On the scattering theory of massless Nelson models

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Abstract

We study the scattering theory for a class of non-relativistic quantum field theory models describing a confined non-relativistic atom interacting with a relativistic bosonic field. We construct invariant spaces $\mathcal{H}_{\rm c}^{\pm}$ which are defined in terms of propagation properties for large times and which consist of states containing a finite number of bosons in the region $\{|x| \geq ct\}$ for $t \to \pm\infty$. We show the existence of asymptotic fields and we prove that the associated asymptotic CCR representations preserve the spaces $\mathcal{H}_{\rm c}^{\pm}$ and induce on these spaces representations of Fock type. For these induced representations, we prove the property of *geometric asymptotic completeness*, which gives a characterization of the vacuum states in terms of propagation properties. Finally we show that a positive commutator estimate imply the *asymptotic completeness* property, ie the fact that the vacuum states of the induced representations coincide with the bound states of the Hamiltonian.

1 Introduction

In this section we describe the class of models that we will consider in this paper, discuss the hypotheses and describe the main results.

1.1 Massless Nelson models

We will consider in this paper a quantum field theory model which describes a confined atom interacting with a field of massless scalar bosons. This model is usually called the *Nelson model* (see [Ne], [A], [Ar], [LMS]). It was originally introduced in [Ne] as a phenomenological model of non-relativistic particles interacting with a quantized scalar field.

The atom is described with the Hilbert space

$$\mathcal{K} := L^2(\mathbb{R}^{3P}, {\rm dx}),$$

where ${\rm x} = ({\rm x}_1, \dots, {\rm x}_P)$, ${\rm x}_i$ is the position of particle i, and the Hamiltonian:

$$K := \sum_{i=1}^{P} \frac{-1}{2m_i}\Delta_i + \sum_{i<j} V_{ij}({\rm x}_i - {\rm x}_j) + W({\rm x}_1, \dots, {\rm x}_P),$$

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where m_i is the mass of particle i, V_{ij} is the interaction potential between particles i and j and W is an external confining potential.

We will assume

$$(H0)\quad \begin{array}{l} V_{ij} \text{ is } \Delta - \text{ bounded with relative bound } 0, \\ W \in L^2_{\mathrm{loc}}(\mathbb{R}^{3N}),\ W(\mathrm{x}) \geq c_0|\mathrm{x}|^{2\alpha} - c_1,\ c_0 > 0,\ \alpha > 0. \end{array}$$

It follows from *(H0)* that K is symmetric and bounded below on $C_0^\infty(\mathbb{R}^{3P})$. We still denote by K its Friedrichs extension. Moreover we have $\mathcal{D}((K+b)^{\frac{1}{2}}) \subset H^1(\mathbb{R}^{3P}) \cap D(|\mathrm{x}|^\alpha)$, which implies that

$$|\mathrm{x}|^\alpha (K+b)^{-\frac{1}{2}} \text{ is bounded.} \tag{1.1}$$

Note also that *(H0)* implies that K has compact resolvent on $L^2(\mathbb{R}^{3P})$. The one particle space for bosons is

$$\mathfrak{h} := L^2(\mathbb{R}^3, \mathrm{d}k),$$

where the observable k is the boson momentum. An important role will be played by the observable

$$x := \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k},\ \text{acting on } \mathfrak{h}.$$

The observable x has the interpretation of the *Newton-Wigner* position. In fact the one-particle space for relativistic massless scalar bosons can be written as $L^2(\mathbb{R}^3, \frac{\mathrm{d}k}{|k|})$. In this representation the selfadjoint operator

$$x_{\mathrm{NW}} := \mathrm{i}|k|^{\frac{1}{2}}\frac{\partial}{\partial k}|k|^{-\frac{1}{2}}$$

is called the *Newton-Wigner position observable* (see eg [Sch, Chap. 3c]). By the unitary map $h(k) \mapsto |k|^{-\frac{1}{2}}h(k)$ between $L^2(\mathbb{R}^3, \frac{\mathrm{d}k}{|k|})$ and $L^2(\mathbb{R}^3, \mathrm{d}k)$ the observable x_{NW} is sent onto the observable x. Hence x has the interpretation of the Newton-Wigner position.

The bosonic field is described with the Fock space $\Gamma(\mathfrak{h})$ and the Hamiltonian $\mathrm{d}\Gamma(|k|)$.

The non-interacting system is described with the Hilbert space

$$\mathcal{H} := \mathcal{K} \otimes \Gamma(\mathfrak{h})$$

and the Hamiltonian

$$H_0 := K \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(|k|)$$

We assume that the interaction is of the form

$$V := \sum_{j=1}^{N} \phi(\check{v}_j(\mathrm{x}_j)), \tag{1.2}$$

for

$$\phi(\check{v}_j(\mathrm{x}_j)) = \frac{1}{\sqrt{2}} \int v_j(k)\mathrm{e}^{-\mathrm{i}k.\mathrm{x}_j} a^*(k) + \bar{v}_j(k)\mathrm{e}^{\mathrm{i}k.\mathrm{x}_j} a(k)\mathrm{d}k,$$

where $\check{f}$ denotes the inverse Fourier transform of f and the functions v_j satisfy

$$(I0)\ \int (1+|k|^{-1})|v_j(k)|^2 \mathrm{d}k < \infty,\ 1 \leq j \leq P.$$

The Hamiltonian describing the interacting system is now:

$$H := H_0 + V.$$

The assumption *(I0)* implies, using Prop. A.1, that $\phi(\check{v}_j(\mathrm{x}_j))$ is H_0-bounded with infinitesimal bound and hence that H is selfadjoint and bounded below on $\mathcal{D}(H_0)$.

Note that the interaction is translation invariant (although the full Hamiltonian H is not because of the confining potential W). Note also that using the notation introduced in (2.1) we can write:

$$V = \phi(v),$$

where $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ is defined by

$$v\psi(\mathrm{x}_1, \dots, \mathrm{x}_P) = \sum_{j=1}^{P} \mathrm{e}^{-\mathrm{i}k.x_j} v_j(k)\psi(\mathrm{x}_1, \dots, \mathrm{x}_P). \tag{1.3}$$

1.2 Results

We now describe the results of this paper. We will also comment on related works and on remaining open questions.

The mathematical framework of scattering theory for quantum field theory models rests on the *asymptotic fields*. Once constructed they define the *asymptotic CCR representations*. In very broad terms, the basic goal of scattering theory is to study the nature of these representations and in particular to understand the nature of their Fock sub-representations (if they exist).

For the class of models considered in this paper, describing a confined atom interacting with a bosonic field, the existence of bound states of the Hamiltonian (in particular the existence of a ground state in Hilbert space), easily implies that the asymptotic CCR representations admit a non trivial subrepresentation of Fock type (ie unitarily equivalent to a direct sum of Fock representations). In fact the bound states of the Hamiltonian are vacua for the asymptotic representations.

One defines then isometric operators $\Omega^{\pm}$ called the *wave operators* between a direct sum of copies of Fock spaces and subspaces $\mathcal{H}^{\pm}$ of $\mathcal{H}$.

One can then ask the following two fundamental questions:

1) are the asymptotic CCR representations entirely of Fock type?

if this property holds the wave operators are unitary.

2) are the spaces $\mathcal{K}^{\pm}$ of vacua for the asymptotic CCR representations identical to the space of bound states of the Hamiltonian?

this second property is called the *asymptotic completeness property*.

Properties 1 and 2 together give a complete understanding of the scattering theory.

For models analogous to those considered in this paper, with a *massive* bosonic field, these two questions were answered in the affirmative in [DG2]. In the massive case, the answer to the first question is rather easy, and relies on the fact that the total number of particles is dominated by the energy.

The second question is more difficult, even in the massive case. In [DG2], this problem was solved in two steps: first a direct geometric characterization of the asymptotic vacua, in terms of their propagation properties for large times, is obtained: one shows that the asymptotic vacua coincide with the states having no particles in $\{|x| \geq \epsilon t\}$ for large t and $\epsilon > 0$ arbitrarily small. This property is called in [DG2] the *geometric asymptotic completeness*. In a second step this geometric characterization of the asymptotic vacua is combined with a Mourre estimate to obtain the asymptotic completeness.

In this paper we give some partial answers to the second problem for models describing a confined atom interacting with a massless bosonic field.

There are several conceptual and technical problems to extend the results of [DG2] to the massless case. First of all the existence of a ground state of the Hamiltonian in the Hilbert space beco,es a non-trivial question whose answer depends on the infrared behavior of the interaction for small momenta. If the interaction is not too singular for small momenta (condition *(I3)*), then it was shown in [G] that the Hamiltonian admits a ground state in Hilbert space. On the contrary if *(I3)* is violated, it was shown in [LMS] that no ground state exists in Hilbert space. In this case the definition of the asymptotic fields has to be modified.

In most of our paper we will assume that condition *(I3)* is satisfied. In this case we expect that properties 1 and 2 should also hold in the massless case.

On the technical level, the observables used in [DG2] to show geometric asymptotic completeness are unbounded observables dominated only by the number operator. This was not a problem in the massive case, as these observables are then also bounded by the energy. In the massless case, this is no longer true and we have to use different methods to prove corresponding propagation estimates.

The main results of this paper can be summarized as follows:

for each constant $0 < \mathrm{c} < 1$, we construct and describe in terms of propagation properties for $t \to \pm\infty$, spaces $\mathcal{H}_{\mathrm{c}}^{\pm}$ which are invariant under the evolution and which contain the states having a finite number of particles in $\{|x| \geq \mathrm{c}'t\}$ for each $\mathrm{c} < \mathrm{c}'$. We show that the spaces $\mathcal{H}_{\mathrm{c}}^{\pm}$ have the following properties:

- $\mathcal{H}_{\mathrm{c}}^{\pm}$ are non trivial if the Hamiltonian has bound states;

- the asymptotic CCR representations preserve $\mathcal{H}_{\mathrm{c}}^{\pm}$ and are of Fock type when restricted to $\mathcal{H}_{\mathrm{c}}^{\pm}$. Recalling that the ranges of the wave operators $\Omega^{\pm}$ are denoted by $\mathcal{H}^{\pm}$ this property means that $\mathcal{H}_{\mathrm{c}}^{\pm} \subset \mathcal{H}^{\pm}$.

- on $\mathcal{H}_{\mathrm{c}}^{\pm}$ the geometric asymptotic completeness holds: the asymptotic vacua in $\mathcal{H}_{\mathrm{c}}^{\pm}$ are exactly the states in $\mathcal{H}_{\mathrm{c}}^{\pm}$ having no particles in $\{|x| \geq \mathrm{c}'t\}$ for all $\mathrm{c} < \mathrm{c}'$, $t \to \pm\infty$.

- if a Mourre estimate holds on an energy interval Δ with the generator of dilations as conjugate operator, then a restricted version of asymptotic completeness holds on Δ: the asymptotic vacua in $\mathcal{H}_{\mathrm{c}}^{\pm}$ with energy in Δ coincide with the bound states of the Hamiltonian in Δ.

There are two questions which remain open: first of all one would like to show that the spaces $\mathcal{H}_{\mathrm{c}}^{\pm}$ are equal to the whole Hilbert space $\mathcal{H}$. This would imply that the asymptotic CCR representations are of Fock type and that the wave operators are unitary. We believe that it should be easier to show that $\mathcal{H}_{\mathrm{c}}^{\pm} = \mathcal{H}$ than to show that $\mathcal{H}^{\pm} = \mathcal{H}$ since we have a geometric description of $\mathcal{H}_{\mathrm{c}}^{\pm}$ instead of the algebraic description of $\mathcal{H}^{\pm}$ given by the asymptotic number operators.

A more modest question would be to show that the spaces $\mathcal{H}_{\mathrm{c}}^{\pm}$ are all identical, which is very likely since the speed of propagation for massless bosons is equal to 1, so no particles should be found in the intermediate regions $\{\mathrm{c}_1 t \leq |x| \leq \mathrm{c}_2 t\}$ for $0 < \mathrm{c}_1 < \mathrm{c}_2 < 1$.

The second remaining open problem is to show a Mourre estimate for the Hamiltonian H outside of a discrete set of points. Up to now a Mourre estimate has been shown only for sufficiently small coupling constant g and outside some intervals whose size depend on g (see

[Sk], [BFSS], citeDJ).

Finally let us make some comments on the scattering theory for the *translation invariant* Nelson model. For simplicity we will restrict our discussion to the case of one nucleon. Using the translation invariance, one can write the Hamiltonian H as a direct integral over the total momentum p:

$$H = \int_{\mathbb{R}^d}^{\oplus} H(p)\mathrm{d}p.$$

One has to replace the space of bound states $\mathcal{H}_{\mathrm{pp}}(H)$ by the space

$$\mathcal{H}_{\mathrm{bd}} = \int^{\oplus_{\mathbb{R}^d}} \mathcal{H}_{\mathrm{pp}}(H(p))\mathrm{d}p,$$

describing the dressed nucleon states.

The basic framework of scattering theory for translation invariant models is similar to the confined case. The first problem is the existence of the asymptotic fields. For massive bosons, this property is known (see [HK2]) in space dimension $d \geq 3$. It is unknown if this property holds for $d \leq 2$. For massless bosons, the existence of the asymptotic fields depends on the nucleon dispersion relation. If one uses the non-relativistic dispersion relation $\frac{1}{2}D_{\mathrm{x}}^2$, then the existence of asymptotic fields on the whole Hilbert space holds in space dimension $d \geq 4$, but is still unknown in the physical case $d = 3$. If one uses the relativistic dispersion relation $(D_{\mathrm{x}}^2 + m^2)^{\frac{1}{2}}$ for the nucleon, the existence of asymptotic fields was shown if $d = 3$ in [FGS].

Assuming that the asymptotic fields exist, one can conjecture that the spaces $\mathcal{K}^{\pm}$ of asymptotic vacua coincide with $\mathcal{H}_{\mathrm{bd}}$. However even the inclusion $\mathcal{H}_{\mathrm{bd}} \subset \mathcal{K}^{\pm}$ is non trivial and seems to require implicit assumptions on the smoothness in p of the eigenvalues $E_n(p)$ of $H(p)$. Finally let us mention the papers [Fr1], [Fr2], [P], where the case when the Hamiltonians $H(p)$ do not have a ground state in Hilbert space is studied.

1.3 Hypotheses

Let us now state the various hypotheses that we will impose on the coupling functions v_j in the sequel. In the formulation of conditions *(I2)* and *(I5)* one introduces polar coordinates $\tilde{\sigma} = |k|$, $\omega = \frac{k}{|k|}$ (see (3.1)).

In Sect. 4, we will impose:

$$(I1)\ \int (1 + |k|^{-1-2\epsilon_0})|v_j(k)|^2\mathrm{d}k < \infty,\ 1 \leq j \leq P,\ \epsilon_0 > 0.$$

This condition will be needed to obtain sharp estimates on the growth of the total number of particles along the evolution.

In Sect. 5, we will impose:

$$(I2)\ \tilde{\sigma}v_j(\tilde{\sigma}\omega) \in H_0^{\mu}(\mathbb{R}^+) \otimes L^2(S^2),\ 1 \leq j \leq P,\ \mu > 0,$$

where the space $H_0^{\mu}(\mathbb{R}^+)$ is the closure of $C_0^{\infty}(]0, +\infty[)$ in the topology of $H^{\mu}(\mathbb{R})$. This condition will allow us to construct $H-$invariant spaces $\mathcal{H}_{\mathrm{c}}^{+}$ containing a finite number of particles in the region $|x| \geq \mathrm{c}t$, for $0 < \mathrm{c} < 1$.

In Subsect. 3.2, we impose:

$$(I3)\ \int (1 + |k|^{-2})|v_j(k)|^2\mathrm{d}k < \infty,\ 1 \leq j \leq P.$$

Conditions *(I3)* and *(H0)* for $\alpha > 0$ imply that H admits a ground state in the Hilbert space $\mathcal{H}$. This fact has two important consequences: firstly the CCR representation given by the asymptotic Weyl operators $W^{+}(h)$ constructed in Sect. 8 admits a Fock subrepresentation (see Subsect. 8.2). Secondly the spaces $\mathcal{H}_{\rm c}^{+}$ are non trivial (see Thm. 5.6).

In Sect. 8 we impose:

$$(I4)\ v_j \in H^{\mu_1}_{\rm loc}(\mathbb{R}^3),\ 1 \leq j \leq P,\ \mu_1 > 0.$$

This condition will allow us to construct the asymptotic fields.

In Subsect. 4.5 we impose:

$$(I5)\ (1 + |\tilde{\sigma}|^{-\frac{1}{2}})(1 - \frac{\Delta_\omega}{\tilde{\sigma}^2})^{\mu_2}\tilde{\sigma}v_j(\tilde{\sigma}\omega) \in L^2(\mathbb{R}^+) \otimes L^2(S^2),\ 1 \leq j \leq P,\ \mu_2 > 0.$$

This assumption will be needed to control the angular part of the observable $|x| = -\Delta_k^{\frac{1}{2}}$.

To illustrate the meaning of these various conditions, let us consider a rotationally invariant coupling function v_j of the form:

$$v_j(k) = |k|^{\rho}\chi(|k|), \tag{1.4}$$

where $\chi \in C_0^\infty(\mathbb{R})$ is an ultraviolet cutoff. Then:

(I0) is satisfied if $\rho > -1$.

(I1) is satisfied if $\rho > -1 + \epsilon_0$.

(I2) is satisfied if $\rho > \mu - 1$. In fact it is easy to see that $\tilde{\sigma}^{\rho+1}\chi(\tilde{\sigma}) \in H_0^{\mu}(\mathbb{R}^+)$ if $\mu < \rho + 1$.

(I3) is satisfied if $\rho > -\frac{1}{2}$.

(I4) and *(I5)* are satisfied for all values of ρ.

The main results of the paper, formulated in Sects. 8 and 12, hold under *(H0)*, *(I0)*, *(I2)*, *(I5)* for $\alpha > 1$, $\mu > 1$, $\mu_2 > 1$. Hence we see that for a coupling function of the form (1.4), the results of the paper hold for $\rho > 0$.

1.4 Plan of the paper

Let us now describe the plan of the paper. In Sect. 2 we define some notation and recall some notions introduced in [DG2].

In Sect. 3 we describe the abstract framework in which we will work for most of the paper. In this framework, the original Hilbert space and Hamiltonian are denoted by $\mathcal{H}$ and H respectively. We introduce the so-called *expanded objects*, which corresponds to the addition of non-physical bosons of negative energy to the system. In particular the *expanded Hilbert space* $\mathcal{H}^{\rm e}$ and the *expanded Hamiltonian* $H^{\rm e}$ will play an important role. The one-particle space is now $L^2(\mathbb{R}, {\rm d}\sigma) \otimes L^2(S^2)$ and the one-particle kinetic energy for the expanded Hamiltonian is simply the operator of multiplication by σ. This allows us to circumvent the problem caused by the lack of smoothness of the dispersion relation $|k|$ of the original Hamiltonian at $k = 0$. A number of basic technical estimates are also proved in this section.

Sect. 4 is devoted to estimating the growth of the number observable along the evolution. We show that if the interaction is of size $O(|k|^{\epsilon_0})$ near $k = 0$, then the number of particles is bounded by t^{ρ_0} when $t \to +\infty$, where ρ_0 depends on ϵ_0. We also prove some estimates on the growth of the 'angular' part of $|x|$ along the evolution which will be useful in Sect. 11.

Most of the analytical work will be done on the expanded objects. In Sect. 5, we construct the spaces $\mathcal{H}_{\rm c}^{\rm e+}$, which are analogous to the spaces $\mathcal{H}_{\rm c}^{+}$ for $H^{\rm e}$.

In Sect. 6, we construct an asymptotic partition of unity on $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$. Using this partition of unity, we can split a state in $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ into pieces having a fixed number of particles in $\{|s| \geq \mathrm{c}'t\}$ for $\mathrm{c} < \mathrm{c}'$, where s is the operator canonically conjugate to σ.

In Sect. 7, we construct geometric inverse wave operators on $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$. The asymptotic fields and the wave operators both for H and H^{e} are constructed in Sect. 8 and their relationship is studied. In Sect. 9 we prove the geometric asymptotic completeness on the spaces $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ for H^{e}.

Sects. 10 and 11 are devoted to a reinterpretation of the spaces $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$. Originally these spaces are described in terms of the observable s. This description is not convenient to obtain corresponding spaces for H, which is the reason why another description with a different observable is given.

In Sect. 12 we prove the main results of this paper for the original Hamiltonian H. The construction and properties of the spaces $\mathcal{H}_{\mathrm{c}}^{+}$ are obtained from the results of Sects. 5, 8, 9 and from functorial arguments, using the alternative description of $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ in Sect. 11. Finally in Sect. 13, we study the consequences of a Mourre estimate for H and show that it implies the asymptotic completeness restricted to $\mathcal{H}_{\mathrm{c}}^{+}$.

Various technical results are collected in an Appendix.

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2 Notation

2.1 General notation

We collect some notation that will be used throughout the paper.

Function spaces

We will denote by $C_\infty(\mathbb{R}^n)$ the space of continuous functions on $\mathbb{R}^n$ tending to 0 at infinity. We set

$$S^0(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | |\partial_x^\sigma f(x)| \leq C_\alpha,\ \alpha \in \mathbb{N}^n\}.$$

We denote by $H^s(\mathbb{R}^n)$ the Sobolev space of order $s \in \mathbb{R}$ on $\mathbb{R}^n$.

Hilbert spaces

If $\mathcal{H}$ is a Hilbert space, we denote by $\mathcal{B}(\mathcal{H})$ resp. $\mathcal{U}(\mathcal{H})$ the set of bounded resp. unitary operators on $\mathcal{H}$. If H is a bounded below selfadjoint operator on $\mathcal{H}$, we will denote by the letter b a constant such that $H + b \geq \mathbb{1}$.

If H is a selfadjoint operator on $\mathcal{H}$ and $\mathbb{R} \ni t \mapsto \Phi(t) \in \mathcal{B}(\mathcal{H})$ is an operator-valued function, we denote by $\mathbf{D}\Phi(t)$ the Heisenberg derivative:

$$\mathbf{D}\Phi(t) = \partial_t \Phi(t) + [H, \mathrm{i}\Phi(t)].$$

For $u \in \mathcal{H}$, we set $u_t = \mathrm{e}^{-\mathrm{i}tH}u$.

Often the Hamiltonian H can be written as a sum $H = H_0 + V$, where H_0 is a 'free' Hamiltonian and V an interaction term. In this case we denote by $\mathbf{D}_0$ the free Heisenberg derivative associated to H_0:

$$\mathbf{D}_0\Phi(t) = \partial_t \Phi(t) + [H_0, \mathrm{i}\Phi(t)].$$

If $\mathbb{R} \ni t \mapsto \Phi(t)$ is a map with values in linear operators on $\mathcal{H}$ and N is a positive selfadjoint operator on $\mathcal{H}$ we will say that

$$\Phi(t) \in O(N^{\alpha})t^{\mu} \text{ for } \alpha \in \mathbb{R}^{+}, \mu \in \mathbb{R}$$

if $\mathcal{D}(N^{\alpha}) \subset \mathcal{D}(\Phi(t))$ for $t \in \mathbb{R}$ and $\|\Phi(t)(N+1)^{-\alpha}\| \in O(t^{\mu})$. The notation $\Phi(t) \in o(N^{\alpha})t^{\mu}$ is defined similarly. If A, B are two selfadjoint operators, we denote by $\mathrm{ad}_A B$ the expression $\mathrm{ad}_A B = [A, B]$. Usually the commutator $[A, B]$ is first defined as a quadratic form on $\mathcal{D}(A) \cap \mathcal{D}(B)$ and then extended as an operator on some domain. The precise meaning of $\mathrm{ad}_A B$ will either be specified or clear from the context.

Finally we recall (see [ABG]) that if A is a selfadjoint operator and $B \in \mathcal{B}(\mathcal{H})$, one says that $B \in C^1(A)$ if the map

$$\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA} B \mathrm{e}^{-\mathrm{i}sA} \in \mathcal{B}(\mathcal{H})$$

is C^1 for the strong topology. If H is a selfadjoint operator, one says that $H \in C^1(A)$ if for some $z \in \mathbb{C}\backslash\sigma(H)$, $(z-H)^{-1} \in C^1(A)$. If $H \in C^1(A)$ then the quadratic form $[(z-H)^{-1}, \mathrm{i}A]$ extend from $\mathcal{D}(A)$ to a bounded quadratic form on $\mathcal{H}$ and

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{\mathrm{i}sA}(z-H)^{-1}\mathrm{e}^{-\mathrm{i}sA}_{|s=0} = [A, \mathrm{i}(z-H)^{-1}] = (z-H)^{-1}[A, \mathrm{i}H](z-H)^{-1}.$$

For $0 < \epsilon < 1$, we say that $H \in C^{1+\epsilon}(A)$ if $H \in C^1(A)$ and the map

$$\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA}[(z-H)^{-1}, \mathrm{i}A]\mathrm{e}^{-\mathrm{i}sA} \in \mathcal{B}(\mathcal{H})$$

is C^{ϵ} for the norm topology.

2.2 Fock space notation

Fock spaces

Let $\mathfrak{h}$ be a Hilbert space, which we will call the *one-particle space*. Let $\otimes_{\mathrm{s}}^{n}\mathfrak{h}$ denote the symmetric nth tensor power of $\mathfrak{h}$. Let $\mathcal{S}_n$ denote the orthogonal projection of $\otimes^{n}\mathfrak{h}$ onto $\otimes_{\mathrm{s}}^{n}\mathfrak{h}$. The Fock space over $\mathfrak{h}$ is the direct sum

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n}\mathfrak{h}.$$

Ω will denote the vacuum vector – the vector $1 \in \mathbb{C} = \otimes_{\mathrm{s}}^{0}\mathfrak{h}$. The number operator N is defined as

$$N\Big|_{\bigotimes_{\mathrm{s}}^{n}\mathfrak{h}} = n\mathbb{1}.$$

The space of finite particle vectors, for which $\mathbb{1}_{[n,+\infty]}(N)u = 0$ for some $n \in \mathbb{N}$, will be denoted by $\Gamma_{\mathrm{fin}}(\mathfrak{h})$.

For $h \in \mathfrak{h}$ we denote by $a^*(h)$, $a(h)$, the creation annihilation operators, by $\phi(h) = \frac{1}{\sqrt{2}}(a^*(h) + a(h))$ the field operators and by $W(h) = \mathrm{e}^{\mathrm{i}\phi(h)}$ the Weyl operators(see eg [DG2, Sect.2]).

It is convenient to extend the definition of $a^*(v)$, $a(v)$ in the following way:

suppose that $\mathcal{K}$ is a Hilbert space. If $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, then we can define $a^*(v)$, $a(v)$, $\phi(v)$ as unbounded operators on $\mathcal{K} \otimes \Gamma(\mathfrak{h})$ by:

$$\begin{aligned} & a^*(v)\Big|_{\mathcal{K}\otimes\bigotimes_{\rm s}^n \mathfrak{h}} := \sqrt{n+1}\Big(1\!\!1_{\mathcal{K}} \otimes \mathcal{S}_{n+1}\Big)\Big(v \otimes 1\!\!1_{\bigotimes_{\rm s}^n \mathfrak{h}}\Big), \\ & a(v) := (a^*(v))^*, \\ & \phi(v) := \tfrac{1}{\sqrt{2}}(a(v) + a^*(v)). \end{aligned} \tag{2.1}$$

They satisfy the estimates

$$\|a^\sharp(v)(N+1)^{-\frac{1}{2}}\| \leq \|v\|, \tag{2.2}$$

where $\|v\|$ is the norm of v in $\mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$.

If b is an operator on $\mathfrak{h}$, we define the operator

$$\begin{aligned} & \mathrm{d}\Gamma(b) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}), \\ & \mathrm{d}\Gamma(b)\Big|_{\bigotimes_{\rm s}^n \mathfrak{h}} := \sum_{j=1}^n \underbrace{1\!\!1 \otimes \cdots \otimes 1\!\!1}_{j-1} \otimes b \otimes \underbrace{1\!\!1 \otimes \cdots \otimes 1\!\!1}_{n-j}. \end{aligned}$$

If $\mathfrak{h}_i, i = 1, 2$ are Hilbert spaces, $q : \mathfrak{h}_1 \mapsto \mathfrak{h}_2$ is a bounded linear operator, one defines

$$\begin{aligned} & \Gamma(q) : \Gamma(\mathfrak{h}_1) \mapsto \Gamma(\mathfrak{h}_2) \\ & \Gamma(q)\Big|_{\bigotimes_{\rm s}^n \mathfrak{h}_1} := q \otimes \cdots \otimes q. \end{aligned}$$

If q, r are operators from $\mathfrak{h}_1$ to $\mathfrak{h}_2$ one defines

$$\begin{aligned} & \mathrm{d}\Gamma(q, r) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2), \\ & \mathrm{d}\Gamma(q, r)\Big|_{\bigotimes_{\rm s}^n \mathfrak{h}_1} := \sum_{j=1}^n \underbrace{q \otimes \cdots \otimes q}_{j-1} \otimes r \otimes \underbrace{q \otimes \cdots \otimes q}_{n-j}. \end{aligned}$$

Let us now introduce some notation related to Heisenberg derivatives. Let ω be a selfadjoint operator on $\mathfrak{h}$. We denote by $\mathbf{d}_0$ the Heisenberg derivative associated to ω:

$$\mathbf{d}_0 = \frac{\partial}{\partial t} + [\omega, \mathrm{i}\cdot], \text{ acting on } \mathcal{B}(\mathfrak{h}).$$

Let $\mathbf{D}_0$ be the Heisenberg derivative associated to the Hamiltonian $H_0 = \mathrm{d}\Gamma(\omega)$. Then for a function $\mathbb{R} \ni t \mapsto b(t) \in \mathcal{B}(\mathfrak{h})$, we have:

$$\mathbf{D}_0 \mathrm{d}\Gamma(b(t)) = \mathrm{d}\Gamma(\mathbf{d}_0 b(t)).$$

Operators $P_k(f)$ and $Q_k(f)$

We now recall some objects introduced in [DG2] which will play an important role in the sequel.

Let f_0, f_∞ be operators on $\mathfrak{h}$. Let $f := (f_0, f_\infty)$. We define the operators $P_k(f) = P_k(f_0, f_\infty)$ and $Q_k(f) = Q_k(f_0, f_\infty)$ for $k \in \mathbb{N}$ by setting

$$
\begin{aligned}
&P_k(f) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}),\\
&P_k(f)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}_1} := \sum_{\sharp\{i|\epsilon_i=\infty\}=k} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_n},
\end{aligned}
$$

where $\epsilon_i = 0, \infty$ and

$$
Q_k(f) := \sum_{j=0}^{k} P_j(f).
$$

We will sometimes denote $P_k(f)$ by $P_k(f_0, f_\infty)$ if $f = (f_0, f_\infty)$. For $f = (f_0, f_\infty)$ and $g = (g_0, g_\infty)$ we define

$$
\begin{aligned}
\mathrm{d}P_k(f,g) : \quad & \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}),\\
\mathrm{d}P_k(f,g)\Big|_{\bigotimes_{\mathrm{s}}^n \mathfrak{h}} \quad & := \sum_{\sharp\{i|\epsilon_i=\infty\}=k} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_{j-1}} \otimes g_0 \otimes f_{\epsilon_{j+1}} \otimes \cdots \otimes f_{\epsilon_n}\\
& + \sum_{\sharp\{i|\epsilon_i=\infty\}=k-1} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_{j-1}} \otimes g_\infty \otimes f_{\epsilon_{j+1}} \otimes \cdots \otimes f_{\epsilon_n},
\end{aligned}
$$

and

$$
\mathrm{d}Q_k(f,g) := \sum_{j=0}^{k} \mathrm{d}P_j(f,g).
$$

Canonical map

Let $\mathfrak{h}_i$, $i = 1, 2$ be Hilbert spaces. Let p_i be the projection of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ onto $\mathfrak{h}_i$, $i = 1, 2$. We define

$$
U : \Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2),
$$

by

$$
\begin{aligned}
&U\Omega = \Omega \otimes \Omega,\\
&U a^\sharp(h) = \left(a^\sharp(p_1 h) \otimes 1\!\!1 + 1\!\!1 \otimes a^\sharp(p_2 h)\right) U, \ h \in \mathfrak{h}_1 \oplus \mathfrak{h}_2.
\end{aligned}
\tag{2.3}
$$

Since the vectors $a^*(h_1) \cdots a^*(h_n)\Omega$ form a total family in $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$, and since U preserves the canonical commutation relations, U extends as a unitary operator from $\Gamma(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ to $\Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2)$.

Operators $\check{\Gamma}(j)$ and $\mathrm{d}\check{\Gamma}(j,k)$

Let $j_0, j_\infty \in \mathcal{B}(\mathfrak{h})$. Set $j = (j_0, j_\infty)$. We identify j with the operator

$$
\begin{aligned}
&j : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h},\\
&jh := (j_0 h, j_\infty h).
\end{aligned}
$$

We have

$$
\begin{aligned}
&j^* : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h},\\
&j^*(h_0, h_\infty) = j_0^* h_0 + j_\infty^* h_\infty,
\end{aligned}
$$

and

$$j^*j = j_0^*j_0 + j_\infty^*j_\infty.$$

By second quantization, we obtain the map

$$\Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h} \oplus \mathfrak{h}).$$

Let U denote the canonical map between $\Gamma(\mathfrak{h} \oplus \mathfrak{h})$ and $\Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$ introduced above . We define

$$\check{\Gamma}(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}),$$

$$\check{\Gamma}(j) := U\Gamma(j).$$

Another formula defining $\check{\Gamma}(j)$ is

$$\check{\Gamma}(j)\Pi_{i=1}^n a^*(h_i)\Omega := \Pi_{i=1}^n \left(a^*(j_0h_i) \otimes 1\!\!1 + 1\!\!1 \otimes a^*(j_\infty h_i)\right) \Omega \otimes \Omega, h_i \in \mathfrak{h}. \tag{2.4}$$

Let $N_0 = N \otimes 1\!\!1$, $N_\infty = 1\!\!1 \otimes N$ acting on $\Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$. Then if we denote by I_k the natural isometry between $\bigotimes^n \mathfrak{h}$ and $\bigotimes^{n-k} \mathfrak{h} \otimes \bigotimes^k \mathfrak{h}$, then we have:

$$1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j)\Big|_{\bigotimes_s^n \mathfrak{h}} = I_k\sqrt{\tfrac{n!}{(n-k)!k!}}\,\underbrace{j_0 \otimes \cdots \otimes j_0}_{n-k} \otimes \underbrace{j_\infty \otimes \cdots \otimes j_\infty}_{k}.$$

Finally we set

$$\check{\Gamma}_k(j) := 1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j).$$

Let $j = (j_0, j_\infty)$, $k = (k_0, k_\infty)$ be maps from $\mathfrak{h}$ to $\mathfrak{h} \oplus \mathfrak{h}$. We set

$$\mathrm{d}\check{\Gamma}(j,k) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}),$$

$$\mathrm{d}\check{\Gamma}(j,k) := U\mathrm{d}\Gamma(j,k).$$

The operator $\mathrm{d}\check{\Gamma}(1,k) = U\mathrm{d}\Gamma(k)$ will be denoted simply by $\mathrm{d}\check{\Gamma}(k)$.

Scattering identification operator

Let

$$i : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h},$$

$$(h_0, h_\infty) \mapsto h_0 + h_\infty.$$

An important role in scattering theory is played by the following identification operator (see [HuSp1]):

$$I := \Gamma(i)U^* = \check{\Gamma}(i^*)^* : \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}).$$

Note that since $\|i\| = \sqrt{2}$, the operator $\Gamma(i)$ is unbounded.

Another formula defining I is:

$$I \prod_{i=1}^n a^*(h_i)\Omega \otimes \prod_{i=1}^p a^*(g_i)\Omega := \prod_{i=1}^p a^*(g_i) \prod_{i=1}^n a^*(h_i)\Omega, \qquad h_i, g_i \in \mathfrak{h}. \tag{2.5}$$

If $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}k)$, then we can write still another formula for I:

$$Iu \otimes \psi = \frac{1}{(p!)^{\frac{1}{2}}} \int \psi(k_1, \cdots, k_p)a^*(k_1)\cdots a^*(k_p)u\mathrm{d}k, \quad u \in \Gamma(\mathfrak{h}),\ \psi \in \otimes_{\mathrm{s}}^p\mathfrak{h}. \tag{2.6}$$

We deduce from (2.5) that

$$I(N+1)^{-k/2}\otimes 1\!\!1 \text{ restricted to } \Gamma(\mathfrak{h})\otimes\otimes_{\rm s}^k\mathfrak{h} \text{ is bounded.} \tag{2.7}$$

Let $j_0, j_\infty \in \mathcal{B}(\mathfrak{h})$ such that $0\leq j_0\leq 1, 0\leq j_\infty\leq 1$, and $j_0+j_\infty=1$. Let $j=(j_0,j_\infty):\mathfrak{h}\to\mathfrak{h}\oplus\mathfrak{h}$, as above. Clearly $0\leq j^*j\leq 1$, hence $\|j\|\leq 1$, and therefore $\check{\Gamma}(j)$ is a bounded operator. We have $ij=1$, hence

$$I\check{\Gamma}(j)=1\!\!1.$$

We also have

$$\begin{aligned} I1\!\!1_{\{1,\dots,k\}}(N_\infty)\check{\Gamma}(j) &= Q_k(j),\\ I1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(j) &= P_k(j). \end{aligned} \tag{2.8}$$

Use of sub- and superscripts

To help the reader with the notation, we briefly describe the use of various sub- and superscripts in the paper.

Asymptotic observables obtained by letting the time t tend to $+\infty$ will be denoted by the superscript $+$. Observables depending on a constant c, which has the meaning of a speed of propagation, will be denoted with the subscript c.

In addition to the original objects, eg Hilbert spaces, Hamiltonians, asymptotic observables, wave operators, etc we will consider two other families of associated objects:

Expanded objects, which correspond to the addition of non-physical bosons of negative energy to the system, and which will be denoted by adding a superscript e to the corresponding original objects. Sometimes an object defined in the expanded framework has no counterpart in the original framework, in which case we will omit the superscript e.

Extended objects, which correspond to the addition of asymptotically free bosons, and which will be denoted with a subscript ext, (with some exceptions).

3 Massless Pauli-Fierz Hamiltonians

We describe in this section an abstract framework introduced in [DG2] in which we will work for most of the paper. We also define the *expanded objects*, which correspond to adding bosons of negative energy to the system. Finally we prove various basic estimates which will be needed in the sequel.

3.1 The abstract setup

We describe now the abstract framework in which we will work for most of the paper. The models that we will introduce describe a small system (eg an atom or a spin) interacting with a scalar bosonic field. Using the terminology of [DG2] we can call this class of models *massless Pauli-Fierz models.*

The small system is described with a separable Hilbert space $\mathcal{K}$ and a bounded below self-adjoint operator K on $\mathcal{K}$. Without loss of generality we will assume that K is positive.

Let $\mathfrak{h} := L^2(\mathbb{R}^+, {\rm d}\tilde{\sigma})\otimes\mathfrak{g}$, where $\mathfrak{g}$ is some auxiliary separable Hilbert space, be the one-particle boson space. The Hilbert space of the interacting system is

$$\mathcal{H} := \mathcal{K}\otimes\Gamma(\mathfrak{h}).$$

The one-particle energy is the operator $\tilde{\sigma}$ of multiplication by $\tilde{\sigma}$ on $\mathfrak{h}$.

The free Hamiltonian describing the non interacting system is

$$H_0 := K \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\tilde{\sigma}).$$

The interaction is described by an operator $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$. Note that since $\mathcal{K}$ and $\mathfrak{g}$ are separable, we can consider v as a function

$$\mathbb{R}^+ \ni \tilde{\sigma} \mapsto v(\tilde{\sigma}) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{g}),$$

defined a.e $\tilde{\sigma}$ by setting

$$v(\tilde{\sigma})\psi := (v\psi)(\tilde{\sigma}),\ \psi \in \mathcal{K},$$

and identifying $\mathcal{K} \otimes L^2(\mathbb{R}^+, \mathrm{d}\tilde{\sigma}) \otimes \mathfrak{g}$ with $L^2(\mathbb{R}^+, \mathrm{d}\tilde{\sigma}; \mathcal{K} \otimes \mathfrak{g})$.

The Hamiltonian describing the interacting system is now

$$H := H_0 + \phi(v),$$

acting on $\mathcal{H}$, where $\phi(v)$ is defined in (2.1).

We will assume that

$$(I'0)\ (1 + \tilde{\sigma}^{-\frac{1}{2}})v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}),$$

which implies by Prop. A.1 that $\phi(v)$ is H_0- bounded with infinitesimal bound and hence that H is selfadjoint and bounded below on $\mathcal{D}(H_0)$.

In terms of the function $v(\tilde{\sigma})$ *(I'0)* is equivalent to

$$\int_0^{+\infty} (1 + |\tilde{\sigma}|^{-1}) \|v(\tilde{\sigma})\|^2 \mathrm{d}\tilde{\sigma} < \infty.$$

We will denote by N the number operator on $\mathcal{H}$

$$N = 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(1\!\!1).$$

We now explain how to cast the massless Nelson Hamiltonian into this framework. Let H be a massless Nelson Hamiltonian as introduced in Subsect. 1.1.

On the one-particle space $L^2(\mathbb{R}^3, \mathrm{d}k)$ we introduce polar coordinates by the unitary map:

$$\begin{aligned} &\mathrm{u} : L^2(\mathbb{R}^3, \mathrm{d}k) \to L^2(\mathbb{R}^+, \mathrm{d}\tilde{\sigma}) \otimes \mathfrak{g}, \\ &\mathrm{u}\psi(\tilde{\sigma}, \omega) := \tilde{\sigma}\psi(\tilde{\sigma}\omega), \end{aligned} \tag{3.1}$$

for $\mathfrak{g} = L^2(S^2)$. We lift the unitary map u to a map $1\!\!1_{\mathcal{K}} \otimes \Gamma(\mathrm{u})$ from $\mathcal{K} \otimes \Gamma(L^2(\mathbb{R}^3, dk))$ into $\mathcal{K} \otimes \Gamma(\mathfrak{h})$ and the free Hamiltonian $K \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(|k|)$ becomes $K \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\tilde{\sigma})$. The interaction $\phi(v)$ becomes $\phi(\mathrm{u}v)$, which we will still denote by $\phi(v)$. If we represent as in Subsect. 1.1 $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$ by a function

$$\mathbb{R}^3 \ni k \mapsto v(k) \in \mathcal{B}(\mathcal{K})\ \text{a.e.}\ k,$$

then uv is represented by the function

$$v(\tilde{\sigma}) = \tilde{\sigma} v(\tilde{\sigma}\omega),\ \text{a.e}\ \tilde{\sigma}.$$

where $v(\tilde{\sigma}\omega)$ for fixed $\tilde{\sigma}$ is an element of $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{g})$. The Pauli-Fierz Hamiltonian (still denoted by H) obtained in this way is said *associated* to the Nelson Hamiltonian H.

3.2 Existence of bound states

The existence of bound states of H in the Hilbert space $\mathcal{H}$ is an important property of the Hamiltonian H. In particular it implies that the CCR representation given by the asymptotic fields constructed in Sect. 8.1 admits a Fock subrepresentation. In this subsection we recall a result of [G] proving the existence of a ground state for H under appropriate condition on the interaction v. For related results see [AH], [BFS],[Ar], [GLL],[LMS]. We introduce the following conditions

$$(H'0) \quad (K+\mathrm{i})^{-1} \text{ is compact on } \mathcal{K}.$$

$$(I'3)\ (1+\tilde{\sigma}^{-1})v(K+1)^{-\frac{1}{2}} \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}).$$

In terms of the function $v(\tilde{\sigma})$ *(I3')* is equivalent to

$$\int_0^{+\infty}(1+\frac{1}{\tilde{\sigma}^2})\|v(\tilde{\sigma})(K+1)^{-\frac{1}{2}}\|^2_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{g})}\mathrm{d}\tilde{\sigma} < \infty.$$

The following result is shown in [G, Thm. 1].

Theorem 3.1 *Assume hypotheses (H'0), (I'0), (I'3). Then* $\inf\mathrm{spec}(H)$ *is an eigenvalue of* H*. In other words* H *admits a ground state in* $\mathcal{H}$*.*

The condition corresponding to *(I'3)* for the concrete Nelson Hamiltonian is *(I3)* introduced in Subsect. 1.1. Hence we obtain:

Theorem 3.2 *Assume hypotheses (H0) for* $\alpha > 0$*, (I0), (I3). Then* $\inf\mathrm{spec}(H)$ *is an eigenvalue of* H*. In other words* H *admits a ground state in* $\mathcal{H}$*.*

3.3 Expanded objects

We describe in this subsection the *expanded objects*, corresponding to the addition of non physical bosons of negative energy. This idea appeared first in [DJ]. We use the notation in Subsect. 3.1. Let

$$\tilde{\mathcal{H}}^{\mathrm{e}} := \mathcal{K}\otimes\Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h}),$$

$$\tilde{H}^{\mathrm{e}} := H\otimes 1\!\!1_{\Gamma(\mathfrak{h})} - 1\!\!1_{\mathcal{K}\otimes\Gamma(\mathfrak{h})}\otimes\mathrm{d}\Gamma(\tilde{\sigma}),$$

acting on $\tilde{\mathcal{H}}^{\mathrm{e}}$. As the sum of two commuting selfadjoint operators $\tilde{H}^{\mathrm{e}}$ is selfadjoint on its natural domain and essentially selfadjoint on $\mathcal{D}(H)\otimes\mathcal{D}(\mathrm{d}\Gamma(\tilde{\sigma}))$.

We set

$$\mathfrak{h}^{\mathrm{e}} := L^2(\mathbb{R}, \mathrm{d}\sigma)\otimes\mathfrak{g},\ \mathcal{H}^{\mathrm{e}} := \mathcal{K}\otimes\Gamma(\mathfrak{h}^{\mathrm{e}}),$$

and consider the unitary map

$$\begin{array}{l} w : \mathfrak{h}\oplus\mathfrak{h} \to \mathfrak{h}^{\mathrm{e}}, \\ h_1\oplus h_2 \mapsto h \text{ with } h(\sigma) := \left\{\begin{array}{l} h_1(\sigma), \sigma \geq 0, \\ h_2(-\sigma), \sigma < 0. \end{array}\right. \end{array}$$

If

(3.2) $$U : \Gamma(\mathfrak{h}\oplus\mathfrak{h}) \to \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$$

is the canonical map defined in Subsect. 2.2, we set

$$\begin{aligned} &\mathcal{W}:\tilde{\mathcal{H}}^{\mathrm{e}}\to\mathcal{K}\otimes\Gamma(\mathfrak{h}^{\mathrm{e}})=\mathcal{H}^{\mathrm{e}},\\ &\mathcal{W}:=1\!\!1_{\mathcal{K}}\otimes\Gamma(w)U^{-1}. \end{aligned}\tag{3.3}$$

We set also

$$v^{\mathrm{e}}:=1\!\!1_{\mathcal{K}}\otimes w(v\oplus 0)\in\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h}^{\mathrm{e}}),$$

where $v\oplus 0$ is an element of $\mathcal{B}(\mathcal{K},\mathcal{K}\otimes(\mathfrak{h}\oplus\mathfrak{h}))$. In terms of operator-valued functions, we have

$$v^{\mathrm{e}}(\sigma)=v(\sigma)1\!\!1_{\{\sigma\geq 0\}}.\tag{3.4}$$

Note also that

$$w(\tilde{\sigma}\oplus-\tilde{\sigma})w^{*}=\sigma,$$

where σ is the operator of multiplication by σ on $\mathfrak{h}^{\mathrm{e}}=L^{2}(\mathbb{R},\mathrm{d}\sigma)\otimes\mathfrak{g}$.

Using the tensorial properties of U (see eg [DG2, Sect. 2.7]), we obtain:

$$\mathcal{W}\tilde{H}^{\mathrm{e}}\mathcal{W}^{*}=:H^{\mathrm{e}},$$

where

$$H^{\mathrm{e}}=K\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\mathrm{e}})}+1\!\!1_{\mathcal{K}}\otimes\mathrm{d}\Gamma(\sigma)+\phi(v^{\mathrm{e}}).$$

On $\mathcal{H}^{\mathrm{e}}$, we denote by $N^{\mathrm{e}}=1\!\!1_{\mathcal{K}}\otimes\mathrm{d}\Gamma(1\!\!1)$ the number operator and by

$$H_{0}^{\mathrm{e}}=K\otimes_{\Gamma(\mathfrak{h}^{\mathrm{e}})}+1\!\!1_{\mathcal{K}}\otimes\mathrm{d}\Gamma(\sigma)$$

the 'free' expanded Hamiltonian.

3.4 Conversion of asymptotic observables

In this subsection we explain how to deduce results for the scattering theory of H from corresponding results for the scattering theory of H^{e}.

We start by describing the canonical embedding of $\mathcal{H}$ into $\mathcal{H}^{\mathrm{e}}$. Let

$$I_{\Omega}:\begin{array}{l}\mathcal{H}\to\mathcal{H}^{\mathrm{e}}=\mathcal{H}\otimes\Gamma(\mathfrak{h})\\ u\mapsto u\otimes\Omega,\end{array}$$

where $\Omega\in\Gamma(\mathfrak{h})$ is the vacuum vector. We have

$$\begin{aligned} &I_{\Omega}^{*}I_{\Omega}=1\!\!1_{\mathcal{H}},\ I_{\Omega}I_{\Omega}^{*}=1\!\!1_{\mathcal{H}}\otimes|\Omega\rangle\langle\Omega|,\\ &I_{\Omega}\mathrm{e}^{-\mathrm{i}tH}=\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}I_{\Omega}. \end{aligned}$$

If we set

$$j:\begin{array}{l}\mathfrak{h}\to\mathfrak{h}^{\mathrm{e}}\\ h\mapsto 1\!\!1_{\{\sigma\geq 0\}}h,\end{array}\tag{3.5}$$

then

$$\mathcal{W}I_{\Omega}=1\!\!1_{\mathcal{K}}\otimes\Gamma(j)$$

is an isometry from $\mathcal{H}$ into $\mathcal{H}^{\mathrm{e}}$ and

$$\mathcal{W}I_{\Omega}\mathrm{e}^{-\mathrm{i}tH}=\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}\mathcal{W}I_{\Omega}.$$

Let us now describe how to convert various asymptotic observables. Let $b \in \mathcal{B}(\mathfrak{h}^{\mathrm{e}})$, $b = b^*$ such that

$$\mathbb{1}_{\{\sigma \leq 0\}} b \mathbb{1}_{\{\sigma \geq 0\}} = 0. \tag{3.6}$$

We set then

$$b_\pm := \mathbb{1}_{\{\pm\sigma \geq 0\}} b \mathbb{1}_{\{\pm\sigma \geq 0\}}.$$

Note that b_+ can be identified with $j^* b j \in \mathcal{B}(\mathfrak{h})$.

Lemma 3.3 *Let $b \in \mathcal{B}(\mathfrak{h}^{\mathrm{e}})$, $b = b^*$ with $\mathbb{1}_{\{\sigma \leq 0\}} b \mathbb{1}_{\{\sigma \geq 0\}} = 0$. Then*

$$\begin{array}{rl} & I_\Omega^* \mathcal{W}^{-1} \Gamma(b) = \Gamma(b_+) I_\Omega^* \mathcal{W}^{-1}, \\ i) & \Gamma(b) \mathcal{W} I_\Omega = \mathcal{W} I_\Omega \Gamma(b_+) \\ & \Gamma(b_+) = I_\Omega^* \mathcal{W}^{-1} \Gamma(b) \mathcal{W} I_\Omega. \end{array}$$

$$ii)\ Iom^* \mathcal{W}^{-1} f(\mathrm{d}\Gamma(b)) \mathcal{W} I_\Omega = f(\mathrm{d}\Gamma(b_+)),\ f \in C^\infty(\mathbb{R})..$$

Proof. Because of the hypothesis on b we have $w^{-1} b w = b_+ \oplus b_-$. Hence

$$\begin{array}{l} \mathcal{W}^{-1} \Gamma(b) \mathcal{W} = U \Gamma(w^{-1} b w) U^{-1} \\ = U \Gamma(b_+ \oplus b_-) U^{-1} = \Gamma(b_+) \otimes \Gamma(b_-). \end{array}$$

This easily implies *i)*.

