From classical to semiclassical non-trapping behaviour: a new proof.

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Abstract

For the semiclassical Schrödinger operator with smooth long-range potential, it is well known that the boundary values of the resolvent at non-trapping energies exist and are bounded by $O(h^{-1})$ (h being the semiclassical parameter). We present here a new proof of this result, which avoids the semiclassical Mourre theory and makes use of semiclassical measures.

Keywords: Schrödinger operators, semiclassical resolvent estimates, resonances, semiclassical measure, global escape function.

1 Introduction.

At the beginning of the eighties, E. Mourre (in [16]) introduced the now well-known Mourre theory in order to prove asymptotic completeness in N-body quantum scattering. It is indeed an essential tool to reach this result (see [4, 5, 21]). The theory is a powerful tool to get boundary values of the resolvent (acting on weighted spaces), yielding important propagations estimates.
On the semiclassical level for Schrödinger operators, this theory gives the bound $O(h^{-1})$ (h being the semiclassical parameter) for the boundary values of the resolvent at non-trapping energy (see [20, 7, 22]) and this semiclassical non-trapping behaviour is even equivalent to the classical one (see [23]).
Using a global escape function, whose existence reflects the non-trapping condition, one can derive the semiclassical Mourre estimate and then follow the Mourre theory (see [7]), keeping the h-dependance in sight, to arrive at the bound $O(h^{-1})$. Our purpose here is to provide a new proof of the last step, that is to avoid the semiclassical version of

Mourre theory. In fact, we want to extend the proof ad absurdum by N. Burq (see [3]), obtained for compactly supported potential, to the case of smooth long-range potentials. In contrast to [3], we do use semiclassical Mourre (type) estimates, which we derive from the existence of escape functions via Gårding inequality.
Notice that a global escape function with constant sign is used in [22] but the Mourre theory is completely avoided there. Direct estimates based on microlocal ideas and Gårding inequality lead to the desired resolvent estimates. Although similarities with the present paper occur, the strategy of the proof in [22] is different.
Before entering precise statements, let us mention three motivations for this project. The original one concerns the matricial version of the result (i.e. for matricial Schrödinger operators). Constating difficulties to apply Mourre theory in this case (see [12]), it is interesting to investigate another strategy. Secondly, the proof we are about to present is rather elementary as soon as pseudodifferential calculus, Gårding inequality, and the notion of semiclassical measure are known (remember that we do not show the existence of the boundary values of the resolvent). Finally, since we consider the case where the bound $O(h^{-1})$ does not hold, we derive some information (more precise than in [3]) about this situation. Considering a sequence $h_n \to 0$ and a sequence of normalized functions (a_n) such that the action of one boundary value of the resolvent on them blows up faster than h_n^{-1}, we show that any "semiclassical measure" of another sequence (b_n) of functions, explicitly related to the first one (in fact, b_n is roughly the boundary value of the resolvent applied to a_n conveniently normalized), is invariant along the classical flow (see Proposition 4 and Remark 5) and has compact support (see Proposition 7). Since we believe that such a situation appears when one considers a resonance (see [8, 14, 15]), which is semiclassically relevant (i.e. of width $O(h)$), we call it a "semiclassically resonant" situation. The previous results could be integrated to and be useful for the semiclassical theory of resonances.

Let us now introduce some notation and present the result. Recall that the semiclassical Schrödinger operator, in dimension $d \geq 1$, is given by $P := -h^2\Delta_x + V(x)$, acting in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C})$ equiped with $\|\cdot\|$, its usual norm, where the semiclassical parameter $h \in]0; h_0]$ for some $h_0 > 0$, Δ_x denotes the Laplacian in $\mathbb{R}^d$, and where V is the multiplication operator by a real-valued function V. We require that V is C^∞ on $\mathbb{R}^d$ and that there exists $\rho > 0$ such that

