spectrum of $H_{\omega,\lambda,\theta}$ for *all* θ.
- An explicit estimate on λ_0 is given in a combination of (2.13),(2.14),(2.18),(2.19). Therefore, λ_0 depends on f in an explicit way, through μ, $\|f\|_1$, $\|f\|_2$, $\|f\|_\infty$.

Previous proofs of absolutely continuous spectrum for quasiperiodic operators were mostly KAM-type perturbative schemes, first developed in this context by Dinaburg-Sinai [6]. Early results [1, 5] were restricted to the case of almost-Mathieu type ($f = \cos$) operators. Eliasson [7] proved purely ac spectrum for analytic f in the continuous case.

While we do not claim that our constant λ_0 is optimal, there is, besides a simple proof, an important difference between our (nonperturbative) result and the (perturbative) theorems above in that λ_0 does not depend on ω and also depends on f in a constructive way.

The first nonperturbative results on the ac spectrum were obtained, for the case of the almost Mathieu operator, by Last [15], who proved the existence of a large absolutely continuous component for any $\lambda > 1/2$ for any ω (and all θ, as shown by Gestesy-Simon [10] or as follows from [17]). The result of Last extends to a general class of operators (1.1) with $f \in C^{3/2+\epsilon}$ and even [16].

In [11] Gordon et al. proved the strong version of Aubry duality theorem: that pure point spectrum for the dual model implies purely ac spectrum for a.e. θ. While the theorem in [11] was stated for almost Mathieu operators only, it holds in higher generality, which will be discussed below. The only previous nonperturbative result on pure ac spectrum was established in [14] for the almost Mathieu operator for any $\lambda > 1/2$ and Diophantine ω, using the duality theorem, as a corollary of localization for $\lambda < 1/2$.

In this paper we follow the same strategy in that we deduce the purely ac spectrum by duality from the pure point spectrum for the dual model. In case of general f the dual model is going to be long-range which creates certain obstacles in applying previously developed nonperturbative arguments [14, 3] based on Lyapunov exponents and transfer-matrices. While perturbative analysis typically extends without substantial problems to the long-range operators, it is not the case with nonperturbative proofs. The main technical achievement of this paper is precisely in developing a nonperturbative argument for localization in long-range Hamiltonians. We note that a related technique was used in [4] for the proof of localization in the band model.

We will now recall the Aubry duality setup. Let $\mathcal{H} = L^2([0,1) \times \mathbb{Z})$. Let $A_{\omega,\lambda}$ be the direct integral in θ of $\tilde{H}_{\omega,\lambda,\theta}$:

$$(A_{\omega,\lambda}\phi)(\theta, n) = \phi(\theta, n+1) + \phi(\theta, n-1) + \frac{1}{\lambda} f(\omega n + \theta)\Phi(\theta, n) \tag{1.3}$$

The duality transform U is a unitary operator on $\mathcal{H}$ acting by

$$(U\phi)(\beta, m) = \hat{\phi}(m, \beta + \omega m) \tag{1.4}$$

where $\hat{\phi}$ is the Fourier transform.

It is straightforward to check that $U^{-1}\lambda A_{\omega,\lambda}U = B_{\omega,\lambda}$, where $B_{\omega,\lambda}$ is the direct integral in θ of $H_{\omega,\lambda,\theta}$, where $H_{\omega,\lambda,\theta}$ is given by

$$(H_{\omega,\lambda,\theta}\Phi)(n) = \sum_{k=-\infty}^{\infty} a_k \Phi(n-k) + 2\lambda cos(\omega n + \theta)\Phi(n) \tag{1.5}$$

We will say that H is dual to $\tilde{H}$.