By the same argument

$$\begin{array}{l} I_\Omega^* \mathcal{W}^{-1} \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(b)} \mathcal{W} I_\Omega = I_\Omega^* \mathcal{W} \Gamma(\mathrm{e}^{-\mathrm{i}tb}) \mathcal{W} I_\Omega \\ = \Gamma(\mathrm{e}^{-\mathrm{i}tb_+}) = \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(b_+)}. \end{array}$$

This proves *ii)* for $f(\lambda) = \mathrm{e}^{-\mathrm{i}t\lambda}$. By a density argument *ii)* holds for all $f \in C_\infty(\mathbb{R})$.$\Box$

The following proposition describe how to deduce existence of asymptotic observables for H from corresponding results for H^{e}.

Proposition 3.4 *Let $\mathbb{R} \ni t \mapsto b_t \in \mathcal{B}(\mathfrak{h}^{\mathrm{e}})$, with $b_t = b_t^*$, $b_t \geq 0$, $\sup_{t \in \mathbb{R}} \|b_t\| < \infty$ and $\mathbb{1}_{\{\sigma \leq 0\}} b_t \mathbb{1}_{\{\sigma \geq 0\}} = 0$. Let $b_{+t} = \mathbb{1}_{\{\sigma \geq 0\}} b_t \mathbb{1}_{\{\sigma \geq 0\}}$*

I) Assume that

$$\text{s-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} \Gamma(b_t) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} = \Gamma^{\mathrm{e}+} \ \textit{exists},$$

$$[H^{\mathrm{e}}, \Gamma^{\mathrm{e}+}] = 0.$$

Then

$$\begin{array}{l} i)\ \text{s-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH} \Gamma(b_{+t}) \mathrm{e}^{-\mathrm{i}tH} = \Gamma^+ \ \textit{exists}, \\ ii)\ [H, \Gamma^+] = 0, \\ iii)\ \Gamma^{\mathrm{e}+} \mathcal{W} I_\Omega = \mathcal{W} I_\Omega \Gamma^+,\ \Gamma^+ = I_\Omega^* \mathcal{W}^{-1} \Gamma^{\mathrm{e}+} \mathcal{W} I_\Omega, \\ iv)\ I_\Omega^* \mathcal{W}^{-1} \Gamma^{\mathrm{e}+} = \Gamma^+ I_\Omega^* \mathcal{W}^{-1}. \end{array}$$

II) Assume that

$$\text{s-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} (\mathrm{d}\Gamma(b_t) + \lambda)^{-1} \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} =: R^{\mathrm{e}+}(\lambda) \ \textit{exists for}\ \lambda \in \mathbb{C} \backslash \mathbb{R}^-,$$

$$[H^{\mathrm{e}}, R^{\mathrm{e}+}(\lambda)] = 0.$$

Then

$$
\begin{aligned}
&i)\ \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}(\mathrm{d}\Gamma(b_{+t})+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH} =: R^{+}(\lambda) \ \textit{exists for}\ \lambda\in\mathbb{C}\backslash\mathbb{R}^{-},\\
&ii)\ R^{+}(\lambda) = I_\Omega^{*}\mathcal{W}^{-1}R^{\mathrm{e}+}(\lambda)\mathcal{W}I_\Omega,\\
&iii)\ [H, R^{+}(\lambda)] = 0.
\end{aligned}
$$

III) By Prop. A.6 the limits

$$
\begin{aligned}
P^{\mathrm{e}+} &:= \text{s-}\lim_{\epsilon\to 0}\epsilon^{-1}R^{\mathrm{e}+}(\epsilon^{-1}),\\
P^{+} &:= \text{s-}\lim_{\epsilon\to 0}\epsilon^{-1}R^{+}(\epsilon^{-1})
\end{aligned}
$$

exist and are orthogonal projections. Then

$$
\begin{aligned}
&i)\ P^{+} = I_\Omega^{*}\mathcal{W}^{-1}P^{\mathrm{e}+}\mathcal{W}I_\Omega,\\
&ii)\ P^{\mathrm{e}+} = \mathcal{W}P^{+}\otimes 1\!\!1_{\Gamma(\mathfrak{h})}\mathcal{W}^{-1}.
\end{aligned}
$$

Proof. I) follows from Lemma 3.3 *i)* and the identity $\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}WI_\Omega = WI_\Omega\mathrm{e}^{-\mathrm{i}tH}$. II) follows from exactly the same arguments, using Lemma 3.3 *ii)* instead. III) *i)* follows directly from II) *ii)*. To prove III) *ii)* is equivalent to show that

$$
\mathcal{W}^{-1}P^{\mathrm{e}+}\mathcal{W} = P^{+}\otimes 1\!\!1_{\Gamma(\mathfrak{h})}.
$$

We have:

$$
\begin{aligned}
\mathcal{W}^{-1}P^{\mathrm{e}+}\mathcal{W} &= \text{s-}\lim_{\epsilon\to 0}\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(1\!\!1+\epsilon\mathrm{d}\Gamma(b_{+t})\otimes 1\!\!1+\epsilon 1\!\!1\otimes\mathrm{d}\Gamma(b_{-t}))^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}},\\
P^{+}\otimes 1\!\!1_{\Gamma(\mathfrak{h})} &= \text{s-}\lim_{\epsilon\to 0}\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(1\!\!1+\epsilon\mathrm{d}\Gamma(b_{+t})\otimes 1\!\!1)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}.
\end{aligned}
$$

Using that

$$
[1\!\!1_{\mathcal{H}}\otimes(N+1)^{-1}, \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}] = 0,
$$

and

$$
\|\Big((1\!\!1+\epsilon\mathrm{d}\Gamma(b_{+t})\otimes 1\!\!1+\epsilon 1\!\!1\otimes\mathrm{d}\Gamma(b_{-t}))^{-1} - (1\!\!1+\epsilon\mathrm{d}\Gamma(b_{+t})\otimes 1\!\!1)^{-1}\Big)1\!\!1\otimes(N+1)^{-1}\| \leq C\epsilon,
$$

we obtain that

$$
(\mathcal{W}^{-1}P^{\mathrm{e}+}\mathcal{W} - P^{+}\otimes 1\!\!1_{\Gamma(\mathfrak{h})})1\!\!1_{\mathcal{H}}\otimes(N+1)^{-1} = 0,
$$

which proves III) *ii)*. □

3.5 Properties of the expanded Hamiltonian

We use the notation of Subsects. 1.1, 3.3. The main problem encountered when working with the Hamiltonian H^{e} is that it is not bounded below. To overcome this difficulty we will use the fact that H^{e} commutes with other observables. For example H^{e} commutes with the Hamiltonians

$$
H_{+}^{\mathrm{e}} := K\otimes 1\!\!1 + 1\!\!1\otimes\mathrm{d}\Gamma(\sigma_{+}) + \phi(v^{\mathrm{e}})
$$

and

$$
H_{-}^{\mathrm{e}} := 1\!\!1\otimes\mathrm{d}\Gamma(\sigma_{-}),
$$

for $\sigma_\pm = 1\!\!1_{\{\pm\sigma\geq 0\}}\sigma$. Note that $H^{\rm e} = H^{\rm e}_+ + H^{\rm e}_-$ and that $H^{\rm e}_+$ is selfadjoint on $\mathcal{D}(K\otimes 1\!\!1 + 1\!\!1\otimes {\rm d}\Gamma(\sigma_+))$, using *(I'0)*. As a consequence $H^{\rm e}$ commutes with the Hamiltonian

$$L := H^{\rm e}_+ - H^{\rm e}_- = K \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})} + 1\!\!1_{\mathcal{K}} \otimes {\rm d}\Gamma(|\sigma|) + \phi(v^{\rm e}).$$

We deduce as in Subsect 3.1 from hypothesis *(I'0)* and (3.4) that L is selfadjoint and bounded below on $\mathcal{D}(L_0)$, for

$$L_0 = K \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})} + 1\!\!1_{\mathcal{K}} \otimes {\rm d}\Gamma(|\sigma|).$$

It is easy to see that $\mathcal{D}(L) = \mathcal{D}(H^{\rm e}_+) \cap \mathcal{D}(H^{\rm e}_-) =$, $H^{\rm e}$ is essentially selfadjoint on $\mathcal{D}(L)$ and that

$$H^{\rm e} = H^{\rm e}_0 + \phi(v^{\rm e}) \text{ on } \mathcal{D}(L).$$

For later use, we collect below various basic properties of L.

Lemma 3.5 *Assume (I'0). Then*
i) $(z-L)^{-1} \in C^1(N^{\rm e})$ *for* $z \in \mathbb{C}\backslash\sigma(L)$,
ii) $(z-L)^{-1}N^{\rm e} = N^{\rm e}(z-L)^{-1} + {\rm i}(z-L)^{-1}\phi({\rm i}v^{\rm e})(z-L)^{-1}$, *for* $z \in \mathbb{C}\backslash\sigma(L)$, *as an identity on* $\mathcal{D}(N^{\rm e})$,
iii) $\chi(L)$ *preserves* $\mathcal{D}((N^{\rm e})^r)$ *for* $r \in \mathbb{R}^+$, $\chi \in C_0^\infty(\mathbb{R})$ *and* $(L+{\rm i})(N^{\rm e})^r\chi(L)(N^{\rm e}+1)^{-r}$ *is bounded for* $r \in \mathbb{R}^+$.

Proof. We have

$$L_s := {\rm e}^{{\rm i}sN^{\rm e}} L {\rm e}^{-{\rm i}sN^{\rm e}} = L_0 + \phi({\rm e}^{{\rm i}s}v^{\rm e}).$$

Since $\phi({\rm e}^{{\rm i}s}v)$ is $L_0^{\frac{1}{2}}$ bounded, we have $\mathcal{D}(L_s) = \mathcal{D}(L_0)$ and

$$\|L_0(L_s - z)^{-1}\| \leq C|{\rm Im}z|^{-1},\ z \in K \Subset \mathbb{C}\backslash\mathbb{R},$$

uniformly for $|s| \leq 1$. Then

$$\begin{aligned} &s^{-1}((z-L_s)^{-1} - (z-L)^{-1}) = s^{-1}(z-L_s)^{-1}(L_s - L)(z-L)^{-1} \\ &= s^{-1}(z-L_s^{-1})\phi(({\rm e}^{{\rm i}s}-1)v^{\rm e})(z-L)^{-1}. \end{aligned}$$

Using Prop. A.1 in the Appendix, we see that

$$(L_0+1)^{-\frac{1}{2}}\phi(s^{-1}({\rm e}^{{\rm i}s}-1)v^{\rm e})(L_0+1)^{-\frac{1}{2}} \to (L_0+1)^{-\frac{1}{2}}\phi({\rm i}v^{\rm e})(L_0+1)^{-\frac{1}{2}} \text{ in norm},$$

and

$$((z-L_s)^{-1} - (z-L)^{-1})(L_0+1)^{\frac{1}{2}} \to 0 \text{ in norm}$$

when $s \to 0$. Hence

$$s^{-1}((z-L_s)^{-1} - (z-L)^{-1}) \to (z-L)^{-1}\phi({\rm i}v)(z-L)^{-1} \text{ in norm}$$

when $s \to 0$. This proves *i)* and *ii)*.

To prove *iii)* we use the identity

$$(N^{\rm e}+1)(z-L)^{-1}(N^{\rm e}+1)^{-1} = (z-L)^{-1} - {\rm i}(z-L)^{-1}\phi({\rm i}v^{\rm e})(z-L)^{-1}(N^{\rm e}+1)^{-1}.$$

By induction, using the fact that $\mathrm{ad}_N^k\phi(v^{\mathrm{e}}) = \mathrm{i}^{-k}\phi(\mathrm{i}^k v^{\mathrm{e}})$, we obtain that

$$\|(L+i)(N^{\mathrm{e}}+1)^k(z-L)^{-1}(N^{\mathrm{e}}+1)^{-k}\| \in O(|\mathrm{Im}z|^{-C_k}),\ z\in K \Subset \mathbb{C}\backslash\mathbb{R},\ k\in\mathbb{N}.$$

Using then the functional calculus formula (see eg [HS], [DG1]):

$$\chi(A) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z-A)^{-1}\mathrm{d}z\wedge\mathrm{d}\,\overline{z}, \tag{3.7}$$

where A is a selfadjoint operator and $\tilde{\chi}\in C_0^\infty(\mathbb{C})$ is an almost-analytic extension of χ satisfying

$$\begin{aligned}&\tilde{\chi}_{|\mathbb{R}} = \chi,\\ &|\partial_{\overline{z}}\tilde{\chi}(z)| \leq C_n|\mathrm{Im}z|^n, \quad n\in\mathbb{N},\end{aligned}$$

we obtain *iii)* for $r\in\mathbb{N}$. Then we extend the result to $r\in\mathbb{R}^+$ by interpolation. $\Box$

Lemma 3.6 *Assume (I'0). Let $b=b(\sigma)$ be a bounded real function supported in $\{|\sigma|\geq\epsilon_0\}$, $\epsilon_0>0$, and $B=\mathrm{d}\Gamma(b)$. Then*

i) $(z-L)^{-1}\in C^1(B)$,

ii) $(z-L)^{-1}B = B(z-L)^{-1}+\mathrm{i}(z-L)^{-1}\phi(\mathrm{i}bv^{\mathrm{e}})(z-L)^{-1}$, for $z\in\mathbb{C}\backslash\sigma(L)$, as an identity on $\mathcal{D}(B)$.

iii) $B^k(L+\mathrm{i})^{-k}$ is bounded for $k\in\mathbb{N}$.

Proof. *i)* and *ii)* can be shown as in Lemma 3.5, introducing $L_s = \mathrm{e}^{\mathrm{i}sB}L\mathrm{e}^{-\mathrm{i}sB} = L_0+\phi(\mathrm{e}^{\mathrm{i}sb}v^{\mathrm{e}})$.

Since $\mathrm{supp}\, b\subset\{|\sigma|\geq\epsilon_0\}$, $B(L+\mathrm{i})^{-1}$ is bounded, which proves *iii)* for $k=1$. To prove *iii)* for arbitrary k, we commute repeatedly factors of B through $(L+\mathrm{i})^{-1}$, using *ii)*, until each factor of B is followed by a factor of $(L+\mathrm{i})^{-1}$. Commutation of B with $(L+\mathrm{i})^{-1}$ produces an extra factor of $(L+\mathrm{i})^{-1}\phi(\mathrm{i}bv)(L+\mathrm{i})^{-1}$. Moreover $\mathrm{ad}_B^k\phi(v^{\mathrm{e}}) = \mathrm{i}^{-k}\phi(\mathrm{i}^k b^k v^{\mathrm{e}})$ is L_0-bounded. The details are left to the reader. $\Box$

The Hamiltonians $H_\pm^{\mathrm{e}}$ have similar properties.

Lemma 3.7 *Assume (I'0). Then*

i) $(z-H_\pm^{\mathrm{e}})^{-1}\in C^1(N^{\mathrm{e}})$ for $z\in\mathbb{C}\backslash\sigma(L)$,

ii) $(z-H_\pm^{\mathrm{e}})^{-1}N^{\mathrm{e}} = N^{\mathrm{e}}(z-H_\pm^{\mathrm{e}})^{-1}+\mathrm{i}(z-H_\pm^{\mathrm{e}})^{-1}\phi(\mathrm{i}v^{\mathrm{e}})(z-H_\pm^{\mathrm{e}})^{-1}$, for $z\in\mathbb{C}\backslash\sigma(H_\pm^{\mathrm{e}})$, as an identity on $\mathcal{D}(N^{\mathrm{e}})$,

iii) $\chi(H_\pm^{\mathrm{e}})$ preserves $\mathcal{D}((N^{\mathrm{e}})^r)$ for $r\in\mathbb{R}^+$, $\chi\in C_0^\infty(\mathbb{R})$ and $(H_\pm^{\mathrm{e}}+\mathrm{i})(N^{\mathrm{e}})^r\chi(H_\pm^{\mathrm{e}})(N^{\mathrm{e}}+1)^{-r}$ is bounded for $r\in\mathbb{R}^+$.

A consequence of Lemma 3.7 is

$$\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}\chi(L) \text{ preserves } \mathcal{D}((N^{\mathrm{e}})^r), \text{ for } \chi\in C_0^\infty(\mathbb{R}),\ r\in\mathbb{R}^+. \tag{3.8}$$

In fact we can write $\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}\chi(L)$ as $\mathrm{e}^{-\mathrm{i}tH_+^{\mathrm{e}}}\chi_1(H_+^{\mathrm{e}})\mathrm{e}^{-\mathrm{i}tH_-^{\mathrm{e}}}\chi_2(H_-^{\mathrm{e}})\chi(L)$, for some $\chi_1,\chi_2\in C_0^\infty(\mathbb{R})$ and apply Lemma 3.7 *iii)*.

A consequence of Lemma 3.5 and (3.8) is

Proposition 3.8 *Assume (I'0). Then*

$$\|(N^{\mathrm{e}}+1)^r\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}\chi(L)(N^{\mathrm{e}}+1)^{-r}\| \leq C_r\langle t\rangle^r,\ \chi\in C_0^\infty(\mathbb{R}),\ r\in\mathbb{R}^+. \tag{3.9}$$

Proof. We will prove the proposition for $r \in \mathbb{N}$ by induction and then argue by interpolation. Let $u_1 \in \mathcal{D}(N), u_2 \in \mathcal{D}(L)$ and consider

$$f(t) = (u_{2t}, N^{\mathrm{e}}\chi(L)u_{1t})$$

(note that $f(t)$ is finite by (3.8)). We have

$$\begin{aligned} f'(t) = & \ \mathrm{i}(H^{\mathrm{e}}u_{2t}, N^{\mathrm{e}}\chi(L)u_{1t}) - \mathrm{i}(u_{2t}, N^{\mathrm{e}}H^{\mathrm{e}}\chi(L)u_{1t}) \\ = & \ \mathrm{i}(H_0^{\mathrm{e}}u_{2t}, N^{\mathrm{e}}\chi(L)u_{1t}) - \mathrm{i}(u_{2t}, N^{\mathrm{e}}H_0^{\mathrm{e}}\chi(L)u_{1t}) \\ & +\mathrm{i}(\phi(v^{\mathrm{e}})u_{2t}, N^{\mathrm{e}}\chi(L)u_{1t}) - \mathrm{i}(u_{2t}, N^{\mathrm{e}}\phi(v^{\mathrm{e}})\chi(L)u_{1t}) \\ = & \ -(u_{2t}, \phi(\mathrm{i}v^{\mathrm{e}})\chi(L)u_{1t}). \end{aligned}$$

By Prop. A.1, we obtain

$$|f'(t)| \leq C\|u_2\|\|u_1\|,$$

which proves (3.9) for $r = 1$.

Assume now that (3.9) holds for all $r' < r$. Let $u_1 \in \mathcal{D}((N^{\mathrm{e}})^r), u_2 \in \mathcal{D}(L)$. Again we differentiate

$$f(t) = (u_{2t}, (N^{\mathrm{e}})^r\chi(L)u_{1t}),$$

and obtain

$$f'(t) = (u_{2t}, [\phi(v^{\mathrm{e}}), \mathrm{i}(N^{\mathrm{e}})^r]u_{1t}).$$

The commutator $[\phi(v^{\mathrm{e}}), \mathrm{i}(N^{\mathrm{e}})^r]$ can be written as a sum of terms of the form $\phi(\mathrm{i}^{\alpha}v^{\mathrm{e}})(N^{\mathrm{e}})^{\beta}$ for $\beta \leq r-1$. We write

$$\begin{aligned} & \phi(\mathrm{i}^{\alpha}v^{\mathrm{e}})(N^{\mathrm{e}})^{\beta}\chi(L) \\ = \ & \phi(\mathrm{i}^{\alpha}v^{\mathrm{e}})(L+\mathrm{i})^{-1}(L+\mathrm{i})(N^{\mathrm{e}})^{\beta}\chi_1(L)(N^{\mathrm{e}}+1)^{-\beta}(N^{\mathrm{e}}+1)^{\beta}\chi(L), \end{aligned}$$

for $\chi_1\chi = \chi$. By Prop. A.1 and Lemma 3.5 *iii)*, we obtain

$$|f'(t)| \leq C\|u_2\|\|(N^{\mathrm{e}}+1)^{r-1}\chi(L)u_{1t}\| \leq C\langle t\rangle^{r-1}\|u_2\|\|(N^{\mathrm{e}}+1)^{r-1}u_1\|,$$

by the induction hypothesis. This proves (3.9) for r. □

Finally we state a lemma analogous to Lemma 3.5 for the Hamiltonian H.

Lemma 3.9 *Assume (I'0). Then*
i) $(z-H)^{-1} \in C^1(N)$ *for* $z \in \mathbb{C}\backslash\sigma(H)$,
ii) $(z-H)^{-1}N = N(z-H)^{-1} + \mathrm{i}(z-H)^{-1}\phi(\mathrm{i}v)(z-H)^{-1}$, *for* $z \in \mathbb{C}\backslash\sigma(H)$, *as an identity on* $\mathcal{D}(N)$,
iii) $\chi(H)$ *preserves* $\mathcal{D}(N^r)$ *for* $r \in \mathbb{R}^+$, $\chi \in C_0^{\infty}(\mathbb{R})$ *and* $(H+\mathrm{i})N^r\chi(H)(N+1)^{-r}$ *is bounded for* $r \in \mathbb{R}^+$.

The proof is completely similar to Lemma 3.5.

3.6 Bounds on field operators

Lemma 3.10 *Assume (I'0) and let* $h_i \in \mathcal{D}(\tilde{\sigma}^{\frac{1}{2}} + \tilde{\sigma}^{-\frac{1}{2}})$, $1 \le i \le n$. *Then:*

$$\|\prod_1^n \phi(h_i)(H+b)^{-n/2}\| \le C_n \prod_1^n \|(1+\tilde{\sigma}^{\frac{1}{2}} + \tilde{\sigma}^{-\frac{1}{2}})h_i\|.$$

Lemma 3.11 *Assume (I'0) and let* $h_i \in \mathcal{D}(|\sigma|^{\frac{1}{2}} + |\sigma|^{-\frac{1}{2}})$, $1 \le i \le n$. *Then:*

$$\|\prod_1^n \phi(h_i)(L+b)^{-n/2}\| \le C_n \prod_1^n \|(1+|\sigma|^{\frac{1}{2}} + |\sigma|^{-\frac{1}{2}})h_i\|.$$

The proofs of Lemmas 3.10 and 3.11 being completely similar, we prove only Lemma 3.11.

Proof. Let first $B \ge 1, A$ be two selfadjoint operators with $\mathcal{D}(B^{\frac{1}{2}}) \subset \mathcal{D}(A)$. Then

$$AB^{-\frac{1}{2}} = \pi^{-1} \int_0^{+\infty} s^{-\frac{1}{2}} A(s+B)^{-1} \mathrm{d}s, \tag{3.10}$$

where the integral is norm convergent on $\mathcal{D}(B^{\epsilon})$ for any $\epsilon > 0$. As bounded operators on $\mathcal{H}^{\mathrm{e}}$, we have:

$$A(s+B)^{-1} = (s+B)^{-1}A - (s+B)^{-1}[A,B](s+B)^{-1}. \tag{3.11}$$

If we assume that $[A,B]$ extends from a bounded quadratic form on $\mathcal{D}(B)$ to a bounded quadratic form on $\mathcal{D}(B^{\frac{1}{2}})$, we deduce from (3.10), (3.11) that

$$[A, B^{-\frac{1}{2}}] = -\pi^{-1} \int_0^{+\infty} s^{-\frac{1}{2}} (s+B)^{-1}[A,B](s+B)^{-1} \mathrm{d}s \tag{3.12}$$

satisfies

$$\|[A, B^{-\frac{1}{2}}]\| \le C\|B^{-\frac{1}{2}}[A,B]B^{-\frac{1}{2}}\|. \tag{3.13}$$

For $B = L + b$, $A = \phi(h)$, $h \in \mathcal{D}(|\sigma|^{\frac{1}{2}})$, we have

$$[A,B] = -\mathrm{i}\phi(\mathrm{i}|\sigma|h) + \mathrm{i}\mathrm{Im}(h, v^{\mathrm{e}})_{\mathfrak{h}^{\mathrm{e}}}.$$

By Prop. A.1 and *(I'0)*, we obtain

$$\|B^{-\frac{1}{2}}[A,B]B^{-\frac{1}{2}}\| \le C\|\langle\sigma\rangle^{\frac{1}{2}}h\|,$$

and hence using (3.13)

$$\|[\phi(h), (L+b)^{-\frac{1}{2}}]\| \le C\|\langle\sigma\rangle^{\frac{1}{2}}h\|. \tag{3.14}$$

Similarly we have

$$\|\mathrm{ad}_{\phi(h_1)}\mathrm{ad}_{\phi(h_2)}L\| \le C\|\langle\sigma\rangle^{\frac{1}{2}}h_1\|\|\langle\sigma\rangle^{\frac{1}{2}}h_2\|, \tag{3.15}$$

and

$$\mathrm{ad}_{\phi(h_1)} \dots \mathrm{ad}_{\phi(h_l)}L = 0, \text{ for } l \ge 3. \tag{3.16}$$

We deduce easily from the identity (3.12) that

$$\|\mathrm{ad}_{\phi(h_1)}\dots\mathrm{ad}_{\phi(h_l)}(L+b)^{-\frac{1}{2}}\|\leq C_l\Pi_1^l\|\langle\sigma\rangle^{\frac{1}{2}}h_i\|. \tag{3.17}$$

Let us now prove the lemma. We consider more generally products of factors of

$$\phi(h_i),\ \mathrm{ad}_{\phi(h_1)}\dots\mathrm{ad}_{\phi(h_l)}R \text{ and } R$$

for $R=(L+b)^{-\frac{1}{2}}$. If a product P contains n factors of $\phi(h_i)$ (for different i) and p factors of R, we define its *degree* $d(P)$ to be equal to n and its *weight* $w(P)$ to be equal to $n-p$. Note that $d(P_1P_2)=d(P_1)+d(P_2)$, $w(P_1P_2)=w(P_1)+w(P_2)$.

We claim that a product P of zero weight is a bounded operator, which in particular implies the lemma. The claim is clearly true in two cases: if the degree of P is zero and if each factor of $\phi(h_i)$ in P is followed by a factor of R and the weight of P is zero. In this last case we say that P is *controlled.*

Commuting $\phi(h)$ with a factor R produces an extra term $ad_{\phi(h)}R$ of zero weight and commuting $\phi(h)$ with a factor $\mathrm{ad}_{\phi(h_1)}\dots\mathrm{ad}_{\phi(h_l)}R$ also. Hence we can move around the factors of $\phi(h_i)$ in a product P of zero weight until we get a controlled product of zero weight, producing error terms of zero weight and strictly lower degrees. Iterating this procedure, we see that P is a bounded operator. The fact that

$$\|\prod_1^n\phi(h_i)(L+b)^{-n/2}\|\leq C_n\prod_1^n\|(1+|\sigma|^{\frac{1}{2}}+|\sigma|^{-\frac{1}{2}})h_i\|,$$

follows then from (3.17) and Prop. A.1. □

4 Number estimates

In this section we prove some bounds on the growth of the number observable along the evolution which take into account the infrared behavior of the interaction. We consider abstract Pauli-Fierz Hamiltonians as introduced in Subsect. 3.1.

We introduce the following strengthened version of *(I'0)*:

$$(I'1)\ \int_0^{+\infty}(1+|\tilde\sigma|^{-1-2\epsilon_0})\|v^*(\tilde\sigma)v(\tilde\sigma)\|_{\mathcal{B}(\mathcal{K})}\mathrm{d}\tilde\sigma<\infty,\ \epsilon_0>0.$$

Note that *(I'1)* implies that

$$\int_{|\tilde\sigma|\leq r}(1+|\tilde\sigma|^{-1})\|v^*(\tilde\sigma)v(\tilde\sigma)\|_{\mathcal{B}(\mathcal{K})}\mathrm{d}\tilde\sigma\leq Cr^{2\epsilon_0},\ 0<r\leq 1. \tag{4.1}$$

For the Nelson Hamiltonians, the condition *(I1)* in Subsect. 1.1 implies that the associated Pauli-Fierz Hamiltonian satisfies *(I'1)*.

4.1 Case of H

We consider first the Hamiltonian H introduced in Subsect. 3.1. In Prop. 4.3 below we show that under hypothesis *(I'1)* the number operator grows at most like $t^{(1+\epsilon_0)^{-1}}$ along the evolution. In the sequel we will use Prop. 4.2 which contains essentially the same information.

Let $f \in C_0^\infty(\mathbb{R})$ be an even function with $f(\lambda) \equiv 1$ near 0, $0 \leq f \leq 1, \lambda f'(\lambda) \leq 0$. Let

$$r_t := f(t^{\rho_0}\tilde{\sigma}),\ N_t := \mathrm{d}\Gamma(r_t),\ \text{for } \rho_0 = (1+\epsilon_0)^{-1}.$$

Lemma 4.1 *Let $\chi \in C_0^\infty(\mathbb{R})$, $F \in S^0(\mathbb{R})$. Then*

$$\|[\chi(H), F(\frac{N_t}{t^\delta})] \in O(t^{-\delta-\rho_0\epsilon_0}).$$

Proof. Using formula (3.7), we write

$$[\chi(H), F(\frac{N_t}{t^\delta})] = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z-H)^{-1}[H, F(\frac{N_t}{t^\delta})](z-H)^{-1}\mathrm{d}z \wedge \mathrm{d}\,\overline{z}.$$

We have $[H, F(\frac{N_t}{t^\delta})] = [\phi(v), iF(\frac{N_t}{t^\delta})]$. To estimate this term, we use a commutator expansion lemma (see eg [DG1, Lemma C.3.1]). We have $ad^j_{N_t}\phi(v) = (-\mathrm{i})^j\phi(\mathrm{i}^j r_t^j v)$. This is an unbounded operator, but the remainder terms in the commutator expansion can be estimated using

$$\|\phi(\mathrm{i}^j r_t^j v)(H_0+1)^{-\frac{1}{2}}\| \leq C_j\left(\int(1+|\tilde{\sigma}|^{-1})|r_t(\tilde{\sigma})|\|v^*(\tilde{\sigma})v(\tilde{\sigma})\|_{\mathcal{B}(\mathcal{K})}\mathrm{d}\tilde{\sigma}\right)^{\frac{1}{2}} \leq Ct^{-\rho_0\epsilon_0}, \tag{4.2}$$

by (4.1), and using the fact that H_0 commutes with N_t. We obtain that

$$[\phi(v), \mathrm{i}F(\frac{N_t}{t^\delta})](H_0+\mathrm{i})^{-1} \in O(t^{-\delta-\rho_0\epsilon_0}). \tag{4.3}$$

This implies the lemma by the standard argument, using the properties of $\tilde{\chi}$. $\Box$

Proposition 4.2 *Assume (I'0), (I'1) and let $\delta > (1+\epsilon_0)^{-1}$. Then*

i) for $G \in C_0^\infty(]0,+\infty[)$, $\chi \in C_0^\infty(\mathbb{R})$ we have:

$$\int_1^{+\infty}\|G(\frac{N_t}{t^\delta})\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2\frac{\mathrm{d}t}{t} \leq C\|u\|^2,\ u \in \mathcal{D}(N).$$

ii) for $F \in C_0^\infty(\mathbb{R}), 0 \leq F \leq 1$, $F(s) \equiv 1$ near 0

$$\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH}F(\frac{N_t}{t^\delta})\mathrm{e}^{-\mathrm{i}tH} = \mathbb{1}.$$

Proof. We pick a function $F(\lambda) \in C^\infty(\mathbb{R})$, with $\mathrm{supp}\,F \subset]0,+\infty[$, $F'(\lambda) = G^2(\lambda)$ for $G \in C_0^\infty(]0,+\infty[)$. For $\chi \in C_0^\infty(\mathbb{R})$, we set

$$\Phi(t) = \chi(H)F(\frac{N_t}{t^\delta})\chi(H).$$

Note that by Lemma 3.9 $\mathrm{e}^{-\mathrm{i}tH}\chi(H)$ preserves $\mathcal{D}(N)$. We compute the Heisenberg derivative of $\Phi(t)$ as a quadratic form on $\mathcal{D}(N)$:

$$\begin{array}{lll} & & \mathbf{D}\Phi(t) \\ & = & -\delta\chi(H)F'(\frac{N_t}{t^\delta})\frac{N_t}{t^{\delta+1}} \\ (4.4) & & +\frac{1}{t^\delta}\chi(H)F'(\frac{N_t}{t^\delta})\mathrm{d}\Gamma(d_t)\chi(H) \\ & & +\chi(H)[\phi(v),\mathrm{i}F(\frac{N_t}{t^\delta})]\chi(H),\end{array}$$

for

$$d_t = \mathbf{d}_0 r_t = \rho_0 t^{\rho_0-1}\tilde{\sigma}f'(t^{\rho_0}\tilde{\sigma}) \leq 0.$$

To estimate $[\phi(v),\mathrm{i}F(\frac{N_t}{t^\delta})]$, we use the commutator expansion lemma as in the proof of Lemma 4.1. We obtain using (4.3):

$$\begin{array}{lll} & & \|\chi(H)[\phi(v),\mathrm{i}F(\frac{N_t}{t^\delta})]\chi(H)\| \\ (4.5) & \leq & C\|[\phi(v),\mathrm{i}F(\frac{N_t}{t^\delta})](H_0+\mathrm{i})^{-1}\| \\ & & \in O(t^{-\delta-\rho_0\epsilon_0}).\end{array}$$

Plugging (4.5) into (4.4), we obtain

$$\mathbf{D}\Phi(t) \leq -\frac{\delta}{t}\chi(H)(\lambda F')(\frac{N_t}{t^\delta})\chi(H) + O(t^{-\delta-\rho_0\epsilon_0}).$$

We pick $\delta > (1+\epsilon_0)^{-1}$ so that $\delta+\rho_0\epsilon_0 > 1$. By Prop. A.2 this proves *i)*.

let us now prove *ii)*. Let $u = \chi(H)v$, $\chi \in C_0^\infty(\mathbb{R})$, $v \in \mathcal{D}(N)$. By Lemma 3.9 $u_t \in \mathcal{D}(H)\cap\mathcal{D}(N)$. We have

$$\begin{array}{l} \partial_t(u_t, N_t u_t) = (u_t, \mathrm{d}\Gamma(d_t)u_t) - (u_t, \phi(\mathrm{i}r_t v)u_t) \\ \leq C_0\|(H+b)^{\frac{1}{2}}u\|^2\|\phi(\mathrm{i}r_t v)(H_0+1)^{-\frac{1}{2}}\| \\ \leq C_0 t^{-\rho_0\epsilon_0}\|(H+b)^{\frac{1}{2}}u\|^2,\end{array}$$

using (4.2) and the fact that $d_t \leq 0$. Integrating from 1 to t we obtain

$$(u_t, N_t u_t) \leq C_0 t^{1-\rho_0\epsilon_0}\|(H+b)^{\frac{1}{2}}u\|^2 + C_1\|(N+1)^{\frac{1}{2}}u\|^2. \tag{4.6}$$

Hence for $\delta > (1+\epsilon_0)^{-1}$, $F \in C^\infty(]0,+\infty[)$, F bounded, we have:

$$(u_t, F(\frac{N_t}{t^\delta})u_t) \leq C_1 t^{-\delta}(u_t, N_t u_t) \in o(1).$$

By a density argument this proves *ii)*. $\Box$

Let us state the following corollary of Prop. 4.2, which will not be used in the sequel:

Proposition 4.3 *Assume (I'0), (I'1) and let* $\delta > (1+\epsilon_0)^{-1}$. *Then for* $F \in C_0^\infty(\mathbb{R}), 0 \leq F \leq 1$, $F(s) \equiv 1$ *near* 0*:*

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}F(\frac{N}{t^\delta})\mathrm{e}^{-\mathrm{i}H} = 1\!\!1.$$

Proof. As above we write for $u \in \mathcal{D}(H) \cap \mathcal{D}(N)$:

$$(u_t, N u_t) = (u_t, N_t u_t) + (u_t, (N - N_t) u_t).$$

We have

$$N - N_t = \mathrm{d}\Gamma((1 - r_t)) \leq t^{\rho_0}(H_0 + 1). \tag{4.7}$$

Since $\rho_0 = 1 - \rho_0\epsilon_0$, we deduce from (4.6) that

$$(u_t, N u_t) \leq C_0 t^{1-\rho_0\epsilon_0} \|(H + b)^{\frac{1}{2}} u\|^2 + C\|(N + 1)^{\frac{1}{2}} u\|^2. \tag{4.8}$$

Then we argue as in the proof of Prop. 4.2 *ii)*. □

4.2 Case of H^{e}

The results of Subsect. 4.1 extend trivially to the case of the expanded Hamiltonian H^{e}. We set again

$$r_t = f(t^{\rho_0}\sigma),\ N_t^{\mathrm{e}} := \mathrm{d}\Gamma(r_t). \tag{4.9}$$

We observe that if $\mathcal{W}$ is the unitary map introduced in Subsect 3.3 then, using the fact that r_t is even, we have

$$\begin{aligned}
\mathcal{W}^* H^{\mathrm{e}} \mathcal{W} &= H \otimes \mathbb{1} - \mathbb{1} \otimes \mathrm{d}\Gamma(\tilde{\sigma}), \\
\mathcal{W}^* L \mathcal{W} &= H \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\tilde{\sigma}), \\
\mathcal{W}^* N_t^e \mathcal{W} &= N_t \otimes \mathbb{1} + \mathbb{1} \otimes N_t.
\end{aligned}$$

This allows to deduce directly the results of this subsection from those of Subsect. 4.1. The details are left to the reader.

The analog of Lemma 4.1 is:

Lemma 4.4 *Assume (I'0), (I'1). Let $\chi, \chi_1 \in C_0^\infty(\mathbb{R})$, $F \in S^0(\mathbb{R})$. Then*

$$[\chi(H^{\mathrm{e}}), F(\frac{N_t^{\mathrm{e}}}{t^\delta})]\chi_1(L),\ [\chi(L), F(\frac{N_t^{\mathrm{e}}}{t^\delta})]\ \in O(t^{-\delta-\rho_0\epsilon_0}).$$

Proposition 4.5 *Assume (I'0), (I'1) and let $\delta > (1 + \epsilon_0)^{-1}$. Then*

i) for $G \in C_0^\infty(]0, +\infty[)$, $\chi \in C_0^\infty(\mathbb{R})$ we have:

$$\int_1^{+\infty} \|G(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u\|^2 \frac{\mathrm{d}t}{t} \leq C\|u\|^2,\ u \in \mathcal{D}(N^{\mathrm{e}}).$$

ii) for $F \in C_0^\infty(\mathbb{R}), 0 \leq F \leq 1$, $F(s) \equiv 1$ near 0

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} F(\frac{N_t^{\mathrm{e}}}{t^\delta})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} = \mathbb{1}.$$

The following lemma will be used in later sections to control the number operator along the evolution.

Lemma 4.6 *Let N^{e}_t be the operator introduced above. Let $F \in C_0^\infty(\mathbb{R})$. Then for $\delta > (1+\epsilon_0)^{-1}$:*

$$(N^{\mathrm{e}})^\alpha F(\frac{N^{\mathrm{e}}_t}{t^\delta})(L+b)^{-\alpha} \in O(t^{\delta\alpha}),\ 0 \le \alpha \le 1.$$

Proof. By interpolation it suffices to consider the case $\alpha = 1$. We have $N^{\mathrm{e}} = N^{\mathrm{e}}_t + \mathrm{d}\Gamma((1-r_t))$ and as in (4.7), $\mathrm{d}\Gamma(1-r_t)^2 \le t^{2\rho_0}(L_0+b)^2 \le Ct^{2\rho_0}(L+b)^2$, since $(L+b)^{-1}(L_0+b)$ is bounded. This yields:

$$\begin{aligned}
& (L+b)^{-1}F(\tfrac{N^{\mathrm{e}}_t}{t^\delta})(N^{\mathrm{e}})^2F(\tfrac{N^{\mathrm{e}}_t}{t^\delta})(L+b)^{-1} \\
\le\ & (L+b)^{-1}F(\tfrac{N^{\mathrm{e}}_t}{t^\delta})(N^{\mathrm{e}}_t)^2F(\tfrac{N^{\mathrm{e}}_t}{t^\delta})(L+b)^{-1} + (L+b)^{-1}F(\tfrac{N^{e}_t}{t^\delta})\mathrm{d}\Gamma(1-r_t)^2F(\tfrac{N^{\mathrm{e}}_t}{t^\delta})(L+b)^{-1} \\
\le\ & Ct^{2\delta} + Ct^{2\rho_0}. \ \square
\end{aligned}$$

4.3 Sharper estimates for H

In this subsection, we prove sharper estimates on the localization of bosons of small momenta. We pick a cutoff function $g \in C^\infty(\mathbb{R})$ with

$$\begin{aligned}
& g(s) = 0, \text{ for } |s| \le \tfrac{1}{2}, \\
& g(s) = 1 \text{ for } |s| \ge 1, \qquad (4.10) \\
& s.g'(s) \ge 0.
\end{aligned}$$

We set

$$g^t := g(t^\delta R\tilde{\sigma}), \qquad (4.11)$$

for an exponent $\delta > 0$ and a constant $R \ge 1$.

Lemma 4.7 *Assume (I'0), (I'1). Then for $\chi \in C_0^\infty(\mathbb{R})$:*

$$[\chi(H), \Gamma(g^t)] \in O(t^{-\delta\epsilon_0}).$$

Proof. We write

$$[\chi(H), \Gamma(g^t)] = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\chi}(z)(z-H)^{-1}[H, \Gamma(g^t)](z-H)^{-1}\mathrm{d}z \wedge \mathrm{d}\overline{z},$$

where $\tilde{\chi}$ is an almost-analytic extension of χ. On $\mathcal{D}(H)$, $H = H_0 + \phi(v)$, and $[H_0, \Gamma(g^t)] = 0$,

$$[\phi(v), \Gamma(g^t)] = \frac{1}{\sqrt{2}}a^*((1-g^t)v)\Gamma(g^t) - \frac{1}{\sqrt{2}}\Gamma(g^t)a((1-g^t)v).$$

By Prop. A.1, we obtain:

$$\|(H_0+1)^{-\frac{1}{2}}[\phi(v), \Gamma(g^t)](H_0+1)^{-\frac{1}{2}} \le C\|(1-g^t)\tilde{\sigma}^{-\frac{1}{2}}v\|, \qquad (4.12)$$

and hence

$$\|(z-H)^{-1}[H,\Gamma(g^t)](z-H)^{-1}\| \leq C\|(1-g^t)\tilde{\sigma}^{-\frac{1}{2}}v\||\mathrm{Im}z|^{-2},\ z\in \mathrm{supp}\,\tilde{\chi}.$$

By (4.1) we have

$$\|(1-g^t)\tilde{\sigma}^{-\frac{1}{2}}v\| \in O(R^{-\epsilon_0}t^{-\delta\epsilon_0}).$$

This implies the lemma. □

Proposition 4.8 *Assume hypotheses (I'0), (I'1) and $\delta\epsilon_0>1$. Then for $\chi\in C_0^\infty(\mathbb{R})$:*

$$i)\ \int_1^{+\infty}\|\mathrm{d}\Gamma(g^t,\mathbf{d}_0g^t)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^2\mathrm{d}t \leq C\|u\|^2,\ u\in\mathcal{D}(N),$$

$$ii)\ \text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH}\Gamma(g^t)\mathrm{e}^{-\mathrm{i}tH} =: \Gamma^+(g,R)\ \textit{exists},$$

$$iii)\ [\Gamma^+(g,R),H]=0.$$

Proof. Let $\Phi(t)=\chi(H)\Gamma(g^t)\chi(H)$. By Lemma 3.9 $\chi(H)\mathrm{e}^{-\mathrm{i}tH}$ preserves $\mathcal{D}(N)$ and for $u\in\mathcal{D}(N)$ the function

$$\mathbb{R}\ni t\mapsto (u_t,\Phi(t)u_t)$$

is C^1 with derivative $(u_t,\chi(H)\mathbf{D}\Gamma(g^t)\chi(H)u_t)$, where the Heisenberg derivative $\mathbf{D}\Gamma(g^t)$ equals

$$\mathbf{D}\Gamma(g^t)=\mathrm{d}\Gamma(g^t,\mathbf{d}_0g^t)+[\phi(v),\mathrm{i}\Gamma(g^t)].$$

We have

$$\mathbf{d}_0g^t=R\delta t^{\delta-1}\tilde{\sigma}g'(t^\delta R\tilde{\sigma})\geq 0,$$

and hence $\mathrm{d}\Gamma(g^t,\mathbf{d}_0g^t)\geq 0$. Next by (4.12), we have

$$\chi(H)[\phi(v),\mathrm{i}\Gamma(g^t)]\chi(H)\in O(t^{-\delta\epsilon_0}).$$

Hence if $\delta\epsilon_0>1$ we obtain *i)* by Prop. A.2 with $\mathcal{D}=\mathcal{D}(N)$.

To prove *ii)* we write for $\chi\in C_0^\infty(\mathbb{R})$

$$\mathrm{e}^{\mathrm{i}tH}\Gamma(g^t)\mathrm{e}^{-\mathrm{i}tH}\chi^2(H)u=\mathrm{e}^{\mathrm{i}tH}\chi(H)\Gamma(g^t)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u+o(1),$$

by Lemma 4.7 and argue by density. *iii)* follows similarly from Lemma 4.7. □

Theorem 4.9 *Assume hypotheses (I'0), (I'1) and $\delta\epsilon_0>1$. Then $\Gamma^+(g,1)=\mathbb{1}$, ie:*

$$\mathrm{e}^{-\mathrm{i}tH}u=\Gamma(g(t^\delta\tilde{\sigma}))\mathrm{e}^{-\mathrm{i}tH}u+o(1),\ u\in\mathcal{H}.$$

Thm. 4.9 means that for large times no particles are found with momentum smaller than $t^{-\delta}$ for $\delta > \epsilon_0^{-1}$, while Prop. 4.3 means that for large times the number of particles with momentum smaller than $t^{-\delta}$ for $\delta > (1+\epsilon_0)^{-1}$ is less than t^{δ}.

Proof. We claim first that

$$\mathrm{w}-\lim_{R\to+\infty}\Gamma^{+}(g,R)=1\!\!1. \tag{4.13}$$

To prove (4.13) we will apply Prop. A.5. For $u\in\mathcal{D}(N)$, we have:

$$\begin{aligned} &\partial_t(u_t,\chi(H)\Gamma(g^t)\chi(H)u_t)\\ =\ &(u_t,\chi(H)\mathrm{d}\Gamma(g^t,\mathbf{d}_0g^t)\chi(H)u_t)\\ &+(u_t,\chi(H)[\phi(v),\mathrm{i}\Gamma(g^t)]\chi(H)u_t)\end{aligned} \tag{4.14}$$

The first term in the r.h.s of (4.14) is positive and by (4.12) the second term is bounded by $CR^{-\epsilon_0}t^{-\delta\epsilon_0}$. Clearly we have

$$\mathrm{w}-\lim_{R\to+\infty}\Gamma(g(t^{\delta}R\tilde{\sigma}))=1\!\!1,\ t\in\mathbb{R}.$$

Applying then Prop. A.5 we obtain that if $u=\chi(H)u$,

$$\lim_{R\to+\infty}(u,\Gamma^{+}(g,R)u)=\|u\|^2.$$

By density this implies (4.13).

Now we use the fact that for $\delta>\delta'$

$$g(t^{\delta}\tilde{\sigma})\geq g(t^{\delta'}R\tilde{\sigma}),\ \text{for fixed } R \text{ and } t\geq T_R.$$

Hence if we denote by $\Gamma'^{+}(g,R)$ the observable in Prop. 4.8 with the exponent δ', we have:

$$\Gamma'^{+}(g,R)\leq\Gamma^{+}(g,1)\leq 1\!\!1.$$

Letting $R\to+\infty$ and using (4.13) we obtain that $\Gamma^{+}(g,1)=1\!\!1$. $\Box$

We now state a lemma which will be used to control the number operator along the evolution, using the cutoffs $\Gamma(g^t)$.

Lemma 4.10

$$i)\ N^{\alpha}\Gamma(g^t)\chi(H)\in O(t^{\delta\alpha}),\ 0\leq\alpha\leq 1,\ \chi\in C_0^{\infty}(\mathbb{R}).$$

ii) Let $g_1^t=g_1(t^{\delta}\tilde{\sigma})$ *where* $g_1\in C^{\infty}(\mathbb{R})$ *is such that* $0\notin\operatorname{supp}g_1$, $g_1g=g$. *Then*

$$N^2\Gamma(g^t)\chi(H)\Gamma(g_1^t)\chi(H)\in O(t^{2\delta}),\ \chi\in C_0^{\infty}(\mathbb{R}).$$

Proof. Let us first prove *i)*. Since $\mathcal{D}(H)=\mathcal{D}(H_0)$, it suffices to estimate $N^\alpha\Gamma(g^t)(\mathrm{d}\Gamma(\tilde\sigma)+1)^{-\alpha}$. On the $n-$particle sector, we have:

$$n^\alpha\prod_1^n g(t^\delta\tilde\sigma_i)(\sum_1^n|\tilde\sigma_i|+1)^{-\alpha}\leq Ct^{\delta\alpha},$$

which proves *i)*.

To prove *ii)* we write:

$$\begin{array}{rl} & N^2\Gamma(g^t)\chi(H)\Gamma(g_1^t)\chi(H)\\ = & \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde\chi(z)N^2\Gamma(g^t)(z-H)^{-1}\Gamma(g_1^t)\chi(H)\mathrm{d}z\wedge\mathrm{d}\,\overline{z}\\ = & -\frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde\chi(z)N\Gamma(g^t)(z-H)^{-1}\phi(\mathrm{i}v)(z-H)^{-1}\Gamma(g_1^t)\chi(H)\mathrm{d}z\wedge\mathrm{d}\,\overline{z}\\ & +N\Gamma(g^t)\chi(H)N\Gamma(g_1^t)\chi(H),\end{array}$$

using Lemma 3.9 *ii)*. The second term is $O(t^{2\delta})$ by *i)*. Using the fact that $\mathcal{D}(H)=\mathcal{D}(H_0)$, we write:

$$\begin{array}{rl} & \|N\Gamma(g^t)(z-H)^{-1}\phi(\mathrm{i}v)(z-H)^{-1}\|\\ \leq & C\|N\Gamma(g^t)(H_0+1)^{-1}\|\|\phi(\mathrm{i}v)(H_0+1)^{-1}\|\|(H_0+1)(z-H)^{-1}\|\\ \leq & Ct^\delta|\mathrm{Im}z|^{-2},\end{array}$$

for $z\in\mathrm{supp}\,\tilde\chi$. This proves *ii)*. □

4.4 Sharper estimates for H^{e}

Let g be as in (4.10) and set

$$g^t=g(t^\delta\sigma).$$

Then by exactly the same arguments as in Subsect. 4.3, replacing cutoffs in H by cutoffs in L, we obtain:

Lemma 4.11 *Assume (I'0), (I'1). Then for $\chi,\chi_1\in C_0^\infty(\mathbb{R})$:*

$$[\chi(L),\Gamma(g^t)],\ [\chi(H^{\mathrm{e}}),\Gamma(g^t)]\chi_1(L)\in O(t^{-\delta\epsilon_0}).$$

Theorem 4.12 *Assume hypotheses (I'0), (I'1) and $\delta\epsilon_0>1$.*

$$\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u=\Gamma(g(t^\delta\sigma))\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u+o(1),\ u\in\mathcal{H}^{\mathrm{e}}.$$

The proofs are analogous to Lemma 4.7 and Thm. 4.9 and left to the reader.

We now state a lemma analogous to Lemma 4.10 for the Hamiltonian H^{e}.

