Stationary Solutions of the Navier-Stokes Equations in a Half-Plane Down-Stream of an Object: Universality of the Wake

Guillaume van Baalen*
Département de Physique Théorique
Université de Genève
Switzerland
guillaume.vanbaalen@physics.unige.ch

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Abstract

We consider stationary solutions of the incompressible Navier-Stokes equations for an exterior domain in two dimensions. We prove that asymptotically in the down–stream direction, the leading order deviation from the constant flow is *universal*, i.e. independent of the details of the domain. To get this result, we show that the (elliptic) Navier–Stokes equations can be interpreted as a dynamical system, the down–stream direction playing the role of time, which shares some aspects with a parabolic PDE.

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1 Introduction

In this paper we consider the time independent incompressible Navier-Stokes equations in dimensionless form in the domain $\Omega_+ = \{(x,y) \in \mathbf{R}^2 \mid x \geq 1\}$. We assume that there is an infinite flux through $\Sigma_+ = \{(x,y) \in \mathbf{R}^2 \mid x = 1\}$, the boundary of Ω_+, meaning that at infinity the fluid flow is parallel to the x axis, with a constant speed $u_\infty = 1$. This leads to the following set of equations for the vector field $\mathbf{u}$ [1]:

$$-(\mathbf{u}\cdot\nabla)\mathbf{u} + \triangle\mathbf{u} - \nabla p = 0\ , \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0\ , \tag{1.2}$$

$$\lim_{\substack{x\geq 1\\ x^2+y^2\to\infty}} \mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty \equiv \begin{pmatrix}1\\0\end{pmatrix}\ , \tag{1.3}$$

$$\mathbf{u}|_{\Sigma_+} = \mathbf{u}_+\ , \tag{1.4}$$

with

$$\lim_{|y|\to\infty} \mathbf{u}_+(y) = \mathbf{u}_\infty\ .$$

As was noted in [Amick] (see also [Galdi]), the Navier–Stokes equations (in exterior domains) have better properties for symmetric flows around the x–axis, that is when u_1 (the component parallel to $\mathbf{u}_\infty$) is even in the 'y' direction, and u_2 is odd. We will restrict ourselves to this particular case.

The choice $x = 1$ for the position of Σ_+ is arbitrary. In a second part of this introduction, we will indicate what our results imply for the Navier–Stokes equations in an exterior domain with physical parameters different from the ones we choose, that is when the dynamic viscosity, the density and u_∞ are different from 1.

We are going to show existence and asymptotic properties of the solutions to (1.1)–(1.4) under certain conditions on the boundary value $\mathbf{u}_+$. These restrictions are best expressed in terms of the inverse Fourier transform of $\mathbf{u}_+$ with respect to y. To fix notation, we define the Fourier transform f of a function $\hat{f}$ by

$$f(x,y) = \frac{1}{2\pi}\int dk\, e^{iky}\hat{f}(x,k)\ ,$$

where $\int dk$ is a shorthand notation for $\int_{-\infty}^{\infty} dk$. To formulate our theorem, we define the two following Banach spaces.

Definition 1.1 *Let $\alpha > 3$, $\gamma_1, \gamma_E \in [0,1)$, $\gamma_2 \in [0,\frac{1}{2}]$ and $\delta > -1$. We denote by $\mathcal{W}_{\alpha,\gamma_1,\gamma_2}$ the Banach space obtained by completing $C_0^\infty(\mathbf{R},\mathbf{R}^2)$, the space of smooth compactly supported functions $\mathbf{R}\to\mathbf{R}^2$, with respect to the norm*

$$||\mathbf{u}||_{\alpha,\gamma_1,\gamma_2} = \max_{i=1,2}\left\{\sup_{k\in\mathbf{R}}\big(1+|k|\big)^{\alpha}\,|\hat{u}_i(k)|\ ,\ \sup_{k\in\mathbf{R}}\big(1+|k|\big)^{\alpha-\gamma_i}\,|k|^{\gamma_i}\,|\partial_k\hat{u}_i(k)|\right\}\ ,$$

[1] We will adopt the following notation, except otherwise stated. Vectors will be denoted by boldface letters, like $\mathbf{u}$, and components by the same letter with a subscript, like u_1.

and by $\mathcal{A}_{\alpha,\gamma_E,\delta}$ the Banach space obtained by completing $C_0^\infty(\mathbf{R},\mathbf{R})$ with respect to the norm

$$|f|_{\alpha,\gamma_E,\delta} = \max\{\sup_{k\in\mathbf{R}} e^{|k|(1+\delta)}|f(k)| \, , \, (1+\delta)^{\gamma_E-1} \sup_{k\in\mathbf{R}} |k|^{\gamma_E} e^{|k|(1+\delta)}|\partial_k f(k)|\} \, .$$

We can now state our condition on $\mathbf{u}_+$.

Condition 1.2 *Let $f \in \mathcal{A}_{\alpha,\gamma_E,\delta}$ and $\mathbf{u}_f$ be defined as the Fourier transform of*

$$\hat{\mathbf{u}}_f(k) = \begin{pmatrix} f(k) \\ -i \ \mathrm{sign}(k) f(k) \end{pmatrix} \, . \tag{1.5}$$

We say that $\mathbf{u}_+$ satisfies Condition 1.2, if $\mathbf{u}_+$ can be written as $\mathbf{u}_+(y) = \mathbf{u}_\infty + \mathbf{u}_f(y) + \mathbf{v}_+(y)$, with $f \in \mathcal{A}_{\alpha,\gamma_E,\delta}$ and $\mathbf{v}_+ \in \mathcal{W}_{\alpha,\gamma_1,\gamma_2}$ satisfying $|f|_{\alpha,\gamma_E,\delta} \le \varepsilon$ and $\|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \le \varepsilon$ for ε sufficiently small.

Let us comment briefly this condition[2]. Assume first that $\|\mathbf{u}_+ - \mathbf{u}_\infty\|_{\alpha,\gamma_1,\gamma_2} \le \varepsilon$, meaning that Condition 1.2 is satisfied with $f = 0$. Roughly speaking, this implies that $\mathbf{u}_+ - \mathbf{u}_\infty$ has derivatives of order $\alpha - 1$ and the i^{th} component of $\mathbf{u} - \mathbf{u}_\infty$ decays at infinity at least like $|y|^{\gamma_i - 2}$, see Appendix A. Thus, the first component should decay faster than $|y|^{-1}$, whereas the second should decay at least like $|y|^{-3/2}$. In view of applications for the exterior problem, $|y|^{-3/2}$ is too restrictive (see below). If the second component of the boundary condition decays slower than $|y|^{-3/2}$, even like $|y|^{-1}$, our technique can still be applied, but we have to capture all this slow decay in a non–zero $\mathbf{u}_f$. From a more technical point of view, we note that $\hat{u}_{+,2}$ is an odd function. This implies that $f(0)$ has to be equal to $\hat{u}_{+,2}(0^+)$, or, in other words,

$$f(0) = \lim_{k\to 0^+} \int dy \sin(ky) \ u_{+,2}(y) \, ,$$

because if $f(0)$ is different, one would have $\hat{v}_{+,2}(k) \sim \mathrm{sign}(k)$ as $k \to 0$, which is not allowed in $\mathcal{W}_{\alpha,\gamma_1,\gamma_2}$.

An explicit 'solution' of (1.1)–(1.3) can be written down which however in the general case fails to satisfy (1.4). This solution is the so–called Euler's solution:

Lemma 1.3 *Let $f \in \mathcal{A}_{\alpha,\gamma_E,\delta}$ and $\mathbf{u}_f$ be defined by (1.5) in terms of f. Let $\mathbf{u}_E$ be defined as the Fourier transform of*

$$\hat{\mathbf{u}}_E(x,k) = e^{-|k|(x-1)}\hat{\mathbf{u}}_f(k) \, . \tag{1.6}$$

Then the functions $\tilde{\mathbf{u}}_E$ and $\tilde{p}_E$ defined as

$$\begin{aligned} \tilde{\mathbf{u}}_E &= \mathbf{u}_\infty + \mathbf{u}_E \, , \\ \tilde{p}_E &= -\frac{1}{2}\left(\tilde{\mathbf{u}}_E \cdot \tilde{\mathbf{u}}_E\right) \, , \end{aligned}$$

solve (1.1)–(1.3) in Ω_+.

Proof. See Appendix B. ■

Remark 1.4 *In general, the function $\tilde{\mathbf{u}}_E$ fails to satisfy (1.4), unless $\mathbf{v}_+ = 0$. In this case $\tilde{\mathbf{u}}_E$ is a complete solution of the problem. This solution is not very interesting from the physical point of view, since the two components of $\mathbf{u}_+ - \mathbf{u}_\infty$ are strongly related (their inverse Fourier transforms is of the form of equation (1.5)) and analytic (they are the Fourier transforms of exponentially decaying functions). However, this $\tilde{\mathbf{u}}_E$ will be used later on in an ansatz of the form $\mathbf{u} = \tilde{\mathbf{u}}_E + \mathbf{v}$.*

For the case $\mathbf{v}_+ \neq 0$, we have the following lemma.

Lemma 1.5 *For each $\mathbf{u}_+$ satisfying Condition 1.2 with ε small enough, there exists a vector field $\mathbf{u} = (u_1, u_2)$ and a function p satisfying the Navier-Stokes equations (1.1)–(1.4). The vector field $\mathbf{u}$ and the function p are unique in the class of functions satisfying $\|\mathbf{u}\| \le C\varepsilon$. The norm $\|\cdot\|$ will be defined in section 4.1.*

Proof. See Section 4. ■

[2]As it will be apparent in the course of the proofs, our technique also works when $f = f_1 + \ldots + f_n$, where $|f_i|_{\alpha,\gamma_{E,i},\delta_i} \le \varepsilon$ for possibly different $\gamma_{E,i}$, δ_i.

Remark 1.6 *No boundary condition on p is needed on Σ_+. The solution is uniquely determined by the vector field $\mathbf{u}_+$ alone.*

The following theorem is our main result.

Theorem 1.7 *For each $\mathbf{u}_+$ satisfying Condition 1.2 with ε small enough, the solution $\mathbf{u}$ of the Navier–Stokes equations with boundary condition $\mathbf{u}_+$ described in Lemma 1.5 satisfies the following properties. There exists a constant C such that for all $y \in \mathbf{R}$,*

$$\sup_{x \geq 1} x^{1/2+\tilde{\gamma}} \left| u_1(x,y) - u_\infty - c \, \frac{e^{-y^2/4x}}{\sqrt{4\pi \, x}} \right| \leq C \, \varepsilon \; , \tag{1.7}$$

$$\sup_{x \geq 1} x \, |u_2(x,y)| \leq C \, \varepsilon \; , \tag{1.8}$$

where

$$\tilde{\gamma} = \min \left\{ \frac{1}{4} \, , \, \frac{1-\gamma_1}{2} \right\} \; ,$$

$$c = -f(0) + \int dy \; u_{+,1}(y) \; . \tag{1.9}$$

Proof. See Section 4. ■

This result is very similar to scaling results in the long–time asymptotics of solution to nonlinear parabolic PDE's. The main feature of our approach is that we can let the *spatial* variable 'x' play the role similar to that of the *time* variable 't' in a setting which shares some aspects with a nonlinear parabolic PDE in one dimension. The conditions on the boundary value are similar to the one used in [Wayne]. In that paper, the author shows how the decay at infinity of the initial condition determines the uncovering of isolated eigenvalues of the Laplacean through a displacement of the continuous spectrum. The eigenfunctions associated to these eigenvalues describe the long time asymptotics of the solution to a parabolic PDE. In our case, the decay at infinity of the boundary condition $\mathbf{u}_+$ is captured by the exponents γ_i, so that it can be expected that they control the rate of convergence of the solution to the 'heat–kernel' asymptotics for large x.

We would also like to mention here that techniques similar to the ones exposed in this paper can be applied to two other problems. The first one is when the boundary condition $\mathbf{u}_+$ is periodic in time, in which case the same bounds (1.7) and (1.8) hold. The second one is probably less interesting, we can handle the 'up–stream' problem, that is when Ω_+ is replaced by the domain $\Omega_- = \{(x,y) \in \mathbf{R}^2 \mid x \leq -1\}$, with the same kind of boundary conditions.

1.1 Interpretation of the Results

Consider the fully dimensional time independent incompressible Navier-Stokes equations in two dimensions,

$$-\varrho \, (\mathbf{u} \cdot \nabla)\mathbf{u} + \mu \, \triangle \mathbf{u} - \nabla p = 0 \; , \tag{1.10}$$

$$\nabla \cdot \mathbf{u} = 0 \; , \tag{1.11}$$

$$\lim_{x^2+y^2 \to \infty} \mathbf{u} = \begin{pmatrix} u_\infty \\ 0 \end{pmatrix} \equiv \mathbf{u}_\infty \; , \tag{1.12}$$

$$\mathbf{u} \, |_{\partial\Omega} = 0 \; , \tag{1.13}$$

in an exterior domain Ω (the complement of a compact subset of $\mathbf{R}^2$ with sufficiently smooth boundary). Consider that the density ϱ, the dynamic viscosity μ, and u_∞ are given (in particular, they are constant in Ω). There exist in the literature results of existence and uniqueness for the solution of these equations (see below, [Galdi] and the references cited therein). To link the exterior problem with the results of Theorem 1.7, we define the number

$$R = \frac{u_\infty \; \varrho}{\mu} \; ,$$

and we define $\tilde{\mathbf{x}}, \tilde{\mathbf{u}}$ and $\tilde{p}$ by the equations $\tilde{\mathbf{x}} = R \; \mathbf{x}, \mathbf{u}(\mathbf{x}) = u_\infty \tilde{\mathbf{u}}(\tilde{\mathbf{x}})$ and $p(\mathbf{x}) = \left(\varrho u_\infty^2\right) \tilde{p}(\tilde{\mathbf{x}})$. Then the equations (1.10)–(1.12) become identical to (1.1)–(1.3), namely

$$-(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} + \triangle \tilde{\mathbf{u}} - \nabla \tilde{p} = 0 \; , \tag{1.14}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0 \,, \tag{1.15}$$

$$\lim_{x^2+y^2\to\infty} \tilde{\mathbf{u}} = \begin{pmatrix}1\\0\end{pmatrix} \,. \tag{1.16}$$

Making some additional hypotheses, we can formulate Theorem 1.7 in the case of the exterior problem.

Theorem 1.8 *Let $\mathbf{u}$ be the solution of (1.10)–(1.13) in an exterior domain Ω, with $u_\infty > 0$. Assume that $\tilde{\mathbf{u}}(x_+, \cdot) = \mathbf{u}_+$ for some $x_+ \geq 0$ and some function $\mathbf{u}_+$ satisfying Condition 1.2 for ε sufficiently small. Then, for $x \gg x_+$,*

$$\tilde{u}_1(x,y) = 1 + \frac{c}{\sqrt{4\pi\, x}}\, e^{-y^2/4x} + \mathcal{O}(x^{-1/2-\tilde{\gamma}/2}) \,, \tag{1.17}$$

$$\tilde{u}_2(x,y) = \frac{c}{4\sqrt{\pi}}\frac{y}{x^{3/2}}\, e^{-y^2/4x} + \mathcal{O}(x^{-1}) \,, \tag{1.18}$$

in the sense of (1.7) and (1.8). The number c is the same as the one defined in (1.9).

Remark 1.9 *The introduction of the heat kernel asymptotic in (1.18) can be misleading, since it is $\mathcal{O}(x^{-3/2})$ for fixed y. We put it there only to make the asymptotics explicitly divergence free.*

Remark 1.10 *In the fully dimensional case, the condition ε small enough of Condition 1.2 can be replaced by the condition $\frac{R}{u_\infty} = \frac{\varrho}{\mu}$ small enough. This can be seen by taking the inverse Fourier transform of $\tilde{\mathbf{u}}$, which is proportional to $\frac{R}{u_\infty}$. Since the Reynolds number associated to our problem would be Re $= RL$ for some characteristic length L, our theorem applies only for small Reynolds number.*

Theorem 1.8 proves the existence of a parabolic wake, within which at least the leading order deviation from the constant flow is *universal*, i.e., independent of the details of the shape of the body. On a heuristic level this is a well known fact (see [Batchelor]), and our paper gives a rigorous proof of this fact, provided that Condition 1.2 holds somewhere downstream. The questions of existence, uniqueness and asymptotic properties of the solutions of the Navier–Stokes equations in exterior domains have been widely studied in the literature (see [Galdi] and the references therein). In the remainder of this section, we will give some arguments in favor of applicability of our theorem based on results of these authors. However, we will not be able to formulate any definitive conclusions.