Let H_θ be a measurable (in θ) family of self-adjoint operators on $\ell^2(\mathbb{Z})$ (weak measurability, e.g. Borel measurability of matrix elements $(H_\theta\phi, \psi)$ is sufficient here.) Suppose H_θ has pure point spectrum for a.e. θ. We will say that H_θ admits a measurable enumeration of eigenelements if there exist Borel measurable functions $e_\ell(\theta) : [0,1) \to \mathbb{R}$ and $u_\ell(n,\theta) : [0,1) \to \ell^2(\mathbb{Z})$ so that $u_\ell(n,\theta)$ is an eigenfunction of H_θ with an eigenvalue $e_\ell(\theta)$ and functions $u_\ell(n,\theta)$ form an orthonormal basis for $\ell^2(\mathbb{Z})$ for a.e. θ.

The following is a part of a general theorem in [12] (see also a related result in [8]):

THEOREM 2 *Every measurable family of selfadjoint operators with a.e. pure point spectrum admits a measurable enumeration of eigenelements.*

Let $(T\phi)(n) = \phi(n+1)$ be a shift operator on $\ell^2(\mathbb{Z})$. Let (Θ, S) be an ergodic dynamical system. A measurable family $H_\theta,\ \theta \in \Theta$, of self-adjoint operators on $\ell^2(\mathbb{Z})$ will be called ergodic if $T^{-1}H_\theta T = H_{S\theta}$. We will say that an ergodic family of selfadjoint operators with a.e. pure point spectrum has a covariant spectral representation if there exist Borel measurable functions $e_\ell(\theta) : [0,1) \to \mathbb{R}$ and $v_\ell(n,\theta) : [0,1) \to \ell^2(\mathbb{Z})$ so that $v_\ell(n,\theta)$ is an eigenfunction of H_θ with an eigenvalue $e_\ell(\theta)$ and functions $\{v_\ell(n-j, S^j\theta)\}_{\ell,j:v_\ell(S^j\theta)\neq 0}$ form an orthonormal basis for $\ell^2(\mathbb{Z})$, in particular,

$$\sum_n \overline{v_m(n-j, S^j\theta)}v_\ell(n,\theta) = 0$$

for all $m, \ell \in \mathbb{Z}$ and $j \neq 0$.

Covariant spectral representations have been obtained constructively for random Schrödinger operators with Markov potentials and quasiperiodic cos-type potentials at large coupling in, correspondingly, [18, 20]. In [11] it was shown that families of operators with simple spectrum always have such representations.

As an immediate corollary of Theorem 2 we obtain

THEOREM 3 *Every ergodic family of selfadjoint operators with a.e. pure point spectrum has a covariant spectral representation.*

Proof: Set $v_\ell(\theta) = u_\ell(\theta)$ if $u_\ell(\theta)$ has it's leftmost maximum at zero. Set $v_\ell(\theta) = 0$ otherwise. Since $v_\ell(n-j, S^j\theta)$ is an eigenfunction of H_θ with it's leftmost maximum at j, the theorem follows. ∎

Finally we will need the following duality theorem proved in [11]:

THEOREM 4 *Suppose for some ω, λ operator $H_{\omega,\lambda,\theta}$ given by (1.5) has pure point spectrum for a.e. θ, and the family $H_{\omega,\lambda,\theta}$ has a covariant spectral representation. Then dual operators $\tilde{H}_{\omega,\lambda,\theta}$ given by (1.1) have purely absolutely continuous spectrum for a.e. θ.*

Given Theorems 3 and 4, Theorem 1 follows immediately from

THEOREM 5 *For any real-analytic f there exists $\lambda_0(f)$ such that for any Diophantine ω there is a set $\Theta \subset [0, 2\pi)$ of zero Lebesgue measure such that for $\theta \notin \Theta,\ \lambda > \lambda_0$, operator $H_{\omega,\lambda,\theta}$ given by (1.5) has only pure point spectrum with exponentially decaying eigenfunctions. Moreover, the family $H_{\omega,\lambda,\theta}$ is strongly dynamically localized.*

Remark. The set of resonant phases Θ may be given explicitly by

$$\Theta = \{\theta : \text{ the relation } |\sin 2\pi(\theta+(j/2)\omega)| < \exp(-k^{\frac{1}{2r(\omega)}}) \text{ holds for infinitely many } k'\text{s}\}. \tag{1.6}$$

Combining the technique of this paper with the method of [3], as adapted in [4], one can obtain a measure-theoretic version (that is a result for a.e. ω, θ, without precise arithmetic conditions) of Theorem 5 for operators (1.5) with cos replaced by an analytic function. This, however, will not lead to a theorem on pure ac spectrum for the dual model (which would be operator (1.1) with a long-range Laplacian) since Theorem 4 is not known in such setting. A conclusion one can make for the dual model based on localization in this situation is continuity of the spectrum for all phases. We also would like to mention that current techniques do not extend to operators (1.5) with multi-frequency potentials.