$$\forall \alpha \in \mathbb{N}^d\,,\, \forall x \in \mathbb{R}^d\,, \quad |\partial_x^\alpha V(x)| \;=\; O_\alpha\Big(\langle x\rangle^{-\rho-|\alpha|}\Big) \tag{1}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$. It is well known that P is self-adjoint on the domain of the Laplacian (see [18]). Denoting its resolvent by $R(z) := (P-z)^{-1}$, for z in the resolvent set of P, we are interested in its boundary values on $\mathbb{R}^+$ as bounded operators from $\mathrm{L}^2_s(\mathbb{R}^d;\mathbb{C})$ to $\mathrm{L}^2_{-s}(\mathbb{R}^d;\mathbb{C})$, for $s > 1/2$. By $\mathrm{L}^2_s(\mathbb{R}^d;\mathbb{C})$ we mean the weighted L^2 space of mesurable, $\mathbb{C}$-valued functions f on $\mathbb{R}^d$ such that $x \mapsto \langle x\rangle^s f(x)$ belongs to $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C})$. The previously mentioned Mourre theory ensures that, for any $\lambda > 0$ outside the pure point spectrum of P, the boundary values $R(\lambda \pm i0)$ exist (see [16, 13]).
We denote by $p(x,\xi) := |\xi|^2 + V(x)$, $(x,\xi) \in T^*\mathbb{R}^d$, the symbol of P and by ϕ^t the

associated Hamilton flow on $T^*\mathbb{R}^d$. Recall that an energy λ is non-trapping for p if

$$\forall (x,\xi)\in p^{-1}(\lambda),\ \lim_{t\to -\infty} |\phi^t(x,\xi)| \;=\; +\infty \text{ and } \lim_{t\to +\infty} |\phi^t(x,\xi)| \;=\; +\infty\,. \tag{2}$$

Notice that this property is "open": if λ is non-trapping, there exists some interval I about λ such that each $\mu\in I$ is non-trapping. Such an interval is called an interval of non-trapping energies. We can now state our main result.

Theorem 1. *([20, 7, 22]) Under the previous assumptions, let $I\subset]0;+\infty[$ be a compact interval of non-trapping energies. Then, for small enough h_0, the boundary values $R(\lambda\pm i0)$ exist for $\lambda\in I$ and, for any $s>1/2$, there exists $C_s>0$ such that, uniformly for $\lambda\in I$ and $h\in]0;h_0]$,*

$$\|\langle x\rangle^{-s}\, R(\lambda\pm i0)\, \langle x\rangle^{-s}\| \;\leq\; C_s\, h^{-1}\,. \tag{3}$$

The paper is organized as follows. In Section 2, we derive some properties of the "semiclassically resonant" situation, in particular the results mentioned above, and consider the interaction with the non-trapping condition. The new proof of Theorem 1 is given in Section 3.

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2 "Semiclassically resonant" situation.

In this section, we want to consider the "semiclassically resonant" situation. Precisely, we assume the existence of a sequence $(f_n)_n$ of nonzero functions of the domain $\mathcal{D}(P)$ of P, of a sequence $(h_n)_n\in]0;h_0]^{\mathbb{N}}$ tending to zero, and of a sequence $(z_n)_n\in\mathbb{C}^{\mathbb{N}}$ with $\Re(z_n)\to\lambda>0$ and $\Im(z_n)/h_n\to r\geq 0$, such that $\|\langle x\rangle^{-s}f_n\|=1$ and $\|\langle x\rangle^{s}(P(h_n)-z_n)f_n\|=o(h_n)$, where $P(h_n):=-h_n^2\Delta_x+V(x)$. As we shall see in Section 3, such a situation appears when one assumes, in view of the proof ad absurdum of Theorem 1, that the bound (3) with $s>1/2$ for "+" is false. However, we do not need here to require $s>1/2$ but only $s\geq 0$. The aim of this section is to show that $\langle x\rangle^{2s}$ times any semiclassical measure of $(\langle x\rangle^{-s}f_n)_n$ is invariant under the flow ϕ^t, has compact support, and that, for $s>1/2$, this semiclassical measure is nonzero. The situation here is similar to those in [3], the strategy of which we follow. However, new ingredients and new results appear in the present paper.