The main difficulty in trying to prove that Condition 1.2 holds is that the estimate of Condition 1.2 has to be taken along a 'vertical' line, whereas the bounds in the literature are mainly estimates along lines starting in the body itself. In particular, such estimates may not be optimal in the direction perpendicular to $\mathbf{u}_\infty$, as they have to take into account the slow decay in the direction parallel to $\mathbf{u}_\infty$. The estimates are the following (see [Galdi], Theorems X.5.4 and X.6.1, Remark X.6.1 and [Smith], Theorem 5):

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_\infty + \mathbf{m} \cdot \mathbf{E}(\mathbf{x}) + \mathcal{V}(\mathbf{x}) \,,$$

where $\mathbf{m}$ are two real numbers, $\mathbf{E}$ is the so–called Oseen Tensor (the fundamental solution of the Navier–Stokes equations linearized around $\mathbf{u}_\infty$) and $\mathcal{V}$ is some error term. Let us ignore $\mathcal{V}$ for the moment. The Fourier transform of $\mathbf{u}_\infty + \mathbf{m} \cdot \mathbf{E}(\mathbf{x})$ can be computed explicitly, and satisfies Condition 1.2 on any line $x = x_+ \gg 1$ with $\alpha \geq 3$, $\gamma_i = 0, \gamma_E = 0$, and $f(k) = m_1\, e^{-|k|(1+\delta)}$, where $\delta = x_+ - 1$. This is in good agreement with the following physical observation one can find in [Batchelor] for example. Once we subtract the constant vector $\mathbf{u}_\infty$, we should see a parabolic region extending in the downstream direction where the fluid flows towards the object (the 'wake'). This inflow is scattered radially, as if there was a 'source' of fluid inside the object. Such a source would be described by $\mathbf{u}_s = m_1 \nabla \ln((x+\delta)^2 + y^2)$, whose Fourier transform is exactly of the form of $\mathbf{u}_f$ with $f(k) = m_1\, e^{-|k|(1+\delta)}$. In particular, $u_{s,2} \sim y^{-1}$ as $y \to \infty$, meaning that without the introduction of $\mathbf{u}_f$, already the first term of the asymptotic expression would *not* satisfy Condition 1.2.

To apply our theorem to the exterior problem, we should be able to bound the contribution of $\mathcal{V}$. The problem is to find for which values of the parameters $\alpha, \gamma_1, \gamma_2$ (and possibly another f), $\mathcal{V}$ satisfies Condition 1.2. First of all, $\mathcal{V}$ is bounded, then (see [Smith], 1965, Theorem 5) there exists a constant C such that for all $y \gg x \gg 1$,

$$|\mathcal{V}_1(x,y)| \leq C|y|^{-3/2}\ln(|y|) \,, \tag{1.19}$$

$$|\mathcal{V}_2(x,y)| \leq C|y|^{-3/2} \,. \tag{1.20}$$

Inequalities (1.19) and (1.20) imply that the Fourier transform of $\mathcal{V}$ is bounded, since $\mathcal{V} \in L^1(\mathbf{R}, dy)$. Let $\hat{\mathcal{V}}$ denotes this Fourier transform. Inequalities (1.19) and (1.20) indicate also that if $\mathcal{V}$ approaches the asymptotic form (1.19) and (1.20) sufficiently fast, then (see Appendix A) $\left(\frac{|k|}{1+|k|}\right)^{\gamma_i} \left|\partial_k \hat{\mathcal{V}}_i\right| < \infty$.

The question of the rate of decay of $\hat{\mathcal{V}}$ and $\partial_k \hat{\mathcal{V}}$ at infinity is simpler. The optimal rate can be estimated as the largest (integer) α for which $\partial_y^{\alpha-1}\mathcal{V} \in L^1(\mathbf{R}, dy)$ and $\partial_y^{\alpha}\mathcal{V} \in L^1(\mathbf{R}, (1+|y|)dy)$, because, for integer α, the Fourier transform of $(ik)^\alpha \hat{f}$ is $\partial_y^\alpha f$, and the one of $(ik)^\alpha \partial_k \hat{f}$ is proportional to $\alpha\partial_y^{\alpha-1} f + y\partial_y^\alpha f$. For $|x|$ sufficiently large, $\mathcal{V}$ obeys the following representation (see [Galdi], Theorem X.5.4)

$$\mathcal{V}_j(x) = \int_\Omega E_{i,j}(x-y)\, v_l(y) D_l v_i(y)\, dy + s_j(x)$$

with $\mathbf{v} = \mathbf{u} - \mathbf{u}_\infty$ and $\partial_y^i s_j(x) = \mathcal{O}(|x|^{-1-i/2})$. Since $\partial_y^{i-1} s_j \in L^1(\mathbf{R}, dy)$ and $\partial_y^i s_j \in L^1(\mathbf{R}, (1+|y|)dy)$ for all $i > 2$, the rate of decay α is determined by the rate of decay of the Fourier transform of $\int_\Omega E_{i,j}(x-y)\, v_l(y) D_l v_i(y)\, dy$.

It can be shown (see [Galdi]) that $v_l D_l v_i \in L^1(\Omega, dxdy)$ and $v_l D_l v_i \in L^1(\mathbf{R}, dy)$ (for x sufficiently large). Using these two arguments, together with the explicit formula for the Fourier transform of $E_{i,j}$, we have checked that the Fourier transform of $\int_\Omega E_{i,j}(x-y)\, v_l(y) D_l v_i(y)\, dy$ decays faster than $|k|^{-3}$ at infinity, and the same holds for the derivative with respect to k. For those terms, the rough estimate above, $\gamma_1 > 1/2$ and $\gamma_2 = 1/2$ seems to hold, but there is no bound so far for the contribution of s_j to the γ_i's.

Since we have no precise bounds on $\alpha, \gamma_1, \gamma_2$, we will keep them as free parameters satisfying the bounds of Condition 1.2.

2 Reformulation as a Dynamical System

In this section, we transform (1.1) and (1.2) to a 4–dimensional system of first order in ∂_x, of the form $\partial_x \mathbf{w} = L\mathbf{w} + F(\mathbf{w})$, where L is a differential operator in ∂_y and F a nonlinear map depending on $\mathbf{w}$ and $\partial_y \mathbf{w}$.

Suppose that Condition 1.2 is satisfied. Lemma 1.3 provides a solution $\tilde{\mathbf{u}}_E$ of (1.1)–(1.3) which, however, does not satisfy the boundary condition (1.4). To remedy this, we make an ansatz of the form

$$\begin{aligned} \mathbf{u} &= \tilde{\mathbf{u}}_E + \mathbf{v}\ , \\ p &= \tilde{p}_E + \tilde{p}\ . \end{aligned}$$

Then we find for $\mathbf{v}$ the equations

$$\partial_x \mathbf{v} = \triangle \mathbf{v} - \mathbf{Q} - \nabla \tilde{p}\ , \tag{2.1}$$

$$\nabla \cdot \mathbf{v} = 0\ , \tag{2.2}$$

$$\lim_{x\to 1} \mathbf{v}(x,y) = \mathbf{u}_+(y) - \mathbf{u}_\infty - \mathbf{u}_E(1,y) \equiv \mathbf{v}_+(y)\ , \tag{2.3}$$

$$\lim_{x^2+y^2\to\infty} \mathbf{v}(x,y) = 0\ , \tag{2.4}$$

with

$$\mathbf{Q} = (\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{u}_E \cdot \nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{u}_E\ . \tag{2.5}$$

Let $\mathbf{v} = (u, v)$, $\mathbf{u}_E = (u_E, v_E)$, $\mathbf{Q} = (Q_1, Q_2)$. For Q_1 and Q_2 we have, using the definition (2.5), and the incompressibility relation $\partial_x u + \partial_y v = 0$,

$$\begin{aligned} Q_1 &= -\partial_y q_0 + h_1\ , \\ Q_2 &= \partial_x q_0 + h_2\ , \end{aligned}$$

where

$$q_0 = uv + u_E v + u v_E\ , \tag{2.6}$$

$$h_1 = 2\ (v\partial_y u + v_E \partial_y u + v\partial_y u_E)\ , \tag{2.7}$$

$$h_2 = \partial_y (v^2 + 2vv_E)\ . \tag{2.8}$$

Exploiting again the incompressibility equation $\partial_x u + \partial_y v = 0$, the two components of (2.1) read

$$\partial_y\left(-\partial_x v + v + \partial_y u + q_0\right) = \quad h_1 + \partial_x \tilde{p} \ , \tag{2.9}$$
$$\partial_x\left(-\partial_x v + v + \partial_y u + q_0\right) = -h_2 - \partial_y \tilde{p} \ . \tag{2.10}$$

This suggest that instead of using an auxiliary variable for $\partial_x v$ to reduce the problem to a first order system, it is more advantageous to introduce a new variable $\tilde{\eta}$ defined by

$$\tilde{\eta} = \partial_x v - v - \partial_y u - q_0 \ . \tag{2.11}$$

One shows easily now that (2.1) and (2.2), together with (2.11), are equivalent to

$$\partial_x u = -\partial_y v \ , \tag{2.12}$$
$$\partial_x v = v + \partial_y u + q_0 + \tilde{\eta} \ , \tag{2.13}$$
$$\partial_x \tilde{p} = -\partial_y \tilde{\eta} - h_1 \ , \tag{2.14}$$
$$\partial_x \tilde{\eta} = \partial_y \tilde{p} + h_2 \ . \tag{2.15}$$

These equations will be interpreted as a dynamical system, where the coordinate parallel to the flow, x, plays the role of time. Indeed, as is easily seen from (2.6), (2.7) and (2.8), the right hand side of these equations does not contain x derivatives anymore.

There is another argument for the introduction of a new variable for $\partial_x v - v - \partial_y u - q_0$. A priori, we expect $-\partial_x v + v + \partial_y u$ to be of order ε, with ε small, whereas q_0, h_1 and h_2 should be of order ε^2. Equations (2.9) and (2.10) indicate that there is a compensation between $\partial_x v$, v and $\partial_y u$. The introduction of $\tilde{\eta}$ as new variable allows us to explicitly control this compensation.

3 Linear Inhomogeneous Equations

In this section, we discuss in detail the equations (2.12)–(2.15). In the first subsection, we analyze their general form, and we explain how the strategy of the proof of existence of the solutions for them will work. This will lead us to consider (2.12) and (2.13) as linear inhomogeneous equations for u, v with given functions q_0 and $\tilde{\eta}$, and (2.14) and (2.15) also as linear inhomogeneous equations for $\tilde{\eta}, \tilde{p}$ with given functions h_1 and h_2. The discussion of these two linear inhomogeneous equations will take place in the Sections 3.2 and 3.3 where we will start by making very few assumptions on the ‘inhomogeneous terms’. Finally, we will make additional hypotheses on them in Section 3.4 and see their implications for the solutions of the Sections 3.2 and 3.3.

3.1 Preamble

If we try to interpret (2.12)–(2.15) as a dynamical system where x plays the role of time, we are faced with the following problem. For the moment, we did not specify any boundary condition for $\tilde{p}$ (let alone for $\tilde{\eta}$) on Σ_+. A priori, there is no obvious reason why we should be able to impose exactly two out of the four possible boundary conditions, neither why we should be able to satisfy those for u, v, and not, say, those of u and p only. We will now motivate by a linear analysis that we can impose u and v on Σ_+. To do this, we set $q_0, h_1, h_2 = 0$ in (2.12)–(2.15), which gives

$$\partial_x u = -\partial_y v \ , \tag{3.1}$$
$$\partial_x v = v + \partial_y u + \tilde{\eta} \ , \tag{3.2}$$
$$\partial_x \tilde{p} = -\partial_y \tilde{\eta} \ , \tag{3.3}$$
$$\partial_x \tilde{\eta} = \partial_y \tilde{p} \ . \tag{3.4}$$

Suppose now that $\tilde{p}(x, y) = a\, e^{\lambda(x-1)+iky}$, then by (3.4), we get $\tilde{\eta}(x, y) = ik\lambda^{-1}\tilde{p}(x, y)$. Substituting in (3.3), we get the characteristic equation $\lambda^2 - k^2 = 0$, whose solutions are $\lambda = \pm|k|$. Since $e^{|k|(x-1)}$ is unbounded as $x \to \infty$, we get

$$\tilde{p}(x, y) = a\, e^{-|k|(x-1)+iky} \ ,$$
$$\tilde{\eta}(x, y) = i \ \operatorname{sign}(k)\ a\, e^{-|k|(x-1)+iky} \ .$$

Suppose now that $u(x,y) = b\, e^{-|k|(x-1)+iky} + c\, e^{\Lambda(x-1)+iky}$, then by (3.1), we get

$$v(x,y) = -i\ \mathrm{sign}(k)\ b\, e^{-|k|(x-1)+iky} + i\ c\,\Lambda\, k^{-1}\ e^{\Lambda(x-1)+iky}\ .$$

Substituting in (3.2), we get

$$i\ k\ (b-a)\ e^{-|k|(x-1)+iky} = -i\ c\, k^{-1}\ (\Lambda^2 - \Lambda - k^2)\ e^{\Lambda(x-1)+iky}\ .$$

To solve this equation, we set $a = b$, and get the characteristic equation $\Lambda^2 - \Lambda - k^2 = 0$, whose solutions are

$$\Lambda_{\pm}(k) = \frac{1 \pm \Delta(k)}{2}\ ,$$

where

$$\Delta(k) = \sqrt{1+4k^2}\ .$$

As is easily seen, $\Lambda_+(k) \geq 1$, $\Lambda_-(k) \leq 0$ and $\Lambda_{\pm}(k) \sim \pm|k|$ as $k \to \infty$. The property which leads to a behavior similar to the one of a parabolic PDE is that $\Lambda_-(k) \sim -k^2$ for $k \sim 0$, which is exactly the spectrum of the one–dimensional Laplacean. Since $e^{\Lambda_+(k)(x-1)}$ is unbounded as $x \to \infty$, we get that $u, v, \tilde{p}$ and $\tilde{\eta}$ will be the Fourier transforms of

$$\hat{u}(x,k) = a(k)\ e^{-|k|(x-1)} + b(k)\ e^{\Lambda_-(k)(x-1)}\ , \tag{3.5}$$
$$\hat{v}(x,k) = -i\ \mathrm{sign}(k)\ a(k)\ e^{-|k|(x-1)} + i\ b(k)\ \Lambda_-(k)\ k^{-1}\ e^{\Lambda_-(k)(x-1)}\ , \tag{3.6}$$
$$\hat{\tilde{p}}(x,k) = a(k)\ e^{-|k|(x-1)}\ , \tag{3.7}$$
$$\hat{\tilde{\eta}}(x,k) = i\ \mathrm{sign}(k)\ a(k)\ e^{-|k|(x-1)}\ , \tag{3.8}$$

for some even functions $a, b : \mathbf{R} \to \mathbf{R}$ to be determined. By Condition 1.2, there exist two even functions $c, d : \mathbf{R} \to \mathbf{R}$ such that $u(1,y)$ is the Fourier transform of $c(k)$ and $v(1,y)$ of $\hat{v}_+(k) = i\ \mathrm{sign}(k)\ d(k)$. Evaluating (3.5) and (3.6) at $x = 1$, we get the relation

$$\begin{pmatrix} c(k) \\ d(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & \Lambda_-(k)|k|^{-1} \end{pmatrix} \begin{pmatrix} a(k) \\ b(k) \end{pmatrix}\ , \tag{3.9}$$

which can be inverted. This shows (at least at the linear level) that the boundary conditions for u and v can be satisfied, provided $\tilde{p}(1,y)$ and $\tilde{\eta}(1,y)$ are chosen in an appropriate way. At the linear level, $\tilde{p}(1,y)$ and $\tilde{\eta}(1,y)$ are obviously linked to each other through (3.7) and (3.8). Since only $\tilde{\eta}$ appears in (2.12) and (2.13), at the nonlinear level it will turn out to be more convenient to specify $\tilde{\eta}(1,y) = \eta_+(y)$ and to express $\tilde{p}(1,y)$ in terms of $\eta_+(y)$. This suggests the definition

$$\tilde{p}(x,y) = p_0(x,y) + p(x,y)\ ,$$
$$\tilde{\eta}(x,y) = \eta_0(x,y) + \eta(x,y)\ ,$$

where p_0 and η_0 are defined as the Fourier transforms of

$$\hat{\eta}_0(x,k) = e^{-|k|(x-1)}\hat{\eta}_+(k)\ , \tag{3.10}$$
$$\hat{p}_0(x,k) = -i\ \mathrm{sign}(k)\ e^{-|k|(x-1)}\hat{\eta}_+(k)\ , \tag{3.11}$$

for some function $\hat{\eta}_+(k)$ to be determined[3]. Now, p and η satisfy the same equations as $\tilde{p}$ and $\tilde{\eta}$, but the boundary condition for η is $\eta(1,y) = 0$. As a consequence of this discussion, we see that two linear inhomogeneous problems appear naturally. The first one is

$$\partial_x u = -\partial_y v\ , \tag{3.12}$$
$$\partial_x v = v + \partial_y u + q + \eta_0\ , \tag{3.13}$$
$$\lim_{x\to 1} \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u_+(y) \\ v_+(y) \end{pmatrix}\ , \tag{3.14}$$

[3]At the linear level, $\hat{\eta}_+(k) = i\ \mathrm{sign}(k)\ a(k)$, where $a(k)$ is determined by (3.9), and in the nonlinear case, we will give the good expression for $\hat{\eta}_+(k)$ later on.

where η_0 is defined by (3.10) for a given $\hat{\eta}_+$, and $q : \Omega_+ \to \mathbf{R}$ is some given function, and the second one is

$$\partial_x p = -\partial_y \eta - h_1 \ , \tag{3.15}$$

$$\partial_x \eta = \partial_y p + h_2 \ , \tag{3.16}$$

$$\lim_{x\to 1} \eta(x,y) = 0 \ , \tag{3.17}$$

where $h_1, h_2 : \Omega_+ \to \mathbf{R}$ are some given functions. The strategy of the proof of existence, uniqueness and different properties of the solutions of the equations (2.12)–(2.15) will be as follows. We will first show that under relatively weak assumptions on q, h_1 and h_2, the two linear inhomogeneous problems (3.12)–(3.14) and (3.15)–(3.17) admit strong solutions. Then we will construct a sequence of functions $\mathbf{v}_n, p_n$ and η_n such that $\mathbf{v}_n, \tilde{p}_n = p_0 + p_n$ and $\tilde{\eta}_n = \eta_0 + \eta_n$ converges to the solution of the nonlinear equations (2.12)–(2.15). We will essentially ask that $\mathbf{v}_{n+1}, p_{n+1}$ and η_{n+1} satisfy (3.12)–(3.14) and (3.15) –(3.17) where the q, h_1, h_2, are given in terms of $\mathbf{v}_n, p_n$ and η_n. We will see how the sequence $\mathbf{v}_n$ can be interpreted as the iteration of a contractive map in a Banach space leading to a fixed point $\mathbf{v}_\star$, and corresponding $\tilde{\eta}_\star$ and $\tilde{p}_\star$ which will be the solution of the nonlinear problem.