Lemma 4.13 *i)*

$$(N^{\mathrm{e}})^\alpha\Gamma(g^t)\chi(L)\in O(t^{\delta\alpha}),\ 0\leq\alpha\leq 1,\ \chi\in C_0^\infty(\mathbb{R}).$$

ii) Let $g_1^t=g_1(t^\delta\sigma)$ where $g_1\in C^\infty(\mathbb{R})$ is such that $0\notin\mathrm{supp}\,g_1$, $g_1g=g$. Then

$$(N^{\mathrm{e}})^2\Gamma(g^t)\chi(L)\Gamma(g_1^t)\chi(L)\in O(t^{2\delta}),\ \chi\in C_0^\infty(\mathbb{R}).$$

The proof is identical to the proof of Lemma 4.10, replacing cutoffs in H by cutoffs in L.

4.5 Some auxiliary estimates for H

In this subsection we consider a positive selfadjoint operator C acting on $\mathfrak{h}$ such that $[C, \tilde{\sigma}] = 0$ and we prove some estimates on the growth of C along the evolution. These estimates will be used for the Nelson Hamiltonian in Sect. 12 for the observable $C = \frac{-\Delta_\omega}{\tilde{\sigma}^2}$.

We assume

$(I'5)$ $(1+|\tilde{\sigma}|^{-\frac{1}{2}})\langle C\rangle^{\mu_2} v(K+1)^{-\frac{1}{2}}$, $(1+|\tilde{\sigma}|^{-\frac{1}{2}})\langle C\rangle^{\mu_2}(K+1)^{-\frac{1}{2}} v \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$, $\mu_2 > 0$.

Note that *(I'5)* implies

$$\begin{array}{rl} & \|(1+|\tilde{\sigma}|^{-\frac{1}{2}})F(C\geq R)v(K+1)^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} \\ & +\|(1+|\tilde{\sigma}|^{-\frac{1}{2}})F(C\geq R)(K+1)^{-\frac{1}{2}}v\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} \\ \leq & C_0 R^{-\mu_2}. \end{array} \tag{4.15}$$

The corresponding assumption for the Nelson Hamiltonian is *(I5)* introduced in Subsect. 1.1. In fact let $P(\omega, \partial_\omega)$ be the expression of $-\Delta_\omega$ in some local coordinates on S^2. Then

$$-\frac{\Delta_\omega}{\tilde{\sigma}^2}(\mathrm{e}^{-\mathrm{i}\tilde{\sigma}\mathrm{x}.\omega}\tilde{\sigma}v_j(\tilde{\sigma}\omega)) = \mathrm{e}^{-\mathrm{i}\tilde{\sigma}\mathrm{x}.\omega}\frac{1}{\tilde{\sigma}^2}P(\omega, \partial_\omega - \mathrm{dx}.\omega)\tilde{\sigma}v_j(\tilde{\sigma}\omega),$$

where $\mathrm{dx}.\omega$ is the differential of the function $\omega \mapsto \mathrm{x}.\omega$. Since $\mathrm{dx}.\omega \in O(|\mathrm{x}|)$, we obtain, using (1.1) to control powers of x, that if *(H0)* holds for $\alpha > 0$ and *(I5)* holds for $\mu_2 > 0$ then *(I'5)* holds for the associated Pauli-Fierz Hamiltonian with the exponent μ_2 replaced by $\inf(\alpha, \mu_2)$.

Let $F \in C_0^\infty(\mathbb{R})$, $0 \leq F \leq 1$, $F(\lambda) \equiv 1$ for $|\lambda| \leq \frac{1}{2}$, $F(\lambda) \equiv 0$ for $|\lambda| \geq 1$. and $\lambda F'(\lambda) \leq 0$. We set for $\rho, R > 0$:

$$c_t := F(\frac{C}{Rt^\rho}). \tag{4.16}$$

Lemma 4.14 *Assume (I'0), (I'5). Then for $\chi \in C_0^\infty(\mathbb{R})$:*

$$[\chi(H), \Gamma(c_t)] \in O(t^{-\mu_2\rho}).$$

Using almost-analytic extensions, we are reduced to estimate

$$[(z-H)^{-1}, \Gamma(c_t)] = (z-H)^{-1}[H, \Gamma(c_t)](z-H)^{-1}.$$

We have

$$[H, \Gamma(c_t)] = \frac{1}{\sqrt{2}}a^*((1-c_t)v)\Gamma(c_t) - \frac{1}{\sqrt{2}}\Gamma(c_t)a((1-c_t)v).$$

Using *(I'5)* and Prop. A.1,we obtain

$$\|(H+\mathrm{i})^{-1}[H, \Gamma(c_t)](H+\mathrm{i})^{-1}\| \leq C_0 t^{-\rho\mu_2}, \tag{4.17}$$

uniformly in $R \geq 1$. This implies that

$$\|(z-H)^{-1}[H, \Gamma(c_t)](z-H)^{-1}\| \leq C_0|\mathrm{Im} z|^{-2}t^{-\rho\mu_2},\ z \in K \Subset \mathbb{C}\backslash\mathbb{R},$$

which implies the lemma. □

Proposition 4.15 *Assume (I'0), (I'5). Assume ρ in (4.16) is such that $\rho\mu_2 > 1$. Then*

$$i)\ \int_1^{+\infty} \|\mathrm{d}\Gamma(c_t, \mathbf{d}_0 c_t)^{\frac{1}{2}}\chi(H)u_t\|^2 \mathrm{d}t \leq C\|u\|^2,\ u \in \mathcal{D}(N),\ \chi \in C_0^\infty(\mathbb{R}),$$

$$ii)\ \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(c_t)\mathrm{e}^{-\mathrm{i}tH} =: P^+(R)\ \textit{exists},$$

$$iii)\ [P^+(R), H] = 0.$$

By the standard argument, we compute for $u, v \in \mathcal{D}(N)$ the derivative of the function

$$t \mapsto (v_t, \chi(H)\Gamma(c_t)\chi(H)u_t),$$

which equals

$$(v_t, \chi(H)\mathrm{d}\Gamma(c_t, \mathbf{d}_0 c_t)\chi(H)u_t) + (v_t, \chi(H)[\phi(v), \mathrm{i}\Gamma(c_t)]\chi(H)u_t).$$

By (4.17) the second term is integrable in norm if $\rho\mu_2 > 1$. We have

$$d_0^t c_t = -\rho F'(\frac{C}{Rt^\rho})\frac{C}{Rt^{\rho+1}} \geq 0,$$

hence $\mathrm{d}\Gamma(c_t, \mathrm{d}_0^t c_t) \geq 0$. The estimate *i)* follows then from Prop. A.2. The existence of the limit *ii)* follows from *i)*, Prop. A.3 and Lemma 4.14. Property *iii)* follows from Lemma 4.14. □

Theorem 4.16 *Assume (I'0), (I'5). Assume ρ in (4.16) is such that $\rho\mu_2 > 1$. Then $P^+(1) = \mathbb{1}$, ie:*

$$\mathrm{e}^{-\mathrm{i}tH}u = \Gamma(F(\frac{C}{t^\rho}))\mathrm{e}^{-\mathrm{i}tH}u + o(1),\ u \in \mathcal{H}.$$

Proof. We first claim that

(4.18) $$\mathrm{w} - \lim_{R\to\infty} P^+(R) = \mathbb{1}.$$

To prove (4.18) we apply Prop. A.5. We have for $u \in \mathcal{D}(N)$, $\chi \in C_0^\infty(\mathbb{R})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_t, \chi(H)\Gamma(c_t)\chi(H)u_t) \geq -Ct^{-\rho\mu_2}\|u\|^2,\ \text{uniformly in } R.$$

On the other hand $\Gamma(c_t) \leq \mathbb{1}$ and

$$\mathrm{w} - \lim_{R\to\infty} \Gamma(c_t) = \mathbb{1},\ \forall t \in \mathbb{R}.$$

Hence (4.18) follows from Prop. A.5. Finally we use the fact that for $\rho' > \rho$:

$$F(\frac{C}{Rt^\rho}) \leq F(\frac{C}{t^{\rho'}}),\ \text{for fixed } R \text{ and } t \geq T(R).$$

If we denote by $\Gamma'(c_t), P'^+(R)$ the same observables with the exponent ρ', we obtain $\Gamma(c_t) \leq \Gamma'(c_t) \leq \mathbb{1}$ and hence

$$P^+(R) \leq P'^+(1) \leq \mathbb{1}.$$

Letting $R \to \infty$ and using (4.18) we get that $P'^+(1) = \mathbb{1}$. This proves the theorem. □

5 Number of asymptotically free particles

In this section we consider the expanded Hamiltonian H^{e} introduced in Subsect. 3.3. On $\mathfrak{h}^{\mathrm{e}} = L^2(\mathbb{R}, \mathrm{d}\sigma) \otimes \mathfrak{g}$, we denote by

$$s = \mathrm{i}\frac{\partial}{\partial\sigma}$$

the observable conjugate to σ, which we interpret as a position. We construct for $0 < \mathrm{c} < 1$ $H^{\mathrm{e}}-$invariant subspaces $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ describing states which contain a finite number of particles in the region $\{s \geq \mathrm{c}t\}$.

Let us fix $f \in C^\infty(\mathbb{R})$, such that

$$0 \leq f \leq 1,\ f' \geq 0,\ f \equiv 0 \text{ for } s \leq \alpha_0,\ f \equiv 1 \text{ for } s \geq \alpha_1, \tag{5.1}$$

for $\alpha_0 < \alpha_1$. We set

$$b_{\mathrm{c}\,t} := f\Big(\frac{s - \mathrm{c}t}{t^\rho}\Big),\ B_{\mathrm{c}\,t} = \mathrm{d}\Gamma(b_{\mathrm{c}\,t}), \tag{5.2}$$

where the constants $0 < \mathrm{c} \leq 1$, $0 < \rho < 1$ will be fixed later.

We assume in this section the following hypothesis:

$$(I'2)\ (K+1)^{-\frac{1}{2}} v^{\mathrm{e}}(\cdot),\ v^{\mathrm{e}}(\cdot)(K+1)^{-\frac{1}{2}} \in H_0^\mu(\mathbb{R}^+) \otimes \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{g}),$$

where for $\mu > 0$ the space $H_0^\mu(\mathbb{R}^+)$ is the closure of $C_0^\infty(]0, +\infty[)$ in the topology of $H^\mu(\mathbb{R})$. Note that *(I'2)* implies

$$\begin{aligned} &\|F(|s| \geq R) v^{\mathrm{e}} (K+1)^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}^{\mathrm{e}})} \leq CR^{-\mu}, \\ &\|F(|s| \geq R)(K+1)^{-\frac{1}{2}} v^{\mathrm{e}}\|_{\mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}^{\mathrm{e}})} \leq CR^{-\mu},\ R \geq 1. \end{aligned} \tag{5.3}$$

The corresponding condition for concrete Nelson Hamiltonians is *(I2)*, introduced in Subsect. 1.1. In fact we note that $\partial_{\tilde\sigma} \mathrm{e}^{-\mathrm{i}\tilde\sigma \mathrm{x}.\omega} \tilde\sigma v_j(\tilde\sigma\omega) = \mathrm{e}^{-\mathrm{i}\tilde\sigma \mathrm{x}.\omega}(\partial_{\tilde\sigma} - \mathrm{i}\mathrm{x}.\omega)\tilde\sigma v_j(\tilde\sigma\omega)$. Using then (1.1) to control powers of x we see that if *(H0)* holds for $\alpha > 0$ and *(I2)* holds for $\mu > 0$ then *(I'2)* holds for the associated Pauli-Fierz Hamiltonian with the exponent μ replaced by $\inf(\alpha, \mu)$.

5.1 Technical preparations

Proposition 5.1 *Assume (I'0) for* $\epsilon_0 > 0$, *(I'2) for* $\mu > 1$. *Assume that* $0 < \mathrm{c} < 1$ *or that* $\mathrm{c} = 1$ *and* $\alpha_1 < 0$. *Then for* $\chi \in C_0^\infty(\mathbb{R})$:

$$i)\ \int_1^{+\infty} \|\mathrm{d}\Gamma(\mathbf{d}_0 b_{\mathrm{c}\,t})^{\frac{1}{2}} (B_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L) u_t\|^2 \frac{dt}{t} \leq C\|u\|^2,\ u \in \mathcal{D}(N^{\mathrm{e}}),\ \lambda > 0,$$

$$ii)\ \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} \chi(L)(B_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} \text{ exists },\ \forall \lambda \in \mathbb{C}\backslash\mathbb{R}^-.$$

Proof. Let us first fix $\lambda > 0$ and set

$$\Phi(t) = \chi(L)(B_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L).$$

For $u \in \mathcal{D}(N^{\rm e})$ the function $\mathbb{R}^+ \ni t \mapsto f(t) = (u_t, \Phi(t)u_t)$ is C^1 with derivative

$$\partial_t f(t) = (u_t, \chi(L)\mathbf{D}\Phi(t)\chi(L)u_t).$$

Note that by (3.8) $\mathrm{e}^{-\mathrm{i}tH^{\rm e}}\chi(L)$ preserves $\mathcal{D}(N^{\rm e})$. Since $H^{\rm e} = H_0^{\rm e} + \phi(v^{\rm e})$ on $\mathcal{D}(L)$, we have:

$$[H^{\rm e}, \mathrm{i}(B_{{\rm c}\,t} + \lambda)^{-1}] = [H_0^{\rm e}, \mathrm{i}(B_{{\rm c}\,t} + \lambda)^{-1}] + [\phi(v^{\rm e}), \mathrm{i}(B_{{\rm c}\,t} + \lambda)^{-1}]$$

on $\mathcal{D}(L)$. Since $\mathcal{K} \otimes \Gamma_{\rm fin}(\mathfrak{h}^{\rm e}) \cap \mathcal{D}(L)$ is dense in $\mathcal{D}(L)$ we can compute $\mathbf{D}(B_{{\rm c}\,t} + \lambda)^{-1}$ on finite vectors. We obtain

$$\mathbf{D}_0(B_{{\rm c}\,t} + \lambda)^{-1} = -(B_{{\rm c}\,t} + \lambda)^{-1}\mathrm{d}\Gamma(c_t)(B_{{\rm c}\,t} + \lambda)^{-1},$$

for

$$c_t = \mathbf{d}_0 b_{{\rm c}\,t} = f'(\frac{s - {\rm c}t}{t^\rho})(\frac{1 - {\rm c}}{t^\rho} - \rho\frac{s - {\rm c}t}{t^{\rho+1}}).$$

Similarly

$$[\phi(v^{\rm e}), \mathrm{i}(B_{{\rm c}\,t} + \lambda)^{-1}] = (B_{{\rm c}\,t} + \lambda)^{-1}\phi(\mathrm{i}b_{{\rm c}\,t}v^{\rm e})(B_{{\rm c}\,t} + \lambda)^{-1}.$$

Since $\mathcal{D}(L) = \mathcal{D}(L_0)$, $(K+1)^{\frac{1}{2}}(L + \mathrm{i})^{-1}$ is bounded, and we have:

$$\begin{aligned} &\|[\phi(v^{\rm e}), \mathrm{i}(B_{{\rm c}\,t} + \lambda)^{-1}](L + \mathrm{i})^{-1}\| \\ (5.4) \qquad &\leq \|(B_{{\rm c}\,t} + \lambda)^{-1}\phi(\mathrm{i}b_{{\rm c}\,t}v^{\rm e})(K+1)^{-\frac{1}{2}})(B_{{\rm c}\,t} + \lambda)^{-1}\| \\ &\leq C\|b_{{\rm c}\,t}^{\frac{1}{2}}v^{\rm e}(K+1)^{-\frac{1}{2}}\| \leq Ct^{-\mu}, \end{aligned}$$

using Prop. A.1 and assumption *(I'2)*.

Next we note that if $0 < {\rm c} < 1$ or ${\rm c} = 1$ and $\alpha_1 < 0$

$$c_t \geq \frac{\rm c}{t^\rho}f'(\frac{s - {\rm c}t}{t^\rho}) \text{ for } t \gg 1.$$

Applying then Prop. A.2 with $\mathcal{D} = \mathcal{D}(N^{\rm e})$, we obtain

$$(5.5) \qquad \int_1^{+\infty} \|\mathrm{d}\Gamma(c_t)^{\frac{1}{2}}(B_{{\rm c}\,t} + \lambda)^{-1}\chi(L)u_t\|^2\frac{dt}{t} \leq C\|u\|^2,\ u \in \mathcal{D}(N^{\rm e}),\ \lambda > 0.$$

This proves *i)*.

For $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$, we have, as quadratic forms on $\mathcal{D}(L) \cap \mathcal{D}(N^{\rm e})$:

$$\begin{aligned} &\mathbf{D}^0(B_{{\rm c}\,t} + \lambda)^{-1} = -(B_{{\rm c}\,t} + \lambda)^{-1}\mathrm{d}\Gamma(c_t)(B_{{\rm c}\,t} + \lambda)^{-1} \\ (5.6) \qquad &= -(B_{{\rm c}\,t} + 1)^{-1}\mathrm{d}\Gamma(c_t)^{\frac{1}{2}}R(t)\mathrm{d}\Gamma(c_t)^{\frac{1}{2}}(B_{{\rm c}\,t} + 1)^{-1}, \end{aligned}$$

for $R(t) = (B_{{\rm c}\,t} + 1)^2(B_{{\rm c}\,t} + \lambda)^{-2} \in O(1)$. Moreover (5.4) is still valid for $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$. Applying then (5.6), (5.4) and the estimate (5.5) for $\lambda = 1$, we obtain *ii)* by Prop. A.3. $\square$

Lemma 5.2 *Let $f_t \in C^\infty(\mathbb{R})$ with $|\partial_s^\alpha f_t| \leq C_\alpha t^{-\rho\alpha}$, $\alpha \in \mathbb{N}$. Then*

$$[f_t(s), |\sigma|] \in O(t^{-\rho}).$$

Proof. We have

$$\|[f_t(s),|\sigma|]\| = t^{-\rho}\|[f_t(t^\rho s),|\sigma|]\|.$$

Note that $g_t(s) = f_t(t^\rho s)$ satisfies $|\partial_s^\alpha g_t| \leq C_\alpha$, $\alpha \in \mathbb{N}$.

It remains to check that $[g_t(s),|\sigma|]$ is bounded, which follows by writing $|\sigma|$ as $r_1(\sigma)+r_2(\sigma)$, where $r_1(\sigma)$ is bounded and $r_2(\sigma) \in C^\infty(\mathbb{R})$, $|\partial_\sigma^\alpha r_2(\sigma)| \leq C_\alpha \langle\sigma\rangle^{1-|\alpha|}$. □

Lemma 5.3 *Assume (I'0) for $\epsilon_0 > 0$, (I'2) for $\mu > 0$. Let N_t^{e} be defined in (4.9). Assume the exponent ρ in (5.2) is such that $\rho > \delta > (1+\epsilon_0)^{-1}$. Then for $\chi, F \in C_0^\infty(\mathbb{R}), \lambda \in \mathbb{C}\backslash\mathbb{R}^-$:*

$$(L+\mathrm{i})[(B_{\mathrm{c}\,t}+\lambda)^{-1},\chi(L)]F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L) \in o(1).$$

Proof. We write

$$\begin{aligned}&(L+\mathrm{i})[(B_{\mathrm{c}\,t}+\lambda)^{-1},\chi(L)]F(\tfrac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\\ =\;& \tfrac{\mathrm{i}}{2\pi}\int \tfrac{\partial\tilde{\chi}(z)}{\partial\overline{z}}(L+\mathrm{i})(z-L)^{-1}[(B_{\mathrm{c}\,t}+\lambda)^{-1},L](z-L)^{-1}F(\tfrac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\mathrm{d}z\wedge\mathrm{d}\overline{z}.\end{aligned}$$

By Lemma 3.5, $(z-L)^{-1}$ preserves $\mathcal{D}(N^{\mathrm{e}})$. On $\mathcal{D}(L)\cap\mathcal{D}(N^{\mathrm{e}})$ we have:

$$[(B_{\mathrm{c}\,t}+\lambda)^{-1},L] = [(B_{\mathrm{c}\,t}+\lambda)^{-1},L_0] + [(B_{\mathrm{c}\,t}+\lambda)^{-1},\phi(v^{\mathrm{e}})].$$

By (5.4)

$$[(B_{\mathrm{c}\,t}+\lambda)^{-1},\phi(v^{\mathrm{e}})](z-L)^{-1} \in O(|\mathrm{Im}z|^{-1}t^{-\mu}).$$

Next on $\mathcal{D}(L)\cap\mathcal{D}(N^{\mathrm{e}})$

$$[(B_{\mathrm{c}\,t}+\lambda)^{-1},L_0] = -(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{d}\Gamma([b_{\mathrm{c}\,t},|\sigma|])(B_{\mathrm{c}\,t}+\lambda)^{-1}. \tag{5.7}$$

By Lemma 5.2, $[b_{\mathrm{c}\,t},|\sigma|] \in O(t^{-\rho})$ and hence

$$[(B_{\mathrm{c}\,t}+\lambda)^{-1},L_0] \in O(N^{\mathrm{e}})t^{-\rho}.$$

By Lemma 3.5 we have

$$(N^{\mathrm{e}}+1)(z-L)^{-1}(N^{\mathrm{e}}+1)^{-1} \in O(|\mathrm{Im}z|^{-2}),\ z \in \mathrm{supp}\,\tilde{\chi}, \tag{5.8}$$

and by Lemma 4.6 $N^{\mathrm{e}}F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L) \in O(t^\delta)$. Finally we obtain that for $u \in \mathcal{D}(N^{\mathrm{e}})$:

$$\begin{aligned}&\|(L+\mathrm{i})(z-L)^{-1}[(B_{\mathrm{c}\,t}+\lambda)^{-1},L](z-L)^{-1}F(\tfrac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)u\|\\ \leq\;& C(t^{\delta-\rho}|\mathrm{Im}z|^{-3}+t^{-\mu}|\mathrm{Im}z|^{-1})\|u\|,\ z\in\mathrm{supp}\,\tilde{\chi}.\end{aligned}$$

Using the properties of $\tilde{\chi}$ this proves the lemma. □

Proposition 5.4 *Assume (I'0), (I'1) for $\epsilon_0 > 0$,(I'2) for $\mu > 0$. Then for $\rho > \delta > (1+\epsilon_0)^{-1}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$, $\chi \in C_0^\infty(\mathbb{R})$:*

$$(B_{\mathrm{c}\,t}+\lambda)^{-1}F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi^2(L) = \chi(L)(B_{\mathrm{c}\,t}+\lambda)^{-1}F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L) + o(1).$$

Proof. We combine Lemma 5.3 and Lemma 4.4. □

5.2 Asymptotic projections

Theorem 5.5 *Assume (I'0), (I'1) for* $\epsilon_0 > 0$*, (I'2) with* $\mu > 1$ *and pick* ρ *in (5.2) such that* $\rho(1+\epsilon_0) > 1$*. Then:*

i) for each $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$ *the limit*

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} =: R_{\mathrm{c}}^{\mathrm{e}+}(\lambda) \text{ exists.}$$

ii) $[R_{\mathrm{c}}^{\mathrm{e}+}(\lambda), L] = [R_{\mathrm{c}}^{\mathrm{e}+}(\lambda), H^{\mathrm{e}}] = 0.$

iii) the limit

$$\text{s-}\lim_{\epsilon\to 0} \epsilon^{-1}R_{\mathrm{c}}^{\mathrm{e}+}(\epsilon^{-1}) =: \hat{P}_{\mathrm{c}}^{\mathrm{e}+} \text{ exists}$$

and is an orthogonal projection.

iv)

$$[H^{\mathrm{e}}, \hat{P}_{\mathrm{c}}^{\mathrm{e}+}] = [L, \hat{P}_{\mathrm{c}}^{\mathrm{e}+}] = 0,$$

$$u = \hat{P}_{\mathrm{c}}^{\mathrm{e}+}u \Leftrightarrow \text{s-}\lim_{\epsilon\to 0}\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u = u.$$

The projections $\hat{P}_{\mathrm{c}}^{\mathrm{e}+}$ depend on the choice of the cutoff function f in (5.2). We introduce projections independent on the choice of f in the next theorem.

Theorem 5.6 *Assume (I'0), (I'1) for* $\epsilon_0 > 0$*, (I'2) with* $\mu > 1$ *and pick* ρ *in (5.2) such that* $\rho(1+\epsilon_0) > 1$*. Let for* $0 < \mathrm{c} < 1$*:*

$$P_{\mathrm{c}}^{\mathrm{e}+} := \inf_{\mathrm{c}<\mathrm{c}'} \hat{P}_{\mathrm{c}'}^{\mathrm{e}+},\ \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+} := \mathrm{Ran}P_{\mathrm{c}}^{\mathrm{e}+}.$$

Then:

i) $P_{\mathrm{c}}^{\mathrm{e}+}$ *is an orthogonal projection independent on the choice of the function* f *in (5.2).*

$$ii)\ [H^{\mathrm{e}}, P_{\mathrm{c}}^{\mathrm{e}+}] = [L, P_{\mathrm{c}}^{\mathrm{e}+}] = 0,$$

$$iii)\ \Omega^{\mathrm{e}+}(\mathcal{H}_{\mathrm{pp}}(H^{\mathrm{e}})\otimes\Gamma(\mathfrak{h}^{\mathrm{e}})) \subset \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+},$$

where the wave operator $\Omega^{\mathrm{e}+}$ *is defined in Subsect. 8.5.*

The space $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ can be understood as the space of states having a finite number of particles in the region $\{s \geq \mathrm{c}'t\}$ for all $\mathrm{c}' > \mathrm{c}$. By part *ii)* of Thm. 5.6 we know that if H^{e} has bound states, in particular under the assumptions *(H'0)*, *(I'3)*, then $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ is non trivial.

Proof. Let f_1, f_2 be two functions such that $0 \leq f_i \leq 1$, $f_i' \in C_0^\infty(\mathbb{R})$, $f_i' \geq 0$ and $f_i \equiv 0$ for $s \ll -1$, $f_i \equiv 1$ for $s \gg 1$. Clearly there exists s_0 such that $f_1(s) \leq f_2(s+s')$ for any $s' \geq s_0$. This implies that if $\mathrm{c}_1 > \mathrm{c}_2$

$$f_1(\frac{s-\mathrm{c}_1 t}{t^\rho}) \leq f_2(\frac{s-\mathrm{c}_2 t}{t^\rho}),\ t \geq T. \tag{5.9}$$

Let us denote by $B_{i,\mathrm{c}\,t}$ the observable defined in (5.2) for $f = f_i$ and by $R_{i,\mathrm{c}}^{\mathrm{e}+}(\lambda)$, $\hat{P}_{i,\mathrm{c}}^{\mathrm{e}+}$ the objects constructed in Thm. 5.5 for $f = f_i$. It follows from (5.9) that

$$(B_{1,\mathrm{c}_1\,t}+\lambda)^{-1} \geq (B_{2,\mathrm{c}_2\,t}+\lambda)^{-1} \text{ for } t \geq T,\ \lambda > 0$$

hence

$$R^{\mathrm{e}+}_{1,\mathrm{c}_1}(\lambda) \geq R^{\mathrm{e}+}_{2,\mathrm{c}_2}(\lambda),\ \lambda > 0,$$

and

$$\hat{P}^{\mathrm{e}+}_{1,\mathrm{c}_1} \geq \hat{P}^{\mathrm{e}+}_{2,\mathrm{c}_2} \text{ if } \mathrm{c}_1 > \mathrm{c}_2. \tag{5.10}$$

If we take $f_1 = f_2 = f$, we obtain that the family of projections $\hat{P}^{\mathrm{e}+}_{\mathrm{c}}$ is increasing w.r.t. c, which shows the existence of $P^{\mathrm{e}+}_{\mathrm{c}}$. Using again (5.10) we obtain that $P^{\mathrm{e}+}_{\mathrm{c}}$ does not depend on f.

ii) follow from Thm. 5.5. It remains to prove *iii)*. We first note that $\mathcal{H}_{\mathrm{pp}}(H^{\mathrm{e}}) \subset \mathcal{H}^{\mathrm{e}+}_{\mathrm{c}}$. In fact this is a direct consequence of the fact that for $\epsilon > 0$ $(\epsilon B_{\mathrm{c}\,t} + 1)^{-1}$ tends strongly to $\mathbb{1}$ when $t \to +\infty$. Next we use the fact proved in Thm. 8.7 that the asymptotic Weyl operators $W^{\mathrm{e}+}(h)$ preserve the space $\mathcal{H}^{\mathrm{e}+}_{\mathrm{c}}$. These two observations imply *iii)*. □

Proof of Thm. 5.5 Let us first prove *i)*. By density it suffices to show the existence of

$$\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,t} + \lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u \tag{5.11}$$

for $u = \chi(L)u$, $\chi \in C_0^\infty(\mathbb{R})$. Let us pick an exponent δ with $\rho > \delta > (1+\epsilon_0)^{-1}$, which is possible since $\rho(1+\epsilon_0) > 1$. By Prop. 4.5 *ii)*, we have for $F \in C_0^\infty(\mathbb{R})$, $F \equiv 1$ near 0:

$$\begin{aligned}
& \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,t} + \lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u \\
= \ & \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,t} + \lambda)^{-1}F(\tfrac{N_t}{t^\delta})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u + o(1) \\
= \ & \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,t} + \lambda)^{-1}F(\tfrac{N_t}{t^\delta})\chi^2(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u + o(1) \\
= \ & \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\chi(L)(B_{\mathrm{c}\,t} + \lambda)^{-1}\chi(L)F(\tfrac{N_t}{t^\delta})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u + o(1) \\
= \ & \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\chi(L)(B_{\mathrm{c}\,t} + \lambda)^{-1}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u + o(1),
\end{aligned}$$

where we used Prop. 4.5 and Prop. 5.4. Hence by Prop. 5.1 the limit (5.11) exists. The first statement of *ii)* follows by the arguments above. Let us now prove the second statement of *ii)*. It suffices to show that

$$R^{\mathrm{e}+}_{\mathrm{c}}(\lambda) = \mathrm{e}^{\mathrm{i}t_1H^{\mathrm{e}}}R^{\mathrm{e}+}_{\mathrm{c}}(\lambda)\mathrm{e}^{-\mathrm{i}t_1H^{\mathrm{e}}},\ \forall t_1 \in \mathbb{R},$$

or equivalently

$$\text{s-}\lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}((B_{\mathrm{c}\,t} + \lambda)^{-1} - (B_{\mathrm{c}t-t_1} + \lambda)^{-1})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} = 0. \tag{5.12}$$

We have

$$(B_{\mathrm{c}\,t} + \lambda)^{-1} - (B_{\mathrm{c}\,t-t_1} + \lambda)^{-1} = -(B_{\mathrm{c}\,t} + \lambda)^{-1}(B_{\mathrm{c}\,t} - B_{\mathrm{c}\,t-t_1})(B_{\mathrm{c}\,t-t_1} + \lambda)^{-1},$$

and

$$B_{\mathrm{c}\,t} - B_{\mathrm{c}\,t-t_1} = \mathrm{d}\Gamma(b_{\mathrm{c}\,t} - b_{\mathrm{c}\,t-t_1}).$$

Since $\|b_{\mathrm{c}\,t} - b_{\mathrm{c}\,t-t_1}\| \in O(t^{-\rho})$, this gives

$$\Big((B_{\mathrm{c}\,t} + \lambda)^{-1} - (B_{\mathrm{c}\,t-t_1} + \lambda)^{-1}\Big) \in O(N^{\mathrm{e}})t^{-\rho}. \tag{5.13}$$

Let $u \in \mathcal{H}^{\mathrm{e}}$ with $\chi(L)u = u$ for $\chi \in C_0^\infty(\mathbb{R})$. We pick δ with $\rho > \delta > (1+\epsilon_0)^{-1}$ and write

$$\begin{aligned}
& \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Big((B_{\mathrm{c}\,t} + \lambda)^{-1} - (B_{\mathrm{c}\,t-t_1} + \lambda)^{-1}\Big)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u \\
= \ & \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Big((B_{\mathrm{c}\,t} + \lambda)^{-1} - (B_{\mathrm{c}\,t-t_1} + \lambda)^{-1}\Big)F(\tfrac{N^{\mathrm{e}}_t}{t^\delta})\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u + o(1).
\end{aligned}$$

Combining (5.13) and Lemma 4.6 we obtain (5.12). This completes the proof of *ii)*.

Statements *iii)* and *iv)* follow from Prop. A.6 in the Appendix. □

5.3 Soft propagation estimates

In this subsection we show rather easy propagation estimates. More precisely we show that for any state in $\mathcal{H}^{\mathrm{e}}$ there is no propagation in the region $\{s \leq -\mathrm{c}t\}$ for $0 < \mathrm{c} < 1$.

We fix a cutoff function $f_1 \in C^\infty(\mathbb{R})$ such that for some $\alpha_4 < \alpha_3 < 0$:

$$
\begin{aligned}
&0 \leq f_1 \leq 1, \ \mathrm{supp}\, f_1 \subset]-\infty, \alpha_3], \\
&f_1 \equiv 1 \text{ in }]-\infty, \alpha_4], \ f_1'(s) \leq 0,
\end{aligned} \tag{5.14}
$$

and set for $0 < \rho < 1$, $0 < \mathrm{c} < 1$:

$$
b_{1,t} = f_1\left(\frac{s + \mathrm{c}t}{t^\rho}\right), \ B_{1,t} := \mathrm{d}\Gamma(b_{1,t}). \tag{5.15}
$$

Proposition 5.7 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 1$ and pick ρ in (5.15) such that $\rho(1 + \epsilon_0) > 1$. Then:*

$$
\begin{aligned}
&i)\ R_1^+(\epsilon) := \mathrm{s}\text{-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(1 + \epsilon B_{1,t})^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} \ \textit{exists}, \\
&ii)\ [R_1^+(\epsilon), H^{\mathrm{e}}] = [R_1^+(\epsilon), L] = 0, \\
&iii)\ \mathrm{s}\text{-}\lim_{\epsilon\to 0} R_1^+(\epsilon) = \mathbb{1}.
\end{aligned}
$$

Prop. 5.7 means that any state has a finite number of particles in the region $\{s \leq -\mathrm{c}t\}$.

Proof. we first prove the existence of

$$
\mathrm{s}\text{-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\chi(L)(1 + \epsilon B_{1,t})^{-1}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}. \tag{5.16}
$$

Arguing exactly as in the proof of Prop. 5.1, we obtain for $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$:

$$
\begin{aligned}
\chi(L)\mathbf{D}(B_{1,t} + \lambda)^{-1}\chi(L) = \ & -\chi(L)(B_{1,t} + \lambda)^{-1}\mathrm{d}\Gamma(c_{1,t})(B_{1,t} + \lambda)^{-1}\chi(L) \\
& -\chi(L)(B_{1,t} + \lambda)^{-1}\phi(\mathrm{i}b_{1,t}v^{\mathrm{e}})(B_{1,t} + \lambda)^{-1}\chi(L),
\end{aligned}
$$

for

$$
\begin{aligned}
c_{1,t} = \ & \mathbf{d}_0 b_{1,t} = f_1'\left(\tfrac{s+\mathrm{c}t}{t^\rho}\right)\left(\tfrac{1+\mathrm{c}}{t^\rho} - \rho\tfrac{s+\mathrm{c}t}{t^{\rho+1}}\right) \\
\leq \ & -cf_1'\left(\tfrac{s+\mathrm{c}t}{t^\rho}\right)\tfrac{1}{t^\rho}, \text{ for } t \gg 1.
\end{aligned}
$$

Moreover by Prop. A.1:

$$
\|\chi(L)(B_{1,t} + \lambda)^{-1}\phi(\mathrm{i}b_{1,t}v^{\mathrm{e}})(B_{1,t} + \lambda)^{-1}\chi(L)\| \leq C\|b_{1,t}^{\frac{1}{2}}v^{\mathrm{e}}(K + 1)^{-\frac{1}{2}}\| \leq Ct^{-\mu},
$$

by *(I'2)*. Arguing as in Subsect. 5.1 we obtain the existence of the limit (5.16). As in the proof of Thm. 5.5, using an analog of Prop. 5.4 we obtain then the existence of $R_1^+(\epsilon)$. Property *ii)* can be shown as in Thm. 5.5.

Let us now prove *iii)*. By Prop. A.6 we obtain the existence of $\text{s-}\lim_{\epsilon\to 0} R_1^+(\epsilon)$, and it suffices to show that

$$\mathrm{w}-\lim_{\epsilon\to 0} R_1^+(\epsilon) = \mathbb{1}. \tag{5.17}$$

By density it suffices to consider states $u \in \mathcal{H}^{\mathrm{e}}$ such that $u = \chi(L)u$ for some $\chi \in C_0^\infty(\mathbb{R})$. We will apply Prop. A.5 to $\Phi_\epsilon(t) = \chi(L)(1+\epsilon B_{1,t})^{-1}\chi(L)$. We have:

$$\begin{aligned} & \chi(L)\mathbf{D}(1+\epsilon B_{1,t})^{-1}\chi(L) \\ = \; & -\epsilon\chi(L)(1+\epsilon B_{1,t})^{-1}\mathrm{d}\Gamma(c_{1,t})(1+\epsilon B_{1,t})^{-1}\chi(L) \\ & -\epsilon\chi(L)(1+\epsilon B_{1,t})^{-1}\phi(\mathrm{i}b_{1,t}v^{\mathrm{e}})(1+\epsilon B_{1,t})^{-1}\chi(L). \end{aligned}$$

Using the fact that

$$\|(1+\epsilon B_{1,t})^{-1}(1+B_{1,t})^{-\frac{1}{2}}\| \leq C\epsilon^{-\frac{1}{2}}, \text{ uniformly in } t,$$

and Prop. A.1, we obtain

$$\|\epsilon\chi(L)(1+\epsilon B_{1,t})^{-1}\phi(\mathrm{i}b_{1,t}v^{\mathrm{e}})(1+\epsilon B_{1,t})^{-1}\chi(L)\| \leq C\|b_{1,t}^{\frac{1}{2}}v(K+1)^{-\frac{1}{2}}\| \leq Ct^{-\mu}, \tag{5.18}$$

uniformly in ϵ. Since $c_{1,t} \leq 0$, we obtain

$$\chi(L)\mathbf{D}(1+\epsilon B_{1,t})^{-1}\chi(L) \geq -Ct^{-\mu}, \text{ uniformly in } \epsilon. \tag{5.19}$$

Clearly $\mathrm{w}-\lim_{\epsilon\to 0}(1+\epsilon B_{1,t})^{-1} = \mathbb{1}$ for $t > 0$ and $0 \leq R_1^+(\epsilon) \leq \mathbb{1}$. Applying Prop. A.5 we obtain (5.17). □

Let now $f_0 \in C^\infty(\mathbb{R})$ be a cutoff function such that:

$$\begin{aligned} & f_0 \equiv 1 \text{ in } s \leq \alpha_1, \; f_0 \equiv 0 \text{ in } s \geq \alpha_2, \\ & f_0' \leq 0. \end{aligned} \tag{5.20}$$

Here the constants $\alpha_1 < \alpha_2$ are such that $0 < \alpha_1 < \alpha_2$.

We set:

$$f_R^t = f_0\left(\frac{-s-\mathrm{c}t}{Rt^\rho}\right), \tag{5.21}$$

for $R \geq 1$ and $0 < \rho < 1$ as in (5.2). The following two lemmas are analogous to Lemmas 6.1 and 6.2 for $k = 0$.

Lemma 5.8 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 0$. Assume the constants ρ, δ are chosen so that $\rho > \delta > (1+\epsilon_0)^{-1}$, $\mu > \delta/2$. Then for $\chi_1, \chi_2, F \in C_0^\infty(\mathbb{R})$:*

$$[\Gamma(f_R^t), \chi_1(L)]F\left(\frac{N_t^{\mathrm{e}}}{t^\delta}\right)\chi_2(L) \in o(1).$$

Lemma 5.9 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 0$. Assume the constants ρ, δ are chosen so that $\rho > \delta > (1+\epsilon_0)^{-1}$, $\mu > \delta/2$. Then for $\chi_1, \chi_2, F \in C_0^\infty(\mathbb{R})$:*

$$\Gamma(f_R^t)F\left(\frac{N_t^{\mathrm{e}}}{t^\delta}\right)\chi_1(L)\chi_2(L) = \chi_1(L)\Gamma(f_R^t)F\left(\frac{N_t^{\mathrm{e}}}{t^\delta}\right)\chi_2(L) + o(1).$$

The following lemma is analogous to Prop. 6.3.

Lemma 5.10 *Assume (I'0), (I'2) for $\mu > 1$. Let $B_{1,t}$ defined in (5.15). Then for $\chi \in C_0^\infty(\mathbb{R})$, $\lambda > 0$, $R \geq 1$ large enough:*

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\chi(L)\Gamma(f_R^t)(B_{1,t}+\lambda)^{-1}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} \text{ exists.} \tag{5.22}$$

Proof. As in the proof of Prop. 6.3 we compute

$$\begin{aligned} &\mathbf{D}\chi(L)\Gamma(f_R^t)(B_{1,t}+\lambda)^{-1}\chi(L) \\ = \;& \chi(L)\mathbf{D}\Gamma(f_R^t)(B_{1,t}+\lambda)^{-1}\chi(L) \\ &+\chi(L)\Gamma(f_R^t)\mathbf{D}(B_{1,t}+\lambda)^{-1}\chi(L). \end{aligned} \tag{5.23}$$

By the proof of Prop. 5.7, we have:

$$|(v, \chi(L)\Gamma(f_R^t)\mathbf{D}(B_{1,t}+\lambda)^{-1}\chi(L)u)| \leq C\|R_1(t)u\|\|R_1(t)v\|, \tag{5.24}$$

uniformly in R, where $R_1(t)$ is integrable along the evolution. Let us now consider the first term in (5.23). We have:

$$[\phi(v^{\mathrm{e}}), \mathrm{i}\Gamma(f_R^t)] = \frac{\mathrm{i}}{\sqrt{2}}\Gamma(f_R^t)a((1-f_R^t)v^{\mathrm{e}}) - \frac{\mathrm{i}}{\sqrt{2}}a^*((1-f_R^t)v^{\mathrm{e}})\Gamma(f_R^t).$$

By (5.20),

$$\operatorname{supp}(1-f_R^t) \subset \{s \leq -\mathrm{c}t - \alpha_1 R t^\rho\},$$

and

$$b_{1,t} \equiv 1 \text{ in } \{s \leq -\mathrm{c}t + \alpha_4 t^\rho\}.$$

Since $\alpha_1 > 0$ $b_{1,t} \equiv 1$ on $\operatorname{supp}(1-f_R^t)$ for $t \geq 0$, $R \geq R_0$. Applying then Prop. A.1, we obtain:

$$\begin{aligned} &\|[\phi(v), \Gamma(f_R^t)](B_{1,t}+\lambda)^{-1}\chi(L)\| \\ \leq \;& C\|(1-f_R^t)v^{\mathrm{e}}(K+1)^{-\frac{1}{2}}\| \\ \leq \;& Ct^{-\mu}, \end{aligned} \tag{5.25}$$

uniformly in $R \geq R_0$, by *(I'2)*. Finally

$$\begin{aligned} \mathbf{D}_0\Gamma(f_R^t) &= \mathrm{d}\Gamma(f_R^t, \mathbf{d}_0 f_R^t), \\ \mathbf{d}_0 f_R^t = \;& f_0'(\tfrac{-s-\mathrm{c}t}{Rt^\rho})\left(\tfrac{-1-\mathrm{c}}{Rt^\rho} + \tfrac{\rho}{Rt^{\rho+1}}(s+\mathrm{c}t)\right) \\ \geq \;& \tfrac{\mathrm{c}}{Rt^\rho}|f_0'|(\tfrac{-s-\mathrm{c}t}{Rt^\rho}), \end{aligned} \tag{5.26}$$

uniformly in $R \geq R_0$. From (5.25), (5.26), (5.24) we obtain:

$$|(v, \mathbf{D}\chi(L)\Gamma(f_R^t)(B_{1,t}+\lambda)^{-1}\chi(L)u)| \leq \sum_{i=1}^{3}\|R_i(t)u\|\|R_i(t)v\|, \tag{5.27}$$

uniformly in $R \geq R_0$, where $R_i(t)$ are integrable along the evolution. This implies that the limit (5.22) exists. $\square$

The following proposition is an improvement on Prop. 5.7. It means that asymptotically there are no particles in $\{s \leq -\mathrm{c}t\}$.

Proposition 5.11 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 1$ and pick ρ such that $\rho(1+\epsilon_0) > 1$. Then:*

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Gamma(f_0(\frac{-s-ct}{t^\rho}))\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} = 1\!\mathrm{l}.$$

Proof. We denote by $f^t_{R,\rho}$ the operator in (5.21) to emphasize the dependence on the exponent ρ. Using Lemma 5.9, Prop. 5.4, Prop. 4.5 and a density argument as in the proof of Thm. 5.5 *i)*, we deduce from Lemma 5.10 the existence of

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Gamma(f^t_{R,\rho})(\epsilon B_{1,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}},\ \forall\epsilon > 0. \tag{5.28}$$

By Prop. 5.7 and a density argument, we obtain the existence of

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Gamma(f^t_{R,\rho})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} =: \Gamma^+_{R,\rho},$$

and the fact that the limit (5.28) equals $\Gamma^+_{R,\rho}R^+_1(\epsilon)$.

Next we apply Prop. A.4 to obtain:

$$\mathrm{w}-\lim_{R\to+\infty} \Gamma^+_{R,\rho}R^+_1(\epsilon) = R^+_1(\epsilon). \tag{5.29}$$

In fact the integrability uniformly in R (condition (A.2)) follows from (5.27) and

$$\mathrm{w}-\lim_{R\to+\infty} \Gamma(f^t_{R,\rho}) = 1\!\mathrm{l},\ \forall t > 0$$

since $f^t_{R,\rho} \equiv 1$ in $\{s \geq -ct-\alpha_1 Rt^\rho\}$ and $\alpha_1 > 0$. Applying Prop. A.4 we obtain (5.29). Applying then Prop. 5.7 *iii)* we obtain

$$\mathrm{w}-\lim_{R\to+\infty} \Gamma^+_{R,\rho} = 1\!\mathrm{l}. \tag{5.30}$$

Let now ρ_1 with $\rho_1(1+\epsilon_0) > 1$ and $\rho > \rho_1$. We claim that

$$f^t_{1,\rho} \geq f^t_{R,\rho_1},\ \text{for } t \geq T_R. \tag{5.31}$$

In fact $\operatorname{supp} f^t_{R,\rho_1} \subset \{s \geq -ct-\alpha_2 Rt^{\rho_1}\}$ and $f^t_{1,\rho} \equiv 1$ in $\{s \geq -ct-\alpha_1 t^\rho\}$, so $f^t_{1,\rho} \equiv 1$ on $\operatorname{supp} f^t_{R,\rho_1}$ for $t \geq T_R$, since $0 < \alpha_1 < \alpha_2$ and $\rho > \rho_1$. By (5.31) $\Gamma^+_{R,\rho_1} \leq \Gamma^+_{1,\rho} \leq 1\!\mathrm{l}$, and hence $\Gamma^+_{1,\rho} = 1\!\mathrm{l}$ by (5.30).□

6 Asymptotic partition of unity

In this section we construct an asymptotic partition of unity on the spaces $\mathcal{H}^{\mathrm{e}+}_{\mathrm{c}}$ constructed in Sect. 5. This partition of unity allows to cut a state in $\mathcal{H}^{\mathrm{e}+}_{\mathrm{c}}$ into pieces having a definite number of particles in the region $\{s \geq ct\}$.

6.1 Asymptotic cutoffs

Let us fix two functions $f_0, f_\infty \in C^\infty(\mathbb{R})$ with $0 \le f_\epsilon \le 1$, $\epsilon = 0, \infty$ and

$$
\begin{aligned}
& f_0 \equiv 1 \text{ in } s \le \alpha_1,\ f_0 \equiv 0 \text{ in } s \ge \alpha_2, \\
& f_\infty \equiv 0 \text{ in } s \le \alpha_1,\ f_\infty \equiv 1 \text{ in } s \ge \alpha_2, \\
& f_0' \le 0,\ f_\infty' \ge 0.
\end{aligned}
\tag{6.1}
$$

Here the constants $\alpha_1 < \alpha_2$ are such that $\alpha_0 < \alpha_1 < \alpha_2$ where the constant α_0 is fixed in Sect. 5. We set $f = (f_0, f_\infty)$, $f^t = (f_0^t, f_\infty^t)$ for

$$
f_\epsilon^t := f_\epsilon\Big(\frac{s - \mathrm{c}t}{t^\rho}\Big), \tag{6.2}
$$

for constants $0 < \mathrm{c} \le 1$ and $0 < \rho < 1$.

We consider in this section the localization operators $P_k(f)$, $Q_k(f)$ defined in Subsect. 2.2.

We recall (see [DG2, Lemma 2.9])

$$
\|P_k(f)\| \le 1,\ \|Q_k(f)\| \le 1 \text{ if } f_0 + f_\infty \le 1.
$$

Using then the definition of $P_k(f), Q_k(f)$ we notice that

$$
\|P_k(f)\| \le \alpha^{-k},\ \|Q_k(f)\| \le \frac{1 - \alpha^{-k}}{1 - \alpha}, \tag{6.3}
$$

if

$$
f_0 + \alpha f_\infty \le 1,\ \alpha > 0. \tag{6.4}
$$

Indeed it suffices consider the new cutoffs $\tilde{f} = (f_0, \alpha f_\infty)$ and use that $P_k(f) = \alpha^{-k} P_k(\tilde{f})$.

In this section we will always assume that (f_0, f_∞) satisfy (6.4).

We recall the following identities ([DG2, Lemma 2.11]):

$$
\begin{aligned}
& \mathbf{D}_0 P_k(f^t) = \mathrm{d}P_k(f^t, \mathbf{d}_0 f^t), \\
& [\phi(v), iP_k(f^t)] \\
= \ & \tfrac{\mathrm{i}}{\sqrt{2}}\Big(a^*((1 - f_0^t)v)P_k(f^t) - a^*(f_\infty^t v)P_{k-1}(f^t) - P_k(f^t)a((1 - f_0^t)v) + P_{k-1}(f^t)a(f_\infty^t v)\Big).
\end{aligned}
\tag{6.5}
$$

Lemma 6.1 *Assume (I'0), (I'1) for $\epsilon_0 > 0$,(I'2) for $\mu > 0$. Assume the constants ρ, δ are chosen so that $\rho > \delta > (1 + \epsilon_0)^{-1}$, $\mu > \delta/2$. Then for $\chi_1, \chi_2, F \in C_0^\infty(\mathbb{R})$:*

$$
[P_k(f^t), \chi_1(L)]F\Big(\frac{N_t^{\mathrm{e}}}{t^\delta}\Big)\chi_2(L) \in o(1).
$$

Proof. By the argument above we may assume that $f_0 + f_\infty \le 1$. We have

$$
[P_k(f^t), \chi_1(L)] = \frac{\mathrm{i}}{2\pi}\int \frac{\partial \tilde{\chi_1}}{\partial \overline{z}}(z)(z - L)^{-1}[L, P_k(f^t)](z - L)^{-1}\mathrm{d}z \wedge \mathrm{d}\overline{z}. \tag{6.6}
$$

On $\mathcal{D}(N^{\mathrm{e}})\cap\mathcal{D}(L)$ we have:

$$[L,P_k(f^t)]=\mathrm{d}P_k(f^t,[|\sigma|,f^t])|+[\phi(v),P_k(f^t)].$$

By Lemma 5.2 we have

$$[|\sigma|,f^t_\epsilon]\in O(t^{-\rho}),\epsilon=0,\infty. \tag{6.7}$$

Applying then [DG2, Lemma 2.11], we get

$$\mathrm{d}P_k(f^t,[|\sigma|,f^t])\in O(N^{\mathrm{e}})t^{-\rho}.$$

Using then (6.5) and *(I'2)*, we obtain

$$\|(K+1)^{-\frac{1}{2}}[\phi(v^{\mathrm{e}}),P_k(f^t)](N^{\mathrm{e}}+1)^{-\frac{1}{2}}\|\in O(t^{-\mu}). \tag{6.8}$$

By Lemma 3.5 for $\alpha=1$ and a interpolation argument for $\alpha=\frac{1}{2}$ we obtain

$$\|(N^{\mathrm{e}}+1)^{\alpha}(z-L)^{-1}(N^{\mathrm{e}}+1)^{-\alpha}\|\leq C|\mathrm{Im}z|^{-2},\ \alpha=\frac{1}{2},1,z\in B\Subset\mathbb{C}\backslash\mathbb{R}.$$

Recall also from Lemma 4.6 that $(N^{\mathrm{e}})^{\alpha}F(\frac{N^{\mathrm{e}}_t}{t^\delta})\chi_2(L)\in O(t^{\delta\alpha})$. This yields

$$\begin{aligned}&\|(z-L)^{-1}[L,P_k(f^t)](z-L^{-1})F(\tfrac{N^{\mathrm{e}}_t}{t^\delta})\chi_2(L)\|\\ &\leq C(t^{-\rho+\delta}+t^{-\mu+\delta/2})|\mathrm{Im}z|^{-3}.\end{aligned}$$

Using (6.6) we obtain the lemma. □

The following lemma follows from Lemmas 4.4 and 6.1.