Let us first recall some well known facts about semiclassical measures, which can be found in [6, 17]. Let μ_s be a semiclassical measure of the bounded sequence $(\langle x\rangle^{-s}f_n)_n$ in L^2. Then μ_s is a finite, nonnegative Radon measure on the cotangent space $T^*\mathbb{R}^d$ (its total mass is less than 1). Furthermore, there exists a sequence $h_n\to 0$ such that, for any $a\in C_0^\infty(T^*\mathbb{R}^d)$,

$$\lim_{n\to\infty}\Big\langle a^w(x,h_nD)\langle x\rangle^{-s}f_n,\langle x\rangle^{-s}f_n\Big\rangle \;=\; \int_{T^*\mathbb{R}^d} a(x,\xi)\,\mu_s(dx\,d\xi) \;=:\; \mu_s(a)\,. \tag{4}$$

Here $\langle \cdot, \cdot \rangle$ denotes the L^2 scalar product and $a^w(x, hD)$ the Weyl h-quantization of the symbol $a(x, \xi)$ defined by

$$C_0^\infty(\mathbb{R}^d; \mathbb{C}) \ni u \;\mapsto\; \Big(a^w(x, hD)u\Big)(x) \;=\; (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)/h} a\Big((x+y)/2, \xi\Big) u(y)\, dy d\xi\, .$$

For the Weyl h-pseudodifferential calculus we shall use at many places in this paper, we refer to [19, 17, 5]. We shall also use the functional calculus of Helffer-Sjöstand, which can be found in [9, 5].

Since $\|\langle x\rangle^s (P(h_n) - z_n) f_n\| = o(h_n)$, the sequence $(f_n)_n$ should accumulate microlocally on the energy shell $p^{-1}(\lambda)$. This is the purpose of

Proposition 2. *The measure μ_s is supported in $p^{-1}(\lambda)$.*

Proof: Let $a \in C_0^\infty(T^*\mathbb{R}^d)$ with support disjoint from $p^{-1}(\lambda)$. Since $\Re(z_n) \to \lambda$, $a\langle x\rangle^{-2s}(p - z_n)^{-1}$ is a bounded symbol for n large enough. Therefore

$$\lim_{n\to\infty} \Big\langle a^w(x, h_n D)\langle x\rangle^{-s} f_n, \langle x\rangle^{-s} f_n \Big\rangle \;=\; 0\, ,$$

as $n \to \infty$, since $\|\langle x\rangle^s (P(h_n) - z_n) f_n\| = o(h_n)$. Thus $\mu_s(a) = 0$ by (4). □

Remark 3. *Thanks to this localization property of μ_s, (4) holds true for $a \in C_0^\infty(\mathbb{R}^d)$.*

Now we want to follow Burq's arguments (in [3]) to derive that the Poisson bracket (in the distributional sense) $\{p, \langle x\rangle^{2s}\mu_s\}$ equals $r\langle x\rangle^{2s}\mu_s$. But it turns out that $r = 0$ in our case. Indeed, writing P instead of $P(h_n)$,

$$\begin{aligned} o(h_n) &= \langle \langle x\rangle^s (P - z_n) f_n, \langle x\rangle^{-s} f_n\rangle \;=\; \langle (P - z_n) f_n, f_n\rangle \;=\; \langle f_n, (P - \bar{z_n}) f_n\rangle \\ &= \langle f_n, (P - z_n) f_n\rangle \,+\, \langle f_n, 2i\Im(z_n) f_n\rangle \;=\; o(h_n) \,+\, \langle f_n, 2i\Im(z_n) f_n\rangle\, , \end{aligned}$$

yielding $\|f_n\|^2 \Im(z_n)/h_n = o(1)$. If $r \neq 0$ then $\|f_n\|^2 \to 0$. Since $s \geq 0$, $1 = \|\langle x\rangle^{-s} f_n\|^2 \leq \|f_n\|^2$ and we arrive at a contradiction. Therefore, we get the following stronger result.