Our technique is mainly an extension of the one used in [Bricmont et al.] in the case of Parabolic PDE's, and relies on estimates on the Fourier transform of the semi–groups generated by the linear operator on the right–hand side of (3.12) , (3.13), and (3.15) , (3.16), together with an extension of the space $\mathcal{W}_{\alpha,\gamma_1,\gamma_2}$ to functions $\Omega_+ \to \mathbf{R}$.

We will first concentrate on the equations for u and v, the ones for η and p being simpler are postponed to the next section.

3.2 Linear Inhomogeneous Equations for u and v

In order to prove existence of solutions to (3.12)–(3.14) for u and v, we will give an integral representation of the solution in Fourier space using Duhamel's formula. Under quite general hypotheses on $q : \Omega_+ \to \mathbf{R}$, we will show that the Fourier transform of this integral solution exists and is a strong solution of (3.12)–(3.14). Since the integral formulation involves rather lengthy formulas, we will separate their expression and their properties in the two next lemmas. The first will deal with the part of Duhamel's formula involving the boundary condition (the solution of (3.12)–(3.14) with $q = 0$), and the second with the one involving the function q. Then, we will show how to use the results of these lemmas to build the solution. The following lemma is a more precise statement than the discussion of the preceding section for the case $q = 0$. It consists essentially in inverting the relation (3.9).

Lemma 3.1 *Assume that $\mathbf{v}_+ = (u_+, v_+) \in \mathcal{W}_{\alpha,\gamma_1,\gamma_2}$ for some $\alpha > 3$, $\gamma_1 \in [0,1)$ and $\gamma_2 \in [0, \frac{1}{2}]$ with $\gamma_2 \le \gamma_1$. Assume that u_+ and v_+ are even and odd in y respectively. Define $\eta_{0,0}$, u_0 and v_0 as the Fourier transforms of*

$$\hat{\eta}_{0,0}(x,k) = \hat{\eta}_{0,+}(k) e^{-|k|(x-1)} \ , \tag{3.18}$$

$$\begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \end{pmatrix}(x,k) = \begin{pmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{pmatrix}(x,k) \begin{pmatrix} \hat{u}_+ \\ \hat{v}_+ \end{pmatrix}(k) \ , \tag{3.19}$$

where

$$\hat{\eta}_{0,+}(k) = -ik \frac{\Lambda_+(k) + |k|}{\Lambda_+(k)} \hat{u}_+(k) - (\Lambda_+(k) + |k|)\hat{v}_+(k) \ , \tag{3.20}$$

$$P_{1,1}(x,k) = e^{-|k|(x-1)} + \frac{|k|}{\Lambda_-(k) + |k|} \left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)} \right) \ ,$$

$$P_{1,2}(x,k) = \frac{-ik}{\Lambda_-(k) + |k|} \left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)} \right) \ ,$$

$$P_{2,1}(x,k) = \frac{i \ \mathrm{sign}(k) \Lambda_-(k)}{\Lambda_-(k) + |k|} \left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)} \right) \ ,$$

$$P_{2,2}(x,k) = e^{-|k|(x-1)} + \frac{\Lambda_-(k)}{\Lambda_-(k) + |k|} \left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)} \right) \ .$$

Then the functions $\eta_{0,0}$, u_0, v_0, $\partial_y u_0$, $\partial_y v_0$ are bounded and continuous in x, y, and u_0 is even in y, while $v_0, \eta_{0,0}$ are odd in y.

Proof. See Appendix C. ■

We now proceed to the part of Duhamel's formula containing q. This part involves auxiliary functions N_1, N_2 and $\eta_{0,q}$, whose definition and properties are given in the next lemma. This is an extension of the discussion of the preceding section to the nonlinear case.

Lemma 3.2 *Let $\hat{q} : \Omega_+ \to \mathbf{R}$ be bounded. Assume that $\hat{q}$ is odd in k, that $\hat{q} \in L^1(\Omega_+, |k|\ dk\ dx)$ and that $\sup_{x\geq 1} \int dk\ (1+|k|)|\hat{q}(x,k)| < \infty$. Define N_1 and N_2 and $\eta_{0,q}$ as the Fourier transforms of*

$$\begin{aligned}\hat{N}_1(x,k) &= \frac{ik}{\Delta(k)} \int_1^x dx_0\ e^{\Lambda_-(k)(x-x_0)} \hat{q}(x_0,k) + \frac{ik}{\Delta(k)} \int_x^\infty dx_0\ e^{\Lambda_+(k)(x-x_0)} \hat{q}(x_0,k) \\ &\quad - (\Lambda_+(k)+|k|) \left[\frac{ik}{\Delta(k)(\Lambda_-(k)+|k|)} e^{\Lambda_-(k)(x-1)} - i\ \mathrm{sign}(k) e^{-|k|(x-1)} \right] \tau(k)\ , \qquad (3.21)\\ \hat{N}_2(x,k) &= -\frac{\Lambda_-(k)}{\Delta(k)} \int_1^x dx_0\ e^{\Lambda_-(k)(x-x_0)} \hat{q}(x_0,k) - \frac{\Lambda_+(k)}{\Delta(k)} \int_x^\infty dx_0\ e^{\Lambda_+(k)(x-x_0)} \hat{q}(x_0,k) \\ &\quad + (\Lambda_+(k)+|k|) \left[\frac{\Lambda_-(k)}{\Delta(k)(\Lambda_-(k)+|k|)} e^{\Lambda_-(k)(x-1)} + e^{-|k|(x-1)} \right] \tau(k)\ , \qquad (3.22)\\ \hat{\eta}_{0,q}(x,k) &= \hat{\eta}_{q,+}(k) e^{-|k|(x-1)}\ , \qquad (3.23)\end{aligned}$$

where

$$\begin{aligned}\tau(k) &= \int_1^\infty dx_0\ e^{\Lambda_+(k)(1-x_0)} \hat{q}(x_0,k)\ , \qquad (3.24)\\ \hat{\eta}_{q,+}(k) &= -(\Lambda_+(k)+|k|)\tau(k)\ . \qquad (3.25)\end{aligned}$$

Then the functions N_1, N_2, $\partial_y N_1, \partial_y N_2$ and $\eta_{0,q}$ are bounded and continuous in x, y, and the function N_1 is even in y, while the functions $N_2, \eta_{0,q}$ are odd in y.

Proof. See Appendix C. ■

We have now all we need to write down Duhamel's formula and show that it provides a strong solution of (3.12)–(3.14).

Proposition 3.3 *Let the conditions of Lemmas 3.1 and 3.2 be satisfied. Assume that q is continuous in x, y, and define the functions u and v as the Fourier transforms of*

$$\begin{aligned}\hat{u}(x,k) &= \hat{u}_0(x,k) + N_1(x,k)\ , \qquad (3.26)\\ \hat{v}(x,k) &= \hat{v}_0(x,k) + N_2(x,k)\ . \qquad (3.27)\end{aligned}$$

Then the functions u, v, $\partial_y u$ and $\partial_y v$ are bounded and continuous in x, y. The function u is even in y and the functions $v, \eta_0 + \eta_q$ are odd. They are strong solutions of equations (3.12) – (3.14) with $\eta_0 = \eta_{0,0} + \eta_{0,q}$.

Proof. See Appendix C. ■

The question of convergence of the solution to 0 as $y \to \infty$ or $x \to \infty$ will be a simple consequence of bounds that we will prove in Section 3.4. Without making additional assumptions on the function q, we can extract more information on (u_0, v_0) than what we wrote in Lemma 3.1 by exploiting the fact that $\mathbf{v}_+ \in \mathcal{W}_{\alpha,\gamma_1,\gamma_2}$. As we will see, the contribution of $(\hat{N}_1, \hat{N}_2)$ to (u_0, v_0) will be smaller than the one of $\hat{u}_0$ and $\hat{v}_0$. This will guide us later on for defining the Banach space in which we will prove that the nonlinear problem has a solution. The following lemma gives these informations.

Lemma 3.4 *Let $\mathbf{v}_+ = (u_+, v_+) \in \mathcal{W}_{\alpha,\gamma_1,\gamma_2}$ for some $\alpha > 3$, $\gamma_1 \in [0,1)$ and $\gamma_2 \in [0, \frac{1}{2}]$ with $\gamma_2 \leq \gamma_1$. Let $\hat{u}_0$ and $\hat{v}_0$ be defined by (3.19). Then there exists a constant C such that for all $(x,k) \in \Omega_+$,*

$$\begin{aligned}|\hat{u}_0(x,k)| &\leq C\ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ \frac{1}{(1+|k|\sqrt{x})^\alpha}\ , \qquad (3.28)\\ |\hat{v}_0(x,k)| &\leq C\ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ \frac{1}{\sqrt{x}} \frac{1}{(1+|k|\sqrt{x})^\alpha}\ , \qquad (3.29)\\ |\partial_k \hat{u}_0(x,k)| &\leq C\ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ \frac{x^{\frac{1-\gamma_1}{2}}}{(1+|k|\sqrt{x})^\alpha} \left(\frac{1+|k|}{|k|}\right)^{\gamma_1}\ , \qquad (3.30)\end{aligned}$$

$$|\partial_k \hat{v}_0(x,k)| \leq C\,\|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\,\frac{1}{(1+|k|\sqrt{x})^{\alpha-1}}\frac{\sqrt{x}}{|kx|^{\gamma_2}+|kx|}\;, \tag{3.31}$$

$$\left|\hat{u}_0(x,k)-e^{\Lambda_-(k)x}u_+(0)\right| \leq C\,\|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\,\frac{x^{\frac{\gamma_1-1}{2}}}{(1+|k|\sqrt{x})^{\alpha}}\;. \tag{3.32}$$

Proof. See Appendix C. ■

3.3 Linear Inhomogeneous Equations for p and η

This section is very similar to the previous one. We will give an integral representation in Fourier space of the solution to the equations (3.15)–(3.17) for η and p. We will show that under quite general hypotheses on h_1 and h_2, the Fourier transform of the integral solution is a strong solution of (3.15)–(3.14). Since the boundary condition (3.17) is $\eta(1,y)=0$, the equivalent of Lemma 3.1 is $\eta_0=0, p_0=0$. The following lemma corresponds to Lemma 3.2 of the previous section.

Lemma 3.5 *Let $\hat{h}_1,\hat{h}_2:\Omega_+\to\mathbf{R}$ be bounded, and assume that $\hat{h}_i\in L^1(\Omega_+,(1+|k|)^{\mu}dk\;dx)$, and that $\sup_{x\geq 1}\int dk\;(1+|k|)^{\mu}|\hat{h}_i(x,k)|<\infty$ for some $\mu>0$. Assume furthermore that $\hat{h}_1$ is even in k and $\hat{h}_2$ is odd in k. Define q_1, q_2, $q_{1,p}$ and $q_{2,p}$ as the Fourier transforms of*

$$\begin{aligned}
\hat{q}_1(x,k) &= \int_1^{\infty} dx_0\;\left[-\frac{1}{2}i\;\mathrm{sign}(k)\left(e^{-|x-x_0||k|}-e^{-(x+x_0-2)|k|}\right)\right]\;\hat{h}_1(x_0,k)\;,\\
\hat{q}_2(x,k) &= \int_1^{\infty} dx_0\;\left[\frac{1}{2}\left(e^{-|x-x_0||k|}\,\mathrm{sign}(x-x_0)+e^{-(x+x_0-2)|k|}\right)\right]\;\hat{h}_2(x_0,k)\;,\\
\hat{q}_{1,p}(x,k) &= \int_1^{\infty} dx_0\;\left[-\frac{1}{2}\left(e^{-|x-x_0||k|}\,\mathrm{sign}(x-x_0)-e^{-(x+x_0-2)|k|}\right)\right]\;\hat{h}_1(x_0,k)\;,\\
\hat{q}_{2,p}(x,k) &= \int_1^{\infty} dx_0\;\left[-\frac{1}{2}i\;\mathrm{sign}(k)\left(e^{-|x-x_0||k|}+e^{-(x+x_0-2)|k|}\right)\right]\;\hat{h}_2(x_0,k)\;.
\end{aligned}$$

Then the functions q_i, $q_{i,p}$, $\partial_y q_i$ and $\partial_y q_{i,p}$ are bounded and continuous in x,y, and the functions q_i are even in y while the functions $q_{i,p}$ are odd in y.

Proof. See Appendix C. ■

We can write down Duhamel's formula and show that it provides a strong solution to (3.15)–(3.17).

Proposition 3.6 *Assume that the hypotheses of Lemma 3.5 are satisfied, and that h_1 and h_2 are continuous in x,y, and define the functions η and p as the Fourier transforms of*

$$\hat{\eta}(x,k) = \hat{q}_1(x,k)+\hat{q}_2(x,k)\;, \tag{3.33}$$

$$\hat{p}(x,k) = \hat{q}_{1,p}(x,k)+\hat{q}_{2,p}(x,k)\;. \tag{3.34}$$

Then the functions $\eta,p,\partial_y\eta$ and $\partial_y p$ are bounded, continuous in x,y, even and odd in y respectively. They are strong solutions of (3.15)–(3.17).

Proof. See Appendix C. ■

3.4 Linear Inhomogeneous Equations, Additional Properties

In this section, we prove that the integral solutions of (3.12)–(3.14) and (3.15)–(3.17) of the Sections 3.2 and 3.3 obey certain estimates when q_0, h_1, h_2 and q satisfy additional properties. The bounds on the Fourier transform of q_0, h_1 and h_2 are suggested by the bounds obtained when substituting u,v by the bounds on u_0,v_0 in q_0,h_1 and h_2, see Appendix F. We will prove that when h_1 and h_2 satisfy these bounds, then q_1,q_2 satisfy additional properties. This will suggest appropriate bounds on $q=q_0+\eta=q_0+q_1+q_2$. All these bounds will ensure that u,v,p and η tend to 0 as $x\to\infty$ or $y\to\infty$. Finally, if q_0,h_1 and h_2 satisfy the conditions below, then the assumptions of Lemmas 3.2 and 3.5 are satisfied.

We begin with the conditions on h_1 and h_2.

Condition 3.7 *Let $\alpha > 3$, $\gamma \in [0, 1)$ and $h_1, h_2 : \Omega_+ \to \mathbf{R}$. There exists a constant H such that for all $(x, k) \in \Omega_+$, the functions h_1 and h_2 satisfy*

$$|\hat{h}_1(x,k)| \leq H \, \frac{x^{-3/2}}{(1+|k|\sqrt{x})^{\alpha-1}} \,,$$
$$|\partial_k \hat{h}_1(x,k)| \leq H \, \frac{|k| \, x^{\frac{\gamma-1}{2}}}{(1+|k|\sqrt{x})^{\alpha}} \,,$$
$$|\hat{h}_2(x,k)| \leq H \, \frac{|k| \, x^{-3/2}}{(1+|k|\sqrt{x})^{\alpha}} \,,$$
$$|\partial_k \hat{h}_2(x,k)| \leq H \, \frac{x^{-3/2}}{(1+|k|\sqrt{x})^{\alpha-1}} \,.$$

Under these conditions, we can prove the following lemma.

Lemma 3.8 *Let η and p be defined by (3.33) and (3.34). Assume that h_1 and h_2 satisfy Condition 3.7 for some constants H, $\alpha > 3$ and $\gamma \in [0, 1)$. Then for all $\nu \in (1/2, 1)$, there exists a constant C such that for all $(x, k) \in \Omega_+$,*

$$|\hat{\eta}(x,k)| \leq C\,H \, \frac{|k| \, x^{-1/2}}{(1+|k|\sqrt{x})^{\alpha+1}} \; + \; C\,H \, \frac{|k| \, x^{1/2}}{(1+|k|\sqrt{x})^{\alpha-1}} \frac{1}{1+k^2x^2} \,,$$
$$|\partial_k \hat{\eta}(x,k)| \leq C\,H \, \frac{x^{\frac{\gamma-1}{2}}}{(1+|k|\sqrt{x})^{\alpha}} \; + \; C\,H \, \frac{x^{1/2}}{(1+|k|\sqrt{x})^{\alpha-1}} \frac{1}{1+k^2x^2} \,,$$
$$|\hat{p}(k,t)| \leq C\,H \, \frac{x^{-1/2}}{(1+|k|\sqrt{x})^{\alpha}} \,,$$
$$|\partial_k \hat{p}(x,k)| \leq C\,H \, \frac{x^{\frac{\gamma-1}{2}}}{(1+|k|\sqrt{x})^{\alpha}} \; + \; C\,H \, \frac{x^{1/2}}{(1+|k|\sqrt{x})^{\alpha-1}} \frac{1}{1+k^2x^2} + C\,H \, \frac{x^{1/2-\nu}}{|k|^{\nu}(1+|k|\sqrt{x})^{\alpha-1}} \,.$$

Proof. See Appendix D. ∎

Now we formulate the conditions on q. Recall that $q = q_0 + \eta$, the following requirements are certainly satisfied by η due to Lemma 3.8, so this condition is really a condition on q_0.