Lemma 6.2 *Assume (I'0), (I'1) for $\epsilon_0>0$, (I'2) for $\mu>0$. Assume the constants ρ,δ are chosen so that $\rho>\delta>(1+\epsilon_0)^{-1}$, $\mu>\delta/2$. Then for $\chi_1,\chi_2,F\in C_0^\infty(\mathbb{R})$:*

$$P_k(f^t)F(\frac{N^{\mathrm{e}}_t}{t^\delta})\chi_1(L)\chi_2(L)=\chi_1(L)P_k(f^t)F(\frac{N^{\mathrm{e}}_t}{t^\delta})\chi_2(L)+o(1).$$

We recall that the observable $B_{\mathrm{c}\,t}$ was defined in (5.2). For $f=(f_0,f_\infty),g\in\mathcal{B}(\mathfrak{h}^{\mathrm{e}})$ we define the operator $R_k(f,g)$ as

$$R_k(f,g)_{|\otimes_s^n\mathfrak{h}^{\mathrm{e}}}:=\sum_{j=1}^n\sum_{\sharp\{i|\epsilon_i=\infty\}=k}f_{\epsilon_1}\otimes\cdots\otimes f_{\epsilon_{j-1}}\otimes g\otimes f_{\epsilon_{j+1}}\otimes\cdots\otimes f_{\epsilon_n}. \tag{6.9}$$

If $f_0+\alpha f_\infty\leq 1$, we see as in [DG2, Lemma 2.11] that

$$|(v,R_k(f,g)u)|\leq\alpha^{-k}\|g\|_{\mathcal{B}(\mathfrak{h}^{\mathrm{e}})}\|(N^{\mathrm{e}})^{\frac{1}{2}}u\|\|(N^{\mathrm{e}})^{\frac{1}{2}}v\|.$$

Proposition 6.3 *Assume (I'0), (I'2) for $\mu>1$. Assume $0<\mathrm{c}<1$ or $\mathrm{c}=1$ and $\alpha_2<0$. For $\chi\in C_0^\infty(\mathbb{R}),\lambda>0$:*

$$i)\ \int_1^{+\infty}\|R_k(f^t,|g^t_\epsilon|)^{\frac{1}{2}}(B_{\mathrm{c}\,t}+\lambda)^{-\frac{1}{2}}\chi(L)u_t\|^2\mathrm{d}t\leq C\|u\|^2,\ u\in\mathcal{D}(N^{\mathrm{e}}),\epsilon=0,\infty,$$

if $g^t_\epsilon=\mathbf{d}_0f^t_\epsilon$.

$$ii)\ \mathrm{s}\text{-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\chi(L)P_k(f^t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}\ \textit{exists.}$$

Proof. Let

$$\Phi_k(t) = \chi(L)(B_{\mathrm{c}\,t} + \lambda)^{-1} P_k(f^t)\chi(L), \ \lambda > 0.$$

Note that $[P_k(f^t), B_{\mathrm{c}\,t}] = 0$. For $u \in \mathcal{D}(N^{\mathrm{e}}) \cap \mathcal{D}(L)$ the function $t \mapsto (u_t, \Phi_k(t)u_t)$ is C^1 with derivative $(u_t, \mathbf{D}\Phi_k(t)u_t)$ and

$$\begin{aligned} \mathbf{D}\Phi_k(t) = {} & \chi(L)\mathbf{D}(B_{\mathrm{c}\,t} + \lambda)^{-1} P_k(f^t)\chi(L) \\ & + \chi(L)(B_{\mathrm{c}\,t} + \lambda)^{-1} \Big(\mathrm{d}P_k(f^t, \mathbf{d}_0 f^t) + [\phi(v^{\mathrm{e}}), \mathrm{i}P_k(f^t)] \Big) \chi(L). \end{aligned} \tag{6.10}$$

We observe that $b_{\mathrm{c}\,t}$ defined in (5.2) is equal to 1 on $\mathrm{supp}\,(1 - f_0^t)$ and on $\mathrm{supp}\, f_\infty^t$. Using the fact that $B_{\mathrm{c}\,t}$ commutes with $P_k(f^t)$, we obtain

$$\begin{aligned} & \|\chi(L)(B_{\mathrm{c}\,t} + \lambda)^{-1} [\phi(v^{\mathrm{e}}), \mathrm{i}P_k(f^t)]\chi(L)\| \\ & \leq C\|(B_{\mathrm{c}\,t} + \lambda)^{-1} a^*((1 - f_0^t)(K+1)^{-\frac{1}{2}} v^{\mathrm{e}})\| + C\|(B_{\mathrm{c}\,t} + \lambda)^{-1} a((1 - f_0^t)(K+1)^{-\frac{1}{2}} v^{\mathrm{e}})\| \\ & + C(\|(B_{\mathrm{c}\,t} + \lambda)^{-1} a^*(f_\infty^t (K+1)^{-\frac{1}{2}} v^{\mathrm{e}})\| + \|\|(B_{\mathrm{c}\,t} + \lambda)^{-1} a(f_\infty^t (K+1)^{-\frac{1}{2}} v^{\mathrm{e}})\|). \end{aligned}$$

Applying Prop. A.1, we obtain

$$\begin{aligned} & \|\chi(L)(B_{\mathrm{c}\,t} + \lambda)^{-1} [\phi(v^{\mathrm{e}}), \mathrm{i}P_k(f^t)]\chi(L)\| \\ \leq {} & C\|(1 - f_0^t)(K+1)^{-\frac{1}{2}} v^{\mathrm{e}}\| + C\|f_\infty^t (K+1)^{-\frac{1}{2}} v^{\mathrm{e}}\| \\ \leq {} & Ct^{-\mu}, \end{aligned} \tag{6.11}$$

by *(I'2)*.

On the other hand we have on the $n-$particle sector:

$$\begin{aligned} \mathrm{d}P_k(f, g) = {} & \textstyle\sum_{j=1}^n \sum_{\sharp\{i|\epsilon_i = \infty\} = k} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_{j-1}} \otimes g_0 \otimes f_{\epsilon_{j+1}} \otimes \cdots \otimes f_{\epsilon_n} \\ & + \textstyle\sum_{j=1}^n \sum_{\sharp\{i|\epsilon_i = \infty\} = k-1} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_{j-1}} \otimes g_\infty \otimes f_{\epsilon_{j+1}} \otimes \cdots \otimes f_{\epsilon_n} \\ = {} & R_k(f, g_0) + R_{k-1}(f, g_\infty). \end{aligned}$$

Finally as in the proof of Prop. 5.1:

$$\mathbf{D}(B_{\mathrm{c}\,t} + \lambda)^{-1} = -(B_{\mathrm{c}\,t} + \lambda)^{-1} \mathrm{d}\Gamma(c_t)(B_{\mathrm{c}\,t} + \lambda)^{-1} + [\phi(v^{\mathrm{e}}), \mathrm{i}(B_{\mathrm{c}\,t} + \lambda)^{-1}],$$

and by (5.4)

$$\|\chi(L)[\phi(v^{\mathrm{e}}), \mathrm{i}(B_{\mathrm{c}\,t} + \lambda)^{-1}]\| \in O(t^{-\mu}). \tag{6.12}$$

Note also that

$$\begin{aligned} & (B_{\mathrm{c}\,t} + \lambda)^{-1} \mathrm{d}\Gamma(c_t) P_k(f^t)(B_{\mathrm{c}\,t} + \lambda)^{-1} \\ = {} & (B_{\mathrm{c}\,t} + \lambda)^{-1} \mathrm{d}\Gamma(c_t)^{\frac{1}{2}} P_k(f^t) \mathrm{d}\Gamma(c_t)^{\frac{1}{2}} (B_{\mathrm{c}\,t} + \lambda)^{-1}, \end{aligned} \tag{6.13}$$

since $P_k(f^t)$ commutes with $B_{\mathrm{c}\,t}$ and $\mathrm{d}\Gamma(c_t)$. Using (6.12), (6.13) and Prop. 5.1, we see that the first term on the r.h.s. of (6.10) is integrable along the evolution.

Let us now consider the second term. Assume first that $f_0 + f_\infty = 1$. Then $\mathbf{d}_0 f_0^t = -\mathbf{d}_0 f_\infty^t$, and hence

$$\mathrm{d}P_k(f^t, \mathbf{d}_0 f^t) = R_k(f^t, g_0^t) - R_{k-1}(f^t, g_0^t), \tag{6.14}$$

where we set $R_{-1}(f,g)=0$. Next

$$g_0^t = \mathbf{d}_0 f_0^t = f_0'(\frac{s-\mathrm{c}t}{t^\rho})\left(\frac{1-\mathrm{c}}{t^\rho} - \rho\frac{s-\mathrm{c}t}{t^{\rho+1}}\right).$$

If $0<\mathrm{c}<1$ or $\mathrm{c}=1$ and $\alpha_2<0$ we have $g_0^t \leq 0$ for $t\gg 1$. By (6.11), Prop. 5.1 and Prop. A.2, we obtain

$$\int_1^{+\infty} \|R_0(f^t,|g_0^t|)^{\frac{1}{2}}(B_{\mathrm{c}\,t}+\lambda)^{-\frac{1}{2}}\chi(L)u_t\|^2 \mathrm{d}t \leq C\|u\|^2,\ u\in\mathcal{D}(N^{\mathrm{e}}).$$

Using then (6.14) we obtain by induction on k:

$$\int_1^{+\infty} \|R_k(f^t,|g_0^t|)^{\frac{1}{2}}(B_{\mathrm{c}\,t}+\lambda)^{-\frac{1}{2}}\chi(L)u_t\|^2 \mathrm{d}t \leq C\|u\|^2,\ u\in\mathcal{D}(N^{\mathrm{e}}). \tag{6.15}$$

Let us now assume that $f_0+\alpha f_\infty \leq 1$. Introducing the cutoffs $\tilde{f}=(f_0,\alpha f_\infty)$, we may assume that $f_0+f_\infty\leq 1$. Since $f_0\leq(1-f_\infty)$, $f_\infty\leq 1-f_0$, we have:

$$\begin{aligned} &R_k(f^t,|g_0^t|)\leq R_k(l^t,|g_0^t|) \text{ for } l^t=(f_0^t,1-f_0^t),\\ &R_k(f^t,|g_\infty^t|)\leq R_k(l^t,|g_\infty^t|) \text{ for } l^t=(1-f_\infty^t,f_\infty^t). \end{aligned} \tag{6.16}$$

If we set $l^t=(f_0^t,1-f_0^t)$ then $g_0^t=\mathbf{d}_0 l_0^t$ and if we set $l^t=(1-f_\infty^t,f_\infty^t)$ then $g_\infty^t=-\mathbf{d}_0 l_0^t$.

Hence *i)* follows from (6.16) and the estimates (6.15) for the two choices of l^t above.

Property *ii)* follows from *i)* and Prop. A.3. □

Theorem 6.4 *Assume (I'0), (I'1) for $\epsilon_0>0$, (I'2) for $\mu>1$ and pick ρ in (5.2) such that $\rho(1+\epsilon_0)>1$. Fix $0<\mathrm{c}<1$ and $\mathrm{c}<\mathrm{c}'<1$. Let us denote $f^t=(f_0^t,f_\infty^t)$ defined in (6.2) by f_c^t to indicate the dependence on the constant* c. *Then*

i) the limit

$$P^+_{\mathrm{c}'k}(f_0,f_\infty):=\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}P_k(f^t_{\mathrm{c}'})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} \text{ exists on } \mathcal{H}_\mathrm{c}^{\mathrm{e}+},$$

ii) $[P^+_{\mathrm{c}'k}(f_0,f_\infty),H^{\mathrm{e}}]=0$,

iii) $[P^+_{\mathrm{c}'k}(f_0,f_\infty),L]=0$,

iv) if $f_0+f_\infty=1$ *then*

$$\text{s}-\sum_0^{+\infty}P^+_{\mathrm{c}'k}(f_0,f_\infty)=1\!\!1 \text{ on } \mathcal{H}_\mathrm{c}^{\mathrm{e}+}.$$

For $k=0$ the asymptotic cutoffs $P^+_{\mathrm{c}'k}(f)$ take a simpler form. In fact we have $P_0(f_0,f_\infty)=\Gamma(f_0)$. We denote $P^+_{\mathrm{c}'0}(f_0,f_\infty)$ by $\Gamma^{\mathrm{e}+}_{\mathrm{c}'}(f_0)$ and we have

$$\Gamma^{\mathrm{e}+}_{\mathrm{c}'}(f_0)=\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Gamma(f^t_{\mathrm{c}'0})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} \text{ on } \mathcal{H}_\mathrm{c}^{\mathrm{e}+},\ \text{for } 0<\mathrm{c}<\mathrm{c}'<1. \tag{6.17}$$

Proof. Let us first prove *i)*. By the definition of $\mathcal{H}_\mathrm{c}^{\mathrm{e}+}$ it suffices to prove the theorem on $\mathrm{Ran}\hat{P}^+_{\mathrm{c}'}$ for $\mathrm{c}<\mathrm{c}'$. Changing notation we may replace c' by c. By Thm. 5.5 we may restrict ourselves to vectors $u\in\mathrm{Ran}\hat{P}^+_\mathrm{c}$ such that $u=\chi(L)u$, $\chi\in C_0^\infty(\mathbb{R})$. Moreover for each $u\in\mathrm{Ran}\hat{P}^+_\mathrm{c}$ and $\epsilon_1>0$ there exists $\epsilon>0$ such that

$$\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u=(\epsilon B_{\mathrm{c}\,t}+1)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u+\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}r_\epsilon+o(1), \tag{6.18}$$

with $\|r_\epsilon\| \leq \epsilon_1$. We pick now $\delta > 0$ such that $\rho > \delta$, $\mu > \delta/2$ and $\delta(1+\epsilon_0) > 1$, which is possible since $\rho(1+\epsilon_0) > 1, \mu > 1$, and consider the observable $N_t^{\rm e}$ constructed in Subsect. 4.2. If $F \in C_0^\infty(\mathbb{R}), F \equiv 1$ near 0 we have:

$$
\begin{aligned}
P_k(f^t){\rm e}^{-{\rm i}tH^{\rm e}}u = \; & P_k(f^t)F(\tfrac{N_t^{\rm e}}{t^\delta})\chi^2(L){\rm e}^{-{\rm i}tH^{\rm e}}u + o(1) \\
= \; & \chi(L)P_k(f^t)\chi(L)F(\tfrac{N_t^{\rm e}}{t^\delta}){\rm e}^{-{\rm i}tH^{\rm e}}u + o(1) \\
= \; & \chi(L)P_k(f^t)\chi(L){\rm e}^{-{\rm i}tH^{\rm e}}u + o(1),
\end{aligned}
\tag{6.19}
$$

where we used successively Prop. 4.5, Lemma 4.4, Lemma 6.2 and Prop. 4.5 again. Next we write using (6.18):

$$
\begin{aligned}
& \chi(L)P_k(f^t)\chi(L){\rm e}^{-{\rm i}tH^{\rm e}}u \\
= \; & \chi(L)P_k(f^t){\rm e}^{-{\rm i}tH^{\rm e}}u \\
= \; & \chi(L)P_k(f^t)(\epsilon B_{{\rm c}\,t}+\lambda)^{-1}{\rm e}^{-{\rm i}tH^{\rm e}}u + \chi(L)P_k(f^t){\rm e}^{-{\rm i}tH^{\rm e}}r_\epsilon + o(1) \\
= \; & \chi(L)P_k(f^t)(\epsilon B_{{\rm c}\,t}+\lambda)^{-1}\chi(L){\rm e}^{-{\rm i}tH^{\rm e}}u + \chi(L)P_k(f^t){\rm e}^{-{\rm i}tH^{\rm e}}r_\epsilon + o(1).
\end{aligned}
$$

Hence to prove *i)* it suffices to prove the existence of

$$
\text{s-}\lim_{t\to+\infty} {\rm e}^{{\rm i}tH^{\rm e}}\chi(L)P_k(f^t)(\epsilon B_{{\rm c}\,t}+1)^{-1}\chi(L){\rm e}^{-{\rm i}tH^{\rm e}}u,
$$

which is shown in Prop. 6.3.

iii) follows from the same arguments as in (6.19). In fact using Lemma 6.2 we obtain that if $\chi(L)u = u$ then $\chi_1(L)P_k^+(f)u = P_k^+(f)\chi_1(L)u$, which proves *iii)*.

To prove *ii)*, it suffices to prove that

$$
{\rm e}^{{\rm i}t_1H^{\rm e}}P_k^+(f){\rm e}^{-{\rm i}tH^{\rm e}} = P_k^+(f), \; \forall t_1 \in \mathbb{R},
$$

or equivalently

$$
\text{s-}\lim_{t\to+\infty} {\rm e}^{{\rm i}tH^{\rm e}}(P_k(f^t) - P_k(f^{t-t_1})){\rm e}^{-{\rm i}tH^{\rm e}} = 0.
\tag{6.20}
$$

Using [DG2, Lemma 2.11], we have:

$$
P_k(f^t) - P_k(f^{t-t_1}) = -\int_0^{t_1} {\rm d}P_k(f^t, \partial_t f^{t-r}){\rm d}r.
$$

Since $\partial_t f^t \in O(t^{-\rho})$, we obtain

$$
(P_k(f^t) - P_k(f^{t-t_1})) \in O(N^{\rm e})t^{-\rho}.
\tag{6.21}
$$

For $u \in \mathcal{H}_{\rm c}^{{\rm e}+}$ with $u = \chi(L)u$, $\chi \in C_0^\infty(\mathbb{R})$, we have:

$$
\begin{aligned}
& (P_k(f^t) - P_k(f^{t-t_1})){\rm e}^{-{\rm i}tH^{\rm e}}u \\
= \; & (P_k(f^t) - P_k(f^{t-t_1}))F(\tfrac{N_t^{\rm e}}{t^\delta})\chi(L){\rm e}^{-{\rm i}tH^{\rm e}}u + o(1),
\end{aligned}
$$

by Prop. 4.5. Using then (6.21) and Lemma 4.6, we obtain

$$
(P_k(f^t) - P_k(f^{t-t_1}))F(\frac{N_t^{\rm e}}{t^\delta})\chi(L) \in O(t^{\delta-\rho}),
$$

which proves (6.20) since $\rho > \delta$.

Let us now prove *iv)*. We claim that if $f = (f_0, f_\infty)$, $f_0 + f_\infty \leq 1$ and $b \equiv 1$ on $\mathrm{supp}\, f_\infty$ then

$$\| \sum_m^\infty P_k(f)(\mathrm{d}\Gamma(b) + \lambda)^{-1} \| \leq \frac{1}{m+\lambda}. \tag{6.22}$$

In fact on the $n-$particle sector we have:

$$\begin{aligned} & \textstyle\sum_m^\infty P_k(f)(\mathrm{d}\Gamma(b) + \lambda)^{-1} \\ = \quad & \textstyle\sum_{m \leq \sharp\{i | \epsilon_i = \infty\} \leq n} f_{\epsilon_1} \otimes \cdots \otimes f_{\epsilon_n} (\sum_{i=1}^n b_i + \lambda)^{-1} \\ \leq \quad & (m+\lambda)^{-1}. \end{aligned}$$

Assume now that $f_0 + f_\infty = 1$. Since $\sum_0^m P_k^+(f) \leq \mathbb{1}$, to prove *iv)* it suffices by density to show that

$$\lim_{m \to \infty} \Big(\mathbb{1} - \sum_0^{m-1} P_k^+(f) \Big) \epsilon R_{\mathrm{c}}^{\mathrm{e}+}(\epsilon^{-1}) u = 0, \ \forall \epsilon > 0,$$

where $R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)$ is defined in Thm. 5.5. Now

$$\begin{aligned} & \| \Big(\mathbb{1} - \textstyle\sum_0^{m-1} P_k^+(f) \Big) \epsilon R_{\mathrm{c}}^{\mathrm{e}+}(\epsilon^{-1}) u \| \\ = \quad & \lim_{t \to +\infty} \| \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} \textstyle\sum_m^\infty P_k(f^t)(\epsilon B_{\mathrm{c}\,t} + 1)^{-1} \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u \| \\ \leq \quad & (\epsilon m + 1)^{-1} \|u\|, \end{aligned}$$

by (6.22). This proves *iv)*. $\square$

6.2 Weak limits

We will consider now for technical purposes a specific choice of the cutoffs f_0, f_∞. We set

$$g(t) := \begin{cases} e^{-(t-\alpha_1)^{-1}(\alpha_2 - t)^{-1}}, \ t \in]\alpha_1, \alpha_2[, \\ 0, \ t \notin]\alpha_1, \alpha_2[, \end{cases}$$

$$g_\epsilon(t) := \begin{cases} e^{-(t-\alpha_1-\epsilon)^{-1}(\alpha_2 - t)^{-1}}, \ t \in]\alpha_1 + \epsilon, \alpha_2[, \\ 0, \ t \notin]\alpha_1 + \epsilon, \alpha_2[, \end{cases}$$

for $0 < \epsilon < \frac{1}{2}(\alpha_2 - \alpha_1)$. Clearly $g_\epsilon \leq g$. Let

$$C = \int_{-\infty}^{+\infty} g(t)\mathrm{d}t, \ C_\epsilon = \int_{-\infty}^{+\infty} g_\epsilon(t)\mathrm{d}t,$$

and note that $\lim_{\epsilon \to 0} C_\epsilon = C$.

We set

$$f_\infty(s) = C^{-1} \int_{-\infty}^s g(s')\mathrm{d}s', \ f_{\infty,\epsilon}(s) = C_\epsilon^{-1} \int_{-\infty}^s g_\epsilon(s')\mathrm{d}s'. \tag{6.23}$$

Since $g_\epsilon \leq g$, we have

$$f_{\infty,\epsilon} \leq \frac{C}{C_\epsilon} f_\infty, \ f'_{\infty,\epsilon} \leq \frac{C}{C_\epsilon} f'_\infty,$$

and

$$f_\infty(s) \equiv 1 \text{ for } s \geq \alpha_2,\ f_\infty(s) \equiv 0 \text{ for } s \leq \alpha_1,$$

$$f_{\infty,\epsilon}(s) \equiv 1 \text{ for } s \geq \alpha_2,\ f_{\infty,\epsilon}(s) \equiv 0 \text{ for } s \leq \alpha_1 + \epsilon.$$

Note also that by [DG1, Lemma A.4.1], $f_\infty^{\frac{1}{2}}, f_{\infty,\epsilon}^{\frac{1}{2}} \in C^\infty(\mathbb{R})$. Next we set

$$f_0 := 1 - f_\infty, \tag{6.24}$$

and again by [DG1, Lemma A.4.1] $f_0^{\frac{1}{2}} \in C^\infty(\mathbb{R})$. The following lemma summarizes the properties of $f_0, f_{\infty,\epsilon}$.

Lemma 6.5

$$\begin{array}{l} i)\ f_0 + f_\infty = 1, \\ ii)\ \exists \alpha > 0,\ \forall \epsilon > 0\ f_0 + \alpha f_{\infty,\epsilon} \leq 1, \\ iii)\ \forall \epsilon > 0,\ \exists \alpha > 0\ f_0^{\frac{1}{2}} + \alpha f_{\infty,\epsilon}^{\frac{1}{2}} \leq 1. \end{array}$$

Proof. *i)* is obvious. *ii)* follows from the fact that $C_\epsilon f_{\infty,\epsilon} \leq C f_\infty$.

Since $f_0 \leq 1$ and $f_{\infty,\epsilon} \equiv 0$ in $\{s \leq \alpha_1 + \epsilon\}$, we have:

$$\forall \alpha > 0,\ f_0^{\frac{1}{2}}(s) + \alpha f_{\infty,\epsilon}^{\frac{1}{2}}(s) \leq 1 \text{ in } \{s \leq \alpha_1 + \epsilon\}.$$

So it suffices to verify that

$$\inf_{s > \alpha_1 + \epsilon} \frac{1 - f_0^{\frac{1}{2}}(s)}{f_{\infty,\epsilon}^{\frac{1}{2}}(s)} > 0,$$

or equivalently

$$\inf_{s > \alpha_1 + \epsilon} \frac{(1 - f_0^{\frac{1}{2}}(s))^2}{f_{\infty,\epsilon}(s)} > 0.$$

If $s > \alpha_1 + \epsilon$, $f_\infty(s) \geq r_\epsilon > 0$ hence

$$\frac{(1 - f_0^{\frac{1}{2}}(s))^2}{f_{\infty,\epsilon}(s)} \geq (1 - f_0^{\frac{1}{2}}(s))^2 \geq (1 - (1 - r_\epsilon)^{\frac{1}{2}})^2 > 0.$$

$\square$

Proposition 6.6 *Assume (I'0), (I'1) for* $\epsilon_0 > 0$*, (I'2) for* $\mu > 1$ *and pick* ρ *in (6.2) such that* $\rho(1 + \epsilon_0) > 1$*. Then for* $0 < \mathrm{c} < \mathrm{c}' < 1$

$$P^+_{\mathrm{c}'k}(f_0, f_\infty) = \mathrm{w-}\lim_{\epsilon \to 0} P^+_{\mathrm{c}'k}(f_{0, f_{\infty,\epsilon}}) \ \text{on}\ \mathcal{H}^{\mathrm{e}+}_{\mathrm{c}}.$$

Proof. As in the proof of Thm. 6.4 it suffices to prove the proposition on $\mathrm{Ran}\hat{P}_{\mathrm{c}'}^{\mathrm{e}+}$. Changing notation, we may replace c' by c. By density it suffices to prove that

$$\mathrm{w-}\lim_{\epsilon\to 0}\chi(L)P_{\mathrm{c}k}^{+}(f_0, f_{\infty,\epsilon})R_{\mathrm{c}}^{+}(\lambda)\chi(L) = \chi(L)P_{\mathrm{c}k}^{+}(f_0, f_{\infty})R_{\mathrm{c}}^{+}(\lambda)\chi(L),$$

for $\chi \in C_0^\infty(\mathbb{R}), \lambda > 0$. Let us omit the index c to simplify notation. We will apply Prop. A.4. To do this we need to estimate uniformly in ϵ the Heisenberg derivative of

$$\Phi_\epsilon(t) = \chi(L)P_k(f_0^t, f_{\infty,\epsilon}^t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L).$$

By (6.12) and (6.13) we have:

$$\begin{array}{rl}
& |(u_2, \chi(L)P_k(f_0^t, f_{\infty,\epsilon}^t)\mathbf{D}(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)u_1)| \\
\leq & Ct^{-\mu}\|P_k(f_0^t, f_{\infty,\epsilon}^t)\|\|u_1\|\|u_2\| \\
& +\|P_k(f_0^t, f_{\infty,\epsilon}^t)\|\|\mathrm{d}\Gamma(c_t)^{\frac12}(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)u_2\|\|\mathrm{d}\Gamma(c_t)^{\frac12}(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)u_1\|.
\end{array}$$

By Lemma 6.5 *ii)* we have $\|P_k(f_0^t, f_{\infty,\epsilon}^t)\| \leq \alpha^{-k}$ uniformly in ϵ.

Let us now consider the terms coming from $\mathbf{D}P_k(f_0^t, f_{\infty,\epsilon}^t)$. By (6.11):

$$\begin{array}{rl}
& \|\chi(L)(B_{\mathrm{c}\,t}+\lambda)^{-1}[\phi(v^{\mathrm{e}}), \mathrm{i}P_k(f_0^t, f_{\infty,\epsilon}^t)]\chi(L)\| \\
\leq & C\|(1-f_0^t)(K+1)^{-\frac12}v^{\mathrm{e}}\| + C\|f_{\infty,\epsilon}^t(K+1)^{-\frac12}v^{\mathrm{e}}\| \\
\leq & C\|(1-f_0^t)(K+1)^{-\frac12}v^{\mathrm{e}}\| + C\|f_{\infty}^t(K+1)^{-\frac12}v^{\mathrm{e}}\| \\
\leq & Ct^{-\mu}, \text{ uniformly in } 0<\epsilon<\frac12(\alpha_2-\alpha_1),
\end{array}$$

since $f_{\infty,\epsilon} \leq C_0 f_\infty$.

Finally

$$\mathbf{D}_0P_k(f_0^t, f_{\infty,\epsilon}^t) = R_k((f_0^t, f_{\infty,\epsilon}^t), \mathbf{d}_0 f_0^t) + R_{k-1}((f_0^t, f_{\infty,\epsilon}^t), \mathbf{d}_0 f_{\infty,\epsilon}^t).$$

Since

$$f_{\infty,\epsilon} \leq C_0 f_\infty,\ f'_{\infty,\epsilon} \leq C_0 f'_\infty,$$

uniformly for $0<\epsilon<\frac12(\alpha_2-\alpha_1)$, we have:

$$\begin{array}{l}
|R_k((f_0^t, f_{\infty,\epsilon}^t), \mathbf{d}_0 f_0^t)|^{\frac12} \leq C_0^{k/2}R_k((f_0^t, f_\infty^t), |\mathbf{d}_0 f_0^t|)^{\frac12}, \\
|R_{k-1}((f_0^t, f_{\infty,\epsilon}^t), \mathbf{d}_0 f_{\infty,\epsilon}^t)|^{\frac12} \leq C_0^{k/2}R_{k-1}((f_0^t, f_\infty^t), |\mathbf{d}_0 f_\infty^t|)^{\frac12}.
\end{array}$$

This yields

$$\begin{array}{rl}
& |(u_2, \chi(L)(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathbf{D}_0P_k(f_0^t, f_{\infty,\epsilon}^t)\chi(L)u_1)| \\
\leq & C_0^k\Big(\|R_1(t)\chi(L)u_1\|\|R_1(t)\chi(L)u_2\| + \|R_2(t)\chi(L)u_1\|\|R_2(t)\chi(L)u_2\|\Big),
\end{array}$$

for

$$\begin{array}{l}
R_1(t) = (B_{\mathrm{c}\,t}+\lambda)^{-\frac12}R_k((f_0^t, f_\infty^t), |\mathbf{d}_0 f_0^t|)^{\frac12}, \\
R_2(t) = (B_{\mathrm{c}\,t}+\lambda)^{-\frac12}R_{k-1}((f_0^t, f_\infty^t), |\mathbf{d}_0 f_\infty^t|)^{\frac12}.
\end{array}$$

By *(I'2)*, Prop. 5.1 and Prop. 6.3, hypothesis (A.2) of Prop. A.4 is satisfied. Hypothesis (A.1) is clearly satisfied since

$$\|P_k(f_0^t, f_{\infty,\epsilon}^t)\| \leq \alpha^{-k}, \text{ uniformly in } 0 < \epsilon < \frac{1}{2}(\alpha_2 - \alpha_1),$$

by Lemma 6.5 *i)*. Finally

$$\mathrm{w-}\lim_{\epsilon \to 0} P_k(f_0^t, f_{\infty,\epsilon}^t) = P_k(f_0^t, f_\infty^t), \ \forall t \geq 0,$$

and hence hypothesis (A.3) of Prop. A.4 is satisfied. Applying Prop. A.4 we obtain the proposition. □

7 Geometric inverse wave operators

This section is devoted to the construction of geometric inverse wave operators on the spaces $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$.

7.1 Extended objects

We first define so called *extended objects* which provide a convenient framework for scattering theory (see [DG2, Sect. 3.4]). Let

$$\begin{aligned}
&\mathcal{H}_{\mathrm{ext}}^{\mathrm{e}} := \mathcal{H}^{\mathrm{e}} \otimes \Gamma(\mathfrak{h}^{\mathrm{e}}),\\
&H_{\mathrm{ext}}^{\mathrm{e}} := H^{\mathrm{e}} \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\mathrm{e}})} + 1\!\!1_{\mathcal{H}^{\mathrm{e}}} \otimes \mathrm{d}\Gamma(\sigma), \text{ acting on } \mathcal{H}_{\mathrm{ext}}^{\mathrm{e}},\\
&L_{\mathrm{ext}} := L \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\mathrm{e}})} + 1\!\!1_{\mathcal{H}^{\mathrm{e}}} \otimes \mathrm{d}\Gamma(|\sigma|), \text{ acting on } \mathcal{H}_{\mathrm{ext}}^{\mathrm{e}}.
\end{aligned}$$

We set

$$N_0^{\mathrm{e}} := N^{\mathrm{e}} \otimes 1\!\!1, \ N_\infty^{\mathrm{e}} := 1\!\!1 \otimes N^{\mathrm{e}}, \ N_{\mathrm{ext}}^{\mathrm{e}} := N_0^{\mathrm{e}} + N_\infty^{\mathrm{e}}.$$

The interpretation of the tensor product $\mathcal{H}_{\mathrm{ext}}^{\mathrm{e}}$ is as follows: $\Gamma(\mathfrak{h}^{\mathrm{e}})$ contains the asymptotically free bosons while $\mathcal{H}^{\mathrm{e}}$ contains the atom and the bosons staying close to it.

We define also the *extended Heisenberg derivatives* (see [DG2, Sect. 3.4]):

$$\begin{aligned}
\check{\mathbf{d}}_0 f(t) \quad &:= \tfrac{\partial}{\partial t} f(t) + (\sigma \oplus \sigma)\mathrm{i} f(t) - \mathrm{i} f(t)\sigma,\\
&f(t) \in \mathcal{B}(\mathfrak{h}^{\mathrm{e}}, \mathfrak{h}^{\mathrm{e}} \oplus \mathfrak{h}^{\mathrm{e}}),\\
\check{\mathbf{D}}_0 F(t) \quad &:= \tfrac{\partial}{\partial t} F(t) + (\mathrm{d}\Gamma(\sigma) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(\sigma))\mathrm{i} F(t) - \mathrm{i} F(t)\mathrm{d}\Gamma(\sigma),\\
&F(t) \in \mathcal{B}(\Gamma(\mathfrak{h}^{\mathrm{e}}), \Gamma(\mathfrak{h}^{\mathrm{e}}) \otimes \Gamma(\mathfrak{h}^{\mathrm{e}})),\\
\check{\mathbf{D}} B(t) \quad &:= \tfrac{\partial}{\partial t} B(t) + H^{\mathrm{ext}}\mathrm{i} B(t) - \mathrm{i} B(t) H^{\mathrm{e}},\\
&B(t) \in \mathcal{B}(\mathcal{H}^{\mathrm{e}}, \mathcal{H}^{\mathrm{ext}}).
\end{aligned}$$

Note that with the notation in Subsect. 2.2 we have

$$\check{\mathbf{D}}_0 \mathrm{d}\Gamma(f) = \mathrm{d}\check{\Gamma}(\check{\mathbf{d}}_0 f).$$

In this section, we will use the operators $\check{\Gamma}(j)$ for the following choice of j. We pick cutoff functions j_0, j_∞ satisfying (6.1) and (6.4). We set $j^t = (j_0^t, j_\infty^t)$ for

$$j_\epsilon^t = j_\epsilon\Big(\frac{s - \mathrm{c}t}{t^\rho}\Big), \ 0 < \mathrm{c} \leq 1, \ 0 < \rho < 1, \ \epsilon = 0, \infty. \tag{7.1}$$

7.2 Technical estimates

Lemma 7.1 *Assume (I'0), (I'1) for* $\epsilon_0 > 0$*, (I'2) for* $\mu > 0$ *and let* $\rho > \delta > (1+\epsilon_0)^{-1}$, $\mu > \delta/2$. *Then for* $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R})$*:*

$$\Big(\chi_1(L_{\rm ext})\check{\Gamma}_k(j^t) - \check{\Gamma}_k(j^t)\chi_1(L)\Big)F(\frac{N_t^{\rm e}}{t^\delta})\chi_2(L) \in o(1).$$

Proof. considering $\tilde{j} = (j_0, \alpha j_\infty)$ and noting that $\check{\Gamma}_k(j^t) = \alpha^{-k}\check{\Gamma}_k(\tilde{j}^t)$, we may assume that $j_0 + j_\infty \leq 1$ and hence $j_0^2 + j_\infty^2 \leq 1$. Since $\check{\Gamma}_k(j^t) = 1\!\!1_{\{k\}}(N_\infty^{\rm e})\check{\Gamma}(j^t)$ and $N_\infty^{\rm e}$ commutes with $L_{\rm ext}$, it suffices to prove the lemma for $\check{\Gamma}(j^t)$. We write

$$\begin{array}{rl} & \Big(\chi_1(L_{\rm ext})\check{\Gamma}(j^t) - \check{\Gamma}(j^t)\chi_1(L)\Big)F(\frac{N_t^{\rm e}}{t^\delta})\chi_2(L) \\ = & \frac{i}{2\pi}\int \frac{\partial\tilde{\chi}_1}{\partial\bar{z}}(z)(z-L_{\rm ext})^{-1}(L_{\rm ext}\check{\Gamma}(j^t) - \check{\Gamma}(j^t)L)(z-L)^{-1}F(\frac{N_t^{\rm e}}{t^\delta})\chi_2(L){\rm d}z\wedge{\rm d}\bar{z}. \end{array}$$

On $\mathcal{D}(L)$ we have:

$$L = K\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})} + 1\!\!1_{\mathcal{K}}\otimes{\rm d}\Gamma(|\sigma|) + \phi(v^{\rm e}),$$

and on $\mathcal{D}(L_{\rm ext})$

$$L_{\rm ext} = K\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})} + 1\!\!1_{\mathcal{K}}\otimes{\rm d}\Gamma(|\sigma|)\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})} + \phi(v^{\rm e})\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})} + 1\!\!1_{\mathcal{K}}\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\otimes{\rm d}\Gamma(|\sigma|).$$

By [DG2, Lemma 2.14]:

$$\begin{array}{rl} & \phi(v^{\rm e})\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\check{\Gamma}(j^t) - \check{\Gamma}(j^t)\phi(v^{\rm e}) \\ = & \frac{1}{\sqrt{2}}\Big((a^*((1-j_0^t)v^{\rm e})\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})} - 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\hat{\otimes}a^*(j_\infty^t v^{\rm e}))\check{\Gamma}(j^t) - \check{\Gamma}(j^t)a((1-j_0^t)v^{\rm e})\Big), \end{array} \tag{7.2}$$

where the twisted tensor product $\hat{\otimes}$ is defined as follows: let $T : \mathcal{K}\otimes\Gamma(\mathfrak{h}^{\rm e})\otimes\Gamma(\mathfrak{h}^{\rm e}) \to \Gamma(\mathfrak{h}^{\rm e})\otimes\mathcal{K}\otimes\Gamma(\mathfrak{h}^{\rm e})$ be the unitary operator defined by

$$T\psi\otimes u_1\otimes u_2 = u_1\otimes\psi\otimes u_2.$$

Then if B is an operator on $\mathcal{K}\otimes\Gamma(\mathfrak{h}^{\rm e})$, we set

$$1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\hat{\otimes}B := T^{-1}1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\otimes BT.$$

By [DG2, Lemma 2.16]:

$$L_0^{\rm ext}\check{\Gamma}(j^t) - \check{\Gamma}(j^t)L_0 = {\rm d}\check{\Gamma}(j^t, k^t),$$

for $k^t = (k_0^t, k_\infty^t)$, $k_\epsilon^t = [|\sigma|, j_\epsilon^t]$. By Lemma 5.2, $k_\epsilon^t \in O(t^{-\rho})$ and by [DG2, Lemma 2.16] we have:

$$(L_0^{\rm ext}\check{\Gamma}(j^t) - \check{\Gamma}(j^t)L_0) \in O(N^{\rm e})t^{-\rho}. \tag{7.3}$$

Using then (7.2), Prop. A.1 and hypothesis *(I'2)*, we have:

$$\begin{array}{rl} & \|(L_{\rm ext}+{\rm i})^{-1}(\phi(v^{\rm e})\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\check{\Gamma}(j^t) - \check{\Gamma}(j^t)\phi(v^{\rm e}))(N+1)^{-\frac{1}{2}}\| \\ \leq & C\|(1-j_0^t)(K+1)^{-\frac{1}{2}}v^{\rm e}\| + C\|j_\infty^t(K+1)^{-\frac{1}{2}}v^{\rm e}\| \\ \leq & Ct^{-\mu}. \end{array} \tag{7.4}$$

Now using (5.8) and Lemma 4.6 we obtain

$$\begin{aligned}&(z-L_{\rm ext})^{-1}(L_{\rm ext}\check{\Gamma}(j^t)-\check{\Gamma}(j^t)L)(z-L)^{-1}F(\tfrac{N_t^{\rm e}}{t^\delta})\chi_2(L)\\ \leq\ & C(t^{\delta-\rho}+t^{-\mu+\delta/2})|{\rm Im} z|^{-4},\ z\in{\rm supp}\,\tilde{\chi_1}.\end{aligned}$$

This implies the lemma. □

Lemma 7.2 *Assume (I'0), (I'1) for $\epsilon_0>0$, (I'2) for $\mu>0$ and $\rho>\delta>(1+\epsilon_0)^{-1}$, $\mu>\delta/2$. Then for $\chi_1,\chi_2\in C_0^\infty(\mathbb{R})$:*

$$\check{\Gamma}_k(j^t)F(\frac{N_t^{\rm e}}{t^\delta})\chi_1(L)\chi_2(L)-\chi_1(L_{\rm ext})\check{\Gamma}_k(j^t)F(\frac{N_t^{\rm e}}{t^\delta})\chi_2(L)\in o(1).$$

Proof. we combine Lemma 7.1 and Lemma 4.4. □

In the following lemma we use the operators $R_k(f,g)$ introduced in (6.9).

Lemma 7.3 *Assume $j_0+\alpha j_\infty\leq 1$. Let $r_\epsilon^t={\bf d}_0 j_\epsilon^t$, $\epsilon=0,\infty$. Then for $u\in\mathcal{H}^{\rm e}$, $v\in\mathcal{H}^{\rm e}_{\rm ext}$:*

$$\begin{aligned}|(v,\check{\bf D}_0\check{\Gamma}_k(j^t)u)|\leq\ &\Big((u,R_k(j^t,|r_0^t|)u)+(u,R_{k-1}(j^t,|r_\infty^t|)u)\Big)^{\frac12}\\ &\times\Big(\alpha^{-k}(v,R_0(j^t,|r_0^t|)\otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}v)+(v,1\!\!1_{\mathcal{K}\otimes\Gamma(\mathfrak{h}^{\rm e})}\otimes R_{k-1}(j^t,|r_\infty^t|)v)\Big)^{\frac12}.\end{aligned}$$

Proof. To lighten notation we will suppress the exponent t in $j_\epsilon^t, r_\epsilon^t$. On the $n-$particle sector, we have (see Subsect. 2.2):

$$\check{\Gamma}_k(j)=I_k\begin{pmatrix}k\\ n\end{pmatrix}^{\frac12}\underbrace{j_0\otimes\cdots\otimes j_0}_{n-k}\otimes\underbrace{j_\infty\otimes\cdots\otimes j_\infty}_{k},$$

so

$$\begin{aligned}\check{\bf D}_0\check{\Gamma}_k(j)=\ &I_k\begin{pmatrix}k\\ n\end{pmatrix}^{\frac12}\textstyle\sum_{i=1}^{n-k}j_0\otimes\cdots\otimes\underset{i}{\underset{\smile}{r_0}}\otimes\cdots\otimes j_0\otimes j_\infty\otimes\cdots\otimes j_\infty\\ &+I_k\begin{pmatrix}k\\ n\end{pmatrix}^{\frac12}\textstyle\sum_{i=n-k+1}^{n}j_0\otimes\cdots\otimes j_0\otimes j_\infty\otimes\cdots\otimes\underset{i}{\underset{\smile}{r_\infty}}\otimes\cdots\otimes j_\infty.\end{aligned}$$

Let

$$R=\begin{pmatrix}k\\ n\end{pmatrix}\sum_{i=1}^n(u,j_0\otimes\cdots\otimes\underset{i}{\underset{\smile}{r}}\otimes\cdots\otimes j_\infty u),\ u\in\otimes_{\rm s}^n\mathfrak{h}^{\rm e},$$

where $r_i=r_0$ if $i\leq n-k$, $r_i=r_\infty$ if $i>n-k$. We claim that

(7.5) $$R=(u,{\rm d}P_k(j,r)u).$$

In fact

$$R=\frac{\rm d}{{\rm d}x}I(x)\Big|_{x=0},$$

for

$$I(x) = \begin{pmatrix} k \\ n \end{pmatrix} \sum_{i=1}^{n} (u, \underbrace{j_0(x) \otimes \cdots j_0(x)}_{n-k} \otimes \underbrace{j_\infty(x) \otimes \cdots \otimes j_\infty(x)}_{k} u),$$

and $j_\epsilon(x) = j_\epsilon + x r_\epsilon$. Since $u \in \otimes_{\mathrm{s}}^n \mathfrak{h}^{\mathrm{e}}$ this equals $P_k(j(x))$ (this identity does not hold if u is not symmetric w.r.t. permutations). Hence (7.5) follows from [DG2, Lemma 2.11].

We now write

$$I_k j_0 \otimes \cdots \underset{i}{\smile}{r} \otimes \cdots j_\infty \text{ as } A_i B_i A_i,$$

for

$$A_i = j_0^{\frac{1}{2}} \otimes \cdots \underbrace{|r|^{\frac{1}{2}}}_{i} \otimes \cdots j_\infty^{\frac{1}{2}},$$
$$B_i = I_k \mathbb{1} \otimes \cdots \underbrace{\mathrm{sign} r}_{i} \otimes \cdots \mathbb{1}.$$

Note that $\|B_i\| \leq 1$. We have

$$\begin{array}{rl} & |(v, \check{\mathbf{D}}_0 \check{\Gamma}_k(j) u)| \\ = & \begin{pmatrix} k \\ n \end{pmatrix}^{\frac{1}{2}} |\sum_{i=1}^n (v, A_i B_i A_i u)| \\ \leq & \begin{pmatrix} k \\ n \end{pmatrix}^{\frac{1}{2}} \sum_{i=1}^n \|A_i v\| \|A_i u\| \\ \leq & \left(\begin{pmatrix} k \\ n \end{pmatrix} \sum_{i=1}^n \|A_i u\|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^n \|A_i v\|^2 \right)^{\frac{1}{2}}. \end{array}$$

By the identity (7.5) we have:

$$\begin{array}{rl} & \begin{pmatrix} k \\ n \end{pmatrix} \sum_{i=1}^n \|A_i u\|^2 \\ = & (u, \mathrm{d}P_k(j, |r|) u) \\ = & (u, R_k(j, |r_0|) u) + (u, R_{k-1}(j, |r_\infty|) u), \end{array}$$

where $R_k(f, g)$ is defined in (6.9). On the other hand

$$\begin{array}{rl} & \sum_{i=1}^n \|A_i v\|^2 \\ = & (v, \sum_{i=1}^{n-k} j_0 \otimes \cdots \otimes \underbrace{|r_0|}_{i} \otimes \cdots \otimes j_0 \otimes j_\infty \otimes \cdots \otimes j_\infty v) \\ & +(v, \sum_{i=n-k+1}^n j_0 \otimes \cdots \otimes j_0 \otimes j_\infty \otimes \cdots \otimes \underbrace{|r_\infty|}_{i} \otimes \cdots \otimes j_\infty v) \\ \leq & \alpha^{-k} (v, R_0(j, |r_0|) \otimes \mathbb{1}_{\otimes^k \mathfrak{h}^{\mathrm{e}}} v) + (v, \mathbb{1}_{\otimes^{n-k} \mathfrak{h}^{\mathrm{e}}} \otimes R_{k-1}(j, |r_\infty|) v), \end{array}$$

using the fact that $j_\infty \leq \alpha^{-1}$. This proves the lemma for $u \in \mathcal{K} \otimes \bigotimes_{\mathrm{s}}^n \mathfrak{h}^{\mathrm{e}}$ and $v \in \mathcal{K} \otimes \bigotimes_{\mathrm{s}}^{n-k} \mathfrak{h}^{\mathrm{e}} \otimes \bigotimes_{\mathrm{s}}^k \mathfrak{h}^{\mathrm{e}}$. To prove the lemma for arbitrary $u \in \mathcal{H}^{\mathrm{e}}, v \in \mathcal{H}^{\mathrm{ext}}$ we set

$$\Pi_n = \mathbb{1}_{\{n\}}(N^{\mathrm{e}}), \; \Pi_n^{\mathrm{ext}} = \mathbb{1}_{\{n\}}(N_{\mathrm{ext}}^{\mathrm{e}}),$$

and note that

$$\check{\mathbf{D}}_0\check{\Gamma}_k(j)\Pi_n = \Pi_n^{\mathrm{ext}}\check{\mathbf{D}}\check{\Gamma}_k(j).$$

The estimate for arbitrary u, v follows from the estimate for $\Pi_n u, \Pi_n^{\mathrm{ext}} v$ and the Cauchy-Schwarz inequality. □

7.3 Number of asymptotically free particles

We start with an extension of the results in Sect. 5 to $\mathcal{H}^{\mathrm{e}}_{\mathrm{ext}}$. We set

$$B^{\mathrm{ext}}_{\mathrm{c}\,t} = B_{\mathrm{c}\,t} \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\mathrm{e}})} + 1\!\!1_{\mathcal{H}^{\mathrm{e}}} \otimes B_{\mathrm{c}\,t}, \text{ acting on } \mathcal{H}^{\mathrm{e}}_{\mathrm{ext}},$$

where $B_{\mathrm{c}\,t}$ is defined in Sect. 5. By exactly the same arguments as in Sect. 5, we obtain

Proposition 7.4 *Assume (I'0), (I'2) for $\mu > 1$. Assume that $0 < \mathrm{c} < 1$ or that $\mathrm{c} = 1$ and $\alpha_1 < 0$. Then for $\chi \in C_0^\infty(\mathbb{R})$:*

$$\int_1^{+\infty} \|(\mathrm{d}\Gamma(\mathbf{d}_0 b_t) \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\mathrm{e}})} + 1\!\!1_{\mathcal{H}^{\mathrm{e}}} \otimes \mathrm{d}\Gamma(\mathbf{d}_0 b_t))^{\frac{1}{2}} (B^{\mathrm{ext}}_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L_{\mathrm{ext}}) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}} u\|^2 \frac{dt}{t} \le C\|u\|^2,$$

for $u \in \mathcal{D}(N^{\mathrm{e}}_{\mathrm{ext}})$, $\lambda > 0$.