It therefore remains to prove Theorem 5 which we do in Section 2. We omit the proof of dynamical localization statement as it is a straightforward adaptation of the argument of [9], and is not needed for the proof of Theorem 1. Note, that uniformity of the upper estimates in energy is immediate in the current proof, therefore our Diophantine condition (1.2) on ω is less restrictive than that of [9].

2 Nonperturbative localization for long-range Hamiltonians

A formal solution $\Psi_E(x)$ of the equation $H_{\omega,\lambda,\theta}\Psi_E = E\Psi_E$ is called **a generalized eigenfunction** if $|\Psi_E(x)| \leq C(1+|x|)$ for some $C = C(\Psi_E) < \infty$. It is well known that to prove Theorem 2 it suffices to prove that generalized eigenfunctions decay exponentially [2, 19]. Fix an interval $I \in \mathbb{Z}$. Let Γ be the coupling operator between I and $\mathbb{Z}\backslash I$:

$$\Gamma(i,j) = \begin{cases} a_{i-j}, & \chi_I(i) + \chi_I(j) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can write $\Psi_E = -(H - E - \Gamma)^{-1}\Gamma\Psi_E$ from which for any $x \in I$ we obtain

$$\Psi_E(x) = -\sum_{y \in I, k \notin I} G_I(E)(x,y)a_{y-k}\Psi_E(k), \tag{2.1}$$

where G_I is the Green's function of H restricted to the interval I : $G_I = (R_I(H-E)R_I)^{-1}$.

Fix $E \in \mathbb{R}$, $m \in \mathbb{R}$. A point $y \in \mathbb{Z}$ will be called (m,k)-regular if there exists an interval $I = [x_1, x_2]$, $x_2 = x_1 + k - 1$, containing y, such that

$$\sum_{x \in I, i=1,2} |G_I(y,x) a_{y-x_i}| < e^{-mk}$$

Otherwise, y will be called (m,k)-singular.

From (2.1) we see that if Ψ_E is a generalized eigenfunction, then every point $y \in \mathbb{Z}$ with $\Psi_E(y) \neq 0$ is (m,k)-singular for k sufficiently large. Similarly, if k is (m,k)-regular then $|\Psi_E(k)|$ is bounded by Ce^{-mk} so it suffices to prove that for an appropriate m and sufficiently large k, (m,k)-singular points cannot be at a distance k from, say, zero.

Let Θ be defined by (1.6). We will prove

LEMMA 6 *There exists $\lambda_0(f)$ such that for $\lambda > \lambda_0$, for small enough m, ω satisfying (1.2), and any $\theta \notin \Theta$, we have that for every $y \in \mathbb{Z}$, $\alpha < 2$, there exists $k(\theta, \omega, y, m, \lambda, \alpha)$, such that for all $k > k(\theta, \omega, y, m, \lambda, \alpha)$, if x and y are both $(m,k)-$singular for some E in the spectrum and $|x-y| > \frac{k+1}{2}$, then $|x-y| > k^{\alpha}$.*

Remarks:

1. Relaxing the nonresonant condition on the excluded set Θ (with Θ still having measure 0, namely replacing the subexponential function by polynomial decay in (1.6)) one can obtain exponential in k repulsion of singular clusters in Lemma 6.