Condition 3.9 *Let $\alpha > 3$, $\gamma \in [0, 1)$ and $q : \Omega_+ \to \mathbf{R}$. There exists a constant Q such that for all $(x, k) \in \Omega_+$, the function q satisfies*

$$|\hat{q}(x,k)| \leq Q \; \min\left\{ \frac{|k| \, x^{\frac{\gamma-1}{2}}}{(1+|k|\sqrt{x})^{\alpha+1}} \,, \frac{x^{-1}}{(1+|k|\sqrt{x})^{\alpha}} \right\} \; + \; Q \, \frac{|k| \, x^{1/2}}{(1+|k|\sqrt{x})^{\alpha-1}} \frac{1}{1+k^2x^2} \,, \tag{3.35}$$
$$|\partial_k \hat{q}(x,k)| \leq Q \, \frac{x^{\frac{\gamma-1}{2}}}{(1+|k|\sqrt{x})^{\alpha}} \; + \; Q \, \frac{x^{1/2}}{(1+|k|\sqrt{x})^{\alpha-1}} \frac{1}{1+k^2x^2} \,. \tag{3.36}$$

Under these hypotheses, we can prove the following lemma.

Lemma 3.10 *Let N_1 and N_2 be defined by (3.21) and (3.22). Assume that q satisfies Condition 3.9 for some constants Q, $\alpha > 3$ and $\gamma \in [0, 1)$, then for all $\gamma_2 \in [0, 1)$ there exists a constant C such that for all $(x, k) \in \Omega_+$,*

$$|\hat{N}_1(x,k)| \leq C\,Q \, \frac{t^{-\tilde{\gamma}}}{(1+|k|\sqrt{x})^{\alpha}} \,,$$
$$|\hat{N}_2(k,t)| \leq C\,Q \, \frac{1}{\sqrt{x}} \frac{1}{(1+|k|\sqrt{x})^{\alpha}} \,,$$
$$|\partial_k \hat{N}_1(k,t)| \leq C\,Q \, \frac{x^{\frac{1-\gamma}{2}}}{(1+|k|\sqrt{x})^{\alpha}} \left(\frac{1+|k|}{|k|} \right)^{\gamma} \,,$$
$$|\partial_k \hat{N}_2(k,t)| \leq C\,Q \, \frac{1}{(1+|k|\sqrt{x})^{\alpha-1}} \frac{\sqrt{x}}{|kx|^{\gamma_2} + |kx|} \,,$$

where $\tilde{\gamma} = \min\left\{ \frac{1}{4} \,, \frac{1-\gamma}{2} \right\}$.

Proof. See Appendix E. ■

This lemma implies that the contribution of N_1 and N_2 to u, v in (3.26) and (3.27) are dominated by the one of u_0 and v_0 if q satisfies Condition 3.9:

Corollary 3.11 *Assume that $\hat{q}$ satisfies Condition 3.9 with $\alpha > 3$ and $\gamma = \gamma_1$. Then for all $\gamma_2 \in [0,1)$, there exists a constant C such that for all $(x,k) \in \Omega_+$,*

$$|\hat{u}(x,k)| \leq \tilde{C}\, \frac{1}{(1+|k|\sqrt{x})^{\alpha}}\;, \tag{3.37}$$

$$|\hat{v}(x,k)| \leq \tilde{C}\, \frac{1}{\sqrt{x}}\frac{1}{(1+|k|\sqrt{x})^{\alpha}}\;, \tag{3.38}$$

$$|\partial_k \hat{u}(x,k)| \leq \tilde{C}\, \frac{x^{\frac{1-\gamma_1}{2}}}{(1+|k|\sqrt{x})^{\alpha}} \left(\frac{1+|k|}{|k|}\right)^{\gamma_1}\;, \tag{3.39}$$

$$|\partial_k \hat{v}(x,k)| \leq \tilde{C}\, \frac{1}{(1+|k|\sqrt{x})^{\alpha-1}}\frac{\sqrt{x}}{|kx|^{\gamma_2}+|kx|}\;, \tag{3.40}$$

$$\left|\hat{u}(x,k) - e^{\Lambda_-(k)x}u_+(0)\right| \leq \tilde{C}\, \frac{x^{-\tilde{\gamma}}}{(1+|k|\sqrt{x})^{\alpha}}\;, \tag{3.41}$$

where $\tilde{\gamma} = \min\left\{\frac{1}{4}\,,\,\frac{1-\gamma_1}{2}\right\}$ and $\tilde{C} = C(Q + \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}) < \infty$.

Remark 3.12 *Inequalities (3.37)–(3.40) are the key ingredient for applying the contraction mapping principle. We can define a Banach space by asking that each element satisfies the bounds (3.37)–(3.40), see in the next section. We will show that $\hat{h}_1$, $\hat{h}_2$ and $\hat{q}_0$ satisfy Conditions 3.7 and 3.9 when $\hat{u}$ and $\hat{v}$ satisfy these bounds. Inequality (3.41) will be sufficient to prove Theorem 1.7, see below. We see also that $|u|, |v|, |\eta|, |p| \leq C\, w_{\alpha-1}$. By integrating this bound with respect to k, we get that u, v, p and η tend to 0 at least like $x^{-1/2}$ as $x \to \infty$. Furthermore, since the derivatives of $\hat{u}, \hat{v}, \hat{\eta}$ and $\hat{p}$ with respect to k are in $L^1(\mathbf{R}, dk)$, we get that u, v, η and p decay at least like $|y|^{-1}$ as $y \to \infty$. The boundary conditions at infinity are thus all satisfied.*

4 Existence and Uniqueness for the Nonlinear Problem

In this section, we will use the results of the preceding section to prove that the nonlinear problem (2.12)–(2.15) has a unique solution which satisfies the boundary conditions. We will first define a Banach space $\mathcal{B}$ of (vector valued) functions which is an extension of $\mathcal{W}_{\alpha,\gamma_1,\gamma_2}$ to functions $\Omega_+ \to \mathbf{R}^2$. Then we will show that q_0, h_1 and h_2 satisfy the conditions 3.7 and 3.9 when $(u,v) \in \mathcal{B}$. After this, we will construct a sequence of approximating solutions to (1.1)–(1.4), and show that this sequence converges to the solution of these equations. In the whole section, it is understood that Condition 1.2 on the boundary condition on Σ_+ is satisfied. In particular, the parameters α, $\gamma_1, \gamma_2, \gamma_E$ and the function $f \in \mathcal{A}_{\alpha,\gamma_E,\delta}$ are considered as given and fixed once and for all.

4.1 Choice of Function Spaces

In this section, we construct the Banach space $\mathcal{B}$. As we noted in Remark 3.12, Corollary 3.11 provides a natural way to define a Banach space. We begin with a definition.

Definition 4.1 *For $(x,k) \in \Omega_+$, we define the following functions.*

$$\begin{aligned}
p_{u,1}(x,k) &= (1+|k|\sqrt{x})^{\alpha}\;,\\
p_{v,1}(x,k) &= x^{1/2}\,(1+|k|\sqrt{x})^{\alpha}\;,\\
p_{u,2}(x,k) &= x^{\frac{\gamma_1-1}{2}}\,(1+|k|\sqrt{x})^{\alpha}\left(\frac{|k|}{1+|k|}\right)^{\gamma_1}\;,\\
p_{v,2}(x,k) &= x^{-1/2}\,(1+|k|\sqrt{x})^{\alpha-1}(|kx|^{\gamma_2}+|kx|)\;.
\end{aligned}$$

With these functions, we can define the following Banach space.

Definition 4.2 *We denote by $\mathcal{B}$ the Banach space obtained by completing $C_0^\infty(\Omega_+, \mathbf{R}^2)$ with respect to the norm*

$$\|\mathbf{u}\| = \max\left\{ |\hat{u}\, p_{u,1}|_\infty \ ,\ |\hat{v}\, p_{v,1}|_\infty \ ,\ |D\hat{u}\, p_{u,2}|_\infty \ ,\ |D\hat{v}\, p_{v,2}|_\infty \right\} \ ,$$

where, for a function $g : \Omega_+ \to \mathbf{R}$, $|g|_\infty$ and Dg denote

$$|g|_\infty = \sup_{(x,k)\in\Omega_+} |g(x,k)| \ ,$$
$$Dg(x,k) = \partial_k g(x,k) \ .$$

Note here that the restriction of a function in $\mathcal{B}$ on any line $x = \tilde{x} > 1$ is in $\mathcal{W}_{\alpha,\gamma_1,\gamma_2}$. In this sense, the line $x = 1$ is by no means particular.

4.2 Bounds on Nonlinear Terms

In this section, we prove that the inverse Fourier transforms of the functions q_0, h_1 and h_2 satisfy Conditions 3.7 and 3.9 when $(u, v) \in \mathcal{B}$. We give first the expression for $\hat{q}_0$, $\hat{h}_1$ and $\hat{h}_2$.

Definition 4.3 *Let $\mathbf{v} = (u, v) \in \mathcal{B}$, and let $\hat{\mathbf{u}}_E = (\hat{u}_E, \hat{v}_E)$ denotes the Fourier transform of $\mathbf{u}_E$ defined by (1.6). We define the following maps*

$$\hat{q}_0(\mathbf{v}) = \frac{1}{2\pi}\left(\hat{u} * \hat{v} + \hat{u}_E * \hat{v} + \hat{u} * \hat{v}_E\right) \ , \tag{4.1}$$

$$\hat{h}_1(\mathbf{v}) = \frac{i}{\pi}\left(\hat{v} * K\hat{u} + \hat{v} * K\hat{u}_E + \hat{v}_E * K\hat{u}\right) \ , \tag{4.2}$$

$$\hat{h}_2(\mathbf{v}) = \frac{i}{2\pi} K\left(\hat{v} * \hat{v} + 2\hat{v} * \hat{v}_E\right) \ , \tag{4.3}$$

where $*$ denotes the convolution and for $h : \Omega_+ \to \mathbf{R}$ the function Kh is defined by

$$Kh(x,k) = kh(x,k) \ .$$

The following proposition proves that Conditions 3.7 and 3.9 are satisfied.

Proposition 4.4 *Let Condition 1.2 be satisfied for some f and ε, and let $\mathbf{u}_E$ be defined by (1.6). Let $\hat{q}_0(\mathbf{v})$, $\hat{h}_1(\mathbf{v})$ and $\hat{h}_2(\mathbf{v})$ be defined by (4.1)–(4.3). Then there exists a constant C such that for all $\mathbf{v} \in \mathcal{B}$, the following bounds hold for all $(x,k) \in \Omega_+$*

$$|\hat{q}_0(\mathbf{v})| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\| \min\left\{ \left|KW_{\alpha+1,\frac{1-\gamma_1}{2}}\right| \ , W_{\alpha,1} \right\} \ , \tag{4.4}$$

$$|D\hat{q}_0(\mathbf{v})| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\|\, W_{\alpha,\frac{1-\gamma_1}{2}} \ , \tag{4.5}$$

$$|\hat{h}_1(\mathbf{v})| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\|\, W_{\alpha-1,\frac{3}{2}} \ , \tag{4.6}$$

$$|D\hat{h}_1(\mathbf{v})| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\| \left|KW_{\alpha,\frac{1-\gamma_1}{2}}\right| \ , \tag{4.7}$$

$$|\hat{h}_2(\mathbf{v})| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\| \left|KW_{\alpha,\frac{3}{2}}\right| \ , \tag{4.8}$$

$$|D\hat{h}_2(\mathbf{v})| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\|\, W_{\alpha-1,\frac{3}{2}} \ , \tag{4.9}$$

where

$$W_{\alpha,\beta}(x,k) = \frac{x^{-\beta}}{(1 + |k|\sqrt{x})^{\alpha}} \ .$$

Proof. See Appendix F. ■

4.3 Proof of Lemma 1.5 and Theorem 1.7

In this section, we construct a sequence of functions $\mathbf{v}_n$, η_n and p_n and show that the sequence converges to a solution of the nonlinear problem.

We begin with the construction of $\mathbf{v}_0$, η_0 and p_0. For $\mathbf{v}_0$, we take the one defined by (3.19). By Lemma 3.1, $\mathbf{v}_0$ and $\partial_y \mathbf{v}_0$ are continuous in x, y. Let $\eta_{0,0}$ be defined by (3.18). By Proposition 3.3, $\mathbf{v}_0$ is a strong solution of

(3.12)–(3.14) with $q = 0$ and $\eta_0 = \eta_{0,0}$. Let $\eta_0 = 0$ and $p_0 = 0$. They are strong solutions of (3.15)–(3.17) with $h_i = 0$ (sic!).

Now we construct inductively the sequence of functions $\mathbf{v}_{n+1}$, η_{n+1} and p_{n+1} for $n \geq 0$. Suppose that $\mathbf{v}_n \in \mathcal{B}$ and that $\mathbf{v}_n$ and $\partial_y \mathbf{v}_n$ are continuous in x, y. Evidently, $q_0(\mathbf{v}_n)$, $h_1(\mathbf{v}_n)$ and $h_2(\mathbf{v}_n)$ are also continuous in x, y. Furthermore, Proposition 4.4 ensures that $h_1(\mathbf{v}_n)$, $h_2(\mathbf{v}_n)$ satisfy Condition 3.7. By Lemma 3.5, this implies the existence of $q_1(\mathbf{v}_n)$, $q_2(\mathbf{v}_n)$, $q_{1,p}(\mathbf{v}_n)$ and $q_{2,p}(\mathbf{v}_n)$ which are bounded and continuous, as well as their derivatives with respect to y. By Lemma 3.8 and Proposition 4.4, $q(\mathbf{v}_n) = q_0(\mathbf{v}_n) + q_1(\mathbf{v}_n) + q_2(\mathbf{v}_n)$ is bounded, continuous and satisfies Condition 3.9. By Lemma 3.2, this implies the existence of $N_1(\mathbf{v}_n), N_2(\mathbf{v}_n)$ which are continuous as well as their derivatives with respect to y, and the existence of $\eta_{0,q}(\mathbf{v}_n)$ which is also continuous. From this, we conclude with Propositions 3.3 and 3.6 that

$$u_{n+1} = u_0 + N_1(\mathbf{v}_n) \,, \tag{4.10}$$
$$v_{n+1} = v_0 + N_2(\mathbf{v}_n) \,, \tag{4.11}$$
$$\eta_{n+1} = q_1(\mathbf{v}_n) + q_2(\mathbf{v}_n) \,, \tag{4.12}$$
$$p_{n+1} = q_{1,p}(\mathbf{v}_n) + q_{2,p}(\mathbf{v}_n) \,, \tag{4.13}$$

are strong solutions of (3.12)–(3.14) and (3.15)–(3.17) with $\eta_0 = \eta_{0,0} + \eta_{0,q}(\mathbf{v}_n)$, $q = q(\mathbf{v}_n)$, $h_1 = h_1(\mathbf{v}_n)$ and $h_2 = h_2(\mathbf{v}_n)$.

By Corollary 3.11, the two equations (4.10) and (4.11) defining $\mathbf{v}_{n+1}$ in terms of $\mathbf{v}_n$ can be easily interpreted as the action of some map $\mathcal{F} : \mathcal{B} \to \mathcal{B}$, namely, we have the following lemma:

Lemma 4.5 *There exist constants C_1, C_2 such that for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{B}$ we have*

$$\|\mathcal{F}(\mathbf{v}_i) - \mathbf{v}_0\| \leq C_1\ (\varepsilon + \|\mathbf{v}_i\|)\ \|\mathbf{v}_i\| \,, \tag{4.14}$$
$$\|\mathcal{F}(\mathbf{v}_1) - \mathcal{F}(\mathbf{v}_2)\| \leq C_1\ (\varepsilon + \|\mathbf{v}_1\| + \|\mathbf{v}_2\|)\ \|\mathbf{v}_1 - \mathbf{v}_2\| \,, \tag{4.15}$$
$$\|\mathbf{v}_0\| \leq C_2\ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \,, \tag{4.16}$$

where $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ is given by the above construction.