Proposition 4. *The measure $\langle x\rangle^{2s}\mu_s$ is invariant under ϕ^t, that is $\{p, \langle x\rangle^{2s}\mu_s\} = 0$.*

Proof: We follow [3]. For any $a \in C_0^\infty(T^*\mathbb{R}^d)$, replacing $P(h_n)$ by P again,

$$\begin{aligned} \Big\langle i h_n^{-1} [P, a^w(x, h_n D)] f_n, f_n \Big\rangle &= \langle i h_n^{-1} a^w(x, h_n D) f_n, (P - \bar{z_n}) f_n\rangle \\ &\quad - \langle i h_n^{-1} a^w(x, h_n D)(P - z_n) f_n, f_n\rangle \\ &= (2r + o(1))\langle a^w(x, h_n D) f_n, f_n\rangle \,+\, o(1) \;=\; o(1)\, , \end{aligned} \tag{5}$$

on one hand, and

$$\begin{aligned} \Big\langle i h_n^{-1} [P, a^w(x, h_n D)] f_n, f_n \Big\rangle &= \langle \{p, a\}^w(x, h_n D) f_n, f_n\rangle \,+\, O(h_n) \\ &= \mu_s\Big(\langle x\rangle^{2s}\{p, a\}\Big) \,+\, o(1)\, , \end{aligned} \tag{6}$$

on the other hand. Thus $\{p, \langle x\rangle^{2s}\mu_s\} = 0$. □

Remark 5. *Notice that (4) holds true with $\langle x\rangle^{-s}f_n$ and μ_s replaced by f_n and $\mu := \langle x\rangle^{2s}\mu_s$ respectively. If the sequence $(f_n)_n$ is bounded in L^2 then μ is a semiclassical measure of this sequence (see [6]). This is true for $s=0$ but, so far, $(f_n)_n$ might be unbounded in L^2 if $s>0$ (such a situation appears for $V=0$ and $s\in]0;1/2[$, see (11)). In this case, the measure μ plays the role of a semiclassical measure of $(f_n)_n$ (since (4) holds true) but might not belong to the limit points of the Wigner transform of $(f_n)_n$ (see [6]).*

Looking for other properties of μ_s, we learn from [6] that $\mu_s = 0$ roughly means that $(f_n)_n$ accumulates at infinity (in the position variable x). More precisely, if

$$\lim_{R\to+\infty}\limsup_{n\to\infty}\int_{|x|>R}\langle x\rangle^{-2s}f_n^2\,dx \;=\; 0\,, \tag{7}$$

the total mass of μ_s equals $\lim_{n\to\infty}\|\langle x\rangle^{-s}f_n\|^2$ and, in particular, μ_s is nonzero. Let $\theta\in C_0^\infty(\mathbb{R})$ supported near λ and let $\mathbb{1}_{\{|x|>R\}}$ be the characteristic function of the set $\{(x,\xi)\in T^*\mathbb{R}^d; |x|>R\} =: T^*\mathbb{R}^d\setminus B_R^*$. To bound above the quantity $\int_{|x|>R}\langle x\rangle^{-2s}f_n^2\,dx$, which should be close to $\|\mathbb{1}_{\{|x|>R\}}\theta(P(h_n))\langle x\rangle^{-s}f_n\|^2$ by energy localization of f_n, we want to use a semiclassical Mourre estimate "at infinity"

$$\theta(P)\mathbb{1}_{\{|x|>R\}}ih^{-1}[P,A]\mathbb{1}_{\{|x|>R\}}\theta(P)\;\geq\; c\,\theta(P)\mathbb{1}_{\{|x|>R\}}\theta(P)\,+\,\tilde{O}(h)\,, \tag{8}$$

where $c>0$, A is a well chosen operator, and $\tilde{O}(h)$ stands for a bounded operator on L^2, the norm of which is $O(h)$. Such an inequality follows, via Gårding inequality, from the existence of an escape function at infinity. Under (1), we do have such a function, namely $x\cdot\xi$. Indeed, since $\lambda>0$, there exists $c>0$ such that