2. An explicit estimate on λ_0 is given in a combination of (2.13),(2.14),(2.18),(2.19).

Theorem 5 now follows in the standard way from Lemma 6 and the discussion above. ∎

We proceed to the proof of Lemma 6. Define $I \subset \mathbb{Z}$ by $I = [0, N-1]$, $N \in \mathbb{N}$. Fix $E \in \mathbb{R}$. We will omit the E dependence of various quantities in what follows, and all constants, unless specifically claimed to be dependent on E, will be uniform for all E in the spectrum. We will also often assume $f \not\equiv 0$, as otherwise Theorems 1,5 become trivial. Let $P_N(\theta) = detR_I(H-E)R_I$. $P_N(\theta)$ is an even function of $\theta + \frac{N-1}{2}\omega$ and can be written as a polynomial of degree N in $\cos 2\pi(\theta + \frac{N-1}{2}\omega)$: $P_N(\theta) = \sum_{j=0}^{N} c_j \cos^j 2\pi(\theta + \frac{N-1}{2}\omega)$.

LEMMA 7 $\int_0^1 \frac{1}{N} \ln |P_N(\theta)| d\theta \geq \ln \lambda$.

Proof: It is proved by a standard Herman's subharmonicity argument [13]. ∎

Our key technical statement is the following

LEMMA 8 *There exists $\lambda_0(f)$ such that for $\lambda > \lambda_0$, and small enough m, ϵ, if $|P_N(\theta + x\omega)| > (\lambda - \epsilon)^N$, then all points $y \in x + I$ with $dist(y, \partial(x+I)) \geq \frac{N}{4} - 1$ are $(m,N)-$regular for sufficiently large N..*

For this Lemma ω is assumed to be Diophantine. We will postpone the proof of Lemma 8 to the end of the section, and finish the proof of Lemma 6. The next Lemma, proved in [14], states, essentially, a simple fact that logarithm of an even trigonometric polynomial of degree k cannot be smaller than its mean value, along a uniformly distributed sequence of $k+1$ points. We will formulate here a particular form convenient to us.

Set

$$\theta_j = \left\{ \begin{array}{c} \theta + j\omega, \ j = 0, 1, ..., [\frac{k+1}{2}] - 1 \\ \theta + (j - [\frac{k+1}{2}] + d)\omega, \ j = [\frac{k+1}{2}], \ldots, k \end{array} \right. ,$$

where $|d| > [\frac{k+1}{2}]$, so all θ_j, $j = 0, \ldots, k$, are different.

LEMMA 9 *Let $P_k(\theta) = \sum_{j=0}^{k} b_j \cos^j 2\pi\theta$ be an even trigonometric polynomial. Suppose ω satisfies (1.2) and $\theta \notin \Theta$. Suppose $\ln |P_k(\theta_j)| < \sup_\theta \ln |P_k(\theta)| - \epsilon k$, $j = 0, \ldots, k$. Then for every $\epsilon > 0, \alpha < 2$, for $k > k(\theta, r(\omega), c(\omega), \epsilon, \alpha)$ we have that $|d| > k^\alpha$.*

To finish the proof of Lemma 6 we take $\lambda > \lambda_0$ given by Lemma 8 and observe that if y is $(m,k)-$singular, then, by Lemma 8, $|P_k(\theta + x\omega)| < (\lambda - \epsilon)^k$, for all $x \in [y - [\frac{3}{4}]k, y - [\frac{3}{4}]k + [\frac{k+1}{2}]]$. Therefore, if both x and y are singular we can, taking into account Lemma 7, apply Lemma 9 (replacing θ with $\theta + (\frac{N-1}{2} + y)\omega$) with $d = x - y$, to obtain that $|x - y| > k^\alpha$. ∎

It now remains to prove Lemma 8. Without loss, set $x = 0$. Set $x_1 = -1$, $x_2 = N$. It will be enough to show that $\sum_{z \in I, i=1,2} |G_I(y,z)||a_{z-x_i}| < e^{-mN}$.

By Cramer's rule $G_I(y,z) = \frac{\mu_{y,z}}{P_N(\theta)}$ where $\mu_{y,z}$ is the corresponding minor. The following Lemma is crucial for our nonperturbative argument, as it reduces the study of Green's function to the study of determinants of restrictions of $H-E$. The statement (and the proof) holds for Schrodinger-type operators on any graph.