Proof. Inequality (4.14) is a direct consequence of Lemma 3.10 and Proposition 4.4, inequality (4.16) is a direct consequence of Lemma 3.4. This implies that $\mathcal{F}$ is a well defined map $\mathcal{B} \to \mathcal{B}$. Finally, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{B}$, we have the estimates valid for all $(x, k) \in \Omega_+$

$$|q_0(\mathbf{v}_1) - q_0(\mathbf{v}_2)| \leq\ C\ (\varepsilon + \|\mathbf{v}_1\| + \|\mathbf{v}_2\|)\ \|\mathbf{v}_1 - \mathbf{v}_2\|\ \min\left\{ \left| KW_{\alpha+1,\frac{1-\gamma_1}{2}} \right| \,, W_{\alpha,1} \right\} \,, \tag{4.17}$$
$$|Dq_0(\mathbf{v}_1) - Dq_0(\mathbf{v}_2)| \leq\ C\ (\varepsilon + \|\mathbf{v}_1\| + \|\mathbf{v}_2\|)\ \|\mathbf{v}_1 - \mathbf{v}_2\|\ W_{\alpha,\frac{1-\gamma_1}{2}} \,, \tag{4.18}$$
$$|h_1(\mathbf{v}_1) - h_1(\mathbf{v}_2)| \leq\ C\ (\varepsilon + \|\mathbf{v}_1\| + \|\mathbf{v}_2\|)\ \|\mathbf{v}_1 - \mathbf{v}_2\|\ W_{\alpha-1,\frac{3}{2}} \,, \tag{4.19}$$
$$|Dh_1(\mathbf{v}_1) - Dh_1(\mathbf{v}_2)| \leq\ C\ (\varepsilon + \|\mathbf{v}_1\| + \|\mathbf{v}_2\|)\ \|\mathbf{v}_1 - \mathbf{v}_2\|\ \left| KW_{\alpha,\frac{1-\gamma_1}{2}} \right| \,, \tag{4.20}$$
$$|h_2(\mathbf{v}_1) - h_2(\mathbf{v}_2)| \leq\ C\ (\varepsilon + \|\mathbf{v}_1\| + \|\mathbf{v}_2\|)\ \|\mathbf{v}_1 - \mathbf{v}_2\|\ \left| KW_{\alpha,\frac{3}{2}} \right| \,, \tag{4.21}$$
$$|Dh_2(\mathbf{v}_1) - Dh_2(\mathbf{v}_2)| \leq\ C\ (\varepsilon + \|\mathbf{v}_1\| + \|\mathbf{v}_2\|)\ \|\mathbf{v}_1 - \mathbf{v}_2\|\ W_{\alpha-1,\frac{3}{2}} \,. \tag{4.22}$$

This follows easily from the structure of q_0, h_1 and h_2, since, for example $u_1\partial_y v_1 - u_2\partial_y v_2 = (u_1 - u_2)\partial_y v_1 + u_2\partial_y(v_1 - v_2)$. Inequalities (4.17)–(4.22) involve the same functions and the same constants on the right–hand side as the one of Proposition 4.4, so that inequality (4.15) follows at once. ■

From this lemma, we get that the map $\mathcal{F}$ is a contraction in an appropriate ball of $\mathcal{B}$.

Corollary 4.6 *For all $\sigma \in (0, 1)$ and $\varepsilon > 0$ satisfying $\varepsilon\ C_1\ (1 + 2\ C_2(1 + \sigma)) < 1$ the map $\mathcal{F}$ has a unique fixed point denoted by $\mathbf{v}_\star$ in the open ball of radius $\sigma\|\mathbf{v}_0\|$ centered in $\mathbf{v}_0 \in \mathcal{B}$ provided $\|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} < \varepsilon$.*

Proof. By inequalities (4.14) and (4.15), the map $\mathcal{F}$ is a contraction in the open ball of radius $\sigma\|\mathbf{v}_0\|$ centered in $\mathbf{v}_0 \in \mathcal{B}$, provided $\varepsilon\ C_1\ (1 + 2\ C_2(1 + \sigma)) < 1$. It follows that $\lim_{n\to\infty} \mathbf{v}_n = \mathbf{v}_\star \in \mathcal{B}$ is a fixed point of $\mathcal{F}$. ■

This corollary implies Lemma 1.5 by the following observation. Since $\mathbf{v}_\star \in \mathcal{B}$, we can define

$$\eta_\star = q_1(\mathbf{v}_\star) + q_2(\mathbf{v}_\star) \,,$$
$$p_\star = q_{1,p}(\mathbf{v}_\star) + q_{2,p}(\mathbf{v}_\star) \,,$$

$$\tilde{\eta}_\star = \eta_{0,\star} + \eta_\star \ ,$$
$$\tilde{p}_\star = p_{0,\star} + \eta_\star \ ,$$

where

$$\eta_{0,\star} = \eta_0(\mathbf{v}_\star) \ ,$$
$$\hat{p}_{0,\star}(x,k) = -i \ \mathrm{sign}(k) \ \hat{\eta}_{0,\star}(x,k) \ .$$

By construction, the right–hand sides of equations (2.12) and (2.13), $\mathbf{v}_\star$, $\partial_y \mathbf{v}_\star$ and $q_0(\mathbf{v}_\star) + \tilde{\eta}_\star$ exist and are continuous in x, y, so that $\mathbf{v}_\star$ is a strong solution of (2.12) and (2.13). Furthermore $\mathbf{v}_\star$ obeys the boundary conditions. On the other hand, the right–hand sides of equations (2.14) and (2.15), $\eta_\star$, $p_\star$, $\partial_y \eta_\star$, $\partial_y p_\star$, $h_1(\mathbf{v}_\star)$ and $h_2(\mathbf{v}_\star)$ exist and are continuous, so that $\eta_\star$, $p_\star$ are strong solutions of (2.14) and (2.15) and satisfy the boundary conditions. It only remains to show that $\eta_{0,\star}$ and $p_{0,\star}$ are strong solution of (2.14) and (2.15) with $h_1, h_2 = 0$. Since $|\hat{\eta}_{0,\star}(x,k)| + |\hat{p}_{0,\star}(x,k)| \le (1+|k|)^{1-\alpha} e^{-|k|(x-1)}$ with $\alpha > 3$, $\eta_{0,\star}$ and $p_{0,\star}$ together with their derivatives with respect to y are bounded and continuous in x, y . This shows that $\tilde{\eta}_\star$ and $\tilde{p}_\star$ are strong solutions of (2.14) and (2.15). Finally, since (2.12)–(2.15) and the boundary conditions (2.3) and (2.4) are equivalent to the original Navier–Stokes equations, $\tilde{\mathbf{u}}_E + \mathbf{v}_\star$ and $\tilde{p}_E + \tilde{p}$ is the solution to (1.1)–(1.4).

We now turn to the proof of Theorem 1.7. The following corollary express the result of Theorem 1.7 in the variables u, v.

Corollary 4.7 *There exists a constant C such that for all $(x,k) \in \Omega_+$, the functions $u_\star, v_\star$ satisfy*

$$\left| u_E(x,y) + u_\star(x,y) - c \, \frac{e^{-y^2/4x}}{\sqrt{4\pi \, x}} \right| \le C \, \varepsilon \, x^{-1/2-\tilde{\gamma}} \ , \tag{4.23}$$

$$|v_E(x,y) + v_\star(x,y)| \le C \, \varepsilon \, x^{-1} \ , \tag{4.24}$$

where $\tilde{\gamma} = \min\left\{\frac{1}{4}, \frac{1-\gamma_1}{2}\right\}$ and $c = \int dy \ u_{+,1}(y) - f(0)$.

Proof. We note first that $c = \hat{u}_+(0)$, so that we have

$$\begin{aligned}\left| u_E(x,y) + u_\star(x,y) - c \, \frac{e^{-y^2/4x}}{\sqrt{4\pi \, x}} \right| &\le \int dk \ \left| \hat{u}_E(x,k) + \hat{u}_\star(x,k) - \hat{u}_+(0) e^{-k^2 x} \right| \\ &\le \int dk \ |\hat{u}_E(x,k)| + \int dk \ \left| \hat{u}_\star(x,k) - \hat{u}_+(0) e^{-k^2 x} \right| \ .\end{aligned}$$

Next, we have

$$\int dk \ |\hat{u}_E(x,k)| \le C \, \varepsilon \int dk \ e^{-|k|(x+\delta)} \le C \, \varepsilon \, x^{-1} \ ,$$

and

$$\left| \hat{u}_\star(x,k) - \hat{u}_+(0) e^{-k^2 x} \right| \le \left| \hat{u}_\star(x,k) - \hat{u}_+(0) e^{\Lambda_-(k)x} \right| + \left| \hat{u}_+(0)(e^{\Lambda_-(k)x} - e^{-k^2 x}) \right| \ .$$

Using (3.41), we get

$$\int dk \ \left| \hat{u}_\star(x,k) - \hat{u}_+(0) e^{\Lambda_-(k)x} \right| \le C \, \varepsilon \, x^{-\tilde{\gamma}} \int dk \ w_\alpha(x,k) \le C \, \varepsilon \, x^{-1/2-\tilde{\gamma}} \ .$$

Finally, since $\Lambda_-(k) + k^2 \ge 0$ and $|\hat{u}_+(0)| \le \varepsilon$, we get

$$\int dk \ \left| \hat{u}_+(0)(e^{\Lambda_-(k)x} - e^{-k^2 x}) \right| \le \varepsilon \int dk \ \left(e^{\Lambda_-(k)x} - e^{-k^2 x} \right) \ .$$

Performing the change of variables $\Lambda_-(k) = -\xi^2$, we get

$$\begin{aligned}\int dk \ \left(e^{\Lambda_-(k)x} - e^{-k^2 x} \right) &= \int_{-\infty}^{\infty} d\xi \ e^{-\xi^2 x} \left(\frac{1 - \sqrt{1+\xi^2} + 2\xi^2}{\sqrt{1+\xi^2}} \right) \ , \\ &\le 2 \int_{\infty}^{\infty} d\xi \ \xi^2 \ e^{-\xi^2 x} \le C \, x^{-3/2} \ ,\end{aligned}$$

because $1 - \sqrt{1+\xi^2} + 2\xi^2 \le 2\xi^2$. This achieves the proof of (4.23). The proof of inequality (4.24) is similar. We omit the details. ■

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Appendix A: Regularity of the Fourier Transform

In this section, we show that $\partial_k \hat{f}(k) \leq C(1+|k|^{-\gamma})$ with $0<\gamma<1$ if the function f approaches sufficiently fast $|y|^{\gamma-2}$ as $y \to \infty$:

Lemma A.1 *Let $f:\mathbf{R}\to\mathbf{R}$ be an odd/even function. Assume that there exist $0<\gamma<1$ and $c_1 \in \mathbf{R}$ such that*

$$\sup_{y\in\mathbf{R}} |y|^{2-\gamma}|f(y)| < \infty \, ,$$

$$\int_0^\infty dy \; y \left|f(y) - c_1 y^{\gamma-2}\right| < \infty \, .$$

Then there is a constant C such that

$$\sup_{k\in\mathbf{R}} \left(\frac{|k|}{1+|k|}\right)^\gamma |\partial_k \hat{f}(k)| \leq C \, ,$$

where $\hat{f}$ is the Fourier transform of f.

Proof. First, we note that

$$\begin{aligned} |k|^\gamma |\partial_k \hat{f}(k)| &= 2|k|^\gamma \left| \int_0^\infty dy \; e^{iky} y \; f(y) \right| \\ &\leq 2|k|^\gamma \left| \int_0^{\frac{1}{|k|}} dy \; e^{iky} \; y \; f(y) \right| + 2|k|^\gamma \left| \int_{\frac{1}{|k|}}^\infty dy \; e^{iky} \; y \; f(y) \right| \, , \end{aligned}$$

then

$$|k|^\gamma \left| \int_0^{\frac{1}{|k|}} dy \; e^{iky} y \; f(y) \right| \leq \sup_{y\in\mathbf{R}} |y|^{2-\gamma}|f(y)| \int_0^1 dz \; z^{\gamma-1} < \infty,$$

and

$$\int_{\frac{1}{|k|}}^\infty dy \; e^{iky} \; y \; f(y) = \int_{\frac{1}{|k|}}^\infty dy \; e^{iky} \; y \; (f(y) - c_1 \; y^{\gamma-2}) + c_1 \int_{\frac{1}{|k|}}^\infty dy \; e^{iky} \; y^{\gamma-1} \, .$$

Now, since

$$|k|^\gamma \left| \int_{\frac{1}{|k|}}^\infty dy \; e^{iky} \; y^{\gamma-1} \right| = \left| \int_1^\infty dz \; e^{iz} \; z^{\gamma-1} \right| < \infty \, ,$$

and

$$\left(\frac{|k|}{1+|k|}\right)^\gamma \left| \int_{\frac{1}{|k|}}^\infty dy \; e^{iky} \; y \left(f(y) - c_1 y^{\gamma-2}\right) \right| \leq \left| \int_0^\infty dy \; e^{iky} \; y \left(f(y) - c_1 y^{\gamma-2}\right) \right| < \infty \, ,$$

this achieves the proof. ■

Remark A.2 *In the case $\gamma=0$, $\partial_k \hat{f}$ has a logarythmic divergence at $k=0$ if $\sup_{y\in\mathbf{R}}(1+|y|)^2|f(y)| < \infty$ and $\int_0^\infty dy \; y \left|f(y) - c_1(1+y)^{-2}\right| < \infty$.*

Appendix B: Existence of Euler's Solution

In this section, we prove the following lemma, from which Lemma 1.3 follows.

Lemma B.1 *Let $f \in \mathcal{A}_{\alpha,\gamma_E,\delta}$. There exists a continuous function $\Psi_E : \Omega_+ \to \mathbf{R}$ satisfying (continuously in x, y)*

$$\begin{aligned}
\triangle \Psi_E &= 0\,, \\
\triangle \nabla \Psi_E &= 0\,, \\
\lim_{x^2+y^2\to\infty} \nabla \Psi_E(x,y) &= 0\,, \\
\lim_{x\to 1} \nabla \Psi_E(x,y) &= \mathbf{u}_f(y).
\end{aligned}$$

Proof. Define the function Ψ_E to be the Fourier transform of

$$\hat{\Psi}_E(x,k) = -\frac{f(k)}{|k|} e^{-|k|(x-1)}\ .$$

To see that Ψ_E is well defined, write

$$\hat{\Psi}_E(x,k) = -\frac{e^{-|k|(x-1)}}{|k|}\left(f(k) - f(0)e^{-|k|(1+\delta)}\right) - \frac{e^{-|k|(x+\delta)}}{|k|} f(0)\ .$$

The Fourier transform of $|k|^{-1}e^{-|k|(x+\delta)}f(0)$ is a constant times $\ln(y^2 + (x+\delta)^2)$, and the Fourier transform of the remainder of $\hat{\Psi}_E$ exists because it is in $L^1(\mathbf{R}, dk)$. Namely,

$$\begin{aligned}
\frac{e^{-|k|(x-1)}}{|k|}\left|f(k) - f(0)e^{-|k|(1+\delta)}\right| &\le \frac{e^{-|k|(x-1)}}{|k|}\left|\int_0^k d\xi\ \partial_\xi\left(f(\xi)e^{-|k-\xi|(1+\delta)}\right)\right| \\
&\le C\varepsilon\ (1+|k|^{-\gamma_E})\ e^{-|k|(x+\delta)}
\end{aligned}$$

for some constant C. Since $\gamma_E < 1$, this is in $L^1(\mathbf{R}, dk)$. As is easily shown, $\partial_x^a \hat{\Psi}_E$ and $k^a \hat{\Psi}_E$ are also in $L^1(\mathbf{R}, dk)$ for $a = 1, 2$. Furthermore, Ψ_E and all its derivatives with respect to x and y are continuous in x, y. Namely, let $\mu \in \mathbf{R}$, if $x + \mu \ge 1$, one has $\left|e^{-|k|(x+\mu-1)} - e^{-|k|(x-1)}\right| \le |\mu|\ |k| e^{-|k|(x-1)}$, and if $\mu \in \mathbf{R}$, one has $|e^{ik(y+\mu)} - e^{iky}| \le |\mu|\ |k|$. This means that

$$|\Psi_E(x+\mu, y) - \Psi_E(x,y)| + |\Psi_E(x, y+\mu) - \Psi_E(x,y)| \le C\ |\mu|\ \varepsilon \int dk\ e^{-|k|(x+\delta)} \le C\ |\mu|,$$

so that Ψ_E is even Lipschitz continuous. The same holds for the derivatives with respect to either x or y. This implies that $\nabla\Psi_E$ exists and is continuous, as well as $\triangle\Psi_E$. Since $(\partial_x^2 - k^2)\hat{\Psi}_E = 0$, $\triangle\Psi_E = 0$ and $\triangle\nabla\Psi_E = 0$. Now, let $\mathbf{u}_E = \nabla\Psi_E$, we have $\partial_k \hat{\mathbf{u}}_E \in L^1(\mathbf{R}, dk)$, so that $\mathbf{u}_E \sim |y|^{-1}$ as $y \to \infty$, and $\int dk\ |\hat{\mathbf{u}}_E(x,k)| \le C(x+\delta)^{-1}$, so that $\mathbf{u}_E \to 0$ as $x^2 + y^2 \to \infty$. Finally, $\nabla\Psi_E$ satisfies also the boundary condition at $x = 1$, since it is continuous in x. ■

Lemma 1.3 is a consequence of this lemma. Namely, define

$$\begin{aligned}
\tilde{\mathbf{u}}_E &= \mathbf{u}_\infty + \nabla\Psi_E\ , \\
\tilde{p}_E &= -\frac{1}{2}\left(\tilde{\mathbf{u}}_E \cdot \tilde{\mathbf{u}}_E\right)\ .
\end{aligned}$$

By Lemma B.1, $\tilde{\mathbf{u}}_E$ satisfies (continuously in x, y)

$$\begin{aligned}
\triangle \tilde{\mathbf{u}}_E &= 0\ , \\
-(\tilde{\mathbf{u}}_E \cdot \nabla)\tilde{\mathbf{u}}_E - \nabla\tilde{p} &= 0\ , \\
\lim_{x^2+y^2\to\infty} \tilde{\mathbf{u}}_E(x,y) &= \mathbf{u}_\infty\ , \\
\lim_{x\to 1} \tilde{\mathbf{u}}_E(x,y) &= \mathbf{u}_\infty + \mathbf{u}_f(y)\ ,
\end{aligned}$$

so that $\tilde{\mathbf{u}}_E, \tilde{p}_E$ is a strong solution of (1.1)–(1.3) in Ω_+.

Appendix C: Proofs for Section 3

In this section, we give the proofs for the lemmas and propositions of Sections 3.2 and 3.3. The proofs involve elementary properties of the functions $\Lambda_\pm$. These properties are given in the following subsection.