Theorem 7.5 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 1$ and pick ρ in (5.2) such that $\rho(1+\epsilon_0) > 1$. Then:*

i) for each $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$ the limit

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}} (B^{\mathrm{ext}}_{\mathrm{c}\,t} + \lambda)^{-1} \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}} =: R^+_{\mathrm{c\,ext}}(\lambda) \text{ exists.}$$

ii) $[R^+_{\mathrm{c\,ext}}(\lambda), L_{\mathrm{ext}}] = [R^+_{\mathrm{c\,ext}}(\lambda), H^{\mathrm{e}}_{\mathrm{ext}}] = 0$.

iii) the limit

$$\text{s-}\lim_{\epsilon\to 0} \epsilon^{-1} R^+_{\mathrm{c\,ext}}(\epsilon^{-1}) =: \hat{P}^+_{\mathrm{c\,ext}} \text{ exists}$$

and is an orthogonal projection.

iv)

$$[H^{\mathrm{e}}_{\mathrm{ext}}, \hat{P}^+_{\mathrm{c\,ext}}] = [L_{\mathrm{ext}}, \hat{P}^+_{\mathrm{c\,ext}}] = 0,$$

$$u = \hat{P}^+_{\mathrm{c\,ext}} u \Leftrightarrow \text{s-}\lim_{\epsilon\to 0} \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}} (B^{\mathrm{ext}}_{\mathrm{c}\,t} + \lambda)^{-1} \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}} u = u.$$

Theorem 7.6 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 1$ and pick ρ in (5.2) such that $\rho(1+\epsilon_0) > 1$. Let for $0 < \mathrm{c} < 1$:*

$$P^+_{\mathrm{c\,ext}} := \inf_{\mathrm{c}<\mathrm{c}'} \hat{P}^+_{\mathrm{c}'\mathrm{ext}},\ \mathcal{H}^{\mathrm{e}+}_{\mathrm{c\,ext}} := \mathrm{Ran} P^+_{\mathrm{c\,ext}}.$$

Then:

i) $P^+_{\mathrm{c\,ext}}$ is an orthogonal projection independent on the choice of the function f in (5.2).

$$ii)\ [H^{\mathrm{e}}_{\mathrm{ext}}, P^+_{\mathrm{c\,ext}}] = [L_{\mathrm{ext}}, P^+_{\mathrm{c\,ext}}] = 0.$$

iii) $\mathcal{H}^{\mathrm{e}+}_{\mathrm{c\,ext}} = \mathcal{H}^{\mathrm{e}+}_{\mathrm{c}} \otimes \Gamma(\mathfrak{h}^{\mathrm{e}})$.

Proof. parts *i)* and *ii)* can be shown exactly as in Thm. 5.6. To prove *iii)* we have to show that for $0 < c \leq 1$

$$\hat{P}^+_{\rm c\,ext} = \hat{P}^{\rm e+}_{\rm c} \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})},$$

which means

$$\text{s-}\lim_{\epsilon\to 0} \epsilon^{-1} R^+_{\rm c\,ext}(\epsilon^{-1}) = \text{s-}\lim_{\epsilon\to 0} \epsilon^{-1} R^{\rm e+}_{\rm c}(\epsilon^{-1}) \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}. \tag{7.6}$$

We note that

$$\|\Big((\epsilon B^{\rm ext}_{{\rm c}\,t} + 1\!\!1)^{-1} - (\epsilon B_{{\rm c}\,t} + 1\!\!1)^{-1} \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\Big) 1\!\!1_{\mathcal{H}^{\rm e}} \otimes (N^{\rm e}+1)^{-1}\| \leq C\epsilon.$$

Since $1\!\!1 \otimes N^{\rm e}$ commutes with $H^{\rm e}_{\rm ext}$, we obtain

$$\|\Big(\epsilon^{-1} R^{\rm e+}_{\rm c\,ext}(\epsilon^{-1}) - \epsilon^{-1} R^{\rm e+}_{\rm c}(\epsilon^{-1}) \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})}\Big) 1\!\!1_{\mathcal{H}^{\rm e}} \otimes (N^{\rm e}+1)^{-1}\| \leq C\epsilon.$$

This proves (7.6) by a density argument. □

7.4 Geometric inverse wave operators

Theorem 7.7 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 1$ and pick ρ in (5.2) such that $\rho(1+\epsilon_0) > 1$. Fix $0 < {\rm c} < 1$ and ${\rm c} < {\rm c}' < 1$. Let $j^t = (j^t_0, j^t_\infty)$ be constructed as in (7.1) with the constant ${\rm c}'$. Then:*

i) the limit

$$W^+_k(j) := \text{s-}\lim_{t\to+\infty} {\rm e}^{{\rm i}tH^{\rm e}_{\rm ext}} \check{\Gamma}_k(j^t) {\rm e}^{-{\rm i}tH^{\rm e}} \textit{ exists on } \mathcal{H}^{\rm e+}_{\rm c};$$

ii) for $\chi \in C^\infty_0(\mathbb{R})$

$$\chi(L_{\rm ext}) W^+_k(j) = W^+_k(j)\chi(L);$$

iii) for $\chi \in C^\infty_0(\mathbb{R})$

$$\chi(H^{\rm e}_{\rm ext}) W^+_k(j) = W^+_k(j)\chi(H^{\rm e});$$

iv) let f_0 as in (6.1) with $f_0 j_0 = j_0$. Then

$$W^+_k(j) = \Gamma^{\rm e+}_{{\rm c}'}(f_0) \otimes 1\!\!1_{\Gamma(\mathfrak{h}^{\rm e})} W^+_k(j);$$

v) for all ${\rm c}'' > {\rm c}$, $\lambda > 0$ we have:

$$R^+_{{\rm c}''\,{\rm ext}}(\lambda) W^+_k(j) = W^+_k(j) R^{\rm e+}_{{\rm c}''}(\lambda);$$

vi) $W^+_k(j)\mathcal{H}^{\rm e+}_{\rm c} \subset \mathcal{H}^{\rm e+}_{\rm c\,ext}$;

vii) the limit

$$\text{s-}\lim_{t\to+\infty} {\rm e}^{{\rm i}tH^{\rm e}} \check{\Gamma}_k(j^t)^* {\rm e}^{-{\rm i}tH^{\rm e}_{\rm ext}} \textit{ exists on } \mathcal{H}^{\rm e+}_{\rm c\,ext}$$

and equals $W^+_k(j)^*$.

Proof. Let us first prove *i)*. Note first that since $j_0^2 + \alpha j_\infty^2 \leq 1$ we have $\|\check{\Gamma}_k(j^t)\| \leq \alpha^{-k}$ and hence $\check{\Gamma}_k(j^t)$ is uniformly bounded in t.

By the definition of $\mathcal{H}^{\rm e+}_{\rm c}$ it suffices to show the existence of the limit on ${\rm Ran}\hat{P}^{\rm e+}_{{\rm c}'}$. Changing notation we may replace ${\rm c}'$ by ${\rm c}$. By Thm. 5.5 we may restrict ourselves to vectors $u \in {\rm Ran}\hat{P}^+_{\rm c}$

such that $u = \chi(L)u$, $\chi \in C_0^\infty(\mathbb{R})$. Arguing as in the proof of Thm. 6.4, it suffices to show the existence of

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}\check{\Gamma}_k(j^t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u,$$

for $\lambda > 0$. We pick now $\delta > 0$ such that $\rho > \delta$, $\mu > \delta/2$ and $\delta(1+\epsilon_0) > 1$, and consider the observable N_t^{e} constructed in Subsect. 4.2. If $F \in C_0^\infty(\mathbb{R}), F \equiv 1$ near 0 we have by Prop. 4.5:

$$\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u = \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}\chi^2(L)R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u = F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi^2(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}R_{\mathrm{c}}^{+}(\lambda)u + o(1).$$

Using again Lemmas 7.2, 4.4 and Prop. 4.5 we have:

$$\begin{array}{rl}
& \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}\check{\Gamma}_k(j^t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u \\
= & \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}\chi(L_{\mathrm{ext}})\check{\Gamma}_k(j^t)F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u + o(1) \\
= & \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}\chi(L_{\mathrm{ext}})\check{\Gamma}_k(j^t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u + o(1) \\
= & \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}\chi(L_{\mathrm{ext}})\check{\Gamma}_k(j^t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u + o(1),
\end{array} \tag{7.7}$$

where in the last step we used the definition of $R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)$.

For $u_1 \in \mathcal{D}(L)\cap\mathcal{D}(N^{\mathrm{e}})$, $u_2 \in \mathcal{D}(L_{\mathrm{ext}})\cap\mathcal{D}(N^{\mathrm{e}}_{\mathrm{ext}})$ the function

$$\mathbb{R}^+ \ni t \mapsto (\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}u_2, \chi(L_{\mathrm{ext}})\check{\Gamma}_k(j^t)\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u_1)$$

is C^1 with derivative:

$$\begin{array}{rl}
& (\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}\chi(L_{\mathrm{ext}})u_2, \check{\mathbf{D}}\check{\Gamma}_k(j^t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u_1) \\
+ & (\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}\chi(L_{\mathrm{ext}})u_2, \check{\Gamma}_k(j^t)\mathbf{D}(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u_1) \\
=: & I_1(t) + I_2(t).
\end{array}$$

Let us first estimate $I_2(t)$. As in Prop. 5.1 we have:

$$\mathbf{D}(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L) = -(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{d}\Gamma(c_t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L) + O(t^{-\mu}).$$

From the expression of $\check{\Gamma}_k(j^t)$ in Subsect. 2.2 and the fact that j^t commutes with b_t and c_t we see that

$$\begin{array}{rl}
& \check{\Gamma}_k(j^t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{d}\Gamma(c_t)(B_{\mathrm{c}\,t}+\lambda)^{-1} \\
= & (B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-1}(\mathrm{d}\Gamma(c_t)\otimes 1\!\!1 + 1\!\!1\otimes\mathrm{d}\Gamma(c_t))^{\frac{1}{2}}\check{\Gamma}_k(j^t)\mathrm{d}\Gamma(c_t)^{\frac{1}{2}}(B_{\mathrm{c}\,t}+\lambda)^{-1}.
\end{array}$$

Hence

$$\begin{array}{rl}
|I_2(t)| \leq & \|\check{\Gamma}_k(j^t)\|\|(\mathrm{d}\Gamma(c_t)\otimes 1\!\!1 + 1\!\!1\otimes\mathrm{d}\Gamma(c_t))^{\frac{1}{2}}(B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-1}\chi(L_{\mathrm{ext}})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}_{\mathrm{ext}}}u_2\| \\
& \times\|\mathrm{d}\Gamma(c_t)^{\frac{1}{2}}(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u_1\| \\
& +Ct^{-\mu}\|u_1\|\|u_2\|.
\end{array} \tag{7.8}$$

Let us now estimate $I_1(t)$. We have:

$$\begin{aligned}
& \check{\mathbf{D}}\check{\Gamma}_k(j^t) \\
= \;& \check{\mathbf{D}}_0\check{\Gamma}_k(j^t) + \mathrm{i}\phi(v^{\mathrm{e}}) \otimes 1\!\!1\check{\Gamma}_k(j^t) - \mathrm{i}\check{\Gamma}_k(j^t)\phi(v^{\mathrm{e}}) \\
=: \;& \check{\mathbf{D}}_0\check{\Gamma}_k(j^t) + C_1(t).
\end{aligned}$$

We use the identity (7.2) and the fact that

$$\check{\Gamma}_k(j^t)(B_{\mathrm{c}\,t}+\lambda)^{-1} = (B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-1}\check{\Gamma}_k(j^t),$$

to obtain

$$\begin{aligned}
\|\chi(L_{\mathrm{ext}})C_1(t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)\| \leq \;& C\|a^*((1-j_0^t)v^{\mathrm{e}}(K+1)^{-\frac{1}{2}}) \otimes 1\!\!1(B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-1}\| \\
& +\|a^*(j_\infty^t v^{\mathrm{e}}(K+1)^{-\frac{1}{2}}) \otimes 1\!\!1(B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-1}\| \\
& +C\|a((1-j_0^t)v^{\mathrm{e}}(K+1)^{-\frac{1}{2}})(B_{\mathrm{c}\,t}+\lambda)^{-1}\|.
\end{aligned}$$

Since by (6.1) and (5.1) $b_{\mathrm{c}\,t} \equiv 1$ on $\mathrm{supp}\,(1-j_0^t)$ and on $\mathrm{supp}\, j_\infty^t$, we obtain by Prop. A.1 that

$$\begin{aligned}
& \|\chi(L_{\mathrm{ext}})C_1(t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)\| \\
\leq \;& C(\|(1-j_0^t)v^{\mathrm{e}}(K+1)^{-\frac{1}{2}}\| + \|j_\infty^t v^{\mathrm{e}}(K+1)^{-\frac{1}{2}}\|) \\
\leq \;& Ct^{-\mu},
\end{aligned} \tag{7.9}$$

by *(I'2)*. Next we apply Lemma 7.3 and the fact that

$$\check{\mathbf{D}}_0\check{\Gamma}_k(j^t)(B_{\mathrm{c}\,t}+\lambda)^{-1} = (B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-\frac{1}{2}}\check{\mathbf{D}}_0\check{\Gamma}_k(j^t)(B_{\mathrm{c}\,t}+\lambda)^{-\frac{1}{2}}$$

to obtain

$$\begin{aligned}
& |(u_2, \chi(L_{\mathrm{ext}})\check{\mathbf{D}}_0\check{\Gamma}_k(j^t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)u_1)| \\
\leq \;& \Big((\chi(L)u_1, R_k(j^t,|r_0^t|)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)u_1) + (\chi(L)u_1, R_{k-1}(j^t,|r_\infty^t|)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)u_1)\Big)^{\frac{1}{2}} \\
& \times\Big(\alpha^{-k}(\chi(L_{\mathrm{ext}})u_2, R_0(j^t,|r_0^t|) \otimes 1\!\!1(B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-1}\chi(L_{\mathrm{ext}})u_2) \\
& +(\chi(L_{\mathrm{ext}})u_2, 1\!\!1 \otimes R_{k-1}(j^t,|r_\infty^t|)(B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-1}\chi(L_{\mathrm{ext}})u_2)\Big)^{\frac{1}{2}},
\end{aligned} \tag{7.10}$$

for $r_\epsilon^t = \mathbf{d}_0 j_\epsilon^t$.

We note that by exactly the same proof, estimates similar to those of Prop. 6.3 with $R_k(f^t,|g_\epsilon^t|)$ replaced by either $R_k(f^t,|g_\epsilon^t|) \otimes 1\!\!1$ or $1\!\!1 \otimes R_k(f^t,|g_\epsilon^t|)$, $B_{\mathrm{c}\,t}$ by $B_{\mathrm{c}\,t}^{\mathrm{ext}}$ and L by L_{ext} hold for the evolution $\mathrm{e}^{-\mathrm{i}tH_{\mathrm{ext}}^{\mathrm{e}}}$.

Combining (7.8), (7.9), (7.10) and Props. 5.1, 6.3, 7.4 we obtain the existence of the limit in *i)*, by Prop. A.3.

Property *ii)* follows from Lemma 7.2, arguing as in (7.7).

To prove *iii)* it suffices as in the proof of Thm. 6.4 *ii)* to show that

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH_{\mathrm{ext}}^{\mathrm{e}}}(\check{\Gamma}_k(j^t) - \check{\Gamma}_k(j^{t-t_1}))\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} = 0, \; \forall t_1 \in \mathbb{R}.$$

By [DG2, Lemma 2.16]:

$$\check{\Gamma}_k(j^t) - \check{\Gamma}_k(j^{t-t_1}) = -\int_0^{t_1} \mathrm{d}\check{\Gamma}_k(j^t, \partial_t j^{t-r})\mathrm{d}r,$$

and since $j_0^t + \alpha j_\infty^t \leq 1$, we have by [DG2, Lemma 2.16]:

$$\|\mathrm{d}\check{\Gamma}_k(j^t, \partial_t j^{t-r})(N^{\mathrm{e}}+1)^{-1}\| \leq C\|\partial_t j^{t-r}\| \leq Ct^{-\rho}.$$

Next we argue as in the proof of Thm. 6.4 using that

$$\|(N^{\mathrm{e}}+1)F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\| \in O(t^\delta).$$

iv) follows from the fact that

$$\Gamma(f_0^t) \otimes \mathbb{1}\check{\Gamma}_k(j^t) = \check{\Gamma}_k(j^t), \text{ if } f_0 j_0 = j_0.$$

v) follows from the fact that if $B_{\mathrm{c}\,t}$ is the observable defined in (5.2) for any constant $0 < c' \leq 1$ we have:

$$\check{\Gamma}_k(j^t)(B_{\mathrm{c}\,t}+\lambda)^{-1} = (B_{\mathrm{c}\,t}^{\mathrm{ext}}+\lambda)^{-1}\check{\Gamma}_k(j^t).$$

Property *vi)* follows from *v)* and Thm. 7.6. Finally the existence of the limit *vii)* follows from exactly the same arguments as those used to prove *i)*. □

Finally we prove a result similar to Prop. 6.6.

Proposition 7.8 *Assume the hypotheses of Thm. 7.7. Let $j = (f_0, f_\infty)$, $j_\epsilon = (f_0, f_{\infty,\epsilon})$, where $f_0, f_\infty, f_{\infty,\epsilon}$ are defined in (6.23), (6.24). Then*

$$W_k^+(j) = \mathrm{w} - \lim_{\epsilon\to 0} W_k^+(j_\epsilon).$$

Proof. We apply again Prop. A.4. By density it suffices to show that for $\lambda > 0$, $\chi \in C_0^\infty(\mathbb{R})$:

$$\chi(L_{\mathrm{ext}})W_k^+(j)\chi(L)R_{\mathrm{c}}^{\mathrm{e}+}(\lambda) = \mathrm{w} - \lim_{\epsilon\to 0}\chi(L_{\mathrm{ext}})W_k^+(j_\epsilon)\chi(L)R_{\mathrm{c}}^{\mathrm{e}+}(\lambda).$$

To check hypothesis (A.2) of Prop. A.4 we have to consider the Heisenberg derivative of

$$\Phi_\epsilon(t) = \chi(L_{\mathrm{ext}})\check{\Gamma}_k(j_\epsilon^t)(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L).$$

The estimates (7.8), (7.9), (7.10) and the fact that

$$j_0 + \alpha j_{\infty,\epsilon} \leq 1,\ j_{\infty,\epsilon} \leq Cj_\infty,\ j'_{\infty,\epsilon} \leq Cj'_\infty, \text{ uniformly in } \epsilon$$

show that hypothesis (A.2) is satisfied. Similarly (A.1) holds since $\|\check{\Gamma}_k(j_\epsilon^t)\| \leq \alpha^{-k}$. Finally

$$\mathrm{w} - \lim_{\epsilon\to 0}\check{\Gamma}_k(j_\epsilon^t) = \check{\Gamma}_k(j^t),\ \forall t \in \mathbb{R}.$$

Applying Prop. A.4 we obtain the proposition. □

8 Asymptotic fields and wave operators

8.1 Asymptotic fields for H

In this section we show the existence of asymptotic Weyl operators and asymptotic fields for the Nelson Hamiltonian introduced in Subsect. 1.1. Similar results can be shown under corresponding hypotheses for abstract Pauli-Fierz models introduced in Subsect. 3.1 (see the remark at the beginning of Subsect. 8.4).

We recall that the one-particle space is $\mathfrak{h} = L^2(\mathbb{R}^3, \mathrm{d}k)$. We set

$$h_t := \mathrm{e}^{-\mathrm{i}t|k|}h, \ h \in \mathfrak{h},$$

and

$$\mathfrak{h}_0 := \{h \in \mathfrak{h} | |k|^{-\frac{1}{2}}h \in \mathfrak{h}\}$$

equipped with the graph topology. In this section, we assume condition *(I4)* introduced in Subsect. 1.1. Introducing the operator $v \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$ defined in (1.3) we see that if *(H0)* holds for $\alpha > 0$, then *(I4)* implies:

$$\tag{8.1} \begin{array}{l} \forall\epsilon > 0, \ \exists C \text{ such that} \\ \|F(|x| \geq R)F(\epsilon \leq |k| \leq \epsilon^{-1})(K+1)^{-\frac{1}{2}}v\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} \leq CR^{-\inf(\alpha,\mu_1)}, \\ \|F(|x| \geq R)F(\epsilon \leq |k| \leq \epsilon^{-1})v(K+1)^{-\frac{1}{2}}\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h})} \leq CR^{-\inf(\alpha,\mu_1)}. \end{array}$$

In fact this follows from the fact that $\partial_k(\mathrm{e}^{-\mathrm{i}k.\mathrm{x}}v_j(k)) = \mathrm{e}^{-\mathrm{i}k.\mathrm{x}}(\partial_k - \mathrm{i}\mathrm{x})v_j(k)$, if we use (1.1) to control the powers of x appearing when differentiating $\mathrm{e}^{-\mathrm{i}k.\mathrm{x}}v_j(k)$.

Theorem 8.1 *Assume (H0) for $\alpha > 1$, (I0), (I4) for $\mu_1 > 1$. Then*

i) for $h \in \mathfrak{h}_0$ the asymptotic Weyl operator

$$\tag{8.2} W^+(h) := \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} \ \textit{exists.}$$

ii) the map

$$\mathfrak{h}_0 \ni h \mapsto W^+(h) \in \mathcal{U}(\mathcal{H})$$

is strongly continuous for the topology of $\mathfrak{h}_0$.

iii) $W^+(h)W^+(g) = \mathrm{e}^{-\mathrm{i}\mathrm{Im}(h,g)}W^+(f+g)$.

iv) $\mathrm{e}^{\mathrm{i}tH}W^+(h)\mathrm{e}^{-\mathrm{i}tH} = W^+(h_{-t})$.

Proof. We have

$$\tag{8.3} W(h_t) = \mathrm{e}^{-\mathrm{i}tH_0}W(h)\mathrm{e}^{\mathrm{i}tH_0},$$

which implies that as a quadratic form on $\mathcal{D}(H_0)$ one has:

$$\partial_t W(h_t) = [-H_0, \mathrm{i}W(h_t)].$$

Since on $\mathcal{D}(H_0)$ $H = H_0 + \phi(v)$, we have as quadratic forms on $\mathcal{D}(H) = \mathcal{D}(H_0)$:

$$\begin{aligned} \partial_t \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH} &= \mathrm{e}^{\mathrm{i}tH}[\phi(v), \mathrm{i}W(h_t)]\mathrm{e}^{-\mathrm{i}tH} \\ &= \mathrm{i}\mathrm{e}^{\mathrm{i}tH}\mathrm{Im}(h_t, v)\mathrm{e}^{-\mathrm{i}H}. \end{aligned}$$

Integrating this relation we obtain, first as a quadratic form identity on $\mathcal{D}(H)$, and then by a simple argument as an operator identity:

$$\mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})^{-1} - W(h)(H+\mathrm{i})^{-1} = \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}sH}\mathrm{Im}(h_s, v)(H+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}sH}u\mathrm{d}s.$$

For $h \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$, we obtain by stationary phase arguments and (8.1):

$$\|(h_t, v)(H+\mathrm{i})^{-1}\| \leq C\|(h_t, v(K+1)^{-\frac{1}{2}})\| + C\|(h_t, (K+1)^{-\frac{1}{2}}v)\| \leq Ct^{-\mu_1}.$$

The existence of the limit (8.2) follows for $h \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$.

Next we use the identity:

$$W(h_1) - W(h_2) = W(h_1)(1\!\!1 - \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1,h_2)}) + \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1,h_2)}W(h_1)(1\!\!1 - W(h_1-h_2)).$$

Using that

$$|1\!\!1 - \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}(h_1,h_2)}| \leq C|\mathrm{Im}(h_1,h_2)| \leq C\|h_1 - h_2\|\sqrt{\|h_1\|^2 + \|h_2\|^2},$$

$$\|(1\!\!1 - W(h_1-h_2))u\| \leq \|\phi(h_1-h_2)u\|,$$

we obtain that for $\|h_1\|, \|h_2\| \leq R$,

$$\|(W(h_1) - W(h_2))(H+\mathrm{i})^{-1}\| \leq C_R\|h_1 - h_2\|_{\mathfrak{h}_0}. \tag{8.4}$$

Since $C_0^\infty(\mathbb{R}^3\backslash\{0\})$ is dense in $\mathfrak{h}_0$, we deduce from (8.4) the existence of the limit

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}W(h_t)\mathrm{e}^{-\mathrm{i}tH}\chi(H), \text{ for } \chi \in C_0^\infty(\mathbb{R}),\ h \in \mathfrak{h}_0.$$

By density this proves the existence of the limit (8.2) for all $h \in \mathfrak{h}_0$. Statement *ii)* follows directly from (8.4). Statements *iii)* and *iv)* are immediate. □

Theorem 8.2 *Assume (H0) for $\alpha > 1$, (I0), (I4) for $\mu_1 > 1$. Then:*

i) there exists for $h \in \mathfrak{h}_0$ a selfadjoint operator $\phi^+(h)$ called the asymptotic field *such that $W^+(sh) = \mathrm{e}^{\mathrm{i}s\phi^+(h)}$, $s \in \mathbb{R}$.*

ii) For $h \in \mathfrak{h}_0$, $\mathcal{D}(H+b)^{\frac{1}{2}} \subset \mathcal{D}(\phi^+(h))$ and:

$$\phi^+(h)(H+b)^{-\frac{1}{2}} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\phi(h_t)(H+b)^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}tH},$$

$$\|\phi^+(h)(H+b)^{-\frac{1}{2}}\| \leq C\|(1+|k|^{-\frac{1}{2}})h\|.$$

For $h_i \in \mathfrak{h}_0 \cap \mathcal{D}(|k|^{\frac{1}{2}})$, $1 \leq i \leq n$, $n \geq 2$, $\mathcal{D}((H+b)^{n/2}) \subset \mathcal{D}(\Pi_1^n\phi^+(h_i))$ and

$$\Pi_{i=1}^n\phi^+(h_i)(H+b)^{-n/2} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Pi_{i=1}^n\phi(h_{i,t})(H+b)^{-n/2}\mathrm{e}^{-\mathrm{i}tH}$$

$$\|\Pi_1^n\phi^+(h_i)(H+b)^{-n/2}\| \leq C_n\Pi_1^n\|(1+|k|^{\frac{1}{2}}+|k|^{-\frac{1}{2}})h_i\|.$$

iii) The operators $\phi^+(h)$ satisfy in the sense of quadratic forms on $\mathcal{D}(\phi^+(h_1)) \cap \mathcal{D}(\phi^+(h_2))$ the canonical commutation relations

$$[\phi^+(h_2), \phi^+(h_1)] = \mathrm{i}\mathrm{Im}(h_2|h_1).$$

Note that the estimates on the domain of $\Pi_{i=1}^n \phi^+(h_i)$ described in *ii)* are better for $n = 1$ than for arbitrary $n \geq 2$.

Proof. *i)* and *iii)* follow by general arguments from the fact that $\mathfrak{h}_0 \ni h \mapsto W^+(h)$ is a regular CCR representation (see eg [DG3, Sect. 2.2]).

We will prove *ii)* for arbitrary n and explain then the modifications for the case $n = 1$. We first prove the existence of the norm limit

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Pi_1^n \phi(h_{i,t})(H+b)^{-n/2}\mathrm{e}^{-\mathrm{i}tH} =: R^+(h_1, \dots, h_n), \tag{8.5}$$

for $h_i \in \mathfrak{h}_0 \cap \mathcal{D}(|k|^{\frac{1}{2}})$. We deduce from the identity (8.3) that the Heisenberg derivative of $\prod_1^n \phi(h_{i,t})(H+b)^{-n/2}$ defined as a quadratic form on $\mathcal{D}(H)$ equals

$$[\phi(v), \mathrm{i}\Pi_1^n \phi(h_{i,t})](H+b)^{-n/2}.$$

Since $[\phi(v), \mathrm{i}\phi(h_{i,t})] = \mathrm{Im}(h_{i,t}, v)$ is bounded and using Lemma 3.10, (8.1) and stationary phase arguments as in the proof of Thm. 8.1 we obtain the existence of the limit (8.5) for $h_i \in C_0^\infty(\mathbb{R}^3\backslash\{0\})$. A density argument and the norm continuity of

$$(h_1, \dots, h_n) \in (\mathfrak{h}_0 \cap \mathcal{D}(|k|^{\frac{1}{2}}))^n \mapsto \Pi_1^n \phi(h_i)(H+b)^{-n/2}$$

shown in Lemma 3.10 proves the existence of the limit (8.5) for arbitrary $h_i \in \mathfrak{h}_0 \cap \mathcal{D}(|k|^{\frac{1}{2}})$. It follows then again from Lemma 3.10 that

$$\|R^+(h_1, \dots, h_n)\| \leq C_n \Pi_1^n \|(1 + |k|^{\frac{1}{2}} + |k|^{-\frac{1}{2}})h_i\|. \tag{8.6}$$

Let us now complete the proof of *ii)* by induction on n. The proof of *ii)* for $n = 1$ needed to start the induction argument will be given later. Let $h_i \in \mathfrak{h}_0 \cap \mathcal{D}(|k|^{\frac{1}{2}})$, $1 \leq i \leq n$. We have to show that

$$\mathcal{D}((H+b)^{n/2}) \subset \mathcal{D}(\Pi_1^n \phi^+(h_i)), \tag{8.7}$$

and then that

$$\Pi_1^n \phi^+(h_i)(H+b)^{-n/2} = R^+(h_1, \dots, h_n). \tag{8.8}$$

To prove (8.7) we have to show that for $u \in \mathcal{H}$:

$$\sup_{s\in\mathbb{R}} \|s^{-1}(W^+(sh_1) - \mathbb{1})\Pi_2^n \phi^+(h_i)(H+b)^{-n/2}u\| < \infty.$$

By the induction assumption, $\mathcal{D}(H+b)^{n/2} \subset \Pi_2^n \phi^+(h_i)$ and

$$\Pi_2^n \phi^+(h_i)(H+b)^{-n/2}u = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Pi_2^n \phi(h_{i,t})(H+b)^{-n/2}\mathrm{e}^{-\mathrm{i}tH}u. \tag{8.9}$$

Using (8.9) and the fact that $\mathrm{e}^{\mathrm{i}tH}W(h_{1,t})\mathrm{e}^{-\mathrm{i}tH}$ is uniformly bounded in t, we get:

$$\begin{aligned} & s^{-1}(W^+(sh_1) - \mathbb{1})\Pi_2^n \phi^+(h_i)(H+b)^{-n/2}u \\ = \; & \lim_{t\to+\infty} s^{-1}\mathrm{e}^{\mathrm{i}tH}(W(sh_{1,t}) - \mathbb{1})\Pi_2^n \phi(h_{i,t})(H+b)^{-n/2}\mathrm{e}^{-\mathrm{i}tH}u. \end{aligned} \tag{8.10}$$

Hence

$$\begin{aligned}
& \|s^{-1}(W^+(sh_1) - 1\!\!1)\Pi_2^n\phi^+(h_i)(H+b)^{-n/2}u\| \\
\leq\ & \sup_{t\in\mathbb{R}} \|s^{-1}(W(sh_{1,t}) - 1\!\!1)\Pi_2^n\phi(h_{i,t})(H+b)^{-n/2}\mathrm{e}^{-\mathrm{i}tH}u\| \\
\leq\ & \sup_{t\in\mathbb{R}} \|\phi(h_{1,t})\Pi_2^n\phi(h_{i,t})(H+b)^{-n/2}\|\|u\| \\
\leq\ & C\Pi_1^n\|(1+|k|^{\frac{1}{2}}+|k|^{-\frac{1}{2}})h_i\|\|u\|,
\end{aligned}$$

by Lemma 3.10. This proves (8.7). To prove (8.8), it suffices to show that for $v\in\mathcal{D}$, $\mathcal{D}$ a dense subspace of $\mathcal{H}$:

$$\lim_{s\to 0}(\mathrm{i}s)^{-1}(v,(W^+(sh_1)-1\!\!1)\Pi_2^n\phi^+(h_i)(H+b)^{-n/2}u) = (v, R^+(h_1,\dots,h_n)u).$$

By (8.10) we have:

$$\begin{aligned}
& (\mathrm{i}s)^{-1}(v,(W^+(sh_1)-1\!\!1)\Pi_2^n\phi^+(h_i)(H+b)^{-n/2}u) \\
=\ & \lim_{t\to+\infty}(\mathrm{i}s)^{-1}(\mathrm{e}^{\mathrm{i}tH}(W(sh_{1,t})-1\!\!1)\mathrm{e}^{-\mathrm{i}tH}v, \mathrm{e}^{\mathrm{i}tH}\Pi_2^n\phi(h_{i,t})(H+b)^{-n/2}u).
\end{aligned}$$

Since $|s^{-1}(\mathrm{e}^{\mathrm{i}s\lambda}-1)-\mathrm{i}\lambda|\leq C_0|s||\lambda|^2$, we have using Lemma 3.10:

$$\|\Big((\mathrm{i}s)^{-1}(W(sh_{1,t})-1\!\!1)-\phi(h_{1,t})\Big)(H+b)^{-1}\|\leq C|s|,\ \text{uniformly in } t.$$

Hence for $v\in\mathcal{D}(H)$, we have:

$$\begin{aligned}
& \lim_{s\to 0}(\mathrm{i}s)^{-1}(v,(W^+(sh_1)-1\!\!1)\Pi_2^n\phi^+(h_i)(H+b)^{-n/2}u) \\
=\ & \lim_{s\to 0}\lim_{t\to+\infty}(\mathrm{i}s)^{-1}(\mathrm{e}^{\mathrm{i}tH}(W(sh_{1,t})-1\!\!1)\mathrm{e}^{-\mathrm{i}tH}v, \mathrm{e}^{\mathrm{i}tH}\Pi_2^n\phi(h_{i,t})(H+b)^{-n/2}u) \\
=\ & \lim_{t\to+\infty}(\mathrm{e}^{-\mathrm{i}tH}\phi(h_{1,t})\mathrm{e}^{-\mathrm{i}tH}v, \mathrm{e}^{\mathrm{i}tH}\Pi_2^n\phi(h_{i,t})(H+b)^{-n/2}u) \\
=\ & \lim_{t\to+\infty}(v, \mathrm{e}^{\mathrm{i}tH}\Pi_1^n\phi(h_{i,t})(H+b)^{-n/2}u) \\
=\ & (v, R^+(h_1,\dots,h_n)u),
\end{aligned}$$

as claimed. The fact that

$$\|\Pi_1^n\phi^+(h_i)(H+b)^{-n/2}\|\leq C_n\Pi_1^n\|(1+|k|^{\frac{1}{2}}+|k|^{-\frac{1}{2}})h_i\|$$

follows then from (8.6).

Let us now prove *ii)* in the case $n=1$. The existence of the limit (8.5) for $n=1$ and $h_1\in\mathfrak{h}_0$ follows from the same arguments, using Prop. A.1 instead of Lemma 3.10. The proof of the fact that $R^+(h_1)(H+b)^{-\frac{1}{2}}=\phi^+(h_1)(H+b)^{-\frac{1}{2}}$ is also similar to the general case, using Prop. A.1 instead of Lemma 3.10. □

The following theorem follows directly from Thm. 8.2 and from general properties of regular CCR representations (see eg [DG3, Sect. 3.3]).

Theorem 8.3 *Assume (H0) for $\alpha>1$, (I0), (I4) for $\mu_1>1$. Then*

i) For any $h\in\mathfrak{h}_0$, the asymptotic creation and annihilation operators *defined on*

$$\mathcal{D}(a^{+\sharp}(h)) := \mathcal{D}(\phi^+(h))\cap\mathcal{D}(\phi^+(\mathrm{i}h))$$

by

$$a^{+*}(h) := \frac{1}{\sqrt{2}}\left(\phi^{+}(h) - \mathrm{i}\phi^{+}(\mathrm{i}h)\right),$$
$$a^{+}(h) := \frac{1}{\sqrt{2}}\left(\phi^{+}(h) + \mathrm{i}\phi^{+}(\mathrm{i}h)\right),$$

are closed.
ii) The operators $a^{+\sharp}$ *satisfy, in the sense of forms on* $\mathcal{D}(a^{+\sharp}(h_1)) \cap \mathcal{D}(a^{+\sharp}(h_2))$*, the canonical commutation relations*

$$[a^{+}(h_1), a^{+*}(h_2)] = (h_1|h_2)1\!\!1,$$
$$[a^{+}(h_2), a^{+}(h_1)] = [a^{+*}(h_2), a^{+*}(h_1)] = 0.$$

iii)

$$\mathrm{e}^{\mathrm{i}tH} a^{+\sharp}(h)\mathrm{e}^{-\mathrm{i}tH} = a^{+\sharp}(h_{-t}). \tag{8.11}$$

iv) For $h \in \mathfrak{h}_0$*,* $\mathcal{D}(H+b)^{\frac{1}{2}} \subset \mathcal{D}(a^{+\sharp}(h))$ *and:*

$$a^{+\sharp}(h)(H+b)^{-\frac{1}{2}} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} a^{\sharp}(h_t)(H+b)^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}tH},$$
$$\|a^{+\sharp}(h)(H+b)^{-\frac{1}{2}}\| \leq C\|(1+|k|^{-\frac{1}{2}})h\|.$$

For $h_i \in \mathfrak{h}_0 \cap \mathcal{D}(|k|^{\frac{1}{2}})$*,* $1 \leq i \leq n$*,* $n \geq 2$*,* $\mathcal{D}((H+b)^{n/2}) \subset \mathcal{D}(\Pi_1^n a^{+\sharp}(h_i))$ *and*

$$\Pi_1^p a^{+\sharp}(h_i)(H+b)^{-\frac{n}{2}} = \text{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\Pi_1^p a^{\sharp}(h_{i,t})(H+b)^{-\frac{n}{2}}\mathrm{e}^{-\mathrm{i}tH},$$
$$\|\Pi_1^n a^{+\sharp}(h_i)(H+b)^{-n/2}\| \leq C_n \Pi_1^n \|(1+|k|^{\frac{1}{2}}+|k|^{-\frac{1}{2}})h_i\|.$$

8.2 Asymptotic vacuum spaces and wave operators for H

In this subsection, we recall the construction of the asymptotic vacuum spaces and of the wave operators, see eg [HK1], [DG2, Sect. 5.3], [DG3, Sect. 10.2]. We define the *asymptotic vacuum space*

$$\mathcal{K}^{+} := \{u \in \mathcal{H} \mid a^{+}(h)u = 0,\ h \in \mathfrak{h}_0\}.$$

The *asymptotic space* is defined as

$$\mathcal{H}^{+}_{\mathrm{ext}} := \mathcal{K}^{+} \otimes \Gamma(\mathfrak{h}).$$

Proposition 8.4 *i)* $\mathcal{K}^{+}$ *is a closed* $H-$*invariant space.*
ii) $\mathcal{K}^{+}$ *is included in the domain of* $\Pi_1^p a^{+\sharp}(h_i)$*, for* $h_i \in \mathfrak{h}_0$*.*

iii) $\mathcal{H}_{\mathrm{pp}}(H) \subset \mathcal{K}^{+}$.

Proof. *i)* and *ii)* follow by the general properties of CCR representations (see eg [DG3, Sect. 4]). The fact that $\mathcal{K}^+$ is $H-$invariant follows from (8.11). To prove *iii)* we verify that for $u \in \mathcal{D}(H)$, $Hu = \lambda u$, $h \in \mathfrak{h}_0$, $a(h_t)\mathrm{e}^{-\mathrm{i}tH}u = \mathrm{e}^{-\mathrm{i}t\lambda}a(h_t)u \in o(1)$. $\square$

The *asymptotic Hamiltonian* is defined by

$$H^+ := K^+ \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(|k|), \text{ acting on } \mathcal{H}^+_{\mathrm{ext}}$$

for

$$K^+ := H\Big|_{\mathcal{K}^+}.$$

We also define the *wave operator*

$$(8.12)\qquad \begin{array}{l} \Omega^+ : \mathcal{K}^+ \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}_0) \to \mathcal{H}, \\ \Omega^+\psi \otimes a^*(h_1) \cdots a^*(h_p)\Omega := a^{+*}(h_1) \cdots a^{+*}(h_p)\psi, \qquad h_1, \dots, h_p \in \mathfrak{h}_0, \quad \psi \in \mathcal{K}^+. \end{array}$$

It follows from general properties of CCR representations that Ω^+ is isometric (see eg [DG3, Prop. 4.2]). Hence we can uniquely extend Ω^+ as an isometric map

$$\Omega^+ : \mathcal{H}^+_{\mathrm{ext}} \to \mathcal{H}$$

such that

$$\begin{array}{l} a^{+\sharp}(h)\Omega^+ = \Omega^+ 1\!\!1 \otimes a^\sharp(h), \quad h \in \mathfrak{h}_0, \\ H\Omega^+ = \Omega^+ H^+. \end{array}$$

We set

$$\mathcal{H}^+ := \mathrm{Ran}\Omega^+.$$

Finally we give another description of $\mathcal{H}^+$ using the notion of *asymptotic number operator* (see eg [DG3, Sect. 4.2]). which we now recall. We first recall some facts about quadratic forms. We will assume that a positive quadratic form is defined on the whole space $\mathcal{H}$ and takes values in $[0, \infty]$. The domain of a positive quadratic form b is then defined as

$$\mathcal{D}(b) := \{u \in \mathcal{H} |\ b(u) < \infty\}.$$

The sum of closed forms is a closed form, and the supremum of a family of closed forms is a closed form.

For each finite dimensional space $\mathfrak{f} \subset \mathfrak{h}_0$, one defines

$$n^+_{\mathfrak{f}}(u) := \sum_{i=1}^{\dim\mathfrak{f}} \|a^+(h_i)u\|^2,$$

where $\{h_i\}$ is an orthonormal basis of $\mathfrak{f}$. (If $u \notin \mathcal{D}(a^+(h_i)$ for some i, then $n^+_{\mathfrak{f}}(u) = \infty$). The quadratic form $n_{\mathfrak{f}}$ does not depend on the choice of the basis $\{h_i\}$ of $\mathfrak{f}$. The quadratic form n^+ is defined by

$$n^+(u) := \sup_{\mathfrak{f}} n^+_{\mathfrak{f}}(u), \; u \in \mathcal{H}.$$

We can associate to n^+ a selfadjoint operator (with an a priori non dense domain) denoted by N^+ called the *asymptotic number operator.*

Then as shown in [DG3, Sect. 4.2]:

$$\mathcal{H}^+ = \mathrm{Ran}\Omega^+ = \overline{\mathcal{D}(n^+)}.$$

One can associate a number operator N_π to any regular CCR representation $\mathfrak{h} \ni h \mapsto W_\pi(h) \in \mathcal{U}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ (see eg [DG3, Sect. 4.2]). The regular CCR representation is of *Fock type* if N_π has a dense domain.

8.3 Extended wave operators for H

Let us first define extended objects similar to those introduced in Sect. 6 for $H^{\rm e}$. We set:

$$\mathcal{H}_{\rm ext} := \mathcal{H}\otimes\Gamma(\mathfrak{h}),\ H_{\rm ext} := H\otimes 1\!\!1 + 1\!\!1\otimes {\rm d}\Gamma(|k|),\ \text{acting on } \mathcal{H}_{\rm ext}.$$

Note that $\mathcal{H}_{\rm ext}^{+}\subset\mathcal{H}_{\rm ext}$. By Thm. 8.3 we can define the *extended wave operator* $\Omega_{\rm ext}^{+}$ as follows:

$$\Omega_{\rm ext}^{+} : \psi\otimes\Pi_1^n a^*(h_i)\Omega \mapsto \Pi_1^n a^{*+}(h_i)\psi,$$

for $\psi\in\mathcal{D}((H+b)^{n/2})$, $h_i\in\mathcal{D}(|k|^{-\frac12}+|k|^{\frac12})$, $1\leq i\leq n$. The extended wave operator is then an unbounded operator from $\mathcal{H}_{\rm ext}$ into $\mathcal{H}$ with domain

$$\mathcal{D}(\Omega_{\rm ext}^{+}) = \bigoplus_{n=0}^{\infty}\mathcal{D}((H+b)^{n/2})\otimes\bigotimes_{s}^{n}\mathcal{D}(|k|^{-\frac12}+|k|^{\frac12}).$$

By (8.11) we have $\Omega_{\rm ext}^{+}{\rm e}^{-{\rm i}tH_{\rm ext}} = {\rm e}^{-{\rm i}tH}\Omega_{\rm ext}^{+}$, and

$$\Omega^{+}_{{\rm ext}|\mathcal{H}_{\rm ext}^{+}} = \Omega^{+}.$$

By Thm. 8.3, we can rewrite as in [DG2, Sect. 5.6]:

$$\Omega_{\rm ext}^{+}\psi\otimes u = \lim_{t\to+\infty}{\rm e}^{{\rm i}tH}I{\rm e}^{-{\rm i}tH_{\rm ext}}\psi\otimes u,$$

for $\psi\in\mathcal{D}((H+b)^{n/2})$, $u\in\otimes_{\rm s}^{n}\mathcal{D}(|k|^{-\frac12}+|k|^{\frac12})$, and I the *scattering identification operator* defined in Subsect. 2.2.

In particular if $\psi\in\mathcal{K}^{+}$, $u\in\otimes_{\rm s}^{n}\mathcal{D}(|k|^{-\frac12}+|k|^{\frac12})$ then

$$\Omega^{+}\psi\otimes u = \lim_{t\to+\infty}{\rm e}^{{\rm i}tH}I{\rm e}^{-{\rm i}tH_{\rm ext}}\psi\otimes u.$$

8.4 Asymptotic fields for $H^{\rm e}$

We now collect results similar to those of Subsect. 8.1 for the expanded Hamiltonian $H^{\rm e}$ defined in Subsect. 3.3. If we restrict ourselves to expanded Hamiltonians $H^{\rm e}$ obtained from a massless Nelson Hamiltonian H, then all these results are obtained from the results in Subsect. 8.1 in the following way:

first the results for $\mathcal{H}, H$ immediately give corresponding results for $\tilde{\mathcal{H}}^{\rm e}, \tilde{H}^{\rm e}$, since $\tilde{H}^{\rm e}$ acts as the free Hamiltonian $-{\rm d}\Gamma(|k|)$ on the second component of $\tilde{\mathcal{H}}^{\rm e}$. Then we use functorial properties of the unitary map $\mathcal{W}$ defined in (3.3) to obtain results for $\mathcal{H}^{\rm e}, H^{\rm e}$.

Remark

All the results in this section are also valid for general massless Pauli-Fierz Hamiltonians, In this case it is more convenient to follow the inverse path, *i.e.,* to start with the expanded Hamiltonian $H^{\rm e}$ and then to go back to H. In this case we can for example assume that

$$(I'4)\ v^{\rm e}(\sigma)(K+1)^{-\frac12},\ (K+1)^{-\frac12}v^{\rm e}(\sigma)\in H^{\mu_1}_{\rm loc}(\mathbb{R}\backslash\{0\},\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{g})),\ \mu_1>0.$$

Note that *(I'4)* implies:

$$\begin{array}{ll} & \forall\epsilon>0,\ \exists C \text{ such that} \\ (8.13) & \|F(|s|\geq R)F(\epsilon\leq|\sigma|\leq\epsilon^{-1})(K+1)^{-\frac12}v\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h}^{\rm e})}\leq CR^{\mu_1}, \\ & \|F(|s|\geq R)F(\epsilon\leq|\sigma|\leq\epsilon^{-1})v(K+1)^{-\frac12}\|_{\mathcal{B}(\mathcal{K},\mathcal{K}\otimes\mathfrak{h}^{\rm e})}\leq CR^{-\mu_1}. \end{array}$$

The proofs of Subsects. 8.1, 8.2, 8.3 extend under conditions *(I'0)*, *(I'4)* for $\mu_1 > 1$, if we replace where appropriate cutoffs in H by cutoffs in L. In this way we extend the results of Subsects. 8.4, 8.5 to the case of general expanded Hamiltonians. Using then functorial properties of $\mathcal{W}$ we can extend the results of Subsects. 8.1, 8.2, 8.3 to abstract massless Pauli-Fierz Hamiltonians satisfying *(I'0)* and *(I'4)* for $\mu_1 > 1$. The details are left to the interested reader.

We set

$$h_t := \mathrm{e}^{-\mathrm{i}t\sigma}h, \ h \in \mathfrak{h}^{\mathrm{e}},$$

and

$$\mathfrak{h}_0^{\mathrm{e}} := \{h \in \mathfrak{h}^{\mathrm{e}} | |\sigma|^{-\frac{1}{2}}h \in \mathfrak{h}^{\mathrm{e}}\}.$$

equipped with the graph topology. For $h \in \mathfrak{h}^{\mathrm{e}}$, we set

$$h_\pm = 1\!\!1_{\{\pm\sigma\geq 0\}}h$$

and note that $h_+ \in \mathfrak{h}_0$ if $h \in \mathfrak{h}_0^{\mathrm{e}}$.

By the arguments outlined above, we obtain directly the following results:

Theorem 8.5 *Assume (I'0) and (I'4) for* $\mu_1 > 1$. *Then:*

i) for $h \in \mathfrak{h}_0^{\mathrm{e}}$ *the* asymptotic Weyl operators

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} W(h_t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} =: W^{\mathrm{e}+}(h) \ \textit{exists.}$$

ii) the map

$$\mathfrak{h}_0^{\mathrm{e}} \ni h \mapsto W^{\mathrm{e}+}(h) \in \mathcal{U}(\mathcal{H}^{\mathrm{e}})$$

is strongly continuous for the topology of $\mathfrak{h}_0^{\mathrm{e}}$.

iii) $W^{\mathrm{e}+}(h)W^{\mathrm{e}+}(g) = \mathrm{e}^{-\mathrm{i}Im(h,g)}W^{\mathrm{e}+}(f+g)$.

iv) $\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}W^{\mathrm{e}+}(h)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} = W^{\mathrm{e}+}(h_{-t})$.

Let $\phi^{\mathrm{e}+}(h), a^{\sharp\mathrm{e}+}(h)$ *be the* asymptotic fields *and* creation -annihilation operators *obtained from* $W^{\mathrm{e}+}(h)$. *Then*

$$\begin{aligned}
W^{\mathrm{e}+}(h) &= \mathcal{W}(W^{+}(h_+)\otimes W(h_-))\mathcal{W}^{-1},\\
\phi^{\mathrm{e}+}(h) &= \mathcal{W}(\phi^{+}(h_+)\otimes 1\!\!1 + 1\!\!1\otimes\phi(h_-))\mathcal{W}^{-1},\\
a^{\sharp\mathrm{e}+}(h) &= \mathcal{W}(a^{\sharp+}(h_+)\otimes 1\!\!1 + 1\!\!1\otimes a^{\sharp}(h_-))\mathcal{W}^{-1}.
\end{aligned}$$

Let us note the following consequence of Thm. 8.5

$$\mathrm{e}^{\mathrm{i}tL}W^{\mathrm{e}+}(h)\mathrm{e}^{-\mathrm{i}tL} = W^{\mathrm{e}+}(\mathrm{e}^{\mathrm{i}t|\sigma|}h), \ h \in \mathfrak{h}_0^{\mathrm{e}}. \tag{8.14}$$

In fact this follows from Thm. 8.1 *iv)* and the fact that $\mathcal{W}^{-1}L\mathcal{W} = H\otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{H}}\otimes \mathrm{d}\Gamma(|k|))$.