LEMMA 10

$$\mu_{y,z} = \sum_{\gamma} \alpha_\gamma \det R_{I\setminus\gamma}(H-E)R_{I\setminus\gamma} \prod_{i=1}^{|\gamma|} |a_{\gamma_{i+1}-\gamma_i}| \tag{2.2}$$

where the sum is taken over all ordered subsets $\gamma = (\gamma_1, \dots, \gamma_n)$ of I with $\gamma_1 = y$ and $\gamma_n = z$, $|\gamma| = n-1$, and $\alpha_\gamma \in \{-1, 1\}$.

Remark: The sign α_γ actually only depends on the parity of $|\gamma|$ and of $x-y$. We will not need this here so omit the proof.

Proof. Expand $\mu_{y,z}$ through the $y-$column: $\mu_{y,z} = \sum\limits_{\gamma_2 \neq y} \delta_{\gamma_2} a_{\gamma_2 - y} h_{z,y,\gamma_2}$ where $\delta_{\gamma_2} \in \{-1, 1\}$, and h_{z,y,γ_2} is the determinant of the original matrix with z and y raws and y and γ_2 columns removed. Continue this process with each term until $\gamma_n = z$. The contribution of each term is equal, up to a sign, to $\det R_{I\setminus\gamma}(H-E)R_{I\setminus\gamma} \prod\limits_{i=1}^{|\gamma|} |a_{\gamma_{i+1}-\gamma_i}|$. ∎

Our next Lemma provides an upper bound necessary to estimate $\mu_{y,z}$. Note that already an obvious bound of the form

$$|\det R_{I\setminus\Lambda}(H-E)R_{I\setminus\Lambda}| \leq (2|\lambda| + |E| + \|f\|_\infty)^{|I\setminus\Lambda|} \tag{2.3}$$

would provide a result for sufficiently large μ, however a more delicate estimate is needed for small μ. Assume ω is Diophantine.

LEMMA 11 *For any $\Lambda \subset I$, uniformly for E in the spectrum of H, for sufficiently large N,*

$$|\det R_{I\setminus\Lambda}(H-E)R_{I\setminus\Lambda}| \leq \lambda^N e^{C_f \lambda^{-1/2} N} \tag{2.4}$$

and if $|\Lambda| > \lambda^{-1/10} N$ then

$$|\det R_{I\setminus\Lambda}(H-E)R_{I\setminus\Lambda}| \leq \lambda^{N-1/5|\Lambda|} \tag{2.5}$$

Proof.
By Hadamard's bound,

$$|\det R_{I\setminus\Lambda}(H-E)R_{I\setminus\Lambda}| \leq \prod_{j\in I\setminus\Lambda}((2\lambda\cos 2\pi(\theta+j\omega)-E)^2+\|f\|_2^2)^{1/2} \quad (2.6)$$

Hence

$$\log|\det R_{I\setminus\Lambda}(H-E)R_{I\setminus\Lambda}| \leq \frac{1}{2}\sum_{j\in I}\log|(2\lambda\cos 2\pi(\theta+j\omega)-E)^2+\|f\|_2^2|- \quad (2.7)$$

$$-\frac{1}{2}\sum_{j\in\Lambda}\log|(2\lambda\cos 2\pi(\theta+j\omega)-E)^2+\|f\|_2^2| \quad (2.8)$$

Assuming $\|f\|_2 \geq 1$ (otherwise we will use bound (2.6) with $\|f\|_2$ replaced by 1) we can estimate $|\det R_{I\setminus\Lambda}(H-E)R_{I\setminus\Lambda}|$ by dropping the negative terms in (2.8). Let $u(\theta)=\log|(2\lambda cos 2\pi\theta-E)^2+\|f\|_2^2|$. The right-hand side of (2.7) is a sum of values of u along the trajectory of rotation by ω. We use Denjoy-Koksma type bound (for a particular statement used here see, e.g. [14], Lemma 12) to estimate the difference between (2.7) and integral of u.

$$\sum_{j\in I}u(\theta+j\omega) \leq N\int u(\theta)d\theta + CN^{1-\frac{1}{r(\omega)}}\log N \mathrm{V}aru \quad (2.9)$$

for ω satisfying (1.2) and any continuous function u, where C depends only on ω.