C.1 Preliminary Bounds

Lemma C.1 *For all $\gamma > 0$, $\alpha \geq 0$, there exists a constant C such that for all $x \geq 1$ and $0 \leq |\xi| \leq |k|$, we have*

$$0 \leq \Lambda_-(k) + |k| \leq \frac{|\Lambda_-(k)|}{|k|} \,, \tag{C.1}$$

$$\left(\frac{|k|}{1+|k|}\right)^\gamma e^{-|k|(x-1)/4} \leq \frac{C}{x^\gamma} \,, \tag{C.2}$$

$$\left(\frac{|k|}{1+|k|}\right)^\gamma e^{\Lambda_-(k)(x-1)/4} \leq \frac{C}{x^{\gamma/2}} \,, \tag{C.3}$$

$$|k|^\gamma e^{\Lambda_-(k)x} \leq C\left(\frac{\sqrt{1+x}}{x}\right)^\gamma \,, \tag{C.4}$$

$$\frac{e^{\Lambda_-(k)(x-1)/4}}{(1+|k|)^\alpha} \leq \frac{C}{(1+|k|\sqrt{x})^\alpha} \,, \tag{C.5}$$

$$\frac{e^{-|k|(x-1)/4}}{(1+|k|)^\alpha} \leq \frac{C}{(1+|k|\sqrt{x})^\alpha} \,, \tag{C.6}$$

$$\frac{(1+|k|)^\alpha}{(1+|\xi|)^\alpha} e^{\Lambda_-(k-\xi)} \leq C\, e^{\Lambda_-(k-\xi)/2} \,. \tag{C.7}$$

Proof. Inequalities (C.1) and (C.2) are elementary. Inequality (C.3) follows from

$$\left(\frac{|k|}{1+|k|}\right)^\gamma \leq \left(\frac{|\Lambda_-(k)|}{1+|\Lambda_-(k)|}\right)^{\gamma/2}$$

and (C.2) with $|k|$ replaced by $|\Lambda_-(k)|$. For inequality (C.4), since $|k| \leq \sqrt{|\Lambda_-(k)|(1+|\Lambda_-(k)|)}$, we have that

$$\begin{aligned} |k|^\gamma \left(\frac{x}{\sqrt{1+x}}\right)^\gamma e^{\Lambda_-(k)x} &\leq \left(\frac{x^2}{1+x}|\Lambda_-(k)|(1+|\Lambda_-(k)|)\right)^{\gamma/2} e^{\Lambda_-(k)x} \\ &\leq \left(|\Lambda_-(k)|x + |\Lambda_-(k)|^2 x^2\right)^{\gamma/2} e^{\Lambda_-(k)x} \leq C \,. \end{aligned}$$

Inequalities (C.5) and (C.6) follow from (C.2) and (C.3). Finally, inequality (C.7) follows by considering separately the case $0 \leq |\xi| \leq |k|/2$ and $|k|/2 \leq |\xi| \leq |k|$. In the first case, we have

$$\frac{(1+|k|)^\alpha}{(1+|\xi|)^\alpha} e^{\Lambda_-(k-\xi)} \leq e^{\Lambda_-(k-\xi)/2}(1+|k|)^\alpha e^{\Lambda_-(k/2)/2} \leq C\, e^{\Lambda_-(k-\xi)/2} \,,$$

and in the second case, we have

$$\frac{(1+|k|)^\alpha}{(1+|\xi|)^\alpha} e^{\Lambda_-(k-\xi)} \leq e^{\Lambda_-(k-\xi)/2} \frac{(1+|k|)^\alpha}{(1+|k|/2)^\alpha} \leq C\, e^{\Lambda_-(k-\xi)/2} \,.$$

This completes the proof. ■

Lemma C.2 *Let $h : \mathbf{R} \to \mathbf{R}$. Assume that there exist constants $H > 0$, $\alpha > 0$ and $0 \leq \gamma < 1$ such that for all $k \in \mathbf{R}$,*

$$\begin{aligned} |h(k)| &\leq \frac{H}{(1+|k|)^\alpha} \,, \\ |Dh(k)| &\leq \frac{H}{(1+|k|)^\alpha}\left(\frac{|k|}{1+|k|}\right)^{-\gamma} \,. \end{aligned}$$

Then there exists a constant C such that for all $k \in \mathbf{R}$,

$$\left|h(k) - e^{\Lambda_-(k)}h(0)\right| \leq C\,H \frac{1}{(1+|k|)^\alpha}\left(\frac{|k|}{1+|k|}\right)^{1-\gamma} . \tag{C.8}$$

Proof. Since $\Lambda_-(0) = 0$, we have that

$$\begin{aligned}
\left|h(k) - e^{\Lambda_-(k)}h(0)\right| &\leq \left|\int_0^k d\xi\,\partial_\xi\left(e^{\Lambda_-(k-\xi)}h(\xi)\right)\right| \\
&\leq C\int_0^{|k|} d\xi\, e^{\Lambda_-(|k|-\xi)}\left(\frac{2(|k|-\xi)}{\sqrt{1+4(|k|-\xi)^2}}|h(k)| + |Dh(k)|\right) \\
&\leq \frac{C\,H}{(1+|k|)^\alpha}\int_0^{|k|} d\xi\, e^{\Lambda_-(|k|-\xi)/2}\left(1+\left(\frac{|\xi|}{1+|\xi|}\right)^{-\gamma}\right) .
\end{aligned}$$

Now, for $|k| \leq 1$, we have

$$\int_0^{|k|} d\xi\, e^{\Lambda_-(|k|-\xi)/2}\left(1+\left(\frac{|\xi|}{1+|\xi|}\right)^{-\gamma}\right) \leq C\,|k|^{1-\gamma}(1+|k|^\gamma) ,$$

and for $|k| \geq 1$, we have

$$\int_0^{|k|} d\xi\, e^{\Lambda_-(|k|-\xi)/2}\left(1+\left(\frac{|\xi|}{1+|\xi|}\right)^{-\gamma}\right) \leq C ,$$

because the integral is convergent. If $|k| \leq 1$, $|k|^{1-\gamma}(1+|k|^\gamma) \leq C|k|^{1-\gamma}(1+|k|)^{\gamma-1}$ and the bound (C.8) follows for this case. If $|k| \geq 1$, then $1 \leq |k|^{1-\gamma}(1+|k|)^{\gamma-1}$ and the bound (C.8) follows for this other case. ■

The following functions enter the definition of $\hat{u}_0$ and $\hat{v}_0$,

$$\begin{aligned}
P_{1,1}(x,k) &= e^{-|k|(x-1)} + \frac{|k|}{\Lambda_-(k)+|k|}\left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)}\right) , \\
P_{1,2}(x,k) &= \frac{|k|}{\Lambda_-(k)+|k|}\left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)}\right) , \\
P_{2,1}(x,k) &= \frac{\Lambda_-(k)}{\Lambda_-(k)+|k|}\left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)}\right) , \\
P_{2,2}(x,k) &= e^{-|k|(x-1)} + \frac{\Lambda_-(k)}{\Lambda_-(k)+|k|}\left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)}\right) .
\end{aligned}$$

We summarize their properties in the following lemma.

Lemma C.3 *For all $0 \leq \mu \leq 1$ and for all $i,j = 1,2$, there exists a constant C such that for all $(x,k) \in \Omega_+$, $|\sigma| \leq 1$ with $x+\sigma \geq 1$, one has*

$$\begin{aligned}
|P_{1,1}(x,k)| + |P_{1,2}(x,k)| &\leq Ce^{\Lambda_-(k)(x-1)/2} , \\
|P_{2,1}(x,k)| &\leq C\frac{|k|}{1+|k|}e^{\Lambda_-(k)(x-1)/2} , \\
|P_{2,2}(x,k)| &\leq Ce^{-|k|(x-1)} + C\frac{|k|}{1+|k|}e^{\Lambda_-(k)(x-1)/2} , \\
|P_{i,j}(x+\sigma,k) - P_{i,j}(x,k)| &\leq C|\sigma|^\mu\,(1+|k|)^\mu , \\
|DP_{1,1}(x,k)| &\leq C\sqrt{x}e^{\Lambda_-(k)(x-1)/2} , \\
|DP_{1,2}(x,k)| &\leq C(x-1)e^{-|k|(x-1)} + C\sqrt{x}e^{\Lambda_-(k)(x-1)/2} , \\
|DP_{2,1}(x,k)| &\leq Ce^{\Lambda_-(k)(x-1)/2} , \\
|DP_{2,2}(x,k)| &\leq C(x-1)e^{-|k|(x-1)} + C\frac{|k|}{1+|k|}e^{\Lambda_-(k)(x-1)/2} .
\end{aligned}$$

Proof. All these properties are elementary. As an example, we show the inequalities involving $P_{1,2}$. The others follow by using very similar arguments. By Lemma C.1, we have $0 \leq \Lambda_-(k) + |k| \leq |k|^{-1}|\Lambda_-(k)|$. On the other hand, $\Lambda_-(k) \sim -k^2$ as $k \to 0$ and $\Lambda_-(k) \sim -|k|$ as $k \to \infty$. Using this, we get

$$
\begin{aligned}
|P_{1,2}(x,k)| &\leq C\,(1+|k|)\left|e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)}\right| \\
&\leq C\,e^{\Lambda_-(k)(x-1)} + C\,|k|e^{\Lambda_-(k)(x-1)}\left|1 - e^{-(\Lambda_-(k)+|k|)(x-1)}\right| \\
&\leq C\,e^{\Lambda_-(k)(x-1)} + C\,|k|(\Lambda_-(k)+|k|)\,(x-1)\,e^{\Lambda_-(k)(x-1)} \\
&\leq C\,e^{\Lambda_-(k)(x-1)} + C\,|\Lambda_-(k)|\,(x-1)\,e^{\Lambda_-(k)(x-1)} \\
&\leq C\,e^{\Lambda_-(k)(x-1)/2}\;, \\
|P_{1,2}(x+\sigma,k) - P_{1,2}(x,k)| &\leq \left(1 - e^{\Lambda_-(k)|\sigma|}\right) \sup_{(x,k)\in\Omega_+} |P_{1,2}(x,k)| \\
&\quad + \frac{|k|}{\Lambda_-(k)+|k|} e^{\Lambda_-(k)|\sigma|}\left(1 - e^{-(\Lambda_-(k)+|k|)|\sigma|}\right) \\
&\leq C\,(1+|k|)^{\mu}|\sigma|^{\mu} + C\,|k|\,|\sigma|e^{\Lambda_-(k)|\sigma|} \\
&\leq C\,(1+|k|)^{\mu}|\sigma|^{\mu} + C\,(1+|k|)^{\mu}\,|\sigma|\,|k|^{1-\mu}e^{\Lambda_-(k)|\sigma|} \\
&\leq C\,(1+|k|)^{\mu}\left(|\sigma|^{\mu} + C\,|\sigma|^{\frac{1+\mu}{2}}\right) \\
&\leq C\,(1+|k|)^{\mu}|\sigma|^{\mu}\;.
\end{aligned}
$$

The bound on the derivative of $P_{1,2}$ is similar. We omit the details. ■

The following functions enter the definition of N_1 and N_2.

$$
\begin{aligned}
P_1(x,k) &= -(\Lambda_+(k)+|k|)\left[\frac{ik}{\Delta(k)(\Lambda_-(k)+|k|)}e^{\Lambda_-(k)(x-1)} - i\ \mathrm{sign}(k)e^{-|k|(x-1)}\right]\;, \\
P_2(x,k) &= (\Lambda_+(k)+|k|)\left[\frac{\Lambda_-(k)}{\Delta(k)(\Lambda_-(k)+|k|)}e^{\Lambda_-(k)(x-1)} + e^{-|k|(x-1)}\right]\;.
\end{aligned}
$$

We summarize their properties in the following lemma.

Lemma C.4 *For all $0 \leq \mu \leq 1$, there exists a constant C such that for all $(x,k) \in \Omega_+$ and all $|\sigma| \leq 1$ with $x+\sigma \geq 1$, we have*

$$
\begin{aligned}
|P_1(x,k)| + |P_2(x,k)| &\leq C\,e^{\Lambda_-(k)(x-1)/2}\;, \\
|P_1(x+\sigma,k) - P_1(x,k)| + |P_2(x+\sigma,k) - P_2(x,k)| &\leq C\,|\sigma|^{\mu}\,(1+|k|)^{\mu}\;.
\end{aligned}
$$

Proof. The proof is very similar to the one of Lemma C.3, we omit the details. ■

C.2 Proof of Lemma 3.1

Lemma C.5 (Lemma 3.1) *Assume that $\mathbf{v}_+ = (u_+, v_+) \in \mathcal{W}_{\alpha,\gamma_1,\gamma_2}$ for some $\alpha > 3$, $\gamma_1 \in [0,1)$ and $\gamma_2 \in [0,\frac{1}{2}]$ with $\gamma_2 \leq \gamma_1$. Assume that u_+ and v_+ are even and odd in y respectively. Define $\eta_{0,0}$, u_0 and v_0 as the Fourier transforms of*

$$
\begin{aligned}
\hat{\eta}_{0,0}(x,k) &= \hat{\eta}_{0,+}(k)e^{-|k|(x-1)}\;, \\
\begin{pmatrix}\hat{u}_0\\ \hat{v}_0\end{pmatrix}(x,k) &= \begin{pmatrix}P_{1,1} & P_{1,2}\\ P_{2,1} & P_{2,2}\end{pmatrix}(x,k)\begin{pmatrix}\hat{u}_+\\ \hat{v}_+\end{pmatrix}(k)\;,
\end{aligned}
$$

where $P_{i,j}$ are defined in the previous section and

$$
\hat{\eta}_{0,+}(k) = -ik\frac{\Lambda_+(k)+|k|}{\Lambda_+(k)}\hat{u}_+(k) - (\Lambda_+(k)+|k|)\hat{v}_+(k)\;.
$$

Then the functions $\eta_{0,0}$, u_0, v_0, $\partial_y u_0$, $\partial_y v_0$ are bounded and continuous in x, y, the function u_0 is even in y and the functions $v_0, \eta_{0,0}$ are odd in y.

Proof. First of all, the parity of the functions is trivial. From Lemma C.3, it follows that there exists a constant C such that for all $(x,k)\in\Omega_+$,

$$|\hat{u}_0(x,k)|+|\hat{v}_0(x,k)|\le C\frac{\|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}}{(1+|k|)^{\alpha}}\ . \tag{C.9}$$

Since $\alpha>3$, this implies that the Fourier transforms of $\hat{u}_0$, $K\hat{u}_0$, $\hat{v}_0$ and $K\hat{v}_0$ are bounded. To see that they are continuous in x, use Lemma C.3 to get that for all $|\sigma|\le 1$ with $x+\sigma\ge 1$, and for all $\mu\in\mathbf{R}$ with $0<\mu<\alpha-2$,

$$\int dk\ (1+|k|)\ |\hat{u}_0(x+\sigma,k)-\hat{u}_0(x,k)|\le|\sigma|^{\mu}\ C\ \int dk\ \frac{\|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}}{(1+|k|)^{\alpha-1-\mu}}\le|\sigma|^{\mu}\ C\ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ .$$

This implies that u_0 and $\partial_y u_0$ are Hölder–μ in x for all $0<\mu<\alpha-2$. To see that u_0 and $\partial_y u_0$ are also Hölder–μ in y, use $\left|e^{ik(y+\sigma)}-e^{iky}\right|\le|\sigma|^{\mu}|k|^{\mu}$ to get that

$$\begin{aligned}|\partial_y u_0(x,y+\sigma)-\partial_y u_0(x,y)|+|u_0(x,y+\sigma)-u_0(x,y)|&\le|\sigma|^{\mu}\ C\ \int dk\ \frac{\|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}}{(1+|k|)^{\alpha-1-\mu}}\\&\le|\sigma|^{\mu}\ C\ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ .\end{aligned}$$

For the continuity of v_0, the proof is done along the same lines. Finally, by (C.9), we have

$$\int_{-\infty}^{\infty}dk\ |\hat{\eta}_{0,0}(x,k)|\le\int_{-\infty}^{\infty}dk\ \frac{\|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}}{(1+|k|)^{\alpha-1}}\ ,$$

so that $\eta_{0,0}(x,y)$ is bounded. The continuity of $\eta_{0,0}$ follows along the same lines as the one for u_0. ■

C.3 Proof of Lemma 3.2

Lemma C.6 (Lemma 3.2) *Let $\hat{q}:\Omega_+\to\mathbf{R}$ be bounded. Assume that $\hat{q}$ is an odd function of k, that $\hat{q}\in L^1(\Omega_+,k\ dk\ dx)$ and that $\sup_{x\ge1}\int dk\ (1+|k|)|\hat{q}(x,k)|<\infty$. Define N_1 and N_2 and $\eta_{0,q}$ as the Fourier transforms of*

$$\begin{aligned}\hat{N}_1(x,k)&=\hat{N}_{1,1}(x,k)+\hat{N}_{1,2}(x,k)+P_1(x,k)\ \tau(k)\ ,\\\hat{N}_2(x,k)&=\hat{N}_{2,1}(x,k)+\hat{N}_{2,2}(x,k)+P_2(x,k)\ \tau(k)\ ,\\\hat{\eta}_{0,q}(x,k)&=\hat{\eta}_{q,+}(k)e^{-|k|(x-1)}\ ,\end{aligned}$$

where P_1,P_2 are defined in Section C.1 and

$$\begin{aligned}\hat{N}_{1,1}(x,k)&=\frac{ik}{\Delta(k)}\int_1^x dx_0\ e^{\Lambda_-(k)(x-x_0)}\hat{q}(x_0,k)\ ,\\\hat{N}_{1,2}(x,k)&=\frac{ik}{\Delta(k)}\int_x^{\infty}dx_0\ e^{\Lambda_+(k)(x-x_0)}\hat{q}(x_0,k)\ ,\\\hat{N}_{2,1}(x,k)&=-\frac{\Lambda_-(k)}{\Delta(k)}\int_1^x dx_0\ e^{\Lambda_-(k)(x-x_0)}\hat{q}(x_0,k)\ ,\\\hat{N}_{2,2}(x,k)&=-\frac{\Lambda_+(k)}{\Delta(k)}\int_x^{\infty}dx_0\ e^{\Lambda_+(k)(x-x_0)}\hat{q}(x_0,k)\ ,\\\tau(k)&=\int_1^{\infty}dx_0\ e^{\Lambda_+(k)(1-x_0)}\hat{q}(x_0,k)\ ,\\\hat{\eta}_{q,+}(k)&=-(\Lambda_+(k)+|k|)\tau(k)\ .\end{aligned}$$

Then the functions N_1, N_2, $\partial_y N_1,\partial_y N_2$ and $\eta_{0,q}$ are bounded and continuous in x,y, the function N_1 is even in y, and the functions $N_2,\eta_{0,q}$ are odd in y.