$$\{p, x\cdot\xi\}\;=\;2|\xi|^2 - x\cdot\nabla V(x)\;=\;2p-2V(x)-x\cdot\nabla V(x)\;\geq\; c$$

on $p^{-1}(\lambda)$ for $|x|$ large enough. To control the l.h.s. of the semiclassical Mourre estimate (8), we need a bounded escape function at infinity. If we could do this, we would have a Mourre estimate for P with a bounded conjugate operator A, which is impossible (see [2]). So we seek for a kind of weighted semiclassical Mourre estimate, that is an estimate like (8) where A would be the Weyl h-quantization of a, a bounded "escape function" at infinity, satisfying $\{p,a\}\geq c|x|^{-b}$ with $b>0$ for $|x|$ large. To this end, let $\chi_0,\chi_1\in C^\infty(\mathbb{R})$ with $0\leq\chi_0,\chi_1\leq 1$, $\chi_0(t)=0$ for $t\leq 1/3$, $\chi_1(t)=0$ for $t\leq 1$, $\chi_0(t)=1$ for $t\geq 2/3$, and $\chi_1(t)=1$ for $t\geq 2$. Let $R_0,\epsilon_0,\epsilon>0$ such that

$$\Big((x,\xi)\in p^{-1}(]\lambda-\epsilon;\lambda+\epsilon[)\setminus B_{R_0}^*\Big)\;\Longrightarrow\;|\xi|\geq\epsilon_0\,.$$

For any $\delta\in]0;\min(1;\rho)[$, where ρ discribes the decay of V at infinity by (1), we set

$$a_\infty(x,\xi)\;:=\chi_1\Big(|x|/R_0\Big)\chi_1\Big(|\xi|/\epsilon_0\Big)\left(\hat{x}\cdot\hat{\xi}-|x|^{-\delta}\Big(\chi_0\Big(\hat{x}\cdot\hat{\xi}\Big)-\chi_0\Big(-\hat{x}\cdot\hat{\xi}\Big)\Big)\right), \tag{9}$$

where $\hat{x}:=x/|x|$. It is easy to show that a_∞ is a smooth, bounded function such that

$$\forall(\alpha,\beta)\in\mathbb{N}^{2d},\ \exists C_{\alpha\beta}>0;\ \forall(x,\xi)\in T^*\mathbb{R}^d,\ \Big|\partial_x^\alpha\partial_\xi^\beta a_\infty(x,\xi)\Big|\;\leq\;C_{\alpha\beta}\,\langle x\rangle^{-|\alpha|} \tag{10}$$

and such that, for $\epsilon > 0$ small enough, we can find $c > 0$, $R > \max(1; R_0)$ such that $\{p, a_\infty\} \geq c\langle x\rangle^{-1-\delta}$ on $p^{-1}(]\lambda - \epsilon; \lambda + \epsilon[) \setminus B^*_{R-1}$. Let us mention that the global escape function used in [22] satisfies similar conditions.
Notice that P may be seen as the Weyl h-quantization of the symbol p for which (10) holds true on $p^{-1}(]\lambda - \epsilon; \lambda + \epsilon[)$. In particular, it follows from Helffer-Sjöstrand functional calculus that $\theta(P)$ is a h-pseudodifferential operator whose symbol s is asymtotic to $\sum_{j\geq 0} h^j s_j$, where each s_j satisfies (10), and whose principal symbol s_0 is $\theta(p)$ (see [9, 5]).
Let us now derive consequences of this non-trapping behaviour at infinity.

Lemma 6. $\lim_{n\to\infty} \|1\!\!1_{\{|x|>R\}}\theta(P(h_n))\langle x\rangle^{-(1+\delta)/2} f_n\|^2 = 0$.