$\int u(\theta)d\theta$ can be computed explicitly by writing

$$\int u(\theta)d\theta = 2\log 2\lambda + 2\int \log|\cos 2\pi\theta - z|d\theta$$

where $z=\frac{E+i\|f\|_2^2}{2\lambda}$.

Hence, using that

$$\int \log|e^{2\pi i\theta}-z| = \begin{cases} 0, & |z|<1, \\ \log|z|, & |z|\geq 1, \end{cases}$$

we obtain

$$\int u(\theta)d\theta = 2\log\lambda + 2\log|z+\sqrt{z^2-1}|$$

where the branch of the square root is chosen so that $|z+\sqrt{z^2-1}| \geq 1$. Finally, for E in the spectrum, we have $E \leq 2\lambda + \|f\|_\infty$, and therefore, by the properties of $z+\sqrt{z^2-1}$, obtain $\int u(\theta)d\theta \leq 2\log\lambda + C\lambda^{-1/2}$ where C depends only on f, through $\|f\|_2$ and $\|f\|_\infty$. Combining it with (2.7),(2.9), we obtain

$$\log|\det R_{I\backslash\Lambda}(H-E)R_{I\backslash\Lambda}| \leq N\log\lambda + C_f N\lambda^{-1/2} \tag{2.10}$$

for sufficiently large N, which is (2.4).

To obtain (2.5) we need a lower bound on

$$\sum_{j\in\Lambda} \log|(2\lambda\cos 2\pi(\theta+j\omega)-E)^2 + \|f\|_2^2|, \tag{2.11}$$

where $\Lambda \in [0, N-1]$ is an arbitrary set of size $|\Lambda| > \lambda^{-1/10}N$.

For any θ and $\delta \in (0,1)$ one has

$$\#\{j=0,\ldots,N-1 : \|\frac{j\omega}{2}+\theta\| < \delta\} = \sum_{j=0}^{N-1} \chi_{[-\delta,\delta]}(\theta+\frac{j\omega}{2}) \leq 2\delta N + CN^{1-\frac{1}{r(\frac{\omega}{2})}}\log N \tag{2.12}$$

by (2.9) applied to the (smoothed) χ.

Set

$$\theta_0 = \begin{cases} \arccos\frac{E}{2\lambda}, |E| \leq 2\lambda, \\ 0, \quad E > 2\lambda, \\ \pi, \quad E < -2\lambda, \end{cases}$$

With $\delta = \frac{|\Lambda|}{4N}$ we can estimate

$$(2.11) > \sum_{j\in\Lambda:\|\frac{j\omega}{2}-\theta_0\|>\delta} \geq \#\{j\in\Lambda : \|\frac{j\omega}{2}-\theta_0\| > \delta\}(\log 4\lambda\delta^2 + \|f\|_2^2|) \geq$$

$$\geq (|\Lambda| - \frac{|\Lambda|}{2} - CN^{1-\frac{1}{r(\frac{\omega}{2})}}\log N)\log(\frac{1}{4}\lambda^{4/5}) \geq \frac{3}{10}|\Lambda|\log\lambda, \tag{2.13}$$

for λ large (the largeness condition is only needed for the last inequality). Finally, for λ such that

$$C_f\lambda^{-\frac{1}{2}} < 1/10\lambda^{-\frac{1}{10}}\log\lambda \tag{2.14}$$

estimate (2.5) follows immediately from (2.8),(2.10) and (2.13) ∎

Remark. By choosing $|\Lambda| > \lambda^{-\beta}N$ we can obtain a bound in (2.5) of the form $\lambda^{N-(1-3\beta)|\Lambda|}$ for any $\beta > 0$ and sufficiently large λ. Having smaller β would improve, for small μ, the bounds in what follows ((2.18),(2.19)), however require larger λ for estimates (2.13),(2.14). Larger β may provide a better overall bound in case of large μ. This proof can therefore be, for each individual f, optimized in β to yield a lower constant λ_0.