Proof. First of all, the parity of the functions is trivial. We begin the proof by showing that $\hat{N}_{i,j}\in L^1(\mathbf{R},(1+|k|)^{1+\mu}dk)$ for all $0\le\mu<1$. We begin with $\hat{N}_{i,1}$. We use first that there exists a constant C such that for all $k\in\mathbf{R}$,

$$|\Lambda_-(k)|\le|k|\ ,$$

$$(1+|k|)^{1+\mu} \leq \Delta(k)\ (C + C|k|^{\mu})\ ,$$

to get

$$\int dk\ (1+|k|)^{1+\mu}|\hat{N}_{i,1}(x,k)|) \leq C\int dk\ \Delta(k)|\hat{N}_{i,1}(x,k)| + C\int dk\ \Delta(k)|k|^{\mu}|\hat{N}_{i,1}(x,k)|\ .$$

Since $\Lambda_-(k) \leq 0$, we have

$$\int dk\ \Delta(k)|\hat{N}_{i,1}(x,k)| \leq \int dk\ \int_1^{\infty} dx_0\ |k\ \hat{q}(x_0,k)| < \infty\ .$$

By Lemma C.1, we have

$$|k|^{\mu}e^{\Lambda_-(k)x} \leq C\ \left(\frac{\sqrt{1+x}}{x}\right)^{\mu}\ ,$$

and since for all for $1 \leq x_0 \leq x$ we have $\sqrt{1+x-x_0} \leq \sqrt{3x}$, we get

$$\begin{aligned}\int dk\ \Delta(k)\ |k|^{\mu}\ |\hat{N}_{i,1}(x,k)| &\leq C\int dk\ \int_1^x dx_0\ \frac{x^{\mu/2}}{(x-x_0)^{\mu}}|k\ \hat{q}(x_0,k)| \\ &\leq Cx^{1-\mu/2}\sup_{x_0\geq 1}\int dk\ |k\ \hat{q}(x_0,k)|\ .\end{aligned}$$

This is bounded for x bounded. To show that it is also bounded as $x \to \infty$, suppose that $x \geq 4$. In this case, we split the integration interval, which gives

$$\begin{aligned}\int dk\ \int_1^{x-2} dx_0\ \frac{(1+x-x_0)^{\mu/2}}{(x-x_0)^{\mu}}|k\ \hat{q}(x_0,k)| &\leq C\int dk\ \int_1^{\infty} dx_0\ |k\ \hat{q}(x_0,k)| < \infty\ , \\ \int dk\ \int_{x-2}^{x} dx_0\ \frac{(1+x-x_0)^{\mu/2}}{(x-x_0)^{\mu}}|k\ \hat{q}(x_0,k)| &\leq C\sup_{x_0\geq 1}\int dk\ |k\ \hat{q}(x_0,k)|\int_{x-2}^{x} dy\ \frac{(1+x-y)^{\mu/2}}{(x-y)^{\mu}} \\ &\leq C\sup_{x_0\geq 1}\int dk\ |k\ \hat{q}(x_0,k)|\int_{x-2}^{x} dy\ \frac{1}{(x-y)^{\mu}} \\ &\leq C\sup_{x_0\geq 1}\int dk\ |k\ \hat{q}(x_0,k)| < \infty\ .\end{aligned}$$

This shows that $\hat{N}_{i,1} \in L^1(\mathbf{R},(1+|k|)^{1+\mu}dk)$ for all $0 \leq \mu < 1$. We now show that $\hat{N}_{i,2} \in L^1(\mathbf{R},(1+|k|)^{1+\mu}dk)$ also for all $0 \leq \mu < 1$. We use first that there exists a constant C such that for all $k \in \mathbf{R}$,

$$\frac{|k|}{\Delta(k)} + \frac{|\Lambda_+(k)|}{\Delta(k)} \leq C\ ,$$

and then, since $\Lambda_+(k) \geq 1$, we have

$$\int dk\ (1+|k|^{1+\mu})|\hat{N}_{i,2}(x,k)| \leq CB_1(x) + CB_2(x)\ ,$$

where

$$\begin{aligned}B_1(x) &= \int_x^{\infty} dx_0\ e^{(x-x_0)}\int dk\ |\hat{q}(x_0,k)|\ , \\ B_2(x) &= \int dk\ |k|^{1+\mu}\int_x^{\infty} dx_0\ e^{\Lambda_+(k)(x-x_0)}\ |\hat{q}(x_0,k)|\ .\end{aligned}$$

Now, we have

$$B_1(x) \leq \sup_{x_0\geq 1}\int dk\ |\hat{q}(x_0,k)| < \infty\ .$$

To estimate B_2, we first use that for all $0 \leq \mu \leq 1$, there exists a constant C such that for all $x_0 \geq x \geq 1$ and $k \in \mathbf{R}$, we have

$$|k|^\mu e^{\Lambda_+(k)(x-x_0)} \leq |\Lambda_+(k)|^\mu e^{\Lambda_+(k)(x-x_0)} \leq C(x_0-x)^{-\mu} \ .$$

Splitting the integration interval in two parts gives

$$\begin{aligned}
\int dk \int_x^{x+2} dx_0 \ e^{\Lambda_+(k)(x-x_0)}|k|^{1+\mu}|\hat{q}(x_0,k)| &\leq C \int dk \int_x^{x+2} dx_0 \ \frac{1}{(x_0-x)^\mu}|k \ \hat{q}(x_0,k)| \\
&\leq C \ \sup_{x\geq 1} \int dk \ |k \ \hat{q}(x,k)| \\
\int dk \int_{x+2}^{\infty} dx_0 \ e^{\Lambda_+(k)(x-x_0)}|k|^{1+\mu}|\hat{q}(x_0,k)| &\leq C \int dk \int_{x+2}^{\infty} dx_0 \ \frac{1}{(x_0-x)^\mu}|k\hat{q}(x_0,k)| \\
&\leq C \int_1^\infty dx_0 \int dk \ |k \ \hat{q}(x,k)| \ .
\end{aligned}$$

Since $|\tau(k)| = \frac{\Delta(k)}{\Lambda_+(k)}|\hat{N}_{2,2}(1,k)| \leq |\hat{N}_{2,2}(1,k)|$, we get that $\tau \in L^1(\mathbf{R},(1+|k|)^{1+\mu}dk)$ for all $0 \leq \mu < 1$. This implies that $\hat{N}_1$ and $\hat{N}_2$ are both in $L^1(\mathbf{R},(1+|k|)^{1+\mu}dk)$ for all $0 \leq \mu < 1$, and that $\hat{\eta}_{0,q} \in L^1(\mathbf{R},(1+|k|)^{\mu}dk)$. From this we conclude that N_1, N_2, $\partial_y N_1$, $\partial_y N_2$ and $\eta_{0,q}$ are bounded. We now turn to the continuity. First, we show that the Fourier transform of $P_1\tau$, $P_2\tau$ and their derivatives with respect to y are Hölder–μ in x, y for all $0 < \mu < 1$, and thus continuous in x, y. To do this, let $|\sigma| \leq 1$ with $x+\sigma \geq 1$ and $0 < \mu < 1$. By Lemma C.4, we have

$$\int dk \ (1+|k|) \ |P_i(x+\sigma,k) - P_i(x,k)| \ |\tau(k)| \leq |\sigma|^\mu \ C \int dk \ (1+|k|)^{1+\mu} \ |\tau(k)| \leq |\sigma|^\mu \ C \ .$$

Next we show that the same holds for $\hat{N}_{i,1}$ and $\hat{N}_{i,2}$. Namely, let $|\sigma| \leq 1$ with $x+\sigma \geq 1$, and define $\bar{x} = \min\{x, x+\sigma\}$. We have

$$\begin{aligned}
\left|\hat{N}_{i,1}(x+\sigma,k) - \hat{N}_{i,1}(x,k)\right| &\leq \text{sign}(\sigma) \int_x^{x+\sigma} dx_0 \ |q(x_0,k)| + (1-e^{-|\Lambda_-(k)\sigma|})|\hat{N}_{i,1}(\bar{x},k)| \\
&\leq \text{sign}(\sigma) \int_x^{x+\sigma} dx_0 \ |q(x_0,k)| + C \ |\sigma|^\mu |\Lambda_-(k)|^\mu |\hat{N}_{i,1}(\bar{x},k)| \\
&\leq \text{sign}(\sigma) \int_x^{x+\sigma} dx_0 \ |q(x_0,k)| + C \ |\sigma|^\mu |(1+|k|)|^\mu |\hat{N}_{i,1}(\bar{x},k)| \ .
\end{aligned}$$

From this, we conclude easily that

$$\int dk \ (1+|k|) \left|\hat{N}_{i,1}(x+\sigma,k) - \hat{N}_{i,1}(x,k)\right| \leq (|\sigma| + |\sigma|^\mu) \ C \ .$$

For $\hat{N}_{i,2}$, let $|\sigma| \leq 1$ with $x+\sigma \geq 1$, and define $\bar{x} = \max\{x, x+\sigma\}$. We have

$$\left|\hat{N}_{i,2}(x+\sigma,k) - \hat{N}_{i,2}(x,k)\right| \leq \text{sign}(\sigma) \int_x^{x+\sigma} dx_0 \ |q(x_0,k)| + (1-e^{-|\Lambda_+(k)\sigma|})|\hat{N}_{i,2}(\bar{x},k)| \ .$$

By the same arguments as above, we conclude that

$$\int dk \ (1+|k|) \left|\hat{N}_{i,2}(x+\sigma,k) - \hat{N}_{i,2}(x,k)\right| \leq (|\sigma| + |\sigma|^\mu) \ C \ .$$

We have thus shown that

$$\int dk \ (1+|k|) \left|\hat{N}_i(x+\sigma,k) - \hat{N}_i(x,k)\right| \leq C \ (|\sigma| + |\sigma|^\mu) \ ,$$

so that

$$|N_i(x+\sigma,y) - N_i(x,y)| + |\partial_y N_i(x+\sigma,y) - \partial_y N_i(x,y)| \leq C \ (|\sigma| + |\sigma|^\mu) \ .$$

This implies that N_i and $\partial_y N_i$ are Hölder–μ in x for all $0 < \mu < 1$. To see that they are also continuous in y, use that $\left|e^{ik(y+\sigma)} - e^{iky}\right| \leq |\sigma|^\mu |k|^\mu$ to get that

$$|\partial_y N_i(x, y+\sigma) - \partial_y N_i(x,y)| + |N_i(x,y+\sigma) - N_i(x,y)| \leq |\sigma|^\mu \int dk\ (1+|k|)^{1+\mu} \left|\hat{N}_i(x,k)\right| \leq C|\sigma|^\mu\ .$$

Finally, to see that $\eta_{0,q}$ is continuous in x, y, use that $\hat{\eta}_{+,q} \in L^1(\mathbf{R}, (1+|k|)^\mu dk)$, and that

$$\left|e^{-|k|(x+\sigma-1)} - e^{-|k|(x-1)}\right| \leq |\sigma|^\mu\ (1+|k|)^\mu\ ,$$
$$\left|e^{ik(y+\sigma)} - e^{iky}\right| \leq |\sigma|^\mu\ (1+|k|)^\mu$$

to get that

$$|\eta_{0,q}(x+\sigma, y) - \eta_{0,q}(x,y)| + |\eta_{0,q}(x, y+\sigma) - \eta_{0,q}(x,y)| \leq C|\sigma|^\mu \int dk\ (1+|k|)^{1+\mu} |\tau(k)| \leq C|\sigma|^\mu.$$

This completes the proof. ■

C.4 Proof of Lemma 3.5

Lemma C.7 (Lemma 3.5) *Let $\hat{h}_1, \hat{h}_2 : \Omega_+ \to \mathbf{R}$ be bounded, and assume that $\hat{h}_i \in L^1(\Omega_+, (1+|k|)^\mu dk\ dx)$, $\sup_{x\geq 1} \int dk\ (1+|k|)^\mu |\hat{h}_i(x,k)| < \infty$ for some $\mu > 0$. Assume furthermore that $\hat{h}_1$ is even in k and $\hat{h}_2$ is odd in k. Define q_1, q_2, $q_{1,p}$ and $q_{2,p}$ as the Fourier transforms of*

$$\hat{q}_1(x,k) = \int_1^\infty dx_0\ \left[-\frac{1}{2} i\ \mathrm{sign}(k) \left(e^{-|x-x_0||k|} - e^{-(x+x_0-2)|k|}\right)\right]\ \hat{h}_1(x_0,k)\ ,$$
$$\hat{q}_2(x,k) = \int_1^\infty dx_0\ \left[\frac{1}{2} \left(e^{-|x-x_0||k|}\,\mathrm{sign}(x-x_0) + e^{-(x+x_0-2)|k|}\right)\right]\ \hat{h}_2(x_0,k)\ ,$$
$$\hat{q}_{1,p}(x,k) = \int_1^\infty dx_0\ \left[-\frac{1}{2} \left(e^{-|x-x_0||k|}\,\mathrm{sign}(x-x_0) - e^{-(x+x_0-2)|k|}\right)\right]\ \hat{h}_1(x_0,k)\ ,$$
$$\hat{q}_{2,p}(x,k) = \int_1^\infty dx_0\ \left[-\frac{1}{2} i\ \mathrm{sign}(k) \left(e^{-|x-x_0||k|} + e^{-(x+x_0-2)|k|}\right)\right]\ \hat{h}_2(x_0,k)\ .$$

Then the functions q_i, $q_{i,p}$, $\partial_y q_i$ and $\partial_y q_{i,p}$ are bounded and continuous in x, y, the functions q_i are even in y and the functions $q_{i,p}$ are odd in y.