Another result which follow from the proof of Thm. 8.2, using Lemma 3.11 instead of Lemma 3.10 is:

Theorem 8.6 *For $h \in \mathfrak{h}_0^{\mathrm{e}}$, $\mathcal{D}(L+b)^{\frac12} \subset \mathcal{D}(\phi^{\mathrm{e}+}(h))$ and:*

$$\phi^{\mathrm{e}+}(h)(L+b)^{-\frac12} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\phi(h_t)(L+b)^{-\frac12}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}},$$

$$\|\phi^{\mathrm{e}+}(h)(L+b)^{-\frac12}\| \leq C\|(1+|\sigma|^{-\frac12})h\|.$$

For $h_i \in \mathfrak{h}_0^{\mathrm{e}} \cap \mathcal{D}(|\sigma|^{\frac12})$, $1 \leq i \leq n$, $n \geq 2$, $\mathcal{D}((L+b)^{n/2}) \subset \mathcal{D}(\Pi_1^n \phi^{\mathrm{e}+}(h_i))$ and

$$\Pi_{i=1}^n phi^{\mathrm{e}+}(h_i)(L+b)^{-n/2} = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Pi_{i=1}^n\phi(h_{i,t})(L+b)^{-n/2}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}$$

$$\|\Pi_1^n\phi^{\mathrm{e}+}(h_i)(L+b)^{-n/2}\| \leq C_n\Pi_1^n\|(1+|\sigma|^{\frac12}+|\sigma|^{-\frac12})h_i\|.$$

8.5 Asymptotic spaces and wave operators for H^{e}

As in Subsect 8.2 we define

$$\mathcal{K}^{\mathrm{e}+} := \{u \in \mathcal{H}^{\mathrm{e}} | a^{\mathrm{e}+}(h)u = 0,\ \forall h \in \mathfrak{h}_0^{\mathrm{e}}\},$$

which is a closed $H^{\mathrm{e}}-$ and $L-$invariant vector space. We define the *wave operator*

$$\begin{aligned}&\Omega^{\mathrm{e}+} : \mathcal{K}^{\mathrm{e}+} \otimes \Gamma_{\mathrm{fin}}(\mathfrak{h}_0^{\mathrm{e}}) \to \mathcal{H}^{\mathrm{e}},\\ &\Omega^{\mathrm{e}+}\psi \otimes a^*(h_1)\cdots a^*(h_p)\Omega := a^{*\mathrm{e}+}(h_1)\cdots a^{*\mathrm{e}+}(h_p)\psi, \quad h_1,\dots,h_p \in \mathfrak{h}_0^{\mathrm{e}}, \quad \psi \in \mathcal{K}^{\mathrm{e}+},\end{aligned} \tag{8.15}$$

which uniquely extends as an isometry

$$\Omega^{\mathrm{e}+} : \mathcal{K}^{\mathrm{e}+} \otimes \Gamma(\mathfrak{h}^{\mathrm{e}}) =: \mathcal{H}_{\mathrm{ext}}^{\mathrm{e}+} \to \mathcal{H}^{\mathrm{e}}$$

We set:

$$\mathcal{H}^{\mathrm{e}+} := \mathrm{Ran}\Omega^{\mathrm{e}+}.$$

We also define the unbounded *extended wave operator* $\Omega_{\mathrm{ext}}^{\mathrm{e}+}$ from $\mathcal{H}_{\mathrm{ext}}^{\mathrm{e}}$ into $\mathcal{H}^{\mathrm{e}}$ with domain

$$\mathcal{D}(\Omega_{\mathrm{ext}}^{\mathrm{e}+}) = \bigoplus_{n=0}^{\infty} \mathcal{D}((L+b)^{n/2}) \otimes \bigotimes_{\mathrm{s}}^{n} \mathcal{D}(|\sigma|^{-\frac12}+|\sigma|^{\frac12}),$$

by

$$\Omega_{\mathrm{ext}}^{\mathrm{e}+}\psi \otimes u = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} I \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ext}}^{\mathrm{e}}}\psi \otimes u$$

for $\psi \in \mathcal{D}((L+b)^{n/2})$, $u \in \bigotimes_{\mathrm{s}}^n \mathcal{D}(|\sigma|^{-\frac12}+|\sigma|^{\frac12})$.

In particular if $\psi \in \mathcal{K}^{\mathrm{e}+}$, $u \in \bigotimes_{\mathrm{s}}^n \mathcal{D}(|\sigma|^{-\frac12}+|\sigma|^{\frac12})$ then

$$\Omega^{\mathrm{e}+}\psi \otimes u = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} I \mathrm{e}^{-\mathrm{i}tH_{\mathrm{ext}}^{\mathrm{e}}}\psi \otimes u.$$

8.6 Conversion of scattering objects

In this subsection we describe how to relate scattering objects for H to scattering objects for $H^{\rm e}$, using the canonical embedding $\mathcal{W}I_\Omega$ introduced in Subsect. 3.4. The results below follow easily from the definition of $\mathcal{W}$ and I_Ω in Sect. 3 and the formulas in Thm. 8.5. We have

$$\mathcal{W}I_\Omega W^+(h)u = W^{\rm e+}(jh)\mathcal{W}I_\Omega u,\ h \in \mathfrak{h}_0,\ u \in \mathcal{H}, \tag{8.16}$$

where $j : \mathfrak{h} \to \mathfrak{h}^{\rm e}$ is the isometry defined in (3.5).

Similarly for $\psi \in \mathcal{K}^+$, $h_i \in \mathfrak{h}_0$, $1 \le i \le n$:

$$\Pi_1^n a^{*+}(h_i)\psi = I_\Omega^* \mathcal{W}^{-1}\Pi_1^n a^{*{\rm e}+}(jh_i)\mathcal{W}I_\Omega\psi.$$

Also:

$$u \in \mathcal{K}^+ \Leftrightarrow \mathcal{W}I_\Omega u \in \mathcal{K}^{\rm e+}. \tag{8.17}$$

This implies that

$$\Omega^+ = I_\Omega^*\mathcal{W}^{-1} \times \Omega^{\rm e+} \times \mathcal{W}I_\Omega \otimes \Gamma(j), \tag{8.18}$$

where we consider $\mathcal{W}I_\Omega \otimes \Gamma(j)$ as a map:

$$\mathcal{W}I_\Omega \otimes \Gamma(j) : \mathcal{K}^+ \otimes \Gamma(\mathfrak{h}) \to \mathcal{K}^{\rm e+} \otimes \Gamma(\mathfrak{h}^{\rm e}).$$

Finally let $N^{\rm e+}$ be the asymptotic number operator associated to the representation $\mathfrak{h}_0^{\rm e} \ni h \mapsto W^{\rm e+}(h)$, which is defined as in Subsect. 8.2. Then

$$N^{\rm e+} = \mathcal{W}(N^+ \otimes 1\!\!1_{\Gamma(\mathfrak{h})} + 1\!\!1_{\mathcal{H}} \otimes N)\mathcal{W}^{-1},$$

and hence:

$$\overline{\mathcal{D}(N^{\rm e+})} = \mathcal{W}\overline{\mathcal{D}(N^+)} \otimes \Gamma(\mathfrak{h}^{\rm e}). \tag{8.19}$$

8.7 Properties of the space $\mathcal{H}_{\rm c}^{\rm e+}$

In this subsection we describe properties of the space $\mathcal{H}_{\rm c}^{\rm e+}$ connected with the asymptotic fields.

Note that hypothesis *(I'2)* implies *(I'4)*.

Theorem 8.7 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 1$ and $\rho(1+\epsilon_0) > 1$. Let $0 < {\rm c} < 1$. Then*

$$i)\ \Omega^{\rm e+}\Big(\mathcal{H}_{\rm pp}(H^{\rm e}) \otimes \Gamma(\mathfrak{h}^{\rm e})\Big) \subset \mathcal{H}_{\rm c}^{\rm e+} \subset \mathcal{H}^{\rm e+}.$$

$$ii)\ W^{\rm e+}(h) : \mathcal{H}_{\rm c}^{\rm e+} \to \mathcal{H}_{\rm c}^{\rm e+}\ \textit{for}\ h \in \mathfrak{h}_0^{\rm e}.$$

$$iii)\ \mathfrak{h}_0^{\rm e} \ni h \mapsto W^{\rm e+}(h) \in \mathcal{U}(\mathcal{H}_{\rm c}^{\rm e+})$$

is a regular CCR representation of Fock type.

Proof. We will use the notation of Sect. 5. Let us first prove *i)*. To prove the first inclusion it suffices by density to show that if $u \in \mathcal{D}(H^{\rm e})$ with $H^{\rm e}u = \lambda u$, $u = \chi(L)u$ for $\chi \in C_0^\infty(\mathbb{R})$ and $h_i \in \mathfrak{h}_0^{\rm e} \cap \mathcal{D}(|\sigma|^{\frac{1}{2}})$ for $1 \le i \le n$ we have $v = \Pi_{i=1}^n a^{*{\rm e}+}(h_i)u \in \mathcal{H}_{\rm c}^{\rm e+}$. This will follow from the fact that

$$\text{s-}\lim_{\epsilon\to 0}\epsilon^{-1}R_{{\rm c}'}^{\rm e+}(\epsilon^{-1})v = v,\ \text{for } {\rm c}' > {\rm c}. \tag{8.20}$$

As usual to simplify notation we denote c' by c. By Thm. 8.6, we have:

$$\epsilon^{-1}R_{\mathrm{c}}^{\mathrm{e}+}(\epsilon^{-1})v=\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(\epsilon B_{\mathrm{c}\,t}+\mathbb{1})^{-1}\Pi_{i=1}^{n}a^{*}(h_{i,t})\chi(L)\mathrm{e}^{-\mathrm{i}t\lambda}u,$$

and hence

$$\lim_{\epsilon\to 0}\|v-\epsilon^{-1}R_{\mathrm{c}}^{\mathrm{e}+}(\epsilon^{-1})v\|\leq\lim_{\epsilon\to 0}\sup_{t\in\mathbb{R}^{+}}\|(\epsilon B_{\mathrm{c}\,t}+\mathbb{1})^{-1}\epsilon B_{\mathrm{c}\,t}\Pi_{i=1}^{n}a^{*}(h_{i,t})\chi(L)u\|.$$

Since

$$\|(\epsilon B_{\mathrm{c}\,t}+\mathbb{1})^{-1}\epsilon B_{\mathrm{c}\,t}\|\leq 1,$$

$$\|\Pi_{i=1}^{n}a^{*}(h_{i,t})\chi(L)\|\leq C_{n}\Pi_{i=1}^{n}\|(1+|\sigma|^{-\frac{1}{2}}+|\sigma|^{\frac{1}{2}})h_{i}\|,$$

by Lemma 3.11, it suffices by density to show that

$$\lim_{\epsilon\to 0}\sup_{t\in\mathbb{R}^{+}}\|(\epsilon B_{\mathrm{c}\,t}+\mathbb{1})^{-1}\epsilon B_{\mathrm{c}\,t}\Pi_{i=1}^{n}a^{*}(h_{i,t})\chi(L)u\|=0,$$

for $u\in\mathcal{D}(N^{\mathrm{e}})$. This follows from the fact that

$$\|B_{\mathrm{c}\,t}\Pi_{i=1}^{n}a^{*}(h_{i,t})(N^{\mathrm{e}}+1)^{-n/2-1}\|\leq C_{n}\Pi_{i=1}^{n}\|h_{i}\|,$$

$$\|(N^{\mathrm{e}}+1)^{k}\chi(L)(N^{\mathrm{e}}+1)^{-k}\|<\infty,$$

by Prop. 3.8. This proves (8.20) and hence the first inclusion in *i)*.

To prove the second inclusion, it suffices to show that $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}\subset\overline{\mathcal{D}(N^{\mathrm{e}+})}=\mathcal{H}^{\mathrm{e}+}$, where $N^{\mathrm{e}+}$ is the asymptotic number operator associated to the representation $\mathfrak{h}_{0}^{\mathrm{e}}\ni h\mapsto W^{\mathrm{e}+}(h)$. By a density argument, it suffices to show that if $u\in\mathcal{H}^{\mathrm{e}}$ with $\chi(L)u=u$, $\chi\in C_{0}^{\infty}(\mathbb{R})$ and $\lambda>0$ then $R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u\in\mathcal{D}(N^{\mathrm{e}+})$. We have for $u\in\mathcal{H}^{\mathrm{e}}$, $\chi(L)u=u$ and $h\in\mathfrak{h}_{0}^{\mathrm{e}}$:

$$\begin{aligned}a^{\mathrm{e}+}(h)R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u=&\ a^{\mathrm{e}+}(h)\chi(L)R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u\\=&\ \lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}a(h_{t})\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u\\=&\ \lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}a(h_{t})\chi(L)(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u,\end{aligned}$$

using Thm. 8.6 and the fact that $a(h_{t})\chi(L)$ is uniformly bounded in t, by Prop. A.1. Then

$$\begin{aligned}&\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}a(h_{t})\chi(L)(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u\\=&\ \lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}a(h_{t})\chi(L)(B_{\mathrm{c}\,t}+\lambda)^{-1}F(\tfrac{N_{t}^{\mathrm{e}}}{t^{\delta}})\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u\\=&\ \lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}a(h_{t})(B_{\mathrm{c}\,t}+\lambda)^{-1}\chi(L)F(\tfrac{N_{t}^{\mathrm{e}}}{t^{\delta}})\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u,\end{aligned}$$

by Prop. 4.5 and Lemma 5.3.

Let now $\mathfrak{f}\subset\mathfrak{h}_{0}^{\mathrm{e}}$ be a finite dimensional space, $h_{1},\dots,h_{p}$ an orthonormal basis of $\mathfrak{f}$. Let $h_{i}^{\epsilon}\in C_{0}^{\infty}(\mathbb{R}\backslash\{0\})\otimes\mathfrak{g}$ such that $h_{i}^{\epsilon}\to h_{i}$ in $\mathcal{D}(\mathfrak{h}_{0}^{\mathrm{e}})$ when $\epsilon\to 0$. Since by Thm. 8.6

$$\mathfrak{h}_{0}^{\mathrm{e}}\ni h\mapsto a^{\mathrm{e}+}(h)(L+b)^{-\frac{1}{2}}\in\mathcal{B}(\mathcal{H}^{\mathrm{e}})$$

is norm continuous, we have:

$$\sum_{i=1}^{p}\|a^{\mathrm{e}+}(h_{i})\chi(L)u\|^{2}=\lim_{\epsilon\to 0}\sum_{i=1}^{p}\|a^{\mathrm{e}+}(h_{i}^{\epsilon})\chi(L)u\|^{2},\ u\in\mathcal{H}^{\mathrm{e}}.\tag{8.21}$$

Next we observe that by stationary phase estimates, we have for $h \in C_0^\infty(\mathbb{R}\backslash\{0\}) \otimes \mathfrak{g}$:

$$(1 - b_{\mathrm{c}\,t}^{\frac{1}{2}})h_t \in O(t^{-\infty}). \tag{8.22}$$

Let now

$$P_t^\epsilon = \sum_{i=1}^{p} |h_{i,t}^\epsilon\rangle\langle h_{i,t}^\epsilon| = \mathrm{e}^{-\mathrm{i}t\sigma} P_0^\epsilon \mathrm{e}^{\mathrm{i}t\sigma}.$$

By (8.22) we have:

$$\|(1 - b_{\mathrm{c}\,t}^{\frac{1}{2}})P_t^\epsilon\| \leq C_{\epsilon,n_0} t^{-n_0},\ \forall n_0 \in \mathbb{N},$$

and hence

$$\begin{array}{rl} P_t^\epsilon = & b_{\mathrm{c}\,t}^{\frac{1}{2}} P_t^\epsilon b_{\mathrm{c}\,t}^{\frac{1}{2}} + b_{\mathrm{c}\,t}^{\frac{1}{2}} P_t^\epsilon (1 - b_{\mathrm{c}\,t}^{\frac{1}{2}}) + (1 - b_{\mathrm{c}\,t}^{\frac{1}{2}}) P_t^\epsilon \\ \leq & b_{\mathrm{c}\,t}^{\frac{1}{2}} P_t^\epsilon b_{\mathrm{c}\,t}^{\frac{1}{2}} + C_{\epsilon,n_0} t^{-n_0} \\ \leq & \|P_0^\epsilon\| b_{\mathrm{c}\,t} + C_{\epsilon,n_0} t^{-n_0}. \end{array} \tag{8.23}$$

Hence we obtain

$$\sum_{i=1}^{p} a^*(h_{i,t}^\epsilon) a(h_{i,t}^\epsilon) = \mathrm{d}\Gamma(P_t^\epsilon) \leq \|P_0^\epsilon\| \mathrm{d}\Gamma(b_{\mathrm{c}\,t}) + C_{\epsilon,n_0} N^{\mathrm{e}} t^{-n_0},\ \forall n_0 \in \mathbb{N}. \tag{8.24}$$

We now write for $u \in \mathcal{H}^{\mathrm{e}}, \chi(L)u = u$:

$$\begin{array}{rl} & \sum_{i=1}^{p} \|a^{\mathrm{e}+}(h_i^\epsilon) R_{\mathrm{c}}^{\mathrm{e}+}(\lambda) u\|^2 \\ = & \lim_{t\to+\infty} \sum_{i=1}^{p} \|a(h_{i,t}^\epsilon)(B_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L) F(\frac{N_t^{\mathrm{e}}}{t^\delta}) \chi(L) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u\|^2 \\ \leq & \overline{\lim}_{t\to+\infty} \|P_0^\epsilon\| \|B_{\mathrm{c}\,t}^{\frac{1}{2}} (B_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L) F(\frac{N_t^{\mathrm{e}}}{t^\delta}) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u\|^2 \\ & + \overline{\lim}_{t\to+\infty} C_{\epsilon,n_0} t^{-n_0} \|(N^{\mathrm{e}})^{\frac{1}{2}} (B_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L) F(\frac{N_t^{\mathrm{e}}}{t^\delta}) \chi(L) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u\|^2. \end{array}$$

We have

$$\begin{array}{rl} & \|(N^{\mathrm{e}})^{\frac{1}{2}} (B_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L) F(\frac{N_t^{\mathrm{e}}}{t^\delta}) \chi(L)\| \\ \leq & C \|(N^{\mathrm{e}})^{\frac{1}{2}} \chi(L) (N+1)^{-\frac{1}{2}}\| \|(N^{\mathrm{e}} + 1)^{\frac{1}{2}} F(\frac{N_t^{\mathrm{e}}}{t^\delta}) \chi(L)\| \\ \leq & C_0 t^{\delta/2}, \end{array}$$

by Lemma 3.5 *iii)* and Lemma 4.6. On the other hand:

$$\|B_{\mathrm{c}\,t}^{\frac{1}{2}} (B_{\mathrm{c}\,t} + \lambda)^{-1} \chi(L) F(\frac{N_t^{\mathrm{e}}}{t^\delta}) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u\|^2 \leq \lambda^{-1} \|u\|^2.$$

This yields

$$\sum_{i=1}^{p} \|a^{\mathrm{e}+}(h_i^\epsilon) R_{\mathrm{c}}^{\mathrm{e}+}(\lambda) u\|^2 \leq C(\|P_0^\epsilon\| + \lambda^{-1}) \|u\|^2, \tag{8.25}$$

uniformly in ϵ, p. Note that since $h_i^\epsilon \to h_i$ in $\mathfrak{h}^{\mathrm{e}}$ and $h_1, \ldots, h_p$ is an orthonormal family, we have $\|P_0^\epsilon\| \to 1$ when $\epsilon \to 0$.

Using (8.21) and letting $\epsilon \to 0$ in (8.25), we get:

$$\sum_{i=1}^{p} \|a^{\mathrm{e}+}(h_i) R_{\mathrm{c}}^{\mathrm{e}+}(\lambda) u\|^2 \leq C(1+\lambda^{-1})\|u\|^2,$$

uniformly in p. By the definition of $\mathcal{D}(N^{\mathrm{e}+})$ recalled in Subsect. 8.2 (see also [DG3, Sect. 4.2]), this implies that $R_{\mathrm{c}}^{\mathrm{e}+}(\lambda)u \in \mathcal{D}(N^{\mathrm{e}+})$ for any $\lambda > 0$ and completes the proof of *i)*.

Let us now prove *ii)*. We have to show that for $u \in \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$, $h \in \mathfrak{h}_0^{\mathrm{e}}$ and $\mathrm{c} < \mathrm{c}' < 1$:

$$\lim_{\epsilon \to 0} (1\!\!1 - \epsilon^{-1} R_{\mathrm{c}'}^{\mathrm{e}+}(\epsilon^{-1})) W^{\mathrm{e}+}(h) u = 0.$$

Since $1\!\!1 - \epsilon^{-1} R_{\mathrm{c}'}(\epsilon^{-1})$ is uniformly bounded in ϵ it suffices to show by density that for $\lambda > 0$:

$$\lim_{\epsilon \to 0} (1\!\!1 - \epsilon^{-1} R_{\mathrm{c}'}^{\mathrm{e}+}(\epsilon^{-1})) W^{\mathrm{e}+}(h) R_{\mathrm{c}'}^{\mathrm{e}+}(\lambda) u = 0. \tag{8.26}$$

We set $\mathrm{c}' = \mathrm{c}$ to shorten notation and we have:

$$\begin{array}{rl} & (1\!\!1 - \epsilon^{-1} R_{\mathrm{c}}^{\mathrm{e}+}(\epsilon^{-1})) W^{\mathrm{e}+}(h) R_{\mathrm{c}}^{\mathrm{e}+}(\lambda) u \\ = & \lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} (1\!\!1 - (\epsilon B_{\mathrm{c}\,t} + 1\!\!1)^{-1}) \mathrm{e}^{\mathrm{i}\phi(h_t)} (B_{\mathrm{c}\,t} + \lambda)^{-1} \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u. \end{array}$$

From the identity

$$\mathrm{e}^{-\mathrm{i}\phi(h)} \mathrm{d}\Gamma(b) \mathrm{e}^{\mathrm{i}\phi(h)} = \mathrm{d}\Gamma(b) + \phi(\mathrm{i}bh) - \frac{1}{2} Re(bh, h)$$

for $h \in \mathfrak{h}^{\mathrm{e}}$, $b \in \mathcal{B}(\mathfrak{h}^{\mathrm{e}})$, we obtain:

$$\begin{array}{rl} & (1\!\!1 - (\epsilon B_{\mathrm{c}\,t} + 1\!\!1)^{-1}) \mathrm{e}^{\mathrm{i}\phi(h_t)} (B_{\mathrm{c}\,t} + \lambda)^{-1} \\ = & \epsilon (\epsilon B_{\mathrm{c}\,t} + 1\!\!1)^{-1} \mathrm{e}^{\mathrm{i}\phi(h_t)} \Big(B_{\mathrm{c}\,t} + \phi(\mathrm{i}b_{\mathrm{c}\,t} h_t) - \frac{1}{2} Re(b_{\mathrm{c}\,t} h_t, h_t) \Big) (B_{\mathrm{c}\,t} + \lambda)^{-1}. \end{array}$$

By Prop. A.1 this yields

$$\begin{array}{rl} & \|(1\!\!1 - (\epsilon B_{\mathrm{c}\,t} + 1\!\!1)^{-1}) \mathrm{e}^{\mathrm{i}\phi(h_t)} (B_{\mathrm{c}\,t} + \lambda)^{-1}\| \\ \leq & C(\lambda)\epsilon (\|b_{\mathrm{c}\,t}^{\frac{1}{2}} h_t\| + \|b_{\mathrm{c}\,t} h_t\| + 1). \end{array}$$

Since $\|b_{\mathrm{c}\,t}\| \leq 1$, $\|h_t\| = \|h\|$, we obtain (8.26).

Finally property *iii)* follows from *i)*, *ii)* and Thm. 8.5. $\Box$

We end this section with another similar result.

Proposition 8.8 *Assume (I'0), (I'1) for* $\epsilon_0 > 0$, *(I'2) for* $\mu > 1$ *and* $\rho(1+\epsilon_0) > 1$. *Let* $0 < \mathrm{c} < \mathrm{c}' < 1$. *Let* f_0 *be a cutoff function as in (6.1). Then*

$$\mathrm{Ran}\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_0) \subset \mathcal{K}^{\mathrm{e}+}.$$

Proof. By a density argument using the fact that $\mathcal{K}^{\mathrm{e}+}$ is closed, it suffices to show that if $u \in \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$, $\chi(L)u = u$ for $\chi \in C_0^\infty(\mathbb{R})$

$$(L+\mathrm{i})^{-1}a^{\mathrm{e}+}(h)\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_0)u = 0,\ \forall h \in \mathfrak{h}_0^{\mathrm{e}}. \tag{8.27}$$

Since by Thm. 8.6 the map

$$\mathfrak{h}_0^{\mathrm{e}} \ni h \mapsto a^{*\mathrm{e}+}(h)(L+\mathrm{i})^{-1} \in \mathcal{B}(\mathcal{H}^{\mathrm{e}})$$

is norm continuous, it suffices to prove (8.27) for $h \in C_0^\infty(\mathbb{R}\backslash\{0\}) \otimes \mathfrak{g}$. Let us again set $\mathrm{c}' = \mathrm{c}$ to simplify the notation. We have by Thm. 8.6, Prop. 4.5 and the fact that $(L+\mathrm{i})^{-1}a(h_t)$ is uniformly bounded:

$$\begin{array}{rl} & (L+\mathrm{i})^{-1}a^{\mathrm{e}+}(h)\Gamma_{\mathrm{c}}^{\mathrm{e}+}(f_0)u \\ = & \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(L+\mathrm{i})^{-1}a(h_t)\Gamma(f_0^t)F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u \\ = & \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(L+\mathrm{i})^{-1}a(f_1^t h_t)\Gamma(f_0^t)F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u, \end{array}$$

where $f_1^t = f_1(\frac{s-\mathrm{c}t}{t^\rho})$, f_1 a cutoff function as f_0 with $f_1 f_0 = f_0$.

By stationary phase estimates, since $0 < \mathrm{c} < 1$ $\|f_1^t h_t\|_{\mathfrak{h}^{\mathrm{e}}} \in O(t^{-\infty})$ for $h \in C_0^\infty(\mathbb{R}\backslash\{0\}) \otimes \mathfrak{g}$ and hence

$$\begin{array}{rl} & \|a(f_1^t h_t)\Gamma(f_0^t)F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\| \\ \leq & \|f_1^t h_t\|\|(N^{\mathrm{e}}+1)^{\frac{1}{2}}\Gamma(f_0^t)F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\| \\ \leq & \|f_1^t h_t\|\|(N^{\mathrm{e}}+1)^{\frac{1}{2}}F(\frac{N_t^{\mathrm{e}}}{t^\delta})\chi(L)\| \in O(t^{-\infty}), \end{array}$$

by Lemma 4.6. Hence (8.27) holds for all $h \in C_0^\infty(\mathbb{R}\backslash\{0\}) \otimes \mathfrak{g}$, which completes the proof. □

9 Geometric asymptotic completeness for H^{e}

In this section we prove the *geometric asymptotic completeness* for H^{e}. This property is a geometric characterization of the space $\mathcal{K}_{\mathrm{c}}^{\mathrm{e}+} = \mathcal{K}^{\mathrm{e}+} \cap \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$. The space $\mathcal{K}_{\mathrm{c}}^{\mathrm{e}+}$ is the space of vacuum states in $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$. We show in Thm. 9.5 that those states are localized in the region $\{|s| \leq \mathrm{c}'t\}$, for any $\mathrm{c} < \mathrm{c}'$.

We assume in this section hypotheses *(I'0)*, *(I'1)* for $\epsilon_0 > 0$, *(I'2)* for $\mu > 1$. We pick the constant $0 < \rho < 1$ such that $\rho(1+\epsilon_0) > 1$.

9.1 Technical preparations

Lemma 9.1 *Let $j_0, j_\infty, b \in \mathcal{B}(\mathfrak{h}^{\mathrm{e}})$, $j_0, j_\infty \geq 0$, $j_0 + \alpha j_\infty \leq 1\!\!1$ for $\alpha > 0$. Then for $u_1, u_2 \in \mathcal{H}^{\mathrm{e}}$:*

$$\begin{array}{rl} & |(u_2, P_k(j_0, j_\infty + b)u_1) - (u_2, P_k(j_0, j_\infty)u_1)| \\ \leq & \sum_{r=1}^k \alpha^{r-k}\|\mathrm{d}\Gamma(|b|)^{r/2}u_2\|\|\mathrm{d}\Gamma(|b|)^{r/2}u_1\|. \end{array}$$

Proof. We have on $\otimes_{\mathrm{s}}^{n}\mathfrak{h}^{\mathrm{e}}$:

$$\begin{array}{rl} & P_k(j_0, j_\infty + b) - P_k(j_0, j_\infty) \\ = & \sum_{r=1}^{k}\sum_{1\leq j_1,\dots,j_r\leq n}\sum_{\sharp\{i|\epsilon_i=\infty\}=k-r} j_{\epsilon_1}\otimes\cdots\otimes \underbrace{b}_{j_1}\otimes j_{\epsilon_{j+1}}\otimes\cdots\otimes\underbrace{b}_{j_r}\otimes\cdots\otimes j_{\epsilon_n} \\ = & \sum_{r=1}^{k}\sum_{1\leq j_1,\dots,j_r\leq n} M_{j_1,\dots,j_r}T_{j_1,\dots,j_r}, \end{array}$$

for

$$\begin{array}{l} M_{j_1,\dots,j_r} = \sum_{\sharp\{i|\epsilon_i=\infty\}=k-r} j_{\epsilon_1}\otimes\cdots\otimes\underbrace{1\!\!1}_{j_1}\otimes j_{\epsilon_{j+1}}\otimes\cdots\otimes\underbrace{1\!\!1}_{j_r}\otimes\cdots\otimes j_{\epsilon_n}, \\ T_{j_1,\dots,j_r} = 1\!\!1\otimes\cdots\otimes\underbrace{b}_{j_1}\otimes 1\!\!1\otimes\cdots\otimes\underbrace{b}_{j_r}\otimes\cdots\otimes 1\!\!1. \end{array}$$

Note that

$$\sum_{1\leq j_1,\dots,j_r\leq n}|T_{j_1,\dots,j_r}| = (\mathrm{d}\Gamma|b|)^r. \tag{9.1}$$

Since $j_0, j_\infty \geq 0$, $j_0 + \alpha j_\infty \leq 1\!\!1$, we obtain by replacing j_∞ by αj_∞ that

$$\|M_{j_1,\dots,j_r}\| \leq \alpha^{r-k}.$$

For $u_1, u_2 \in \otimes_{\mathrm{s}}^{n}\mathfrak{h}^{\mathrm{e}}$, we obtain:

$$\begin{array}{rl} & \|(u_2, P_k(j_0, j_\infty + b)u_1) - (u_2, P_k(j_0, j_\infty)u_1) \\ \leq & \sum_{r=1}^{k}\sum_{1\leq j_1,\dots,j_r\leq n}\alpha^{r-k}\||T_{j_1,\dots,j_r}|^{\frac12}u_1\|\||T_{j_1,\dots,j_r}|^{\frac12}u_2\| \\ \leq & \sum_{r=1}^{k}\left(\sum_{1\leq j_1,\dots,j_r\leq n}\alpha^{r-k}(u_1, |T_{j_1,\dots,j_r}|u_1)\right)^{\frac12}\left(\sum_{1\leq j_1,\dots,j_r\leq n}\alpha^{r-k}(u_2, |T_{j_1,\dots,j_r}|u_2)\right)^{\frac12} \\ = & \sum_{r=1}^{k}\alpha^{r-k}\|\mathrm{d}\Gamma(|b|)^{r/2}u_1\|\|\mathrm{d}\Gamma(|b|)^{r/2}u_2\|, \end{array}$$

by (9.1). $\square$

We pick for $0 < \epsilon < 1$ a cutoff function $F_\epsilon \in C_0^\infty([\epsilon, \epsilon^{-1}])$ with $0 \leq F_\epsilon \leq 1$ and set

$$b_\epsilon := F_\epsilon(\sigma).$$

Lemma 9.2 *The operator*

$$(L + \mathrm{i})^{-k/2} I 1\!\!1 \otimes \Gamma(b_\epsilon)$$

is bounded on $\mathcal{H}^{\mathrm{e}} \otimes \bigotimes_{\mathrm{s}}^{k}\mathfrak{h}^{\mathrm{e}}$ *for* $k \in \mathbb{N}$.

Proof. We first claim that if B is an open set included in $\mathbb{R}\backslash\{0\} \times S^2$, $p \in \mathbb{N}$ then

$$\int_{B^p}\|a(\sigma_1,\omega_1)\cdots a(\sigma_p,\omega_p)u\|^2\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_p\mathrm{d}\omega_1\cdots\mathrm{d}\omega_p \leq C_p\|(L+b)^{p/2}u\|^2,\ u \in \mathcal{H}^{\mathrm{e}}. \tag{9.2}$$

In fact we write

$$\mathfrak{h}^{\mathrm{e}} = L^2(B, \mathrm{d}\sigma\mathrm{d}\omega) \oplus L^2(B^{\mathrm{c}}, \mathrm{d}\sigma\mathrm{d}\omega) =: \mathfrak{h}_B^{\mathrm{e}} \oplus \mathfrak{h}_B^{\mathrm{e}\perp}.$$

Let U be the canonical map from $\Gamma(\mathfrak{h}^{\mathrm{e}})$ into $\Gamma(\mathfrak{h}_B^{\mathrm{e}}) \otimes \Gamma(\mathfrak{h}_B^{\mathrm{e}\perp})$. Then:

$$\begin{array}{l} Ua(\sigma,\omega) = (a(\sigma,\omega)\otimes 1\!\!1)U, \text{ for } (\sigma,\omega) \in B, \\ U\mathrm{d}\Gamma(1\!\!1_B) = N^{\mathrm{e}} \otimes 1\!\!1 U. \end{array} \tag{9.3}$$

This yields:

$$\begin{array}{rl} & \int_{B^p} \|\Pi_1^p a(\sigma_i,\omega_i)u\|^2 \mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_p\mathrm{d}\omega_1\cdots\mathrm{d}\omega_p \\ = & \int_{B^p} \|\Pi_1^p a(\sigma_i,\omega_i)\otimes 1\!\!1 Uu\|^2 \mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_p\mathrm{d}\omega_1\cdots\mathrm{d}\omega_p \\ = & (N^{\mathrm{e}}\cdots(N^{\mathrm{e}}-p+1)\otimes 1\!\!1 Uu, Uu) \\ \leq & C_p((N^{\mathrm{e}}+1)^p\otimes 1\!\!1 Uu, Uu) \\ = & C_p\|(\mathrm{d}\Gamma(1\!\!1_B)+1\!\!1)^{p/2}u\|^2, \end{array}$$

by (9.3). Using then Lemma 3.6 *iii)*, we obtain (9.2).

Next for $u, v\in\mathcal{H}^{\mathrm{e}}, \psi\in\otimes_{\mathrm{s}}^n\mathfrak{h}^{\mathrm{e}}$, we have:

$$\begin{array}{rl} & |(u,(L+\mathrm{i})^{-k/2}I1\!\!1\otimes\Gamma(b_\epsilon)v\otimes\psi)| \\ = & \frac{1}{k!^{\frac{1}{2}}}|\int\psi(\sigma_1,\omega_1,\ldots,\sigma_k,\omega_k)\Pi_{i=1}^k F_\epsilon(\sigma_i) \\ & \times\Big(a(\sigma_1,\omega_1)\cdots a(\sigma_k,\omega_k)(L-\mathrm{i})^{-k/2}u,v\Big)_{\mathcal{H}^{\mathrm{e}}}\mathrm{d}\sigma_1\cdots\mathrm{d}\sigma_k\mathrm{d}\omega_1\cdots\mathrm{d}\omega_k| \\ \leq & C_{k,\epsilon}\|\psi\|_{\otimes_{\mathrm{s}}^k\mathfrak{h}^{\mathrm{e}}}\|u\|_{\mathcal{H}^{\mathrm{e}}}\|v\|_{\mathcal{H}^{\mathrm{e}}}, \end{array}$$

by (9.2) for $B=]\epsilon,\epsilon^{-1}[\times S^2$ and Cauchy-Schwarz inequality. □

9.2 Geometric asymptotic completeness

Proposition 9.3 *Let $0\leq \mathrm{c}<\mathrm{c}'\leq 1$. Let j_0, j_∞ satisfying (6.1) and (6.4) with $j_\infty^{\frac{1}{2}}\in C^\infty(\mathbb{R})$ and let j_0^t, j_∞^t be as in (7.1) for the constant c'. Then*

$$\mathrm{w}-\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}P_k(j_0^t,(j_\infty^t)^{\frac{1}{2}}b_\epsilon(j_\infty^t)^{\frac{1}{2}})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}=:P_k^+(j_0,j_\infty^{\frac{1}{2}}b_\epsilon j_\infty^{\frac{1}{2}})$$

exists on $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ and equals $\Omega^{\mathrm{e}+}1\!\!1\otimes\Gamma(b_\epsilon)W_k^+(j)$.

Proof. By the definition of $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ it suffices to show the existence of the limit on $\mathrm{Ran}\hat{P}_{\mathrm{c}'}^{\mathrm{e}+}$. Changing notation we may replace c' by c. We first claim that for $\chi_1\in C_0^\infty(\mathbb{R})$:

(9.4) $$\chi_1(L)\Omega^{\mathrm{e}+}1\!\!1\otimes\Gamma(b_\epsilon)W_k^+(j)=\mathrm{s}\text{-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\chi_1(L)I1\!\!1\otimes\Gamma(b_\epsilon)\check{\Gamma}_k(j^t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}\text{ on }\mathrm{Ran}\hat{P}_{\mathrm{c}}^{\mathrm{e}+}.$$

First since by Lemma 9.2 $\chi_1(L)I1\!\!1\otimes\Gamma(b_\epsilon)$ is bounded, it suffices to prove (9.4) for $u\in\mathrm{Ran}\hat{P}_{\mathrm{c}}^{\mathrm{e}+}$ with $u=\chi(L)u$ for $\chi\in C_0^\infty(\mathbb{R})$.

We note first that by Thm. 7.7 *ii)*:

$$W_k^+(j)u=W_k^+(j)\chi(L)u=\chi(L_{\mathrm{ext}})W_k^+(j)u.$$

Moreover by Thm. 7.7 *iv)* and Prop. 8.8

$$1\!\!1\otimes\Gamma(b_\epsilon)W_k^+(j)u\in\mathcal{K}^{\mathrm{e}+}\otimes\Gamma(\mathfrak{h}^{\mathrm{e}})=\mathcal{D}(\Omega^{\mathrm{e}+}).$$

Also

$$\begin{array}{l} 1\!\!1\otimes\Gamma(b_\epsilon)W_k^+(j)u=\chi(L_{\mathrm{ext}})1\!\!1\otimes\Gamma(b_\epsilon)W_k^+(j)u \\ \in\mathcal{D}((L+\mathrm{i})^{k/2})\otimes\bigotimes_{\mathrm{s}}^k\mathcal{D}(|\sigma|^{-\frac{1}{2}}+|\sigma|^{\frac{1}{2}})\subset\mathcal{D}(\Omega_{\mathrm{ext}}^{\mathrm{e}+}). \end{array}$$

Hence for $\chi_1 \in C_0^\infty(\mathbb{R})$:

$$\begin{aligned} & \chi_1(L)\Omega^+ 1\!\!1 \otimes \Gamma(b_\epsilon) W_k^+(j) u \\ = \;& \chi_1(L)\Omega^{\mathrm{ext}+} 1\!\!1 \otimes \Gamma(b_\epsilon) W_k^+(j) u \\ = \;& \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}} \chi_1(L) I 1\!\!1 \otimes \Gamma(b_\epsilon) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{ext}}} W_k^+(j) u. \end{aligned}$$

Since by Lemma 9.2 $\chi_1(L) I 1\!\!1 \otimes \Gamma(b_\epsilon)$ is bounded, we can apply the chain rule and obtain (9.4).

Next we note that

$$I 1\!\!1 \otimes \Gamma(b_\epsilon) \check{\Gamma}(j^t) = P_k(j_0^t, b_\epsilon j_\infty^t),$$

which implies

$$\|\chi_1(L) P_k(j_0^t, b_\epsilon j_\infty^t)\| \leq C, \text{ uniformly in } t. \tag{9.5}$$

Note also that since $b_\epsilon \leq 1$, we have

$$j_0^t + \alpha (j_\infty^t)^{\frac{1}{2}} b_\epsilon (j_\infty^t)^{\frac{1}{2}} \leq j_0^t + \alpha j_\infty^t \leq 1,$$

and hence

$$\|P_k(j_0^t, (j_\infty^t)^{\frac{1}{2}} b_\epsilon (j_\infty^t)^{\frac{1}{2}})\| \leq C, \text{ uniformly in } t. \tag{9.6}$$

By (9.4), (9.5), (9.6) and a density argument, it suffices to prove the proposition to show that for $u, v \in \mathcal{D}((N^{\mathrm{e}})^\infty)$:

$$\lim_{t\to+\infty} (\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} v, \chi(L) \Big(P_k(j_0^t, b_\epsilon j_\infty^t) - P_k(j_0^t, (j_\infty^t)^{\frac{1}{2}} b_\epsilon (j_\infty^t)^{\frac{1}{2}}) \Big) \chi(L) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u) = 0. \tag{9.7}$$

We have $b_\epsilon j_\infty^t = (j_\infty^t)^{\frac{1}{2}} b_\epsilon (j_\infty^t)^{\frac{1}{2}} + r^t$, for

$$r^t = [b_\epsilon, (j_\infty^t)^{\frac{1}{2}}] (j_\infty^t)^{\frac{1}{2}}, \; r^t \in O(t^{-\rho}).$$

Let $b_{1,\epsilon}$ be a cutoff function similar to b_ϵ such that $b_{1,\epsilon} \equiv 1$ on $\operatorname{supp} b_\epsilon$. By p.d.o. calculus:

$$r^{t*} r^t = b_{1,\epsilon} r^{*t} r^t b_{1,\epsilon} + r_2^t, \; r_2^t \in O(t^{-\infty}),$$

and hence

$$r^{*t} r^t \leq C^2 t^{-2\rho} b_{1,\epsilon}^2 + r_\infty^2(t) 1\!\!1, \; r_\infty(t) \in O(t^{-\infty}).$$

Since the function $\lambda \mapsto \lambda^{\frac{1}{2}}$ is matrix monotone, this gives

$$|r^t| \leq (C^2 t^{-2\rho} b_{1,\epsilon}^2 + r_\infty^2(t) 1\!\!1)^{\frac{1}{2}} \leq C t^{-\rho} b_{1,\epsilon} + r_\infty(t) 1\!\!1. \tag{9.8}$$

Let $f(t)$ be the expression on the l.h.s. of (9.7). By Lemma 9.1, we have:

$$|f(t)| \leq \sum_{r=1}^{k} \alpha^{r-k} \|\mathrm{d}\Gamma(|r^t|)^{r/2} \chi(L) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} v\| \|\mathrm{d}\Gamma(|r^t|)^{r/2} \chi(L) \mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} u\|. \tag{9.9}$$

We note that if $a, b \in \mathcal{B}(\mathfrak{h}^{\mathrm{e}})$, $0 \leq a \leq b$, then

$$\mathrm{d}\Gamma(a)^r \leq \mathrm{d}\Gamma(b)^r, \; \forall r \in \mathbb{N}.$$

This follows from the fact that if $0 \leq a \leq b$ then $a \otimes a \leq b \otimes b$ and from easy computations on each $n-$particle sector. By (9.8), this gives

$$
\begin{array}{rl}
& \|\mathrm{d}\Gamma(|r^t|)^{r/2}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u\|^2 \\
\leq & (\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u, (Ct^{-\rho}\mathrm{d}\Gamma(b_{1,\epsilon}) + r_\infty(t)N)^r\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u) \\
\leq & \sum_{j=0}^{r} C^j t^{-j\rho} r_\infty(t)^{r-j}\|\mathrm{d}\Gamma(b_{1,\epsilon})^j\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u\|\|(N^{\mathrm{e}})^{r-j}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u\|.
\end{array}
$$

By Lemma 3.6 $\mathrm{d}\Gamma(b_{1,\epsilon})^j\chi(L)$ is bounded. By Prop. 3.8 and the fact that $u \in \mathcal{D}((N^{\mathrm{e}})^\infty)$ we know that $\|(N^{\mathrm{e}})^{r-j}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u\| \leq Ct^{r-j}$. Hence

$$
\lim_{t\to+\infty} \|\mathrm{d}\Gamma(|r^t|)^{r/2}\chi(L)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u\| = 0,
$$

which proves (9.7). □

Theorem 9.4 *Let $0 < \mathrm{c} < \mathrm{c}' < 1$. Let f_0, f_∞ be defined in (6.23), (6.24). Let $j = (f_0, f_\infty)$, j^t be as in (7.1) for the constant c'. Then:*

$$
\Omega^{\mathrm{e}+}W_k^+(j) = P_k^+(j) \ \text{on} \ \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}.
$$

Proof. By the same argument as in Prop. 9.3, we set $\mathrm{c}' = \mathrm{c}$ and reduce ourselves to prove the theorem on $\mathrm{Ran}\hat{P}_{\mathrm{c}}^{\mathrm{e}+}$. We first note that

$$
W_k^+(j)\mathrm{Ran}\hat{P}_{\mathrm{c}}^{\mathrm{e}+} \subset \mathcal{K}^{\mathrm{e}+} \otimes \bigotimes_{\mathrm{s}}^{k} \mathfrak{h}^{\mathrm{e}} \subset \mathcal{D}(\Omega^{\mathrm{e}+}),
$$

by Prop. 8.8 and Thm. 7.7 *iv)*. By Prop. 6.6

$$
P_k^+(j) = \mathrm{w} - \lim_{\epsilon\to 0} P_k^+(f_0, f_{\infty,\epsilon}),
$$

and by Prop. 7.8

$$
W_k^+(j) = \mathrm{w} - \lim_{\epsilon\to 0} W_k^+(f_0, f_{\infty,\epsilon}).
$$

Hence it suffices to prove that for any $\epsilon_0 > 0$:

$$
\Omega^{\mathrm{e}+}W_k^+(f_0, f_{\infty,\epsilon_0}) = P_k^+(f_0, f_{\infty,\epsilon_0}). \tag{9.10}
$$

We note the following identity, similar to those in [DG2, Lemma 2.14], valid for $r_0, r_\infty, b \in \mathcal{B}(\mathfrak{h}^{\mathrm{e}})$, $0 \leq r_0 + \alpha r_\infty \leq 1, 0 \leq b \leq 1$, $r = (r_0, r_\infty)$:

$$
\check{\Gamma}(r)^* 1\!\!1 \otimes \Gamma(b) 1\!\!1_{\{k\}}(N_\infty)\check{\Gamma}(r) = P_k(r_0^2, r_\infty b r_\infty). \tag{9.11}
$$

Let us fix the constant ϵ_0. We will apply (9.11) to $r_0 = f_0^{\frac{1}{2}}$, $r_\infty = f_{\infty,\epsilon_0}^{\frac{1}{2}}$, $b = b_\epsilon$, where b_ϵ is defined in Lemma 9.2. Note that by Lemma 6.5 there exists $\alpha > 0$ such that $f_0^{\frac{1}{2}} + \alpha f_{\infty,\epsilon_0}^{\frac{1}{2}} \leq 1$, so we can apply this identity. By Thm. 7.7 *i)* and *vii)*, (9.11) and the chain rule of wave operators, we get:

$$
W_k^{+*}(r) 1\!\!1 \otimes \Gamma(b_\epsilon) W_k^+(r) = P_k^+(f_0, f_{\infty,\epsilon_0}^{\frac{1}{2}} b_\epsilon f_{\infty,\epsilon_0}^{\frac{1}{2}}).
$$

By Prop. 9.3, we obtain:

$$W_k^+(r)^* 1\!\!1 \otimes \Gamma(b_\epsilon) W_k^+(r) = \Omega^{\mathrm{e}+} 1\!\!1 \otimes \Gamma(b_\epsilon) W_k^+(f_0, f_{\infty,\epsilon_0}).$$

Now since $W_k^+(r), \Omega^{\mathrm{e}+}$ are bounded operators:

$$\begin{aligned}
&\text{s-}\lim_{\epsilon\to 0} W_k^+(r)^* 1\!\!1 \otimes \Gamma(b_\epsilon) W_k^+(r) = W_k^+(r)^* W_k^+(r) = P_k^+(f_0, f_{\infty,\epsilon_0}),\\
&\text{s-}\lim_{\epsilon\to 0} \Omega^+ 1\!\!1 \otimes \Gamma(b_\epsilon) W_k^+(f_0, f_{\infty,\epsilon_0}) = \Omega^+ W_k^+(f_0, f_{\infty,\epsilon_0}).
\end{aligned}$$

Hence (9.10) holds and this completes the proof of the theorem. □

The following theorem is the so called *geometric asymptotic completeness.* It provides a geometric characterization of the asymptotic vacuum states.

Theorem 9.5 *Assume (I'0), (I'1) for* $\epsilon_0 > 0$*, (I'2) for* $\mu > 1$ *and let* $\rho(1+\epsilon_0) > 1$*. Let* $0 < \mathrm{c} < \mathrm{c}' < 1$*. Let* $f_1 \in C^\infty(\mathbb{R})$ *be a cutoff function such that* $0 \le f_1 \le 1$*,* $f_1 \equiv 1$ *in* $\{s \le \alpha_1\}$*,* $f_1 \equiv 0$ *in* $\{s \ge \alpha_2\}$ *and* $f_1^t = f_1(\frac{s - \mathrm{c}' t}{t^\rho})$*.*

Then $\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_1)$ *defined in (6.17) is equal to the orthogonal projection on* $\mathcal{K}_{\mathrm{c}}^{\mathrm{e}+} := \mathcal{K}^+ \cap \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$*.*

Proof. By Prop. 8.8 we know that $\mathrm{Ran}\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_1) \subset \mathcal{K}^{\mathrm{e}+}$ and since $\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_1)$ clearly preserves $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$, we have $\mathrm{Ran}\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_1) \subset \mathcal{K}_{\mathrm{c}}^{\mathrm{e}+}$.

Let us prove the converse inclusion. By Thm. 8.7, $\mathfrak{h}_0^{\mathrm{e}} \ni h \mapsto W^{\mathrm{e}+}(h) \in \mathcal{U}(\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+})$ is a regular CCR representation of Fock type. Hence the restriction $\Omega_{\mathrm{c}}^{\mathrm{e}+}$ of the wave operator $\Omega^{\mathrm{e}+}$ to $\mathcal{K}_{\mathrm{c}}^{\mathrm{e}+} \otimes \Gamma(\mathfrak{h}^{\mathrm{e}})$:

$$\Omega_{\mathrm{c}}^{\mathrm{e}+} : \mathcal{K}_{\mathrm{c}}^{\mathrm{e}+} \otimes \Gamma(\mathfrak{h}^{\mathrm{e}}) \to \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$$

is unitary. Let now $j = (f_0, f_\infty)$ as in Thm. 9.4 for a constant c'' with $\mathrm{c} < \mathrm{c}'' < \mathrm{c}'$. Let

$$W^+(j) = \oplus_{k=0}^\infty W_k^+(j).$$

Since $\|W_k^+(j)\| \le 1$ and $\mathrm{Ran} W_k^+(j) \subset \mathcal{H}^{\mathrm{e}} \otimes \bigotimes_{\mathrm{s}}^k \mathfrak{h}^{\mathrm{e}}$, we have $\|W^+(j)\| \le 1$. Next we note that by Thm. 7.6 *iii)* and Thm. 7.7 *vi)* we have

$$W^+(j)\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+} \subset \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+} \otimes \Gamma(\mathfrak{h}^{\mathrm{e}}).$$

Moreover by Thm. 7.7 *iv)* and the fact that $f_0^t f_1^t = f_0^t$ for t large enough, we have:

$$W^+(j) = \Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_1) \otimes 1\!\!1 W^+(j). \tag{9.12}$$

By Prop. 8.8 this implies that

$$W^+(j)\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+} \subset \mathcal{K}_{\mathrm{c}}^{\mathrm{e}+} \otimes \Gamma(\mathfrak{h}^{\mathrm{e}}).$$

Finally by Thm. 9.4 and Thm. 6.4 *iv)*

$$\Omega_{\mathrm{c}}^{\mathrm{e}+} W^+(j) = 1\!\!1 \text{ on } \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+},$$

which means that

$$W^+(j) = (\Omega_{\mathrm{c}}^{\mathrm{e}+})^{-1}.$$

By (9.12) this implies that $\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_1) = 1\!\!1$ on $\mathcal{K}_{\mathrm{c}}^{\mathrm{e}+}$, and hence that $\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_1)$ is the orthogonal projection on $\mathcal{K}_{\mathrm{c}}^{\mathrm{e}+}$.

10 1−particle space estimates

This section is devoted to some estimates on the one-particle space $L^2(\mathbb{R}^3, \mathrm{d}k)$. They will be used in Sects. 11 and 12 to construct the spaces analogous to $\mathcal{H}_\mathrm{c}^{\mathrm{e}+}$ for the Nelson Hamiltonian. The need for these estimates can be understood as follows:

The space $\mathcal{H}_\mathrm{c}^{\mathrm{e}+}$ is constructed using the observable $s = \mathrm{i}\partial_\sigma$ acting on $\mathfrak{h}^\mathrm{e} = L^2(\mathbb{R}, \mathrm{d}\sigma) \otimes \mathfrak{g}$. This observable has the drawback that it does not commute with the projection $1\!\!1_{\mathbb{R}^+}(\sigma)$ and hence does not satisfy the condition (3.6) in Subsect. 3.4. In Subsect. 10.2 we introduce the observable $|s|_0$ which is the square root of the Laplacian $-\frac{\partial^2}{\partial\sigma^2}$ with Dirichlet condition at 0 and satisfies (3.6).