We can now complete the proof of Lemma 8.

Assuming $|P_N(\theta)| > (\lambda-\epsilon)^N$, we estimate, using (2.2):

$$\sum_{z\in I,\, i=1,2} |G_I(y,z)||a_{z-x_i}| \leq (\lambda-\epsilon)^{-N}\sum_{n=1}^{N-1}\sum_{i=1,2;\,\gamma:\,|\gamma|=n} |\det R_{I\setminus\gamma}(H-E)R_{I\setminus\gamma}||a_{x_i-\gamma_{|\gamma|+1}}|\prod_{i=1}^{n}|a_{\gamma_{i+1}-\gamma_i}|$$

$$= (\lambda-\epsilon)^{-N}\Big(\sum_{n=1}^{\lambda^{-\frac{1}{10}}N} + \sum_{n>\lambda^{-\frac{1}{10}}N}\Big) = (\lambda-\epsilon)^{-N}\big(\textstyle\sum_1 + \sum_2\big) \tag{2.15}$$

Using (2.4), we bound $\sum_1$ by

$$\textstyle\sum_1 \leq \lambda^N e^{c_f\lambda^{-\frac{1}{2}}N}\displaystyle\sum_{n=1}^{\lambda^{-\frac{1}{10}}N}\|f\|_1^n\sum_{i=1,2;\,b} e^{-\mu b}\#\{\gamma:\,|\gamma|=n,\,\gamma_1=y,\,|\gamma_{n+1}-x_i|+\sum_{i=1}^{n}|\gamma_{i+1}-\gamma_i|=b\}$$

$$\leq \lambda^N e^{c_f\lambda^{-\frac{1}{2}}N}\sum_{b\geq dist(y,\partial I)} e^{-\mu b}\sum_{n=1}^{\lambda^{-\frac{1}{10}}N}\|f\|_1^n\binom{b}{n}$$

Assume $\|f\|_1 \geq 1$ (otherwise bound it by 1.) For y as in the statement of the Lemma we have $dist(y,\partial I) > N/4-1$, therefore, for large N,

$$\textstyle\sum_1 \leq \lambda^N e^{c_f\lambda^{-\frac{1}{2}}N}\lambda^{-\frac{1}{10}}N\|f\|_1^{\lambda^{-\frac{1}{10}}N}\displaystyle\sum_{b\geq dist(y,\partial I)} e^{-\mu b}\binom{b}{\lambda^{-\frac{1}{10}}N}$$

By Stierling formula, for large N,

$$\binom{b}{\lambda^{-\frac{1}{10}}N} \leq \binom{b}{4\lambda^{-\frac{1}{10}}b} \leq C_\lambda e^{\phi(4\lambda^{-\frac{1}{10}})b}$$

where $\phi(x) = -x\log x - (1-x)\log x$.
Hence,

$$\sum_1 \le \lambda^N C e^{(c_f\lambda^{-\frac{1}{2}}+\lambda^{-\frac{1}{10}}\log\|f\|_1+\frac{1}{4}\phi(4\lambda^{-\frac{1}{10}})-\frac{\mu}{4})N}, \tag{2.16}$$

where C is a constant depending on f, μ, λ. Similarly, using (2.5), we have

$$\sum_2 \le \lambda^N e^{c_f\lambda^{-\frac{1}{2}}N} \sum_n \|f\|_1^n \lambda^{-\frac{1}{5}n} \sum_{b\ge dist(y,\partial I)} e^{-\mu b}\binom{b}{n} \le \lambda^N e^{c_f\lambda^{-\frac{1}{2}}N} \sum_{b\ge dist(y,\partial I)} e^{-\mu b}(1+\|f\|_1\lambda^{-\frac{1}{5}})^b \tag{2.17}$$