Proof: Let $\tau \in C_0^\infty(\mathbb{R})$ with $0 \leq \tau \leq 1$, $\tau = 1$ on $[-R+1; R-1]$, and $\operatorname{supp}\tau \subset [-R; R]$. Let $\chi(x) := \tau(|x|)$. We have $\{p, (1-\chi)^2 a_\infty\} \geq (1-\chi)^2\{p, a_\infty\} \geq c(1-\chi)^2\langle x\rangle^{-1-\delta}$ on $p^{-1}(]\lambda - \epsilon; \lambda + \epsilon[)$. By Gårding inequality (see [11, 5]), we deduce from this that

$$\begin{aligned}\theta(P)\left\{p, (1-\chi)^2 a_\infty\right\}^w(x, h_n D)\,\theta(P) \;\;&\geq\;\; c\langle x\rangle^{-(1+\delta)/2}\,\theta(P)\,(1-\chi)^2\,\theta(P)\,\langle x\rangle^{-(1+\delta)/2}\\ &\quad + \langle x\rangle^{-1}\,\theta(P)\,\tilde{O}(h_n)\,\theta(P)\,\langle x\rangle^{-1}\,,\end{aligned}$$

where P stands for $P(h_n)$. Since

$$ih_n^{-1}\Big[P, ((1-\chi)^2 a_\infty)^w(x, h_n D)\Big] = \left\{p, (1-\chi)^2 a_\infty\right\}^w(x, h_n D) + \langle x\rangle^{-1}\tilde{O}(h_n)\langle x\rangle^{-1}$$

and $\|\chi\theta(P)\langle x\rangle^{s-1}\,\langle x\rangle^{-s} f_n\| = O(1)$, we obtain, using $\delta \leq 1$,

$$\begin{aligned}&\Big\langle ih_n^{-1}\Big[P, \big((1-\chi)^2 a_\infty\big)^w(x, h_n D)\Big]\,\theta(P)\,f_n\,,\,\theta(P)\,f_n\Big\rangle \;\geq\\ &\qquad\Big(c + O(h_n)\Big)\,\Big\|(1-\chi)\,\theta(P)\,\langle x\rangle^{-(1+\delta)/2} f_n\Big\|^2\,.\end{aligned}$$

Since a_∞ is a bounded symbol, the l.h.s of the last inequality tends to zero as in (5), yielding the desired result. □

Notice that the previous proof shows that $(\|\theta(P)\langle x\rangle^{-(1+\delta)/2} f_n\|)_n$ is bounded. Let $\mu_{(1+\delta)/2}$ be a semiclassical measure of this sequence. For all $a \in C_0^\infty(T^*\mathbb{R}^d)$, we have $\mu_{(1+\delta)/2}(a) = \mu_s(\langle x\rangle^{2s-1-\delta} a)$. By Proposition 2, $\mu_{(1+\delta)/2}$ is supported in $p^{-1}(\lambda)$ and Lemma 6 implies that $\mu_{(1+\delta)/2}$ is supported in B^*_R. Thus, $\mu_{(1+\delta)/2}$ and μ_s have compact support included in $p^{-1}(\lambda) \cap B^*_R$. By Lemma 6, (7) holds true for the sequence $(\theta(P)\langle x\rangle^{-(1+\delta)/2} f_n)_n$. Thus, there exists $\lim_{n\to\infty} \|\theta(P)\langle x\rangle^{-(1+\delta)/2} f_n\|^2$ and it is the total mass of $\mu_{(1+\delta)/2}$. For $s > 1/2$, we can choose $\delta \leq s$, yielding $\|\theta(P)\langle x\rangle^{-(1+\delta)/2} f_n\|^2 \geq \|\theta(P)\langle x\rangle^{-s} f_n\|^2$. But, by the energy localization of the f_n, $\|(1-\theta(P))\langle x\rangle^{-s} f_n\| = O(h_n)$. Thus, the total mass of $\mu_{(1+\delta)/2}$, and therefore of μ_s, is positive. We have proven the

Proposition 7. *μ_s has compact support in $p^{-1}(\lambda) \cap B^*_R$. If $s > 1/2$, μ_s is nonzero.*

Remark 8. *Although it is not apparent in the proof of Proposition 7, we did use the invariance enonced in Proposition 4. By (6), this invariance is equivalent to the fact that the l.h.s of (6) tends to zero.*
Of course, Proposition 7 is true, if μ_s is replaced by the invariant measure $\mu := \langle x\rangle^{2s}\mu_s$ considered in Remark 5.