We estimate the difference between some functions of s and $|s|_0$. It will allow us in Sect. 11 to reinterpret the space $\mathcal{H}_\mathrm{c}^{\mathrm{e}+}$ using the observable $|s|_0$. In this way a space $\mathcal{H}_\mathrm{c}^{+}$ can be constructed on $\mathcal{H}$ using the abstract arguments in Subsect. 3.4.

In Sect. 12 we describe the space $\mathcal{H}_\mathrm{c}^{+}$ replacing the observable $|s|_0$ by the more physical position observable $|x|$. We note that $|x|$ is the square root of the Laplacian $-\Delta_k$ acting on $L^2(\mathbb{R}^3, \mathrm{d}k)$. Going to polar coordinates we see that $|x|$ is the square root of $-\frac{\partial^2}{\partial\tilde\sigma^2} - \frac{\Delta_\omega}{\tilde\sigma^2}$ acting on $L^2(\mathbb{R}^+, \mathrm{d}\tilde\sigma)\otimes L^2(S^2)$ with a Dirichlet condition at 0. Again we need to estimate the difference between functions of $|x|$ and functions of $|s|_0$. This is done in Subsect. 10.1, by introducing cutoffs in the angular part $-\frac{\Delta_\omega}{\tilde\sigma^2}$ of the Laplacian. The use of these cutoffs in Sect. 12 will be justified using the results of Subsect. 4.5.

10.1 Case of $\mathfrak{h}$

We use the notation of Subsect. 1.1. We will consider the observable

$$|x| = (-\Delta_k)^{\frac{1}{2}}.$$

Note that $-\Delta_k$ with domain $H^2(\mathbb{R}^3)$ is also the Friedrichs extension of $-\Delta_k$ on $C_0^\infty(\mathbb{R}^3\backslash\{0\})$, since $H_0^1(\mathbb{R}^3\backslash\{0\}) = H^1(\mathbb{R}^3)$. Let

$$\begin{aligned}&\mathrm{u} : L^2(\mathbb{R}^3, \mathrm{d}k) \to L^2(\mathbb{R}^+, \mathrm{d}\tilde\sigma) \otimes L^2(S^2)\\ &\mathrm{u}\phi(\tilde\sigma, \omega) = \tilde\sigma\phi(\tilde\sigma\omega)\end{aligned}$$

be the unitary map introduced in Subsect. 3.1. We have

$$\mathrm{u}C_0^\infty(\mathbb{R}^3\backslash\{0\}) = C_0^\infty(]0, +\infty[) \otimes C^\infty(S^2)$$

and on $C_0^\infty(]0, +\infty[) \otimes C^\infty(S^2)$ we have:

$$\mathrm{u}(-\Delta_k)\mathrm{u}^{-1} = -\frac{\partial^2}{\partial\tilde\sigma^2} - \frac{\Delta_\omega}{\tilde\sigma^2},$$

where Δ_ω is the Laplacian on S^2. By the above remark this means that $\mathrm{u}(-\Delta_k)\mathrm{u}^{-1}$ is the Friedrichs extension of $-\frac{\partial^2}{\partial\tilde\sigma^2} - \frac{\Delta_\omega}{\tilde\sigma^2}$ on $C_0^\infty(]0, +\infty[) \otimes C^\infty(S^2)$.

Let now s_0^2 be the Friedrichs extension of $-\frac{\partial^2}{\partial\tilde\sigma^2}$ on $C_0^\infty(]0, +\infty[) \otimes C^\infty(S^2)$, ie

$$s_0^2 = -\frac{\partial^2}{\partial\tilde\sigma^2},\ \mathcal{D}(s_0^2) = H_0^1(]0, +\infty[) \otimes L^2(S^2) \cap H^2(]0, +\infty[) \otimes L^2(S^2).$$

Then we set:

$$a_0 := (s_0^2)^{\frac{1}{2}},\ a := \mathrm{u}|x|\mathrm{u}^{-1}. \tag{10.1}$$

We note that for $u \in \mathcal{D}(a_0) = H_0^1(]0,+\infty[) \otimes L^2(S^2)$, we have:

$$\|a_0 u\|^2 = \int_0^{+\infty} |\frac{\partial}{\partial\tilde{\sigma}} u|^2 \mathrm{d}\tilde{\sigma},$$

hence for $z \in \mathbb{C}\backslash\mathbb{R}$

$$\|\frac{\partial}{\partial\tilde{\sigma}}(a_0 \pm z)^{-1}\| = \|a_0(a_0 \pm z)^{-1}\| \leq 1. \tag{10.2}$$

By duality we also have

$$\|(a_0 \pm z)^{-1}\frac{\partial}{\partial\tilde{\sigma}}\| \leq 1. \tag{10.3}$$

Let now $f \in C^\infty(\mathbb{R})$ with

$$f(\lambda) \equiv 1 \text{ for } \lambda \gg 1,\ f \equiv 0 \text{ for } \lambda \ll -1. \tag{10.4}$$

We set for $0 < \mathrm{c} \leq 1$, $0 < \rho < 1$:

$$b_{0,t} := f(\tfrac{a_0 - \mathrm{c}t}{t^\rho}) + f(\tfrac{-a_0 - \mathrm{c}t}{t^\rho}),$$
$$b_t := f(\tfrac{a - \mathrm{c}t}{t^\rho}) + f(\tfrac{-a - \mathrm{c}t}{t^\rho}).$$

We also set for $\delta, \rho_1 > 0$:

$$\begin{aligned} g &:= F(\tfrac{-\Delta_\omega}{t^{\rho_1}\tilde{\sigma}^2} \leq 1)F(t^\delta\tilde{\sigma} \geq 1) \\ g_1 &:= F(\tfrac{-\Delta_\omega}{t^{\rho_1}\tilde{\sigma}^2} \leq 2)F(t^\delta\tilde{\sigma} \geq \tfrac{1}{2}), \end{aligned} \tag{10.5}$$

so that $g_1 g = g$.

Lemma 10.1 *Assume $\rho > \delta$. Then*

i) $(1 - g_1)(b_{0,t} + \mu + R)^{-1} \in O(t^{-\infty})$, *for* $\mu \in \mathbb{C}\backslash\mathbb{R}^-$, *uniformly for* $R \geq 0$,

ii) $(b_t - b_{0,t})g \in O(t^{\rho_1 - 2\rho}\log t)$.

Proof. As a preparation for the proof of Lemma 10.1, we first show:

Lemma 10.2 *We have:*

$$(1 - g_1)(z^2 - a_0^2)^{-1} g \in O(|\mathrm{Im} z|^{-N-2} t^{N\delta}),\ N \in \mathbb{N},\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{10.6}$$

$$a_0^2(1 - g_1)(z^2 - a_0^2)^{-1} g \in O(t^{N\delta}|\mathrm{Im} z|^{-N-2}\langle z\rangle) + O(t^{(N+2)\delta}|\mathrm{Im} z|^{-N-1}),\ N \in \mathbb{N}. \tag{10.7}$$

Proof. We have:

$$(z^2 - a_0^2)^{-1} g = g(z^2 - a_0^2)^{-1} + (z^2 - a_0^2)^{-1}[a_0^2, g](z^2 - a_0^2)^{-1},$$
$$[a_0^2, g] = -g'\tfrac{\partial}{\partial\tilde{\sigma}} - \tfrac{\partial}{\partial\tilde{\sigma}} g',\ \text{for } g' = \partial_{\tilde{\sigma}} g,$$

and hence

$$(1-g_1)(z^2-a_0^2)^{-1}g=(1-g_1)(z^2-a_0^2)^{-1}[a_0^2,g](z^2-a_0^2)^{-1}. \tag{10.8}$$

We observe that $\|\partial_{\tilde{\sigma}}^{\alpha}g\|\in O(t^{\delta\alpha})$.

Moreover if g_2 is another cutoff similar to g with $g_2g=g, g_2g_1=g_2$, we have $[a_0^2,g]=g_2[a_0^2,g]$ and

$$(1-g_1)(z^2-a_0^2)^{-1}g=(1-g_1)(z^2-a_0^2)^{-1}[a_0^2,g_2](z^2-a_0^2)^{-1}[a_0^2,g](z^2-a_0^2)^{-1}.$$

If we iterate N times the identity (10.8), we can write $(1-g_1)(z^2-a_0^2)^{-1}g$ as a finite sum of terms of the form

$$(1-g_1)(z^2-a_0^2)^{-1}\Pi_{j=1}^{N}R_j(z^2-a_0^2)^{-1}, \tag{10.9}$$

Using the form of $[a_0^2,g]$ given above and the estimates (10.2), (10.3), we see that we have

$$\|(a_0\pm z)^{-1}R_j\|\leq Ct^{\delta},\ \text{or}\ \|R_j(a_0\pm z)^{-1}\|\leq Ct^{\delta}.$$

We can rewrite (10.9) as:

$$(1-g_1)(z-a_0)^{-1}\left(\Pi_{j=1}^{N}(z+a_0)^{-1}R_j(z-a_0)^{-1}\right)(z+a_0)^{-1}. \tag{10.10}$$

We have

$$\|(z+a_0)^{-1}R_j(z-a_0)^{-1}\|\leq C|\mathrm{Im}z|^{-1}t^{\delta},$$

which proves (10.6).

To prove (10.7), we write:

$$\begin{aligned}
&a_0^2(1-g_1)(z^2-a_0^2)^{-1}g\\
=\ &(1-g_1)a_0^2(z^2-a_0^2)^{-1}g-[a_0^2,g_1](z^2-a_0^2)^{-1}g\\
=\ &z(1-g_1)(z^2-a_0^2)^{-1}g-[a_0^2,g_1](z^2-a_0^2)^{-1}g.
\end{aligned}$$

Now $[a_0^2,g_1]=-2g_1'\partial_{\tilde{\sigma}}-g_1''$. Using the expression analogous to (10.10) with $1-g_1$ replaced by $[a_0^2,g_1]$, and (10.2), we obtain

$$[a_0^2,g_1](z^2-a_0^2)^{-1}g\in O(t^{(N+2)\delta}|\mathrm{Im}z|^{-N-1}),\ \forall N\in\mathbb{N}.$$

Using also (10.6) we obtain (10.7). $\square$

Proof of Lemma 10.1.

Let us first prove *i)*. The function

$$f_t:\lambda\mapsto f(\frac{\lambda-\mathrm{c}t}{t^{\rho}})+f(\frac{-\lambda-\mathrm{c}t}{t^{\rho}})$$

is equal to 1 for $|\lambda|\geq c_0t$ and satisfies $|\partial_{\lambda}^{\alpha}f_t|\leq C_{\alpha}t^{-\rho\alpha}$, $\alpha\in\mathbb{N}$. This implies that the function

$$\chi_t(\lambda)=(f_t(\lambda)+\mu+R)^{-1}-(1+\mu+R)^{-1}$$

satisfies

$$\begin{aligned}
&\operatorname{supp}\chi_t\subset\{|\lambda|\leq c_0t\},\\
&|\partial_{\lambda}^{\alpha}\chi_t(\lambda)|\leq C_{\alpha}t^{-\rho\alpha},\ \alpha\in\mathbb{N},\ \text{uniformly in}\ R\geq 0.
\end{aligned} \tag{10.11}$$

Using the construction in [DG1, Prop. C.2.1], we can find an almost-analytic extension $\tilde{\chi}_t$ of χ_t such that

$$
\begin{aligned}
&\operatorname{supp}\tilde{\chi}_t \subset \{z\in\mathbb{C}||\mathrm{Re}z|\le c_0 t, |\mathrm{Im}z|\le c_0 t^{\rho}\},\\
&|\partial_{\overline{z}}\tilde{\chi}_t(z)| \le C_N|\mathrm{Im}z|^N t^{-\rho(N+1)},\ N\in\mathbb{N},\ \text{uniformly in } R\ge 0.
\end{aligned}
\tag{10.12}
$$

We observe that if $\chi\in C_0^\infty(\mathbb{R})$ is an even function and A a selfadjoint operator, we have:

$$
\begin{aligned}
\chi(A) = &\ \tfrac{\mathrm{i}}{2\pi}\int\partial_{\overline{z}}\tilde{\chi}(z)(z-A)^{-1}\mathrm{d}z\wedge\mathrm{d}\overline{z}\\
= &\ \tfrac{\mathrm{i}}{2\pi}\int\partial_{\overline{z}}\tilde{\chi}(z)\tfrac{1}{2}\left((z-A)^{-1}+(z+A)^{-1}\right)\mathrm{d}z\wedge\mathrm{d}\overline{z}\\
= &\ \tfrac{\mathrm{i}}{2\pi}\int\partial_{\overline{z}}\tilde{\chi}(z)z(z^2-A)^{-1}\mathrm{d}z\wedge\mathrm{d}\overline{z},
\end{aligned}
\tag{10.13}
$$

using the identity $(z-A)^{-1}+(z+A)^{-1}=2z(z^2-A^2)^{-1}$. Applying this identity to the even function χ_t, we have:

$$
\begin{aligned}
&(1-g_1)(b_{0,t}+\mu+R)^{-1}g\\
= &\ (1-g_1)\chi_t(a_0)g\\
= &\ \tfrac{\mathrm{i}}{2\pi}\int\partial_{\overline{z}}\tilde{\chi}_t(z)z(1-g_1)(z^2-a_0^2)^{-1}g\mathrm{d}z\wedge\mathrm{d}\overline{z}.
\end{aligned}
$$

i) follows then from (10.6) and (10.12), since $\rho>\delta$.

Let us now prove *ii)*. We denote again by $\chi_t(\lambda)$ the function $f_t(\lambda)-1$ which is even and satisfies (10.11). We have by (10.13)

$$
\begin{aligned}
&(b_t-b_{0,t})g=(f_t(a)-f_t(a_0))g\\
= &\ \tfrac{\mathrm{i}}{2\pi}\int\partial_{\overline{z}}\tilde{\chi}_t(z)z(z^2-a^2)^{-1}(a^2-a_0^2)(z^2-a_0^2)^{-1}g\mathrm{d}z\wedge\mathrm{d}\overline{z}.
\end{aligned}
$$

Next we write

$$
\begin{aligned}
&(z^2-a^2)^{-1}(a^2-a_0^2)(z^2-a_0^2)^{-1}g\\
= &\ (z^2-a^2)^{-1}(a^2-a_0^2)(1-g_1)(z^2-a_0^2)^{-1}g\\
&+(z^2-a^2)^{-1}(a^2-a_0^2)g_1(z^2-a_0^2)^{-1}g\\
= &\ I_1(z)+I_2(z).
\end{aligned}
$$

and estimate separately the two terms. We have:

$$
\begin{aligned}
\|I_1(z)\| = &\ \|(z^2-a^2)^{-1}(a^2-a_0^2)(1-g_1)(z^2-a_0^2)^{-1}g\|\\
\le &\ \|(z^2-a^2)^{-1}a^2\|\|(1-g_1)(z^2-a_0^2)^{-1}g\|\\
&+\|(z^2-a^2)^{-1}\|\|a_0^2(1-g_1)(z^2-a_0^2)^{-1}g\|\\
\le &\ C_N|\mathrm{Im}z|^{N+2}t^{N\delta}+C_N|\mathrm{Im}z|^{N+4}\langle z\rangle t^{N\delta}+C_N|\mathrm{Im}z|^{N+3}t^{(N+2)\delta},
\end{aligned}
\tag{10.14}
$$

using (10.6), (10.7) and the fact that $\|a^2(z^2-a^2)^{-1}\|\le 1$, $\|(z^2-a^2)^{-1}\|\le|\mathrm{Im}z|^{-2}$. This yields

$$
\|\int\partial_{\overline{z}}\tilde{\chi}_t(z)zI_1(z)\mathrm{d}z\wedge\mathrm{d}\overline{z}\|\in O(t^{-\infty}),
\tag{10.15}
$$

using (10.12) and the fact that $\rho > \delta$. Let us now estimate $I_2(z)$.

We have

$$\|(a^2 - a_0^2)g_1\| \le Ct^{\rho_1}.$$

A sharp estimate for $(z^2 - a^2)^{-1}$ where a is any selfadjoint operator is

$$\|(z^2 - a^2)^{-1}\| \le C \inf(|\mathrm{Im}z|^{-1}|\mathrm{Re}z|^{-1}, |\mathrm{Im}z|^{-2}).$$

Let us now estimate

$$\| \int \partial_{\overline{z}}\tilde{\chi}_t(z) z I_2(z) \mathrm{d}z \wedge \mathrm{d}\overline{z}\|.$$

Recall that $\partial_{\overline{z}}\tilde{\chi}_t$ is supported in $\{z \in \mathbb{C} | |\mathrm{Re}z| \le c_0 t, |\mathrm{Im}z| \le c_0 t^\rho\}$. We cut the integral in three parts:

$$\begin{aligned}
R_1 = & \quad \| \textstyle\int_{|\mathrm{Re}z| \le 1} \partial_{\overline{z}}\tilde{\chi}_t(z) z I_2(z) \mathrm{d}z \wedge \mathrm{d}\overline{z}\| \\
\le & \quad \textstyle\int_{|\mathrm{Re}z| \le 1} |\partial_{\overline{z}}\tilde{\chi}_t(z)| \langle z\rangle |\mathrm{Im}z|^{-4} t^{\rho_1} \mathrm{d}z \wedge \mathrm{d}\overline{z} \\
\le & \quad \textstyle\int_{|\mathrm{Re}z| \le 1, |\mathrm{Im}z| \le c_0 t^\rho} \langle z\rangle t^{\rho_1 - 5\rho} \mathrm{d}z \wedge \mathrm{d}\overline{z} \\
\le & \quad Ct^{\rho_1 - 3\rho}; \\
R_2 = & \quad \| \textstyle\int_{|\mathrm{Re}z| \ge c_1 |\mathrm{Im}z|, |\mathrm{Re}z| \ge 1} \partial_{\overline{z}}\tilde{\chi}_t(z) z I_2(z) \mathrm{d}z \wedge \mathrm{d}\overline{z}\| \\
\le & \quad \textstyle\int_{1 \le |\mathrm{Re}z| \le c_0 t, |\mathrm{Im}z| \le c_0 t^\rho} |\partial_{\overline{z}}\tilde{\chi}_t(z)| \langle z\rangle |\mathrm{Re}z|^{-2} |\mathrm{Im}z|^{-2} t^{\rho_1} \mathrm{d}z \wedge \mathrm{d}\overline{z} \\
\le & \quad \textstyle\int_{|\mathrm{Re}z| \ge c_1 |\mathrm{Im}z|, |\mathrm{Re}z| \ge 1} \langle z\rangle |\mathrm{Re}z|^{-2} t^{\rho_1 - 3\rho} \mathrm{d}z \wedge \mathrm{d}\overline{z} \\
\le & \quad Ct^{\rho_1 - 2\rho} \log t; \\
R_3 = & \quad \| \textstyle\int_{|\mathrm{Re}z| \le c_1 |\mathrm{Im}z|, |\mathrm{Re}z| \ge 1} \partial_{\overline{z}}\tilde{\chi}_t(z) z I_2(z) \mathrm{d}z \wedge \mathrm{d}\overline{z}\| \\
\le & \quad \textstyle\int_{|\mathrm{Re}z| \le c_1 |\mathrm{Im}z|} |\partial_{\overline{z}}\tilde{\chi}_t(z)| \langle z\rangle |\mathrm{Im}z|^{-4} t^{\rho_1} \mathrm{d}z \wedge \mathrm{d}\overline{z} \\
\le & \quad \textstyle\int_{|\mathrm{Re}z| \le c_1 |\mathrm{Im}z|, |\mathrm{Im}z| \le c_0 t^\rho} t^\rho t^{\rho_1 - 5\rho} \mathrm{d}z \wedge \mathrm{d}\overline{z} \\
\le & \quad Ct^{\rho_1 - 2\rho}.
\end{aligned}$$

This yields

$$\| \int \partial_{\overline{z}}\tilde{\chi}_t(z) z I_2(z) \mathrm{d}z \wedge \mathrm{d}\overline{z}\| \le Ct^{\rho_1 - 2\rho} \log t.$$

Using (10.15) this proves *ii)*. □

10.2 Case of $\mathfrak{h}^{\mathrm{e}}$

In this subsection, we prove similar results on the Hilbert space $\mathfrak{h}^{\mathrm{e}} = L^2(\mathbb{R}, \mathrm{d}\sigma) \otimes \mathfrak{g}$. We recall that on $\mathfrak{h}^{\mathrm{e}}$ we defined the observable $s = \mathrm{i}\frac{\partial}{\partial\sigma}$, so that $|s| = (-\frac{\partial^2}{\partial\sigma^2})^{\frac{1}{2}}$. We define the observable $|s|_0$ by

$$\begin{aligned}
& s_0^2 = -\tfrac{\partial^2}{\partial\sigma^2} \text{ with Dirichlet condition at } 0, \\
& \text{ie } \mathcal{D}(s_0^2) = H^2(\mathbb{R}\backslash\{0\}) \cap H_0^1(\mathbb{R}\backslash\{0\}), \\
& |s|_0 := (s_0^2)^{\frac{1}{2}}.
\end{aligned}$$

Let again $f \in C^\infty(\mathbb{R})$ with $f(\lambda) \equiv 1$ for $\gg 1$, $f \equiv 0$ for $\lambda \ll -1$. We set for $0 < \mathrm{c} \leq 1$, $0 < \rho < 1$:

$$b_{0,t} := f(\tfrac{|s|_0 - \mathrm{c}t}{t^\rho}) + f(\tfrac{-|s|_0 - \mathrm{c}t}{t^\rho}),$$

$$b_t := f(\tfrac{s-\mathrm{c}t}{t^\rho}) + f(\tfrac{-s-\mathrm{c}t}{t^\rho}),$$

and for $\delta > 0$:

$$g = F(t^\delta|\sigma| \geq 1),\ g_1 = F(t^\delta|\sigma| \geq \frac{1}{2}),$$

so that $g_1 g = g$. The following lemma is analogous to Lemma 10.1.

Lemma 10.3 *Assume $\rho > \delta$. Then:*

i) $(1-g_1)(b_t + \mu + R)^{-1} g \in O(t^{-\infty})$, *for* $\mu \in \mathbb{C}\backslash\mathbb{R}^-$, *uniformly for* $R \geq 0$,

ii) $(b_t - b_{0,t})g \in O(t^{-\infty})$.

Proof. Let us first prove *ii)*. We apply the identity (10.13) to the even ($t-$dependent) function

$$\chi_t(\lambda) = f(\frac{\lambda - \mathrm{c}t}{t^\rho}) + f(\frac{-\lambda - \mathrm{c}t}{t^\rho}) - 1,$$

and obtain

$$\begin{array}{rl} & b_t - b_{0,t} = \chi_t(|s|) - \chi_t(|s|_0) \\ = & \frac{\mathrm{i}}{2\pi}\int \frac{\partial}{\overline{z}}\tilde{\chi}_t(z) z\left((z^2-s^2)^{-1} - (z^2-s_0^2)^{-1}\right) \mathrm{d}z\wedge \mathrm{d}\overline{z}, \end{array}$$

where $\tilde{\chi}_t$ is an almost-analytic extension of $\tilde{\chi}$ satisfying (10.12). We recall the identity (see [AGHH, Thm. 3.1.2]):

$$(z^2-s^2)^{-1} - (z^2-s_0^2)^{-1} = \frac{\mathrm{i}}{2z}|\phi_z\rangle\langle\phi_{\overline{z}}| \text{ for } \mathrm{Im}z > 0,$$

where $\phi_z(\sigma) = \mathrm{e}^{\mathrm{i}z|\sigma|}$. We have:

$$\|\phi_z\| \leq C|\mathrm{Im}z|^{-\frac{1}{2}},\ \|g\phi_z\| \leq C|\mathrm{Im}z|^{-\frac{1}{2}}\mathrm{e}^{-t^{-\delta}|\mathrm{Im}z|}.$$

This gives

(10.16) $$\|\left((z^2-s^2)^{-1} - (z^2-s_0^2)^{-1}\right)g\| \leq C|\mathrm{Im}z|^{-1}\mathrm{e}^{-t^{-\delta}|\mathrm{Im}z|},\ \mathrm{Im}z \neq 0.$$

We deduce from (10.16) and (10.12) that

$$\begin{array}{rl} & \|(b_t - b_{0,t})g\| \\ \leq & C_N\int_{\mathrm{supp}\,\tilde{\chi}_t}|z||\mathrm{Im}z|^{N-1}t^{-\rho(N+1)}\mathrm{e}^{-t^{-\delta}|\mathrm{Im}z|}\mathrm{d}z\wedge\mathrm{d}\overline{z} \\ \leq & C_N\int_{\mathrm{supp}\,\tilde{\chi}_t}|z|t^{(\delta-\rho)N}\mathrm{d}z\wedge\mathrm{d}\overline{z} \in O(t^{-\infty}), \end{array}$$

since $\rho > \delta$. This proves *ii)*.

To prove *i)*, we write

$$(1-g_1)(b_t+\mu+R)^{-1}g = -(1-g_1)(b_t+\mu+R)^{-1}[b_t, g](b_t+\mu+R)^{-1}.$$

By p.d.o. calculus, $[b_t, g] = g_2[b_t, g]$, where $g_2 = F(t^\delta|\sigma| \geq \frac{3}{2})$, and $[b_t, g] \in O(t^{\delta-\rho})$. Iterating this argument we obtain *i)*. □

11 Reinterpretation of the spaces $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$

In this section we describe the spaces $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ using the observable $|s|_0$ introduced in Subsect. 10.2.

11.1 Preliminary results

In this subsection we show that the spaces $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ can also be defined with a cutoff function in s which is even. This easy result uses the fact shown in Subsect. 5.3 that there is no propagation in the region $\{s \leq -\mathrm{c}t\}$.

Let $f \in C^{\infty}(\mathbb{R})$ satisfying (5.1) for $0 < \alpha_0 < \alpha_1$. We set for $0 < \rho < 1$:

$$b_{\mathrm{c}\,t} := f\Big(\frac{s - \mathrm{c}t}{t^{\rho}}\Big) + f\Big(\frac{-s - \mathrm{c}t}{t^{\rho}}\Big),\ B_{\mathrm{c}\,t} := \mathrm{d}\Gamma(b_{\mathrm{c}\,t}). \tag{11.1}$$

(The reader should compare (11.1) with (5.2)).

Theorem 11.1 *Assume (I'0), (I'1) for $\epsilon_0 > 0$, (I'2) for $\mu > 1$ and pick ρ in (11.1) such that $\rho(1+\epsilon_0) > 1$. Then:*

i) for each $\lambda \in \mathbb{C}\backslash\mathbb{R}^-$ the limit

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,t} + \lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} =: \tilde{R}_{\mathrm{c}}^{+}(\lambda) \ \textit{exists.}$$

ii) $[\tilde{R}_{\mathrm{c}}^{+}(\lambda), L] = [\tilde{R}_{\mathrm{c}}^{+}(\lambda), H^{\mathrm{e}}] = 0$.

$$iii)\ \text{s-}\lim_{\epsilon\to 0} \epsilon^{-1}\tilde{R}_{\mathrm{c}}^{+}(\epsilon^{-1}) =: \hat{P}_{\mathrm{c}}^{\mathrm{e}+},$$

where the orthogonal projection $\hat{P}_{\mathrm{c}}^{\mathrm{e}+}$ is defined in Thm. 5.5.

It follows from Thms. 11.1 and 5.6 that $u \in \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ if and only if there exists $\mathrm{c}' > \mathrm{c}$ such that

$$\lim_{\epsilon\to 0}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(\epsilon B_{\mathrm{c}'\,t} + \mathbb{1})^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u = u.$$

Proof. We set $f_1(s) = f(-s)$ and note that f_1 satisfies (5.14). Let

$$b_{+,t} = f(\tfrac{s-\mathrm{c}t}{t^{\rho}}),\ B_{+,t} := \mathrm{d}\Gamma(b_{+,t}),$$
$$b_{-,t} = f_1(\tfrac{s+\mathrm{c}t}{t^{\rho}}),\ B_{-,t} := \mathrm{d}\Gamma(b_{-,t}),$$

so that $B_{\mathrm{c}\,t} = B_{+,t} + B_{-,t}$. By Thm. 5.5 and Prop. 5.7 we know that for all $\lambda, \lambda' \in \mathbb{C}\backslash\mathbb{R}^+$ the limit

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(\lambda - B_{+,t})^{-1}(\lambda' - B_{-,t})^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} \text{ exists.}$$

Note that the functions $\mathbb{R}^2 \ni (s, s') \mapsto (\lambda - s)^{-1}(\lambda' - s')^{-1}$ for $\lambda, \lambda' \in \mathbb{C}\backslash\mathbb{R}$ are total in $C_{\infty}(\mathbb{R}^2)$. Hence for all $\chi \in C_{\infty}(\mathbb{R}^2)$, the limit

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\chi(B_{+,t}, B_{-,t})\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} \text{ exists.}$$

We claim that

$$\text{s-}\lim_{\epsilon\to 0}\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(\epsilon B_{+,t} + \epsilon B_{-,t} + \mathbb{1})^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} = \hat{P}_{\mathrm{c}}^{\mathrm{e}+}, \tag{11.2}$$

where $\hat{P}_{\mathrm{c}}^{\mathrm{e}+}$ is defined in Thm. 5.5. By density using Prop. 5.7 *iii)*, it suffices to show that

$$\text{(11.3)} \qquad \text{s-}\lim_{\epsilon\to 0}\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Big((\epsilon B_{+,t}+\epsilon B_{-,t}+1\!\!1)^{-1}-(\epsilon B_{+,t}+1\!\!1)^{-1}\Big)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}R_1^+(\epsilon_0)=0,$$

for any $\epsilon_0>0$, where $R_1^+(\epsilon_0)$ is defined in Prop. 5.7. Now

$$\begin{aligned}
&\Big((\epsilon B_{+,t}+\epsilon B_{-,t}+1\!\!1)^{-1}-(\epsilon B_{+,t}+1\!\!1)^{-1}\Big)(\epsilon_0 B_{-,t}+1\!\!1)^{-1}\\
=\;&-\epsilon(\epsilon(B_{+,t}+B_{-,t})+1\!\!1)^{-1}(\epsilon B_{+,t}+1\!\!1)^{-1}B_{-,t}(\epsilon_0 B_{-,t}+1\!\!1)^{-1}\\
=\;&O(\epsilon\epsilon_0^{-1})\text{ uniformly in }t.
\end{aligned}$$

This proves (11.3) and hence (11.2). Statements *i)* and *ii)* follow from Thm. 5.5 and Prop. 5.7. Statement *iii)* follows from (11.2).□

11.2 Reinterpretation of the space $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$

We now want to replace the observable $b_{\mathrm{c}\,t}$ by an observable $b_{\mathrm{c}\,0\,t}$ which commutes with the projections $1\!\!1_{\{\pm\sigma\geq 0\}}$. Let $|s|_0$ be the observable defined in Subsect. 10.2. We set

$$b_{\mathrm{c}\,0\,t}=f(\frac{|s|_0-\mathrm{c}t}{t^\rho})+f(\frac{-|s|_0-\mathrm{c}t}{t^\rho}),\ B_{\mathrm{c}\,0\,t}:=\mathrm{d}\Gamma(b_{\mathrm{c}\,0\,t}).$$

Proposition 11.2 *Assume (I'0), (I'1) for $\epsilon_0>0$, (I'2) for $\mu>1$ and pick ρ in (11.1) such that $\rho\epsilon_0>1$. Then for each $\lambda\in\mathbb{C}\backslash\mathbb{R}^-$ the limit*

$$\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}0,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}$$

exists and equals

$$\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}=\hat{R}_{\mathrm{c}}^{\mathrm{e}+}(\lambda).$$

The following consequence of Thms. 11.1 and 5.6 gives the final description of the space $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$:

Theorem 11.3 *Assume (I'0), (I'1) for $\epsilon_0>0$, (I'2) for $\mu>1$ and pick ρ in (11.1) such that $\rho\epsilon_0>1$. Then $u\in\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ if and only if there exists $\mathrm{c}'>\mathrm{c}$ such that*

$$\lim_{\epsilon\to 0}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(\epsilon B_{\mathrm{c}'\,0\,t}+1\!\!1)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}u=u.$$

Proof of Prop. 11.2. We drop the subscript c to simplify the notation. We will use the notation in Subsect. 10.2. Recall that we have set:

$$g=F(t^\delta|\sigma|\geq 1),\ g_1=F(t^\delta|\sigma|\geq\frac{1}{2}),$$

for $\delta>0$. We fix $\delta<\rho$ with $\delta\epsilon_0>1$ so that the results of Subsect. 4.4 apply.

By a density argument, using Thm. 4.12 and Lemma 4.11, it suffices to prove that for $\chi\in C_0^\infty(\mathbb{R})$:

$$\text{(11.4)} \qquad \Big((\lambda+\mathrm{d}\Gamma(b_t))^{-1}-(\lambda+\mathrm{d}\Gamma(b_{0\,t}))^{-1}\Big)\Gamma(g_1)\chi(L)\Gamma(g)\chi(L)\in o(1).$$

We first claim that

$$(11.5)\qquad (\mathbb{1}-\Gamma(g_1))(\lambda+\mathrm{d}\Gamma(b_t))^{-1}\Gamma(g)\in O(N)t^{-\infty}.$$

In fact on the $n-$particle sector:

$$\mathbb{1}-\Gamma(g_1)=\sum_{j=1}^{n}\mathbb{1}\otimes\cdots\otimes\mathbb{1}\otimes(1-g_{1,j})\otimes g_{1,j+1}\otimes\cdots\otimes g_{1,n},$$

so

$$\begin{aligned}&\|(\mathbb{1}-\Gamma(g_1))(\lambda+\mathrm{d}\Gamma(b_t))^{-1}\Gamma(g)\|\\ \leq\ & n\sup_{R\geq 0}\|(1-g_1)(b_t+\lambda+R)^{-1}g\|\in O(N^{\mathrm{e}})t^{-\infty}.\end{aligned}$$

by Lemma 10.3 *i)*. Now we write:

$$\begin{aligned}&\Big((\lambda+\mathrm{d}\Gamma(b_{0\,t}))^{-1}-(\lambda+\mathrm{d}\Gamma(b_t))^{-1}\Big)\Gamma(g)\\ =\ &(\lambda+\mathrm{d}\Gamma(b_{0\,t}))^{-1}\mathrm{d}\Gamma(b_t-b_{0\,t})(\lambda+\mathrm{d}\Gamma(b_t))^{-1}\Gamma(g)\\ =\ &(\lambda+\mathrm{d}\Gamma(b_{0\,t}))^{-1}\mathrm{d}\Gamma(b_t-b_{0\,t})\Gamma(g_1)(\lambda+\mathrm{d}\Gamma(b_t))^{-1}\Gamma(g)\\ &+(\lambda+\mathrm{d}\Gamma(b_{0\,t}))^{-1}\mathrm{d}\Gamma(b_t-b_{0\,t})(\mathbb{1}-\Gamma(g_1))(\lambda+\mathrm{d}\Gamma(b_t))^{-1}\Gamma(g)\\ =:\ &I_1+I_2.\end{aligned}$$

By Lemma 10.3 *ii)*, we have:

$$(11.6)\qquad \mathrm{d}\Gamma(b_t-b_{0\,t})\Gamma(g_1)\in O(N^{\mathrm{e}})t^{-\infty},$$

and hence

$$\|I_1\chi(L)\Gamma(g_1)\chi(L)\|\leq Ct^{-\infty}\|(N^{\mathrm{e}}+1)\Gamma(g)\chi(L)\|\in O(t^{-\infty}),$$

by Lemma 4.13 *i)*.

Similarly by (11.5), we have:

$$(11.7)\qquad (\lambda+\mathrm{d}\Gamma(b_{0,t}))^{-1}\mathrm{d}\Gamma(b_t-b_{0,t})(\mathbb{1}-\Gamma(g_1))(\lambda+\mathrm{d}\Gamma(b_t))^{-1}\Gamma(g)\in O((N^{\mathrm{e}})^2)t^{-\infty},$$

and hence if g_2 is such that $gg_2=g$, $g_1g_2=g_2$, we have:

$$\begin{aligned}&\|I_2\chi(L)\Gamma(g_1)\chi(L)\|=\|I_2\Gamma(g_2)\chi(L)\Gamma(g_1)\chi(L)\|\\ \leq\ & Ct^{-\infty}\|(N^{\mathrm{e}})^2\Gamma(g_2)\chi(L)\Gamma(g_1)\chi(L)\|\leq Ct^{-\infty},\end{aligned}$$

by Lemma 4.13 *ii)*. This proves (11.4) and completes the proof of the proposition. □

11.3 Reinterpretation of $\Gamma^{+}(f_0)$

Let f_0 be a cutoff function as in (6.1) with $0<\alpha_0<\alpha_1<\alpha_2$. We recall that the observable $\Gamma_{\mathrm{c}}^{\mathrm{e}+}(f_0)$ was defined in (6.17).

Proposition 11.4 *Assume (I'0), (I'1) for $\epsilon_0>0$, (I'2) for $\mu>1$ and pick ρ in (11.1) such that $\rho\epsilon_0>1$. Then for $0<\mathrm{c}<\mathrm{c}'<1$:*

$$\Gamma_{\mathrm{c}'}^{\mathrm{e}+}(f_0)=\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Gamma\Big(f_0\Big(\frac{|s|_0-\mathrm{c}'t}{t^{\rho}}\Big)\Big)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}}\ \textit{on}\ \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}.$$

Proof. Let us replace c' by c to simplify notation. By Prop. 5.11 we have:

$$\Gamma_{\mathrm{c}}^{\mathrm{e}+}(f_0) = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Gamma(c_t)\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}},$$

for

$$c_t = f_0(\frac{s-\mathrm{c}t}{t^{\rho}})f_0(\frac{-s-\mathrm{c}t}{t^{\rho}}).$$

Let

$$c_{0\,t} = f_0(\frac{|s|_0-\mathrm{c}t}{t^{\rho}})f_0(\frac{-|s|_0-\mathrm{c}t}{t^{\rho}}),$$

and note that

$$c_{0\,t} = f_0(\frac{|s|_0-\mathrm{c}t}{t^{\rho}}) \text{ for } t\gg 1,$$

since $|s|_0 \geq 0$. As in the proof of Prop. 11.2, we set $g = F(t^{\delta}|\sigma|\geq 1)$ for $\delta<\rho$ with $\delta\epsilon_0>1$ so that the results of Subsect. 4.4 apply. The function

$$\chi_t(\lambda) = f_0(\frac{\lambda-\mathrm{c}t}{t^{\rho}})f_0(\frac{-\lambda-\mathrm{c}t}{t^{\rho}})$$

is an even function of λ, satisfying (10.11). As in the proof of Lemma 10.3 *ii)*, we have

$$(c_t - c_{0\,t})g \in O(t^{-\infty}), \tag{11.8}$$

As in the proof of Prop. 11.2, it suffices to show that for $\chi\in C_0^{\infty}(\mathbb{R})$:

$$(\Gamma(c_t)-\Gamma(c_{0\,t}))\Gamma(g)\chi(L) \in o(1). \tag{11.9}$$

We claim that if $a,b,g\in\mathcal{B}(\mathfrak{h}^{\mathrm{e}})$ with $0\leq a,b,g\leq 1$ then

$$\|(\Gamma(b)-\Gamma(a))\Gamma(g)(N^{\mathrm{e}}+1)^{-1}\| \leq \|(b-a)g\|. \tag{11.10}$$

To prove (11.10), we write on the $n-$particle sector

$$\Gamma(b)-\Gamma(a) = \sum_{i=1}^{n} b_1\otimes\cdots\otimes b_{i-1}\otimes(b_i-a_i)\otimes a_{i+1}\otimes\cdots\otimes a_n.$$

Using then (11.8) and (11.10) we get:

$$(\Gamma(c_t)-\Gamma(c_{0,t}))\Gamma(g) \in O(N^{\mathrm{e}})t^{-\infty}.$$

Next:

$$\begin{aligned} & \|(\Gamma(c_t)-\Gamma(c_{0\,t}))\Gamma(g)\chi(L)\| \\ = \; & \|(\Gamma(c_t)-\Gamma(c_{0\,t}))\Gamma(g)\Gamma(g_1)\chi(L)\| \\ \leq \; & \|(\Gamma(c_t)-\Gamma(c_{0\,t}))\Gamma(g)(N^{\mathrm{e}}+1)^{-1}\|\|(N^{\mathrm{e}}+1)\Gamma(g_1)\chi(L)\| \in O(t^{-\infty}), \end{aligned}$$

by Lemma 4.13 *i)*. This proves (11.9) and completes the proof of the proposition. □

12 Scattering theory for H

This section contains the main results of this paper. We first construct for $0 < \mathrm{c} < 1$ spaces $\mathcal{H}_{\mathrm{c}}^{+}$ containing a finite number of particles in the region $\{|x| \geq \mathrm{c}' t\}$ for any $\mathrm{c} < \mathrm{c}'$. We show then that the asymptotic Weyl operators $W^{+}(h)$ induce on $\mathcal{H}_{\mathrm{c}}^{+}$ a regular CCR representation of Fock type. Finally we prove the *geometric asymptotic completeness property*, which states that the vacuum states of this induced representation contain no particles in the region $\{|x| \geq \mathrm{c}' t\}$, for any $\mathrm{c} < \mathrm{c}'$.

We start with an easy technical lemma.

Lemma 12.1 *Hypotheses (I2) for $\mu > 0$, (I5) for $\mu_2 > 0$ imply hypothesis (I4) for $\mu_1 = \inf(\mu, 2\mu_2)$.*

From Lemma 12.1 we see that if *(I2)*, *(I5)* are satisfied for $\mu > 1$, $\mu_2 > 1$ then *(I4)* is satisfied for $\mu_1 > 1$.

Proof. Let $\epsilon > 0$ and set $v_{j,\epsilon} = \chi(\epsilon \leq |k| \leq \epsilon^{-1}) v_j$. We drop the index j to simplify notation. Going to polar coordinates as in Sect. 10, we have by *(I2)*:

$$(-\partial_{\tilde{\sigma}}^2 + 1)^{\mu/2} \tilde{\sigma} v_\epsilon(\tilde{\sigma}\omega) \in L^2(\mathbb{R}^+, \mathrm{d}\tilde{\sigma}) \otimes L^2(S^2), \tag{12.1}$$

and by *(I5)*

$$(-\frac{\Delta_\omega}{\tilde{\sigma}^2} + 1)^{\mu_2} \tilde{\sigma} v_\epsilon(\tilde{\sigma}\omega) \in L^2(\mathbb{R}^+, \mathrm{d}\tilde{\sigma}) \otimes L^2(S^2). \tag{12.2}$$

Since v_ϵ has support in $\tilde{\sigma}$ included in $]0, +\infty[$, (12.2) is equivalent to

$$(-\Delta_\omega + 1)^{\mu_2} \tilde{\sigma} v_\epsilon(\tilde{\sigma}\omega) \in L^2(\mathbb{R}^+, \mathrm{d}\tilde{\sigma}) \otimes L^2(S^2). \tag{12.3}$$

Clearly (12.1) and (12.3) imply that

$$(-\partial_{\tilde{\sigma}}^2 - \Delta_\omega + 1)^{\mu_1/2} \tilde{\sigma} v_\epsilon(\tilde{\sigma}\omega) \in L^2(\mathbb{R}^+, \mathrm{d}\tilde{\sigma}) \otimes L^2(S^2), \tag{12.4}$$

for $\mu_1 = \inf(\mu, 2\mu_2)$. Again because of the support of v_ϵ, (12.4) implies that

$$(-\partial_{\tilde{\sigma}}^2 - \frac{\Delta_\omega}{\tilde{\sigma}^2} + 1)^{\mu_1/2} \tilde{\sigma} v_\epsilon(\tilde{\sigma}\omega) \in L^2(\mathbb{R}^+, \mathrm{d}\tilde{\sigma}) \otimes L^2(S^2). \tag{12.5}$$

This can be shown by a direct computation for $\mu_1 \in \mathbb{N}$ and then extended to $\mu_1 \in \mathbb{R}^+$ by interpolation. Going back to the original coordinates we see that (12.5) implies *(I4)* for μ_1. $\square$

12.1 Number of asymptotically free particles

Let $f \in C^\infty(\mathbb{R})$ a cutoff function such that

$$0 \leq f \leq 1,\ f' \geq 0,\ f \equiv 0 \text{ for } s \leq \alpha_0,\ f \equiv 1 \text{ for } s \geq \alpha_1, \tag{12.6}$$

for $0 < \alpha_0 < \alpha_1$. We set

$$b_{\mathrm{c}\,t} := f(\frac{|x| - \mathrm{c}t}{t^\rho}),\ B_{\mathrm{c}\,t} = \mathrm{d}\Gamma(b_{\mathrm{c}\,t}), \tag{12.7}$$

for constants $0 < \mathrm{c} < 1$, $0 < \rho < 1$.

Proposition 12.2 *Assume (H0) for $\alpha > 1$, (I0), (I1) for $\epsilon_0 > 1$, (I2) for $\mu > 1$, (I5) for $\mu_2 > 1$ and choose ρ in (12.7) such that $\rho\epsilon_0 > 1$, $\rho\mu_2 > 1$. Then*

$$i)\ \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}(B_{\mathrm{c}\,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH} =: \hat{R}_{\mathrm{c}}^{+}(\lambda)$$

exists for $\lambda \in \mathbb{C}\backslash\mathbb{R}^{-}$.

$$ii)\ [\hat{R}_{\mathrm{c}}^{+}(\lambda), H] = 0.$$

$$iii)\ \hat{P}_{\mathrm{c}}^{+} := \text{s-}\lim_{\epsilon\to 0}\epsilon^{-1}\hat{R}_{\mathrm{c}}^{+}(\epsilon^{-1})\ \textit{exists}$$

and is an orthogonal projection.

Theorem 12.3 *Assume (H0) for $\alpha > 1$, (I0), (I1) for $\epsilon_0 > 1$, (I2) for $\mu > 1$, (I5) for $\mu_2 > 1$ and choose ρ in (12.7) such that $\rho\epsilon_0 > 1$, $\rho\mu_2 > 1$. Let*

$$P_{\mathrm{c}}^{+} := \inf_{\mathrm{c}<\mathrm{c}'} \hat{P}_{\mathrm{c}'}^{+},\ \mathcal{H}_{\mathrm{c}}^{+} := \mathrm{Ran}P_{\mathrm{c}}^{+},$$

Then

i) P_{c}^{+} is an orthogonal projection independent on the choice of the function f in (12.7).

ii) $[H, P_{\mathrm{c}}^{+}] = 0$.

$$iii)\ u \in \mathcal{H}_{\mathrm{c}}^{+} \Leftrightarrow \mathcal{W}I_{\Omega}u \in \mathcal{H}_{\mathrm{c}}^{\mathrm{e}+},$$

where $\mathcal{H}_{\mathrm{c}}^{\mathrm{e}+}$ is defined in Thm. 5.6.

$$iv)\ \Omega^{+}\Big(\mathcal{H}_{\mathrm{pp}}(H)\otimes\Gamma(\mathfrak{h})\Big) \subset \mathcal{H}_{\mathrm{c}}^{+} \subset \mathcal{H}^{+},$$

where the space $\mathcal{H}^{+}$ is defined in Subsect. 8.2.

$$v)\ W^{+}(h) : \mathcal{H}_{\mathrm{c}}^{+} \to \mathcal{H}_{\mathrm{c}}^{+}\ \textit{for}\ h \in \mathfrak{h}_0.$$

$$vi)\ \mathfrak{h}_0 \ni h \mapsto W^{+}(h) \in \mathcal{U}(\mathcal{H}_{\mathrm{c}}^{+})$$

is a regular CCR representation of Fock type.

Proof of Prop. 12.2. We will use the notation and results in Sect. 11, Sect. 10 and Subsect. 3.4. Note also that to consider a Nelson Hamiltonian as an abstract Pauli-Fierz Hamiltonian one has to introduce polar coordinates using the unitary transformation u defined in Subsect. 3.1. To lighten notation, we will omit this transformation and its extension $\Gamma(\mathrm{u})$ to Fock spaces in the computations below. For example the observable $|x|$ will be identified with the observable $a = \mathrm{u}|x|\mathrm{u}^{-1}$ considered in Subsect. 10.1.

By Prop. 11.2:

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}(B_{\mathrm{c}\,0\,t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} = \hat{R}_{\mathrm{c}}^{\mathrm{e}+}(\lambda),\ \lambda \in \mathbb{C}\backslash\mathbb{R}^{-}. \tag{12.8}$$

Note that because of the Dirichlet condition at 0 in the definition of s_0^2 (see Subsect. 10.1) we have:

$$1\!\!1_{\{\sigma\geq 0\}} b_{c\,0\,t} 1\!\!1_{\{\sigma\leq 0\}} = 0.$$

Hence $b_{\mathrm{c}\,0\,t}$ satisfies property (3.6) in Subsect. 3.4. Moreover

$$\begin{aligned} b_{\mathrm{c}\,+\,t} &= 1\!\!1_{\{\sigma\geq 0\}} b_{\mathrm{c}\,0\,t} 1\!\!1_{\{\sigma\geq 0\}} \\ &= f(\tfrac{a_0-ct}{t^\rho}) + f(\tfrac{-a_0-ct}{t^\rho}), \end{aligned}$$

where a_0 is defined in (10.1). Note also that since $|x|\geq 0$

$$b_{\mathrm{c}\,t} = f(\frac{a-ct}{t^\rho}) + f(\frac{-a-ct}{t^\rho}),$$

where a is defined in (10.1).

We deduce then from Prop. 3.4 that

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}(\mathrm{d}\Gamma(b_{\mathrm{c}+,t})+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH} =: \tilde{R}^+_{\mathrm{c}0}(\lambda)$$

exists for $\lambda\in\mathbb{C}\backslash\mathbb{R}^-$ and

$$[H, \tilde{R}^+_{\mathrm{c}0}(\lambda)] = 0.$$

We claim now that

$$\tilde{R}^+_{\mathrm{c}0}(\lambda) = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}(B_{\mathrm{c}t}+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH}, \tag{12.9}$$

which will prove *i)* and *ii)*. Property *iii)* follows then from Prop. A.6.

Let us now prove (12.9). Let $g, g_1\in\mathcal{B}(\mathfrak{h})$ be defined in (10.5) for exponents ρ_1,δ such that $\rho_1\mu_2>1$, $\delta\epsilon_0>1$. Using Thms. 4.9 and 4.16 for $C=\frac{-\Delta_\omega}{\tilde{\sigma}^2}$, we have:

$$\mathrm{e}^{-\mathrm{i}tH}u = \Gamma(g)\mathrm{e}^{-\mathrm{i}tH}u + o(1),\ u\in\mathcal{H}. \tag{12.10}$$

By a density argument and using Lemmas 4.7 and 4.14, (12.9) will follow from the fact that

$$\Big((\mathrm{d}\Gamma(b_{\mathrm{c}\,t})+\lambda)^{-1} - (\mathrm{d}\Gamma(b_{\mathrm{c}\,+\,t})+\lambda)^{-1}\Big)\Gamma(g)\chi(H)\Gamma(g_1)\chi(H)\in o(1), \tag{12.11}$$

for $\chi\in C_0^\infty(\mathbb{R})$. Let us prove (12.11) following the proof of (11.4).