Substituting (2.16),(2.17) into (2.15) we obtain the desired exponential decay for sufficiently small ϵ and large N provided

$$\log\frac{\lambda}{\lambda-\epsilon} + c_f\lambda^{-\frac{1}{2}} + \lambda^{-\frac{1}{10}}\log\|f\|_1 + \frac{1}{4}\phi(4\lambda^{-\frac{1}{10}}) - \frac{\mu}{4} < 0 \tag{2.18}$$

and

$$\log\frac{\lambda}{\lambda-\epsilon} + c_f\lambda^{-\frac{1}{2}} + \frac{1}{4}\log(1+\|f\|_1\lambda^{-\frac{1}{5}}) - \frac{\mu}{4} < 0 \tag{2.19}$$

∎

References

[1] J. Bellisard, R. Lima, and D. Testard, A metal-insulator transition for the almost Mathieu model. Commun. Math. Phys. **88** (1983), 207–234

[2] Y. Berezanskii, Expansions in eigenfunctions of selfadjoint operators. Transl. Math. Monogr., Vol. 17. Providence, RI: Am. Math. Soc. 1968.

[3] J. Bourgain, M. Goldstein, On nonperturbative localization with quasiperiodic potential. Annals of Math., **152** (2000), 835-879

[4] J. Bourgain, S. Jitomirskaya, Anderson localization for the band model, GAFA (1996-2000), 67-79, Lecture Notes in Math. **1745**, Springer, Berlin, 2000

[5] V. Chulaevsky and F. Delyon, Purely absolutely continuous spectrum for almost Mathieu operators. J. Stat. Phys. **55** (1989), 1279–1284

[6] E. Dinaburg and Ya. Sinai, The one-dimensional Schrödinger equation with a quasi-periodic potential. Funct. Anal. Appl. **9** (1975), 279–289

[7] L. H. Eliasson, Floquet solutions for the one-dimensional quasi-periodic Schrödinger equation. Commun. Math. Phys. **146** (1992), 447–482

[8] M. Foreman, A descriptive view of ergodic theory. Descrptive set theory and dynamical systems (Marseille-Luminy 1996), 87-171, London Math. Soc. Lect. Notes Ser. 277, Cambridge Univ. Press, Cambridge, 2000

[9] F. Germinet, S. Jitomirskaya, Strong dynamical localization for the almost Mathieu model, Rev. Math. Phys., to appear.

[10] F. Gesztesy and B. Simon, The xi function, Acta Math. **176** (1996), 49–71

[11] A. Gordon, S. Jitomirskaya, Y. Last, and B. Simon, Duality and singular continuous spectrum in the almost Mathieu equation. Acta Math., **178** (1997), 169–183

[12] A. Gordon, A.S. Kechris, Measurable enumeration of eigenelements. Appl. Anal. **71** (1999), 41-61,

[13] M. Herman, Une methode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractere local d'un theoreme d'Arnold et de moser sur le tore en dimension 2.Commun. Math. Helv. **58** (1983), 453–502

[14] S. Jitomirskaya, Metal-Insulator transition for the almost Mathieu operator. Annals of Math. **150** (1999), 1159-1175

[15] Y. Last, A relation between absolutely continuous spectrum of ergodic Jacobi matrices and the spectra of periodic approximants. Commun. Math. Phys. **151** (1993), 183–192

[16] Y. Last, 1993, unpublished.

[17] Y. Last and B. Simon, Eigenfuctions, transfermatrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators, Invent. Math. **135** (1999), 329–367

[18] S.A.Molchanov, The structure of eigenfunctions of one-dimensional disordered structures. Math. USSR Izv. **75**, 179-196 (1980).

[19] B. Simon, Schrödinger semigroups, Bull. AMS **7** (1982), 447–526

[20] Ya. Sinai, Anderson localization for one-dimensional difference Schrödinger operator with quasi-periodic potential. J. Stat. Phys. **46** (1987), 861–909