In view of Proposition 7, we expect that the f_n are not localized on non-trapping regions of $p^{-1}(\lambda)$. Therefore, the following proposition is not surprising.

Proposition 9. *Assume that λ is a non-trapping energy. Then $\mu_s = 0$.*

Proof: By Proposition 7, μ_s has compact support in $p^{-1}(\lambda) \cap B_R^*$. Let $g \in C_0^\infty(T^*\mathbb{R}^d)$ with $g = 1$ on $p^{-1}(\lambda) \cap B_R^*$. By the non-trapping condition,

$$a(x,\xi) \;:=\; -\int_0^{+\infty} g \circ \phi^t(x,\xi)\,dt$$

is a well-defined, smooth function on $T^*\mathbb{R}^d$. By Proposition 4,

$$0 \;=\; \mu_s\Bigl(\langle x\rangle^{2s}\{p,a\}\Bigr) \;=\; \mu_s\Bigl(\langle x\rangle^{2s}\Bigr)\,. \qquad \square$$

Remark 10. *Notice that we do not use a global escape function, like in [7, 22]. In fact, since we work on measures, we can decouple the effect of the non-trapping behaviours at infinity (Proposition 7) and on a compact set (Proposition 9).*

For $s > 1/2$ and λ non-trapping, Propositions 7 and 9 give the contradiction we shall use in Section 3 for the proof of Theorem 1.
What happens, if $s \le 1/2$? If λ is non-trapping, $\mu_s = 0$ by Proposition 9 but the proof of Proposition 7 only shows that $\lim_{n\to\infty} \|\theta(P)\langle x\rangle^{-(1+\delta)/2} f_n\|^2 = 0$ which does not contradict a priori $\|\langle x\rangle^{-s} f_n\| = 1$.
This situation really appears for $V = 0$, the free case for which each $\lambda > 0$ is non-trapping, and for $s \in [0; 1/2[$. Consider, for $k \in \mathbb{R}^d \setminus \{0\}$,

$$f_n(x) \;:=\; \frac{1}{\sqrt{m_d}}\, e^{ih_n^{-1}k\cdot x} \langle x\rangle^s |x|^{(1-d)/2}\, \frac{1}{\sqrt{n}}\, \chi\Bigl(|x|/n\Bigr)\,, \tag{11}$$

where $\chi \in C_0^\infty(]1;2[;\mathbb{R})$ with $\int_{\mathbb{R}} \chi^2 = 1$ and m_d is the Lebesgue measure of the $(d-1)$-dimensional unit sphere. By direct calculation, we see that $\|\langle x\rangle^{-s} f_n\| = 1$, $\langle x\rangle^s(-h_n^2\Delta - |k|^2) f_n = o(h_n)$, $\lim_{n\to\infty} \|\chi_0(x)\langle x\rangle^{-s} f_n\| = 0$ for any $\chi_0 \in C_0^\infty(\mathbb{R}^d)$, and $\lim_{n\to\infty} \|\langle x\rangle^{-s'} f_n\| = 0$ for any $s' > s$. The two first properties show that we have a "semiclassically resonant" situation. The third one proves that $\mu_s = 0$, which also follows from Proposition 9. The fourth one implies that, for any $s' > 1/2$, $\lim_{n\to\infty} \|\langle x\rangle^{-s'} f_n\| = 0$, which follows from the proof of Proposition 7. In particular, the second result of Proposition 7 is false for $s < 1/2$. In view of Remark 5, notice that the sequence (11) is unbounded in L^2 for $s \in]0; 1/2[$.