First

$$(1\!\!1-\Gamma(g_1))(\mathrm{d}\Gamma(b_{\mathrm{c}\,t})+\lambda)^{-1}\Gamma(g)\in O(N)t^{-\infty}, \tag{12.12}$$

using Lemma 10.1 *i)* and the same argument as in the proof of (11.5). Next we write:

$$\begin{aligned} &\Big((\mathrm{d}\Gamma(b_{\mathrm{c}\,+\,t})+\lambda)^{-1} - (\mathrm{d}\Gamma(b_{\mathrm{c}\,t})+\lambda)^{-1}\Big)\Gamma(g) \\ = \ &(\mathrm{d}\Gamma(b_{\mathrm{c}\,+\,t})+\lambda)^{-1}\mathrm{d}\Gamma(b_{\mathrm{c}\,t}-b_{\mathrm{c}\,+\,t})(\mathrm{d}\Gamma(b_{\mathrm{c}\,t})+\lambda)^{-1}\Gamma(g) \\ = \ &(\mathrm{d}\Gamma(b_{\mathrm{c}\,+\,t})+\lambda)^{-1}\mathrm{d}\Gamma(b_{\mathrm{c}\,t}-b_{\mathrm{c}\,+\,t})\Gamma(g_1)(\mathrm{d}\Gamma(b_{\mathrm{c}\,t})+\lambda)^{-1}\Gamma(g) \\ &+(\mathrm{d}\Gamma(b_{\mathrm{c}\,+\,t})+\lambda)^{-1}\mathrm{d}\Gamma(b_{\mathrm{c}\,t}-b_{\mathrm{c}\,+\,t})(1\!\!1-\Gamma(g_1))(\mathrm{d}\Gamma(b_{\mathrm{c}\,t})+\lambda)^{-1}\Gamma(g) \\ = \ &I_1+I_2. \end{aligned}$$

By Lemma 10.1 *ii)*:

$$\mathrm{d}\Gamma(b_{\mathrm{c}\,t}-b_{\mathrm{c}\,+\,t})\Gamma(g_1)\in O(N)t^{\rho_1-2\rho}\log t,$$

and hence

$$\|I_1\chi(H)\Gamma(g_1)\chi(H)\|\leq Ct^{\rho_1-2\rho}\log t\|N\Gamma(g)\chi(H)\|.$$

Applying then Lemma 4.10, we obtain:

$$\|I_1\chi(H)\Gamma(g_1)\chi(H)\| \in O(t^{\rho_1-2\rho+\delta}\log t).$$

Similarly by (12.12), we have:

$$I_2 \in O(N^2)t^{-\infty},$$

and hence if g_2 is another operator analogous to g such that $gg_2 = g$, $g_1g_2 = g_1$, then:

$$\begin{aligned} & \|I_2\chi(H)\Gamma(g_1)\chi(H)\| \\ \leq\ & Ct^{-\infty}\|N^2\Gamma(g_2)\chi(H)\Gamma(g_1)\chi(H)\| \\ \leq\ & Ct^{-\infty}, \end{aligned}$$

by the same argument as in the proof of Lemma 4.10 *ii)*.

Since ρ is such that $\rho\epsilon_0 > 1$, $\rho\mu_2 > 1$, we can pick exponents ρ_1, δ in the definition of g with $\delta\epsilon_0 > 1$, $\rho_1\mu_2 > 1$ and $\rho > \delta$, $\rho > \rho_1$. Hence (12.9) holds and the proof is complete. □

Proof of Thm. 12.3. Applying first Prop. 3.4 and using (12.8), (12.9) we obtain that

$$\begin{aligned} \hat{P}_{\rm c}^{+} &= I_\Omega^* \mathcal{W}^{-1}\hat{P}_{\rm c}^{\rm e+}\mathcal{W}I_\Omega, \\ \hat{P}_{\rm c}^{\rm e+} &= \mathcal{W}\hat{P}_{\rm c}^{+} \otimes 1\!\!1_{\Gamma(\mathfrak{h})}\mathcal{W}^{-1}, \end{aligned}$$

where $\hat{P}_{\rm c}^{\rm e+}$ is defined in Thm. 5.6, and hence:

$$\begin{aligned} P_{\rm c}^{+} &= I_\Omega^* \mathcal{W}^{-1}P_{\rm c}^{\rm e+}\mathcal{W}I_\Omega, \\ P_{\rm c}^{\rm e+} &= \mathcal{W}P_{\rm c}^{+} \otimes 1\!\!1_{\Gamma(\mathfrak{h})}\mathcal{W}^{-1}. \end{aligned}$$

Clearly this implies *i)*, *ii)*, *iii)*. We note next that

$$\begin{aligned} & u \in \mathcal{H}_{\rm pp}(H) \Leftrightarrow \mathcal{W}I_\Omega u \in \mathcal{H}_{\rm pp}(H^{\rm e}), \qquad (12.13) \\ & u \in \mathcal{H}^{+} = \overline{\mathcal{D}(N^{+})} \Leftrightarrow \mathcal{W}I_\Omega u \in \mathcal{H}^{\rm e+} = \overline{\mathcal{D}(N^{\rm e+})}, \end{aligned}$$

by (8.19) and

$$u \in \Omega^{+}\Big(\mathcal{H}_{\rm pp}(H)\otimes\Gamma(\mathfrak{h})\Big) \Leftrightarrow \mathcal{W}I_\Omega u \in \Omega^{\rm e+}\Big(\mathcal{H}_{\rm pp}(H^{\rm e})\otimes\Gamma(\mathfrak{h}^{\rm e})\Big),$$

by (8.18) and (12.13). This proves the two inclusions in *iv)*, using the corresponding inclusions in Thm. 8.7. Finally *v)* follows from (8.16) and the corresponding statement in Thm. 8.7. □

12.2 Operators $\Gamma_{\rm c}^{+}(f_0)$

Let $f_0 \in C^\infty(\mathbb{R})$ such that

$$0 \leq f_0 \leq 1,\ f_0' \leq 0,\ f_0 \equiv 1 \text{ for } s \leq \alpha_1,\ f_0 \equiv 0 \text{ for } s \geq \alpha_2, \qquad (12.14)$$

for $0 < \alpha_0 < \alpha_1 < \alpha_2$. Let

$$f_{0\,{\rm c}\,t} := f_0\Big(\frac{|x| - {\rm c}t}{t^\rho}\Big),\ \text{acting on } \mathfrak{h} = L^2(\mathbb{R}^3, {\rm d}k), \qquad (12.15)$$

for constants $0 < {\rm c} < 1$, $0 < \rho < 1$.

Theorem 12.4 *Assume (H0) for $\alpha > 1$, (I0), (I1) for $\epsilon_0 > 1$, (I2) for $\mu > 1$, (I5) for $\mu_2 > 1$ and choose ρ in (12.15) such that $\rho\epsilon_0 > 1$, $\rho\mu_2 > 1$. Then for $0 < \mathrm{c} < \mathrm{c}' < 1$:*

$$i)\ \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(f_{0,\mathrm{c}',t})\mathrm{e}^{-\mathrm{i}tH} =: \Gamma^{+}_{\mathrm{c}'}(f_0) \text{ exists on } \mathcal{H}^{+}_{\mathrm{c}},$$

$$ii)\ [\Gamma^{+}_{\mathrm{c}'}(f_0), H] = 0,$$

$$iii)\ \mathcal{W}I_\Omega\Gamma^{+}_{\mathrm{c}'}(f_0) = \Gamma^{\mathrm{e}+}_{\mathrm{c}'}(f_0)\mathcal{W}I_\Omega,$$

Proof. We use the notation in Sects. 10, 11 and in the proof of Prop. 12.2. By Prop. 11.4 we know that:

$$\Gamma^{\mathrm{e}+}_{\mathrm{c}'}(f_0) = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH^{\mathrm{e}}}\Gamma(f_0(\frac{|s|_0 - \mathrm{c}'t}{t^\rho}))\mathrm{e}^{-\mathrm{i}tH^{\mathrm{e}}} \text{ on } \mathcal{H}^{\mathrm{e}+}_{\mathrm{c}}.$$

Since by Thm. 12.3:

$$u \in \mathcal{H}^{+}_{\mathrm{c}} \Leftrightarrow \mathcal{W}I_\Omega u \in \mathcal{H}^{\mathrm{e}+}_{\mathrm{c}},$$

we deduce from Prop. 3.4 and the fact that $|s|_0$ satisfies (3.6) that:

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(f_0(\frac{a_0 - \mathrm{c}'t}{t^\rho}))\mathrm{e}^{-\mathrm{i}tH} =: \Gamma^{+}_{\mathrm{c}',0}(f_0) \text{ exists on } \mathcal{H}^{+}_{\mathrm{c}}$$

and:

$$[\Gamma^{+}_{\mathrm{c}',0}(f_0), H] = 0,$$
$$\mathcal{W}I_\Omega\Gamma^{+}_{\mathrm{c}',0}(f_0) = \Gamma^{\mathrm{e}+}_{\mathrm{c}'}(f_0)\mathcal{W}I_\Omega.$$

To prove the theorem, it remains to prove that

$$\Gamma^{+}_{\mathrm{c}',0}(f_0) = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Gamma(f_0(\frac{|x| - \mathrm{c}'t}{t^\rho}))\mathrm{e}^{-\mathrm{i}tH} \text{ on } \mathcal{H}^{+}_{\mathrm{c}}. \tag{12.16}$$

Using (12.10) and a density argument, (12.16) will follow from

$$\Big(\Gamma(f_0(\frac{a_0 - \mathrm{c}'t}{t^\rho})) - \Gamma(f_0(\frac{|x| - \mathrm{c}'t}{t^\rho}))\Big)\Gamma(g)\chi(H) \in o(1), \tag{12.17}$$

for $\chi \in C_0^\infty(\mathbb{R})$. Let us replace c' by c to simplify notation. We claim that

$$\Big(f_0(\frac{|x| - \mathrm{c}t}{t^\rho}) - f_0(\frac{a_0 - \mathrm{c}t}{t^\rho})\Big)g \in O(t^{\rho_1 - 2\rho}\log t). \tag{12.18}$$

In fact set $f(s) = 1 - f_0(s)$. The function f satisfies condition (10.4) in Sect. 10. Then

$$f(\frac{s - \mathrm{c}t}{t^\rho}) + f(\frac{-s - \mathrm{c}t}{t^\rho}) = 1 - f_0(\frac{s - \mathrm{c}t}{t^\rho}) \text{ for } s \geq 0, t \gg 1.$$

Applying Lemma 10.1 *ii)* we obtain (12.18). Using (12.18) and (11.10), we obtain:

$$\|\Big(\Gamma(f_0(\frac{a_0 - \mathrm{c}t}{t^\rho})) - \Gamma(f_0(\frac{|x| - \mathrm{c}t}{t^\rho}))\Big)\Gamma(g)(N+1)^{-1}\| \in O(t^{\rho_1 - 2\rho}\log t).$$

By Lemma 4.10, we obtain:

$$\begin{array}{rl} & \|\Big(\Gamma(f_0(\frac{a_0-\mathrm{c}t}{t^\rho}))-\Gamma(f_0(\frac{|x|-\mathrm{c}t}{t^\rho}))\Big)\Gamma(g)\chi(H)\| \\ \leq & Ct^{\rho_1-2\rho}\log t\|(N+1)\Gamma(g_1)\chi(H)\| \\ \leq & Ct^{\rho_1-2\rho+\delta}\log t, \end{array}$$

if g_1 is as g with $g_1 g = g$. As in the proof of Prop. 12.2, we can choose ρ_1, δ such that $\rho_1 - 2\rho + \delta < 0$. This proves (12.17) and completes the proof of the theorem. □

12.3 Geometric asymptotic completeness

Theorem 12.5 *Assume (H0) for $\alpha > 1$, (I0), (I1) for $\epsilon_0 > 1$, (I2) for $\mu > 1$, (I5) for $\mu_2 > 1$. Let $0 < \rho < 1$ such that $\rho\epsilon_0 > 1$, $\rho\mu_2 > 1$ and $0 < \mathrm{c} < \mathrm{c}' < 1$. Let f_0 be a cutoff function satisfying (12.14). Then the operator $\Gamma^+_{\mathrm{c}'}(f_0)$ is equal to the orthogonal projection on the space $\mathcal{K}^+_{\mathrm{c}} := \mathcal{K}^+ \cap \mathcal{H}^+_{\mathrm{c}}$.*

Proof. By Thm. 12.3 *iii)* and identity (8.17) we have:

$$u \in \mathcal{K}^+_{\mathrm{c}} \Leftrightarrow \mathcal{W} I_\Omega u \in \mathcal{K}^{\mathrm{e}+}_{\mathrm{c}}.$$

By Thm. 12.4:

$$\Gamma^+_{\mathrm{c}'}(f_0) = I^*_\Omega \mathcal{W}^{-1}\Gamma^{\mathrm{e}+}_{\mathrm{c}'}(f_0)\mathcal{W} I_\Omega.$$

The theorem follows then from the corresponding result in Thm. 9.5. □

13 The Mourre estimate and its consequences

In this section we study the consequences of a Mourre estimate for the Hamiltonian H for the spaces $\mathcal{H}^+_{\mathrm{c}}$. We show that if a Mourre estimate holds on an energy interval Δ with the generator of dilations as conjugate operator, then the space $1\!\!1_\Delta(H)\mathcal{K}^+_{\mathrm{c}}$ of asymptotic vacua in $\mathcal{H}^\pm_{\mathrm{c}}$ with energy in Δ coincide with the space of bound states of H in Δ.

Let $a := -\frac{1}{2}(k.D_k + D_k.k)$ acting on $\mathfrak{h} = L^2(\mathbb{R}^3, dk)$ be the generator of dilations on the one-particle space. Let $A = 1\!\!1_{\mathcal{K}} \otimes \mathrm{d}\Gamma(a)$. We introduce the following hypothesis on the coupling functions v_j defined in Subsect. 1.1:

$$(I6)\quad \begin{array}{l} \int(1+|k|^{-1})||a|^{1+\epsilon}v_j(k)|^2\mathrm{d}k < \infty, \\ \int|k|^{2+2\epsilon}|v_j(k)|^2\mathrm{d}k < \infty,\ 1 \leq j \leq P,\ 0 \leq \epsilon \leq 1. \end{array}$$

Lemma 13.1 *Assume (I0), (H0) for $\alpha > 1$ and (I6) for $\epsilon > 0$. Then $H \in C^{1+\epsilon'}(A)$ for $\epsilon' = \inf(\alpha - 1, \epsilon)$.*

Proof. Let $v \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h})$ be defined in (1.3). We first claim that under hypothesis *(I6)* we have:

$$(1+|k|^{-\frac{1}{2}})\||a|^{1+\epsilon}\langle \mathrm{x}\rangle^{-1-\epsilon}v\| \in \mathcal{B}(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}).$$

It suffices to prove the claim for $\epsilon = 0, 1$ and then argue by interpolation. The proof of the claim for $\epsilon = 0, 1$ is easy if we note the identity

$$a(\mathrm{e}^{-\mathrm{i}k.\mathrm{x}_j}v_j) = \mathrm{e}^{-\mathrm{i}k.\mathrm{x}_j}(a + k.\mathrm{x}_j)v_j,$$

and use the factor of $\langle \mathrm{x} \rangle$ to control the powers of x_j appearing when computing $a^i v$ for $i = 1, 2$.

We deduce from our claim that under hypothesis *(H0)* for $\alpha > 1$ and *(I6)* for $\epsilon > 1$, we have:

$$(1 + |k|^{-\frac{1}{2}})(K + b)^{-\frac{1}{2}} a^{1+\epsilon'} v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}),\ \epsilon' = \inf(\alpha - 1, \epsilon). \tag{13.1}$$

Another easy observation is that for $v_s = \mathrm{e}^{\mathrm{i}sa} v$ we have:

$$\|(1 + |k|^{-\frac{1}{2}}) v_s\|_{\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})} \leq C, \text{ uniformly in } |s| \leq 1. \tag{13.2}$$

We first claim that the map

$$\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA} (z - H)^{-1} \mathrm{e}^{-\mathrm{i}sA} (H_0 + b)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$$

is C^1 for the norm topology. In fact let

$$H(s) := \mathrm{e}^{\mathrm{i}sA} H \mathrm{e}^{-\mathrm{i}sA} = \mathrm{e}^{-s} H_0 + \phi(\mathrm{e}^{\mathrm{i}sa} v).$$

We have $\mathcal{D}(H(s)) = \mathcal{D}(H_0)$ and $\|(H(s) + \mathrm{i})^{-1}(H_0 + \mathrm{i})\| \leq C$ uniformly for $|s| \leq 1$. We compute

$$\begin{aligned}
& s^{-1}\Big((z - H(s))^{-1} - (z - H)^{-1}\Big)(H_0 + b)^{\frac{1}{2}} \\
= \ & s^{-1}(z - H(s))^{-1}(H(s) - H)(z - H)^{-1}(H_0 + b)^{\frac{1}{2}} \\
= \ & s^{-1}(\mathrm{e}^{-s} - 1)(z - H(s))^{-1} H_0 (z - H)^{-1}(H_0 + b)^{\frac{1}{2}} \\
& + (z - H(s))^{-1} \phi(s^{-1}(\mathrm{e}^{\mathrm{i}sa} - 1\!\!1) v)(z - H)^{-1}(H_0 + b)^{\frac{1}{2}} \\
= \ & s^{-1}(\mathrm{e}^{-s} - 1)(z - H(s))^{-1} H_0 (z - H)^{-1}(H_0 + b)^{\frac{1}{2}} \\
& + (z - H(s))^{-1}(H_0 + b)^{\frac{1}{2}}(H_0 + b)^{-\frac{1}{2}}(K + b)^{\frac{1}{2}} \\
& \times (K + b)^{-\frac{1}{2}} \phi(s^{-1}(\mathrm{e}^{\mathrm{i}sa} - 1\!\!1) v)(z - H)^{-1}(H_0 + b)^{\frac{1}{2}}.
\end{aligned}$$

Using (13.1) and Prop. A.1, we obtain that

$$\lim_{s \to 0} s^{-1}\Big((z - H(s))^{-1} - (z - H)^{-1}\Big)(H_0 + b)^{\frac{1}{2}} = (z - H)^{-1}(-H_0 + \phi(\mathrm{i}av))(z - H)^{-1}(H_0 + b)^{\frac{1}{2}}.$$

in norm, which proves in particular that $H \in C^1(A)$. It remains to prove that the map

$$\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA}(z - H)^{-1}(-H_0 + \phi(\mathrm{i}av))(z - H)^{-1} \mathrm{e}^{-\mathrm{i}sA} \in \mathcal{B}(\mathcal{H})$$

is $C^{\epsilon'}$ for the norm topology. We write:

$$\begin{aligned}
& \mathrm{e}^{\mathrm{i}sA}(z - H)^{-1} H_0 (z - H)^{-1} \mathrm{e}^{-\mathrm{i}sA} \\
= \ & \mathrm{e}^{\mathrm{i}sA}(z - H)^{-1} \mathrm{e}^{-\mathrm{i}sA}(H_0 + b)^{\frac{1}{2}} \\
& \times (H_0 + b)^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}sA} H_0 \mathrm{e}^{-\mathrm{i}sA}(H_0 + b)^{-\frac{1}{2}} \\
& \times (H_0 + b)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}sA}(z - H)^{-1} \mathrm{e}^{-\mathrm{i}sA}.
\end{aligned}$$

The first and third terms in the product are C^1 in norm. The second term is equal to $\mathrm{e}^{-s}H_0(H_0+b)^{-1}$ and hence is also C^1 in norm. This shows that

$$\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}sA}(z-H)^{-1}H_0(z-H)^{-1}\mathrm{e}^{-\mathrm{i}sA}$$

is C^1 in norm. We consider next:

$$\begin{aligned} & \mathrm{e}^{\mathrm{i}sA}(z-H)^{-1}\phi(\mathrm{i}av)(z-H)^{-1}\mathrm{e}^{-\mathrm{i}sA} \\ = \ & \mathrm{e}^{\mathrm{i}sA}(z-H)^{-1}\mathrm{e}^{-\mathrm{i}sA}(H_0+b)^{\frac{1}{2}} \\ & \times(H_0+b)^{-\frac{1}{2}}\phi(\mathrm{e}^{\mathrm{i}sa}\mathrm{i}av)(H_0+b)^{-\frac{1}{2}} \\ & \times(H_0+b)^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}sA}(z-H)^{-1}\mathrm{e}^{-\mathrm{i}sA}. \end{aligned}$$

Again the first and third terms in the product are C^1 in norm. The second term we write as

$$(H_0+b)^{-\frac{1}{2}}(K+b)^{\frac{1}{2}}\phi(\mathrm{e}^{\mathrm{i}sa}(K+b)^{-\frac{1}{2}}\mathrm{i}av)(H_0+b)^{-\frac{1}{2}}.$$

Using (13.1) and Prop. A.1 we see that the second term is $C^{\epsilon'}$ in norm. This completes the proof of the lemma. $\Box$

Lemma 13.2 *Let $f_t(x) \in C^\infty(\mathbb{R}^3)$ with $|\partial_x^\alpha f_t(x)| \leq C_\alpha t^{-\rho|\alpha|}$. Then:*
i) for $F \in C^\infty(\mathbb{R})$ with $\partial_\lambda^\alpha F \in O(\langle\lambda\rangle^{-|\alpha|})$, we have:

$$[\Gamma(f_t), F(\frac{A}{t})] \in O(N)t^{-\rho}.$$

ii) if $\operatorname{supp} f_t \subset \{|x| \leq \mathrm{c}t\}$, $|f_t| \leq 1$ then:

$$\Gamma(f_t)\frac{A}{t}\Gamma(f_t) \leq \mathrm{cd}\Gamma(|k|) + CNt^{-\rho}.$$

Proof. Let us first prove *i)*. We set $F(\lambda) = (\lambda + \mathrm{i})F_{-1}(\lambda)$, with $\partial_\lambda^\alpha F_{-1}(\lambda) \in O(\langle\lambda\rangle^{-1-\alpha})$. We have:

$$[\Gamma(f_t), F(\frac{A}{t})] = [\Gamma(f_t), \frac{A}{t}]F_{-1}(\frac{A}{t}) + (\frac{A}{t}+\mathrm{i})[\Gamma(f_t), F_{-1}(\frac{A}{t})]. \tag{13.3}$$

Now $[\Gamma(f_t), \frac{A}{t}] = \mathrm{d}\Gamma(f_t, [f_t, \frac{a}{t}]) \in O(N)t^{-\rho}$, using [DG2, Lemma 2.8].

Let us estimate the second term in (13.3). Let $\tilde{F}_{-1} \in C^\infty(\mathbb{C})$ be an almost-analytic extension of F_{-1} satisfying (see eg [DG1, Prop. C.2.2]):

$$\begin{aligned} & |\tfrac{\partial \tilde{F}_{-1}}{\partial \overline{z}}(z)| \leq C_N\langle z\rangle^{-2-N}|Imz|^N,\ N \in \mathbb{N}, \\ & \operatorname{supp} \tilde{F}_{-1} \subset \{z||\mathrm{Im}z| \leq C\langle \mathrm{Re}z\rangle\}. \end{aligned}$$

We have:

$$\begin{aligned} & (\tfrac{A}{t}+\mathrm{i})[\Gamma(f_t), F_{-1}(\tfrac{A}{t})] \\ = \ & \tfrac{\mathrm{i}}{2\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{F}_{-1}(z)(\tfrac{A}{t}+\mathrm{i})(z-\tfrac{A}{t})^{-1}[\Gamma(f_t), \tfrac{A}{t}](z-\tfrac{A}{t})^{-1}\mathrm{d}z\wedge\mathrm{d}\,\overline{z} \\ & \in O(N)t^{-\rho}, \end{aligned}$$

using the properties of $\tilde{F}_{-1}$ and the fact that N commutes with A and $\Gamma(f_t)$. This completes the proof of *i)*.

Let us now prove *ii)*. We have:

$$\begin{array}{rl} & \Gamma(f_t)\frac{A}{t}\Gamma(f_t) \\ = & \frac{1}{2}\Gamma(f_t)\mathrm{d}\Gamma(\frac{x}{t}.D_x)\Gamma(f_t)+\frac{1}{2}\Gamma(f_t)\mathrm{d}\Gamma(D_x.\frac{x}{t})\Gamma(f_t) \\ = & \frac{1}{2}\Gamma(f_t^2)\mathrm{d}\Gamma(\frac{x}{t}.D_x)+\frac{1}{2}\mathrm{d}\Gamma(D_x.\frac{x}{t})\Gamma(f_t^2)+O(N)t^{-\rho}. \end{array}$$

Hence on the $n-$particle sector we have:

$$\Gamma(f_t)\frac{A}{t}\Gamma(f_t)=\frac{1}{2}\sum_{i=1}^{n}a_ib_i+b_ia_i+O(N)t^{-\rho},$$

for

$$a_i=\Pi_{j=1}^{n}f_t^2(x_j)\frac{x_i}{t},\ b_i=D_{x_i}.$$

Note the following identity:

$$(ab+ba)^2=4ba^2b+2[[a,b],ab]+[a,b]^2. \tag{13.4}$$

This yields

$$\begin{array}{rl} (a_ib_i+b_ia_i)^2\leq & 4b_ia_i^2b_i+O(t^{-2\rho}) \\ \leq & 4\mathrm{c}^2b_i^2+O(t^{-2\rho}), \end{array}$$

using the properties of f_t. Using the fact that the function $\lambda\to\lambda^{\frac{1}{2}}$ is matrix monotone (see [BR, Sect. 2.2.2]), we obtain:

$$\pm\frac{1}{2}(a_ib_i+b_ia_i)\leq \mathrm{c}|b_i|+O(t^{-\rho}).$$

Summing over i, we get:

$$\pm\Gamma(f_t)\frac{A}{t}\Gamma(f_t)\leq \mathrm{c}\mathrm{d}\Gamma(|k|)+CNt^{-\rho},$$

which proves *ii)*. $\square$

The following theorem is the main result of this section. It means that if a Mourre estimate holds on an energy interval Δ, then on the range of $1\!\!1_\Delta(H)$ the space of asymptotic vacua in $\mathcal{H}_\mathrm{c}^+$ coincide with the space of bound states for c small enough.

Theorem 13.3 *Assume (H0) for $\alpha>1$, (I0), (I1) for $\epsilon_0>1$, (I2) for $\mu>1$, (I5) for $\mu_2>1$ and (I6) for $\epsilon>0$. Let $0<\rho<1$ such that $\rho\epsilon_0>1$, $\rho\mu_2>1$ and $0<\mathrm{c}<\mathrm{c}'<1$. Let $\Delta\subset\mathbb{R}$ be an open interval such that the following Mourre estimate holds on Δ:*

$$1\!\!1_\Delta(H)[H,\mathrm{i}A]1\!\!1_\Delta(H)\geq c_0 1\!\!1_\Delta(H)+R,$$

where $c_0>0$ and $R\in\mathcal{B}(\mathcal{H})$ is compact. Then for $0<\mathrm{c}<c(\Delta,c_0)$ we have:

$$1\!\!1_\Delta(H)\mathcal{K}_\mathrm{c}^+=1\!\!1_\Delta(H)\mathcal{H}_{\mathrm{pp}}(H).$$

Proof. To prove the theorem, it suffices to prove that for c small enough

$$\mathbb{1}_{\Delta}(H)\mathcal{H}_{\mathrm{cont}}(H)\cap\mathcal{K}_{\mathrm{c}}^{+}=\{0\}, \tag{13.5}$$

where $\mathcal{H}_{\mathrm{cont}}(H)$ is the continuous spectral subspace of H. In fact (13.5) implies that $\mathbb{1}_{\Delta}(H)\mathcal{K}_{\mathrm{c}}^{+}\subset\mathbb{1}_{\Delta}(H)\mathcal{H}_{\mathrm{pp}}(H)$. The fact that $\mathcal{H}_{\mathrm{pp}}(H)\subset\mathcal{K}_{\mathrm{c}}^{+}$ is shown in Prop. 8.4.

We first recall that it follows from the fact that $H\in C^{1}(A)$ and that H satisfies a Mourre estimate on Δ that:

i) $\sigma_{\mathrm{pp}}(H)$ is locally finite in Δ,

ii) $\forall\lambda\in\Delta\backslash\sigma_{\mathrm{pp}}(H)$, $\forall\epsilon>0$, there exists $\delta>0$ such that

$$\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H)[H,\mathrm{i}A]\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H)\geq(c_{0}-\epsilon)\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H). \tag{13.6}$$

Let now $f_0\in C^{\infty}(\mathbb{R})$ satisfying (12.14) and let $\lambda\in\Delta\backslash\sigma_{\mathrm{pp}}(H)$. We will show that for δ and c small enough, we have:

$$\|\Gamma_{\mathrm{c}}^{+}(f_0)\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H)\|<1. \tag{13.7}$$

Note that by Thm. 12.5 $\Gamma_{\mathrm{c}'}^{+}(f_0)\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H)$ is equal to the orthogonal projection on the space $\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H)\mathcal{K}_{\mathrm{c}}^{+}$, for $0<\mathrm{c}<\mathrm{c}'$. Hence (13.7) implies that for c small enough

$$\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H)\mathcal{K}_{\mathrm{c}}^{+}=\{0\},$$

which implies (13.5).

Let us now prove (13.7). We deduce first from (13.6) and the fact that $H\in C^{1+\epsilon'}(A)$ for some $\epsilon'>0$ that for any $\epsilon>0$ we have:

$$\mathrm{e}^{-\mathrm{i}tH}\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H)u=F(\frac{A}{t})\mathrm{e}^{-\mathrm{i}tH}\mathbb{1}_{[\lambda-\delta,\lambda+\delta]}(H)u+o(1), \tag{13.8}$$

where $F\in C^{\infty}(\mathbb{R})$, $0\leq F\leq 1$, is supported in $\{\lambda\geq c_0-2\epsilon\}$ and equal to 1 in $\{\lambda\geq c_0-\epsilon\}$. This abstract result is due to [SS2]. A proof under the hypotheses above can be found in [GN].

Let now $u\in D(N^{\frac{1}{2}})$ and $\chi\in C_0^{\infty}(]\lambda-\delta,\lambda+\delta[)$. We recall that it follows from (4.8) and the fact that $\chi(H)$ preserves $D(N^{\frac{1}{2}})$ that

$$\|(N+1)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|\leq Ct^{(1+\epsilon_0)^{-1}}\|(N+1)^{\frac{1}{2}}u\|. \tag{13.9}$$

We have using (13.8), (13.9), Lemma 13.2 *i)* and the fact that $\rho>(1+\epsilon_0)^{-1}$:

$$\begin{array}{rl}
& (\mathrm{e}^{-\mathrm{i}tH}u,\chi(H)\Gamma(f_{0\,\mathrm{c}\,t})^{2}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)\\
= & (\mathrm{e}^{-\mathrm{i}tH}u,\chi(H)\Gamma(f_{0\,\mathrm{c}\,t})^{2}F(\frac{A}{t})\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)+o(1)\\
= & (\mathrm{e}^{-\mathrm{i}tH}u,\chi(H)\Gamma(f_{0\,\mathrm{c}\,t})F(\frac{A}{t})\Gamma(f_{0\,\mathrm{c}\,t})\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)+O(t^{-\rho})\|(N+1)^{\frac{1}{2}}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u\|^{2}+o(1)\\
= & (\mathrm{e}^{-\mathrm{i}tH}u,\chi(H)\Gamma(f_{0\,\mathrm{c}\,t})F(\frac{A}{t})\Gamma(f_{0\,\mathrm{c}\,t})\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)+o(1).
\end{array}$$

Next we have:

$$\begin{array}{rl}
& (\mathrm{e}^{-\mathrm{i}tH}u,\chi(H)\Gamma(f_{0\,\mathrm{c}\,t})^{2}\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)\\
= & (\mathrm{e}^{-\mathrm{i}tH}u,\chi(H)\Gamma(f_{0\,\mathrm{c}\,t})F(\frac{A}{t})\Gamma(f_{0\,\mathrm{c}\,t})\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)+o(1)\\
\leq & (\mathrm{e}^{-\mathrm{i}tH}u,\chi(H)\Gamma(f_{0\,\mathrm{c}\,t})\frac{A}{(c_0-\epsilon)t}\Gamma(f_{0\,\mathrm{c}\,t})\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)+o(1)\\
\leq & (\mathrm{e}^{-\mathrm{i}tH}u,\frac{\mathrm{c}+\epsilon_1}{c_0-\epsilon}\chi(H)\mathrm{d}\Gamma(|k|)\chi(H)\mathrm{e}^{-\mathrm{i}tH}u)+o(1),
\end{array}$$

using Lemma 13.2 *ii)* and the fact that $\mathrm{supp}\, f_{0\,\mathrm{c}\,t} \subset \{|x| \leq (\mathrm{c}+\epsilon_1)t\}$ for all $\epsilon_1 > 0$, $t \geq T(\epsilon_1)$.

Next we have:

$$\chi(H)\mathrm{d}\Gamma(|k|)\chi(H) \leq c_1(\Delta)\chi^2(H).$$

Picking c such that $\mathrm{c}c_1(\Delta) < c_0$, we obtain for ϵ, ϵ_1 small enough:

$$(\mathrm{e}^{-\mathrm{i}tH}u, \chi(H)\Gamma(f_{0\,\mathrm{c}\,t})^2\chi(H)\mathrm{e}^{-\mathrm{i}tH}u) < (u, \chi^2(H)u) + o(1).$$

This yields

$$\|\Gamma_{\mathrm{c}}^{+}(f_0)\chi(H)u\| < \|\chi(H)u\|,\ u \in D(N^{\frac{1}{2}})$$

and hence proves (13.7) by density. This completes the proof of the theorem. □

A Appendix

A.1 Operator bounds

The following proposition is shown in [DJ, Prop. 4.1].

Proposition A.1 *Let $v \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}^{\mathrm{e}})$, ω be a selfadjoint operator on $\mathfrak{h}^{\mathrm{e}}$. Assume that $\omega \geq 0$ and that ω is invertible on the range of v. Then:*

$$\begin{aligned}
&\|a(v)u\|^2 \leq \|v^*\omega^{-1}v\|(u, \mathrm{d}\Gamma(\omega)u),\\
&\|a^*(v)u\|^2 \leq (u, v^*vu) + \|v^*\omega^{-1}v\|(u, \mathrm{d}\Gamma(\omega)u),\\
&\|\phi^*(v)u\|^2 \leq (u, v^*vu) + 2\|v^*\omega^{-1}v\|(u, \mathrm{d}\Gamma(\omega)u),
\end{aligned}$$

for $u \in \mathcal{D}(\mathrm{d}\Gamma(\omega)^{\frac{1}{2}})$.

A.2 Propagation estimates and existence of limits

In this subsection we formulate two generalizations of standard arguments due to Sigal-Soffer [SS1]. Their proofs are analogous to the standard ones.

Proposition A.2 *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a dense subspace, H a selfadjoint operator on $\mathcal{H}$ and*

$$\mathbb{R}^+ \ni t \mapsto \Phi(t) \in \mathcal{B}(\mathcal{H})$$

a function with $\sup_{t\geq 0}\|\Phi(t)\| < \infty$. Assume that for $u \in \mathcal{D}$ the function:

$$f(t) = (u_t, \Phi(t)u_t) \in C^1(\mathbb{R}) \text{ if } u_t = \mathrm{e}^{-\mathrm{i}tH}u,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \geq (u_t, R^*(t)R(t)u_t) - \sum_{i=1}^{n}(u_t, R_i^*(t)R_i(t)u_t)$$

where

$$\int_1^{+\infty}\|R_i(t)u_t\|^2\mathrm{d}t \leq C\|u\|^2,\ u \in \mathcal{D},\ 1 \leq i \leq n.$$

Then

$$\int_1^{+\infty}\|R(t)u_t\|^2\mathrm{d}t \leq C\|u\|^2,\ u \in \mathcal{D}.$$

Proposition A.3 *Let $\mathcal{H}_i, \mathcal{D}_i, H_i$ $i = 1, 2$ be as in Prop. A.2. Let*

$$\mathbb{R}^+ \ni t \mapsto \Phi(t) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$$

a function with $\sup_{t\geq 0} \|\Phi(t)\| < \infty$*. Assume that for $u_i \in \mathcal{D}_i$ the function*

$$f(t) = (u_{2,t}, \Phi(t)u_{1,t}) \in C^1(\mathbb{R})$$

and

$$|\frac{\mathrm{d}f(t)}{dt}| \leq \sum_{i=1}^{n} \|B_{2,j}(t)u_{2,t}\| \|B_{1,j}(t)u_{1,t}\|,$$

where

$$\int_1^{+\infty} \|B_{i,j}(t)u_{i,t}\|^2 \mathrm{d}t \leq C\|u_i\|^2, \; u_i \in \mathcal{D}_i \; i = 1, 2, \; 1 \leq j \leq n.$$

Then

$$\text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH_2}\Phi(t)\mathrm{e}^{-\mathrm{i}tH_1} \textit{ exists.}$$

A.3 Existence of limits of asymptotic observables

In this subsection we give two different methods to show the existence of weak or strong limits of asymptotic observables.

Proposition A.4 *Let $\mathcal{H}_i, \mathcal{D}_i, H_i$ $i = 1, 2$ be as in Prop. A.2. Let for $\epsilon \in [0, 1[$:*

$$\mathbb{R}^+ \ni t \mapsto \Phi_\epsilon(t) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$$

such that:

(A.1)
$$\sup_{t\in\mathbb{R}^+} \|\Phi_\epsilon(t)\| < \infty, \forall\epsilon \geq 0, \; \sup_{\epsilon\in[0,1[} \|\Phi_\epsilon(0)\| < \infty;$$

$$f_\epsilon(t) = (v_t, \Phi_\epsilon(t)u_t) \in C^1(\mathbb{R}^+) \textit{ for } u \in \mathcal{D}_1, v \in \mathcal{D}_2;$$

(A.2)
$$|\tfrac{df_\epsilon(t)}{\mathrm{d}t}| \leq \textstyle\sum_{i=1}^n \|R_{1,j}(t)u_t\| \|R_{2,j}(t)v_t\| \textit{ uniformly in } \epsilon \in [0, 1[$$

$$\textit{with } \textstyle\int_1^{+\infty} \|R_{i,j}(t)u_t\|^2 \mathrm{d}t \leq C\|u\|^2, \; u \in \mathcal{D}_i, \; i = 1, 2, \; 1 \leq j \leq n;$$

(A.3)
$$\mathrm{w} - \lim_{\epsilon\to 0} \Phi_\epsilon(t) = \Phi_0(t), \; \forall t \gg 1,$$

$$\textit{resp.} \quad \text{s-}\lim_{\epsilon\to 0} \Phi_\epsilon(t) = \Phi_0(t), \; \forall t \gg 1.$$

Then:

$$i) \;\; \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Phi_\epsilon(t)\mathrm{e}^{-\mathrm{i}tH} =: \Phi_\epsilon^+ \textit{ exists } \forall\epsilon \in [0, 1[$$

$$ii) \quad \mathrm{w} - \lim_{\epsilon\to 0} \Phi_\epsilon^+ = \Phi_0^+,$$

$$\textit{resp.} \quad \text{s-}\lim_{\epsilon\to 0} \Phi_\epsilon^+ = \Phi_0^+.$$

Proof. *i)* follows from Prop. A.3. It follows from (A.1), (A.2) that Φ_ϵ^+ is uniformly bounded in ϵ. Hence to prove *ii)* it suffices by density to show that

$$\lim_{\epsilon\to 0}(v, \Phi_\epsilon^+ u) - (v, \Phi_0^+ u) = 0,\ v \in \mathcal{D}_2, u \in \mathcal{D}_1,$$

respectively that:

$$\lim_{\epsilon\to 0} \sup_{v\in\mathcal{D}_2, \|v\|\leq 1} |(v, \Phi_\epsilon^+ u) - (v, \Phi_0^+ u)| = 0,\ u \in \mathcal{D}_1.$$

We have:

$$\begin{array}{rl} & (v, \Phi_\epsilon^+ u) - (v, \Phi_0^+ u) \\ = & (v_T, \Phi_\epsilon(T)u_T) - (v_T, \Phi_0(T)u_T) \\ & + \int_T^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}(v_t, \Phi_\epsilon(t)u_t)\mathrm{d}t - \int_T^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}(v_t, \Phi_0(t)u_t)\mathrm{d}t. \end{array}$$

The sum of the last two terms is less than

$$\begin{array}{rl} & 2\sum_{i=1}^n (\int_T^{+\infty} \|R_{1,j}(t)u_t\|^2\mathrm{d}t)^{\frac{1}{2}} (\int_T^{+\infty} \|R_{2,j}(t)v_t\|^2\mathrm{d}t)^{\frac{1}{2}} \\ \leq & C\sum_{i=1}^n (\int_T^{+\infty} \|R_{1,j}(t)u_t\|^2\mathrm{d}t)^{\frac{1}{2}} \|v\|, \end{array}$$

uniformly in ϵ by (A.2). For $T \gg 1$ this is less than $\alpha\|v\|$ for fixed $u \in \mathcal{D}_1$, uniformly in ϵ. Then *ii)* follows from the fact that for fixed T $\Phi_\epsilon(T)$ converges weakly (resp. strongly) to $\Phi_0(T)$ when $\epsilon \to 0$. □

Proposition A.5 *Let $\mathcal{H}, \mathcal{D}, H$ be as in Prop. A.2. Let for $\epsilon \in]0, 1[$*

$$\mathbb{R}^+ \ni t \mapsto \Phi_\epsilon(t) \in \mathcal{B}(\mathcal{H}) \text{ selfadjoint},$$

such that for fixed ϵ $\Phi_\epsilon(t)$ satisfies the hypotheses of Prop. A.3 with $\mathcal{H}_i = \mathcal{H}$, $\mathcal{D}_i = \mathcal{D}$ and $H_i = H$, $i = 1, 2$. It follows that

$$\Phi_\epsilon^+ = \text{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH}\Phi_\epsilon(t)\mathrm{e}^{-\mathrm{i}tH} \text{ exists } \forall \epsilon > 0.$$

Assume that

$$\begin{array}{l} 0 \leq \Phi_\epsilon^+ \leq 1\!\!1; \\ \frac{\mathrm{d}}{\mathrm{d}t}(u_t, \Phi_\epsilon(t)u_t) \geq -\|R(t)u_t\|^2,\ \epsilon \in]0, 1[,\ u \in \mathcal{D}; \\ \text{where } \int_1^{+\infty} \|R(t)u_t\|^2\mathrm{d}t \leq C\|u\|^2,\ u \in \mathcal{D}; \\ \mathrm{w} - \lim_{\epsilon\to 0} \Phi_\epsilon(t) = 1\!\!1,\ \forall t \gg 1. \end{array}$$

Then

$$\mathrm{w} - \lim_{\epsilon\to 0} \Phi_\epsilon^+ = 1\!\!1.$$

Proof. Since Φ_ϵ^+ is uniformly bounded, it suffices by density to show that

$$\lim_{\epsilon\to 0}(u, \Phi_\epsilon^+ u) = (u, u),\ u \in \mathcal{D}.$$

We have:

$$\begin{aligned}(u,\Phi^+_\epsilon u) = {} & (u_T,\Phi_\epsilon(T)u_T)+\int_T^{+\infty}\frac{\mathrm{d}}{\mathrm{d}t}(u_t,\Phi_\epsilon(t)u_t)\mathrm{d}t\\ \geq {} & (u_T,\Phi_\epsilon(T)u_T)-\int_T^{+\infty}\|R(t)u_t\|^2\mathrm{d}t.\end{aligned}$$

For $\alpha>0$ we first choose $T\gg 1$ such that the second term is less than α then ϵ_0 such that for $\epsilon<\epsilon_0$ the first term is greater than $(u,u)-\alpha$. We obtain $\underline{\lim}_{\epsilon\to 0}(u,\Phi^+_\epsilon u)\geq\|u\|^2$. Since $(u,\Phi^+_\epsilon u)\leq\|u\|^2$ this proves the proposition. □

A.4 Existence of some projections

In this subsection we show the existence of some projections, using pseudo-resolvent arguments.

Proposition A.6 *Let $\mathcal{H},H$ be as in Prop. A.2 and let $\mathbb{R}^+\ni t\mapsto B_t$, where B_t is a selfadjoint operator on $\mathcal{H}$, $B_t\geq 0$. Assume that $\forall\lambda\in\mathbb{C}\backslash\mathbb{R}$:*

$$R^+(\lambda)=\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH}(B_t+\lambda)^{-1}\mathrm{e}^{-\mathrm{i}tH}\ \textit{exists.}$$

Then:

i) for $\chi\in C_\infty(\mathbb{R})$ the limit

$$\chi^+=\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH}\chi(B_t)\mathrm{e}^{-\mathrm{i}tH}\ \textit{exists.}$$

ii) if $\chi\in C_0(\mathbb{R})$, $0\leq\chi\leq 1$, χ decreasing, $\chi\equiv 1$ near 0 and $\chi_n(\lambda)=\chi(n^{-1}\lambda)$ then

$$\text{s-}\lim_{n\to+\infty}\chi_n^+=P^+\ \textit{exists}$$

and is an orthogonal projection independent on the choice of χ.

$$iii)\ P^+=\text{s-}\lim_{\epsilon\to 0}\text{s-}\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}tH}(\epsilon B_t+1\!\!1)^{-1}\mathrm{e}^{-\mathrm{i}tH}.$$

Proof. *i)*: the functions $s\mapsto(s+\lambda)^{-1}$ for $\lambda\in\mathbb{C}\backslash\mathbb{R}$ are total in $C_\infty(\mathbb{R})$ by the Stone Weierstrass theorem. Hence the limit χ^+ exists for all $\chi\in C_\infty(\mathbb{R})$.

ii): clearly we have $[\chi_n^+,\chi_m^+]=0\ \forall n,m$ and $\chi_n^+\leq\chi_{n+1}^+\leq 1\!\!1$. Hence $P^+=\mathrm{w}-\lim_{n\to+\infty}\chi_n^+$ exists.

For $m\geq n_0n$ with n_0 large enough, we have $\chi_n^+\chi_m^+=\chi_n^+$. Letting $m\to+\infty$, we obtain $\chi_n^+P^+=\chi_n^+$. Letting then $n\to+\infty$ we obtain $P^{+2}=P^+$, ie P^+ is a projection. We also have $\chi_{m(n)}\leq\chi_n^2\leq\chi_n$, for $m(n)\ll n$, $m(n)\to+\infty$ when $n\to+\infty$. Hence $\chi^+_{m(n)}\leq\chi_n^{+2}\leq\chi_n^+$. Letting $n\to+\infty$, we get $P^+=\mathrm{w}-\lim_{n\to+\infty}\chi_n^{+2}$.

Then we compute

$$\lim_{n\to+\infty}\|(P^+-\chi_n^+)u\|^2=\lim_{n\to+\infty}(u,(\chi_n^{+2}-P^+)u)=0,$$

which shows that $P^+=\text{s-}\lim_{n\to+\infty}\chi_n^+$. To prove that P^+ is independent on the choice of χ, we note that if χ_1,χ_2 are two such functions, we have $\chi_{1,m(n)}\leq\chi_{2,n}$ for $m(n)\to+\infty$ when $n\to+\infty$. This yields $\chi^+_{1,m(n)}\leq\chi^+_{2,n}$ and proves the statement by letting $n\to+\infty$.

To prove *iii)* it suffices to show that if $\chi\in C_\infty(\mathbb{R})$, $0\leq\chi\leq 1$, χ decreasing on $\mathbb{R}^+$ and $\chi(0)=1$ then $\text{s-}\lim_{n\to+\infty}\chi_n^+=P^+$. For such χ and fixed $\epsilon_0>0$, we can find a function χ_1 satisfying the conditions of *ii)* such that $\|\chi-\chi_1\|_\infty\leq\epsilon_0$. Then the statement follows from the fact that $\text{s-}\lim_{n\to+\infty}\chi^+_{1,n}=P^+$, by *ii)*. □

References

[AGHH] Albeverio, S. , Gesztesy F., Høgh-Krohn R., Holden H.: *Solvable models in quantum mechanics*, Texts and Monographs in Physics, Springer, 1988.

[A] Ammari, Z.: Asymptotic completeness for a renormalized non relativistic Hamiltonian in quantum field theory: the Nelson model, preprint mp-arc 00-43 (2000)

[ABG] Amrein, W., Boutet de Monvel, A., Georgescu, W.: *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser, Basel-Boston-Berlin, 1996.

[Ar] Arai, A.: Ground State of the Massless Nelson Model Without Infrared Cutoff in a Non-Fock Representation, preprint mp-arc 00-478 (2000)

[AH] Arai, A., Hirokawa, M.: On the existence and uniqueness of ground states of a generalized spin-boson model. J. Funct.Anal. **151** (1997) 455–503

[BFS] Bach, V., Fröhlich, J., Sigal, I.: Quantum electrodynamics of confined non-relativistic particles, Adv. Math. 137 (1998), no. 2, 299–395

[BFSS] Bach, V., Fröhlich, J., Sigal, I., Soffer, A.: Positive commutators and the spectrum of Pauli-Fierz Hamiltonian of atoms and molecules. Comm. Math. Phys. 207 (1999), 557–587

[BR] Bratteli, O, Robinson D. W.: *Operator Algebras and Quantum Statistical Mechanics*, Texts and Monographs in Physics, Springer Berlin 1981.

[DG1] Dereziński, J., Gérard, C.: *Scattering Theory of Classical and Quantum N-Particle Systems*, Texts and Monographs in Physics, Springer, 1997.

[DG2] Dereziński, J., Gérard, C.: Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians, Rev. Math. Phys. 11 (1999) 383-450.

[DG3] Dereziński, J., Gérard, C.: Spectral and scattering theory of spatially cut-off $P(\varphi)_2$ Hamiltonians, Comm. Math. Phys. 213 (2000) 39-125

[DJ] Dereziński, J. Jaksic, V.: Spectral theory of Pauli-Fierz operators, preprint mp-arc 00-31 (2000)

[Fr1] Fröhlich, J.: Existence of dressed one-electron states in a class of persistent models, Forschritte der Physik 22 (1974) 159-198

[Fr2] Fröhlich, J.: On the infrared problem in a model of scalar electrons and massless scalar bosons, Ann. Inst. Henri Poincaré A 19 (1973) 1-103

[FGS] Fröhlich, J,, Griesemer, M., Schlein B.: Asymptotic electromagnetic fields in models of quantum-mechanical matter interacting with the quantized radiation field, preprint mp-arc 00-374 (2000)

[G] Gérard, C. : On the existence of ground states for massless Pauli-Fierz Hamiltonians, Ann. Henri Poincaré 1 (2000), no. 3, 443–459

[GN] Gérard, C., Nier, F.: Scattering theory for the perturbations of periodic Schrödinger operators, J. Math. Kyoto Univ. 38 (1998) 595-634

[GLL] Griesemer M., Lieb E., Loss M.: Ground states in non-relativistic quantum electrodynamics, preprint mp-arc 00-313 (2000)

[HS] Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et equation de Harper, Springer Lecture Notes in Physics **345** (1989) 118-197

[HK1] Høegh-Krohn, R.: On the spectrum of the space cutoff : $P(\varphi)$: Hamiltonian in 2 space-time dimensions, Comm. Math. Phys. **21** (1971) 256-260

[HK2] Høegh-Krohn R.: Asymptotic limits in some models of quantum field theory, I J. Math. Phys. J. Math. Phys. 9 (1968) 2075-2079

[HuSp1] Hübner, M., Spohn, H.: Radiative decay: non-perturbative approaches, Rev. Math. Phys 7 (1995) 363-387

[HuSp2] Hübner, M., Spohn, H.: Spectral properties of the spin-boson Hamiltonian, Ann. Inst. H. Poincare 62 (1995) 289-323

[LMS] Lorinczi, J., Minlos, R. Spohn, H.: The infrared behavior in Nelson's model of a quantum particle coupled to a massless scalar field, preprint mp-arc 00-467 (2000)

[Ne] Nelson. E: Interaction of non-relativistic particles with a quantized scalar field, J. Math. Phys. 5(1964) 1190-1197

[P] Pizzo A.:One particle (improper) states and Scattering States in Nelson's Massless Model, preprint math-ph 0010043 (2000)

[Sch] Schweber, S.: *An Introduction to Relativistic Quantum Field Theory* New York : Harper and Row, 1961.

[SS1] Sigal, I., Soffer, A.: The $N-$particle scattering problem: asymptotic completeness for short-range quantum systems, Ann. of Math. **125** (1987) 35-108

[SS2] Sigal, I., Soffer, A.: Local decay and velocity bounds, preprint Princeton University (1988)

[Sk] Skibsted, E.: Spectral analysis of N-body systems coupled to a bosonic field. Rev. Math. Phys. 10 (1998), no. 7, 989–1026