3 Proof of the main result.

This section is devoted to the proof of Theorem 1. We start by proving that a non-trapping energy λ must be outside the pure point spectrum of P for h small enough. This allows us to consider the boundary values $R(\lambda \pm i0)$ of the resolvent. Assuming that (3) is false for "+" (for instance), we show that this is a "semiclassically resonant" situation for $s > 1/2$. Now Propositions 7 and 9 yield the desired contradiction.

Suppose that there exists $h_n \to 0$ such that $\lambda \in I$ belongs to the pure point spectrum of $P(h_n)$, for all n. Then, for all n, we can find eigenvalues λ_n of $P(h_n)$ such that $\lambda_n \to \lambda$. For all n, let u_n be a normalized eigenfunction of $P(h_n)$ with eigenvalue λ_n. Thus, we have a "semiclassically resonant" situation with $s = 0$ and we may use the results of Section 2. From Proposition 9, we derive that any semiclassical measure μ_0 of $(u_n)_n$ is zero. Let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ on $[-1;1]$ and, for any $R > 0$, let $\chi_R(x) := \chi(|x|/R)$. For any $R > 0$, this means that $\lim_{n\to\infty}\langle(1-\chi_R)u_n, u_n\rangle = 1$. But, since $\lambda_n \to \lambda$, there exists, by the exponential decay of eigenfunctions (see [1, 10]), $c, C > 0$ such that, for n large enough, $\|\exp(c\langle x\rangle)u_n\| \leq C$. Therefore, uniformly w.r.t. n,

$$\Big\langle (1-\chi_R)u_n, u_n\Big\rangle \;=\; \Big\langle (1-\chi_R)e^{-2c\langle x\rangle}\, e^{c\langle x\rangle}u_n\,,\, e^{c\langle x\rangle}u_n\Big\rangle \;\leq\; C^2 e^{-2c\langle R\rangle}\,.$$

Taking the $\lim_{R\to+\infty}\limsup_{n\to\infty}$ of this term, we arrive at a contradiction.
Thus Mourre theory (see [16, 13]) ensures that, for n large enough, the boundary value $R(\lambda + i0)$ is well defined on I as a bounded operator from L^2_s to L^2_{-s} for $s > 1/2$.
Now, we suppose that (3) is false for "+" and some $s > 1/2$. So, we can find $h_n \to 0$, $N > 0$, $(z_n)_n \in \mathbb{C}^{\mathbb{N}}$, and a sequence $(v_n)_n$ of nonzero L^2 functions, such that $\Re(z_n) \in I$, $0 \leq \Im(z_n) \leq N h_n$, and

$$\|\langle x\rangle^{-s}\, R(z_n, h_n)\, \langle x\rangle^{-s}\, v_n\| \;>\; n\, h_n^{-1}\, \|v_n\|^2\,, \tag{12}$$

where $R(z_n, h_n) := (P(h_n) - z_n)^{-1}$. Furthermore, $\Im(z_n)/h_n \to r \in [0; N]$ and $\Re(z_n) \to \lambda \in I$, possibly after extracting a subsequence. We write the l.h.s of (12) as $(1 + \kappa_n)(n/h_n)\|v_n\|^2 =: \eta_n$, for some $\kappa_n > 0$. Let $w_n := v_n/\eta_n$. Then, $\|w_n\| = o(h_n)$. Now, we set $f_n := R(z_n, h_n)\langle x\rangle^{-s} w_n$, which belongs to the domaine of $P(h_n)$. Clearly, this sequence $(f_n)_n$ satisfies $\|\langle x\rangle^{-s} f_n\| = 1$ and $\|\langle x\rangle^{s}(P(h_n) - z_n)f_n\| = o(h_n)$, that is the properties which characterise the situation we called "semiclassically resonant". Therefore we can use the results of Section 2. Since $s > 1/2$, both Propositions 7 and 9 apply, yielding a contradiction. Thus the resolvent estimates (3) must hold true and the proof of Theorem 1 is complete.

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