Proof. First of all, the parity of the functions is trivial. The proof is very similar to the one of Lemma 3.2. We show that $\hat{q}_i \in L^1(\mathbf{R}, (1+|k|)^{1+\nu} dk)$, $\hat{q}_{i,p} \in L^1(\mathbf{R}, (1+|k|)^{1+\nu} dk)$ for all $0 \leq \nu < \mu$. We first note that there exists a constant C such that for all $k \in \mathbf{R}$, $x, x_0 \geq 1$,

$$(1+|k|)^{1+\nu} \leq C((1+|k|)^\mu + |k|^{1+\nu})\ ,$$
$$\left|e^{-|k|\ |x-x_0|} \pm e^{-|k|(x+x_0-2)}\right| \leq 2\ e^{-|k|\ |x-x_0|}\ .$$

This last inequality shows that the proof will be very similar for q_i and for $q_{i,p}$. For this reason, we consider only q_i in details. We have

$$\begin{aligned}\int dk\ (1+|k|)^{1+\nu} |\hat{q}_i(x,k)| &\leq C \int dk\ \int_1^\infty dx_0\ (1+|k|)^\mu |h_i(x_0,k)| + C \int dk\ |k|^{1+\nu}\ |\hat{q}_i(x,k)| \\ &\leq C + C \int dk\ |k|^{1+\nu}\ |\hat{q}_i(x,k)|\ .\end{aligned}$$

Next, we have

$$\int dk\ |k|^{1+\nu} |\hat{q}_i(x,k)| \leq 2\ H_{i,1}(x,k) + 2\ H_{i,2}(x,k)\ ,$$

where

$$H_{i,1}(x,k) = \int dk \int_1^{x+2} dx_0 \ |k|^{1+\nu} e^{-|k(x-x_0)|} |h_i(x_0,k)| \ ,$$
$$H_{i,2}(x,k) = \int dk \int_{x+2}^{\infty} dx_0 \ |k| e^{-|k(x-x_0)|} |k|^{\mu} |h_i(x_0,k)| \ .$$

Since $|k|^{\sigma} e^{-|k||x-x_0|} \leq C|x-x_0|^{-\sigma}$, we have

$$H_{i,2}(x,k) \leq C|x-2|^{-1} \int_1^{\infty} dx \int dk \ (1+|k|)^{\mu} |h_i(x,k)| < \infty \ ,$$
$$H_{i,1}(x,k) \leq C \sup_{x_0 \geq 1} \int dk \ |k|^{\mu} \ |h_i(x_0,k)| \int_1^{x+2} dx_0 \ |x-x_0|^{\mu-\nu-1}$$
$$\leq C(1+x^{\mu-\nu}) \sup_{x_0 \geq 1} \int dk \ (1+|k|)^{\mu} \ |h_i(x_0,k)| \ .$$

This is bounded for x bounded. To see that it is also bounded as $x \to \infty$, let $x \geq 4$, then

$$H_{i,1}(x,k) = H_{i,3}(x,k) + H_{i,4}(x,k) \ ,$$

where

$$H_{i,3}(x,k) = \int dk \int_1^{x-2} dx_0 \ |k|^{1+\nu} e^{-|k(x-x_0)|} |h_i(x_0,k)| \ ,$$
$$H_{i,4}(x,k) = \int dk \int_{x-2}^{x+2} dx_0 \ |k|^{1+\nu} e^{-|k(x-x_0)|} |h_i(x_0,k)| \ .$$

As above, we have

$$H_{i,3}(x,k) \leq |x-2|^{-1} \int_1^{\infty} dx \int dk \ (1+|k|)^{\mu} |h_i(x,k)| < \infty \ ,$$
$$H_{i,4}(x,k) \leq C 2^{\mu-\nu} \sup_{x \geq 1} \int dk \ |k|^{\mu} \ |h_i(x,k)| < \infty \ .$$

This shows that for all $0 \leq \nu < \mu$, we have

$$\int dk \ (1+|k|)^{1+\nu} |\hat{q}_i(x,k)| < \infty \ ,$$
$$\int dk \ (1+|k|)^{1+\nu} |\hat{q}_{i,p}(x,k)| < \infty \ ,$$

so that q_i, $q_{i,p}$, $\partial_y q_i$ and $\partial_y q_{i,p}$ are bounded. To prove that they are also continuous in x, y, use the same arguments as in the proof of Lemma 3.2 to get that they are Hölder–ν in (x,y) for all $0 \leq \nu < \mu$, and hence, continuous. Since the proof is very similar, we omit the details. ■

C.5 Proof of Lemma 3.4

Lemma C.8 (Lemma 3.4) *Let* $\mathbf{v}_+ = (u_+, v_+) \in \mathcal{W}_{\alpha,\gamma_1,\gamma_2}$ *for some* $\alpha > 3$, $\gamma_1 \in [0,1)$ *and* $\gamma_2 \in [0, \frac{1}{2}]$ *with* $\gamma_2 \leq \gamma_1$. *Let* $\hat{u}_0$ *and* $\hat{v}_0$ *be defined by (3.19). There exists a constant* C *such that for all* $(x,k) \in \Omega_+$, *we have*

$$|\hat{u}_0(x,k)| \leq C \ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \ \frac{1}{(1+|k|\sqrt{x})^{\alpha}} \ ,$$
$$|\hat{v}_0(x,k)| \leq C \ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \ \frac{1}{\sqrt{x}} \frac{1}{(1+|k|\sqrt{x})^{\alpha}} \ ,$$
$$|D\hat{u}_0(x,k)| \leq C \ \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \ \frac{x^{\frac{1-\gamma_1}{2}}}{(1+|k|\sqrt{x})^{\alpha}} \left(\frac{1+|k|}{|k|}\right)^{\gamma_1} \ ,$$

$$|D\hat{v}_0(x,k)| \le C\, \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ \frac{1}{(1+|k|\sqrt{x})^{\alpha-1}} \frac{\sqrt{x}}{|kx|^{\gamma_2}+|kx|}\ ,$$

$$\left|\hat{u}_0(x,k) - e^{\Lambda_-(k)x} u_+(0)\right| \le C\, \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ \frac{x^{\frac{\gamma_1-1}{2}}}{(1+|k|\sqrt{x})^{\alpha}}\ .$$

Proof. By hypothesis, there exists a constant C such that for all $k \in \mathbf{R}$, we have

$$\begin{aligned}
|u_+(k)| &\le C\, \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \frac{1}{(1+|k|)^{\alpha}}\ ,\\
|v_+(k)| &\le C\, \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \frac{1}{(1+|k|)^{\alpha}}\ ,\\
|Du_+(k)| &\le C\, \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \left(\frac{1+|k|}{|k|}\right)^{\gamma_1} \frac{1}{(1+|k|)^{\alpha}}\ ,\\
|Dv_+(k)| &\le C\, \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \left(\frac{1+|k|}{|k|}\right)^{\gamma_2} \frac{1}{(1+|k|)^{\alpha}}\\
&\le C\, \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2} \left(\frac{1+|k|}{|k|}\right)^{\gamma_1} \frac{1}{(1+|k|)^{\alpha}}\ .
\end{aligned}$$

Furthermore, we have that $\lim_{k\to 0} v_+(k) = 0$. This holds because v_+ is odd, and $Dv_+(k)$ diverges only like a power law as $k \to 0$, so that $v_+(k) \sim \text{sign}(k)$ for small k is ruled out. By Lemma C.2, and since $\gamma_2 \le \gamma_1$, we have

$$\begin{aligned}
|v_+(k)| &= |v_+(k) - e^{\Lambda_-(k)} v_+(0)|\\
&\le C \left(\frac{|k|}{1+|k|}\right)^{1-\gamma_2} \frac{1}{(1+|k|)^{\alpha}} \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\\
&\le C \left(\frac{|k|}{1+|k|}\right)^{1-\gamma_1} \frac{1}{(1+|k|)^{\alpha}} \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ ,\\
|u_+(k) - e^{\Lambda_-(k)} u_+(0)| &\le C \left(\frac{|k|}{1+|k|}\right)^{1-\gamma_1} \frac{1}{(1+|k|)^{\alpha}} \|\mathbf{v}_+\|_{\alpha,\gamma_1,\gamma_2}\ .
\end{aligned}$$

Finally, we note that

$$P_{1,u}(x,k) - e^{\Lambda_-(x-1)} = -P_{2,u}(x,k)\ ,$$

so that

$$P_{1,u}(x,k)u_+(k) - e^{\Lambda_-(k)x}u_+(0) = -P_{2,u}(x,k)u_+(k) + e^{\Lambda_-(k)(x-1)}\left(u_+(k) - e^{\Lambda_-(k)}u_+(0)\right)\ .$$

The inequalities (3.28)–(3.32) follow easily by using these inequalities, Lemmas C.1 and C.3. We omit the details. ■

C.6 Proof of Proposition 3.3

In this section, we show that the expressions given in Proposition 3.3 solve the Fourier transform of (3.12)–(3.14), namely

$$\partial_x \begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix}(x,k) = \begin{pmatrix}0 & -ik\\ ik & 1\end{pmatrix}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix}(x,k) + \begin{pmatrix}0\\ \hat{\eta}_+(k)e^{-|k|(x-1)} + \hat{q}(x,k)\end{pmatrix}\ , \tag{C.10}$$

$$\lim_{x\to 1} \hat{\mathbf{u}}(x,k) \equiv \lim_{x\to 1}\begin{pmatrix}\hat{u}(x,k)\\ \hat{v}(x,k)\end{pmatrix} = \begin{pmatrix}\hat{u}_+(k)\\ \hat{v}_+(k)\end{pmatrix} \equiv \hat{\mathbf{v}}_+(k)\ . \tag{C.11}$$

Once we have shown this, the proof is completed by remarking that Lemmas 3.1 and 3.2 ensure that the right–hand sides of (2.12) and (2.13) are continuous in x, y, so that these equations are strongly satisfied. The proof relies mainly in Duhamel's formula. Although we could check directly that the integral expressions we gave are solution of (C.10), we prefer to give a constructive proof. As we noted in section 3.1, the linear operator

$$L(k) = \begin{pmatrix}0 & -ik\\ ik & 1\end{pmatrix}\ ,$$

has both unstable and center–stable eigenvalues $\Lambda_{\pm}$. We will show below that the expression we gave in Proposition 3.3 for η_+ is the good choice to satisfy the initial conditions for u and v. We begin with diagonalyzing the matrix $L(k)$. We have that

$$L(k) = S(k)O(k)S(k)^{-1} ,$$

where

$$S(k) = \begin{pmatrix} 1 & \frac{-ik}{\Lambda_+(k)} \\ \frac{-ik}{\Lambda_+(k)} & 1 \end{pmatrix} ,$$

$$O(k) = \begin{pmatrix} \Lambda_-(k) & 0 \\ 0 & \Lambda_+(k) \end{pmatrix} ,$$

$$S(k)^{-1} = \frac{\Lambda_+(k)}{\Delta(k)} \begin{pmatrix} 1 & \frac{ik}{\Lambda_+(k)} \\ \frac{ik}{\Lambda_+(k)} & 1 \end{pmatrix} .$$

The unstable modes corresponding to Λ_+ can be converted to stable modes by running their evolution backward in x. See for example [Gallay]. To do this, define new variables $(\hat{v}_s, \hat{v}_u)$ (the subscript u is for unstable and the subscript s for center–stable) by

$$\begin{pmatrix} \hat{v}_s \\ \hat{v}_u \end{pmatrix}(x,k) = S(k)^{-1} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}(x,k) . \tag{C.12}$$

For $(\hat{v}_s, \hat{v}_u)$, we get the system

$$\partial_x \begin{pmatrix} \hat{v}_s \\ \hat{v}_u \end{pmatrix}(x,k) = O(k) \begin{pmatrix} \hat{v}_s \\ \hat{v}_u \end{pmatrix}(x,k) + \begin{pmatrix} \hat{n}_s \\ \hat{n}_u \end{pmatrix}(x,k) , \tag{C.13}$$

with

$$\begin{pmatrix} \hat{n}_s \\ \hat{n}_u \end{pmatrix}(x,k) = S(k)^{-1} \begin{pmatrix} 0 \\ \hat{n}(x,k) \end{pmatrix} ,$$

where $\hat{n}(x,k) = \hat{\eta}_+(k)e^{-|k|(x-1)} + \hat{q}(x,k)$. Following [Gallay], we impose for the center-stable modes initial conditions at $x = 1$ and for the unstable modes at $x = \infty$. We set $\hat{v}_s(1,k) = \hat{\nu}_s(k)$ for some (even) function $\hat{\nu}_s$ to be chosen later on, and we set $\lim_{x\to\infty} \hat{v}_u(x,k) = 0$ (in order to satisfy the boundary conditions for $\mathbf{v}$ at infinity). From (C.12) and (C.13), we get the integral equation

$$\hat{v}_s(x,k) = e^{\Lambda_-(k)(x-1)}\hat{\nu}_s(k) + \frac{ik}{\Delta(k)} \int_1^x dx_0 \; e^{\Lambda_-(k)(x-x_0)} \hat{n}(x_0,k) ,$$

$$\hat{v}_u(x,k) = -\frac{\Lambda_+(k)}{\Delta(k)} \int_x^\infty dx_0 \; e^{\Lambda_+(k)(x-x_0)} \hat{n}(x_0,k) .$$

Since $\hat{n}(x,k) = \hat{\eta}_+(k)e^{-|k|(x-1)} + \hat{q}(x,k)$, we find that

$$\begin{aligned}\hat{v}_s(x,k) &= e^{\Lambda_-(k)(x-1)}\hat{\nu}_s(k) + \frac{ik}{\Delta(k)} \frac{1}{\Lambda_-(k)+|k|} \left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)} \right) \hat{\eta}_+(k) \\ &\quad + \frac{ik}{\Delta(k)} \int_1^x dx_0 \; e^{\Lambda_-(k)(x-x_0)} \hat{q}(x_0,k) , \\ \hat{v}_u(x,k) &= -\frac{\Lambda_+(k)}{\Delta(k)} \frac{1}{\Lambda_+(k)+|k|} e^{-|k|(x-1)} \hat{\eta}_+(k) - \frac{\Lambda_+(k)}{\Delta(k)} \int_x^\infty dx_0 \; e^{\Lambda_+(k)(x-x_0)} \hat{q}(x_0,k) .\end{aligned}$$

In matrix notation, we have

$$\begin{pmatrix} \hat{v}_s \\ \hat{v}_u \end{pmatrix}(x,k) = A(x,k) \begin{pmatrix} \hat{\nu}_s \\ \hat{\eta}_+ \end{pmatrix}(k) + b(x,k) , \tag{C.14}$$

where

$$A(x,k) = \begin{pmatrix} e^{\Lambda_-(k)(x-1)} & \frac{ik}{\Delta(k)} \frac{1}{\Lambda_-(k)+|k|} \left(e^{\Lambda_-(k)(x-1)} - e^{-|k|(x-1)} \right) \\ 0 & -\frac{\Lambda_+(k)}{\Delta(k)} \frac{1}{\Lambda_+(k)+|k|} e^{-|k|(x-1)} \end{pmatrix} ,$$

and

$$b(x,k) = \begin{pmatrix} \frac{ik}{\Delta(k)} \int_1^x dx_0 \; e^{\Lambda_-(k)(x-x_0)} \hat{q}(x_0,k) \\ -\frac{\Lambda_+(k)}{\Delta(k)} \int_x^\infty dx_0 \; e^{\Lambda_+(k)(x-x_0)} \hat{q}(x_0,k) \end{pmatrix} .$$

In order to complete the proof, we now have to eliminate from (C.12) $\hat{\eta}_+$ and $\hat{\nu}_s$ in favor of $\hat{u}_+$ and $\hat{v}_+$. Evaluating (C.14) at $x = 1$ gives

$$\begin{pmatrix} \hat{\nu}_- \\ \hat{\nu}_+ \end{pmatrix} (1,k) = A(1,k) \begin{pmatrix} \hat{\nu}_+ \\ \hat{\eta}_+ \end{pmatrix} (k) + b(1,k) .$$

From this, we get

$$\begin{aligned} \begin{pmatrix} \hat{\nu}_+ \\ \hat{\eta}_+ \end{pmatrix} (k) &= A(1,k)^{-1} \left[\begin{pmatrix} \hat{\nu}_- \\ \hat{\nu}_+ \end{pmatrix} (1,k) - b(1,k) \right] \\ &= A(1,k)^{-1} \left[S(k)^{-1} \begin{pmatrix} \hat{u}_+ \\ \hat{v}_+ \end{pmatrix} (k) - b(1,k) \right] , \end{aligned}$$

and therefore

$$\begin{aligned} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} (x,k) &= S(k) \begin{pmatrix} \hat{\nu}_- \\ \hat{\nu}_+ \end{pmatrix} (x,k) \\ &= S(k) \left(A(x,k) A(1,k)^{-1} \left[S(k)^{-1} \begin{pmatrix} \hat{u}_+ \\ \hat{v}_+ \end{pmatrix} (k) - b(1,k) \right] + b(x,k) \right) . \end{aligned}$$

The evaluation of the matrix products is not difficult but lengthy. The results are the equations as stated in the proposition.

C.7 Proof of Proposition 3.6

In this section, we show that the expressions given in Proposition 3.6 solve the Fourier transform of (3.15)–(3.17), namely,

$$\begin{aligned} &\partial_x \begin{pmatrix} \hat{p} \\ \hat{\eta} \end{pmatrix} (x,k) = \begin{pmatrix} 0 & -ik \\ ik & 0 \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{\eta} \end{pmatrix} (x,k) + \begin{pmatrix} -\hat{h}_1 \\ \hat{h}_2 \end{pmatrix} (x,k) , \\ &\lim_{x \to 1} \hat{\eta}(x,k) = 0 . \end{aligned} \tag{C.15}$$

Once we have shown this, we can use Lemma 3.5 to get that the right–hand sides of (2.14) and (2.15) are well defined and continuous in x, y, so that these equations are strongly satisfied. The proof that the integral expression solves (C.15) is very similar to the one of Proposition 3.3, but with $\Lambda_\pm$ replaced by $\pm|k|$. In particular, there are also stable and unstable modes. The proof relies on Duhamel's formula, and will be a constructive proof. The idea is to diagonalize in (C.15) the matrix

$$M(k) = \begin{pmatrix} 0 & -ik \\ ik & 0 \end{pmatrix} .$$

We have that $M = RDR^{-1}$ where

$$R(k)D(k)R^{-1}(k) \equiv \begin{pmatrix} 1 & 1 \\ i \ \mathrm{sign}(k) & -i \ \mathrm{sign}(k) \end{pmatrix} \begin{pmatrix} -|k| & 0 \\ 0 & |k| \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \ \mathrm{sign}(k) \\ \frac{1}{2} & \frac{i}{2} \ \mathrm{sign}(k) \end{pmatrix} .$$

We define the new variables $(\hat{p}_s, \hat{p}_u)$ by

$$\begin{pmatrix} \hat{p}_s \\ \hat{p}_u \end{pmatrix} (x,k) = R^{-1}(k) \begin{pmatrix} \hat{p} \\ \hat{\eta} \end{pmatrix} (x,k) .$$

For $(\hat{p}_s, \hat{p}_u)$, we get the equations

$$\partial_x \begin{pmatrix} \hat{p}_s \\ \hat{p}_u \end{pmatrix} (x,k) = D(k) \begin{pmatrix} \hat{p}_s \\ \hat{p}_u \end{pmatrix} (x,k) + \begin{pmatrix} \hat{h}_s \\ \hat{h}_u \end{pmatrix} (x,k) , \tag{C.16}$$

with

$$\begin{pmatrix}\hat{h}_s\\ \hat{h}_u\end{pmatrix}(x,k) = R^{-1}(k)\begin{pmatrix}-\hat{h}_1\\ \hat{h}_2\end{pmatrix}(x,k)\ . \tag{C.17}$$

From (C.16) and (C.17), we get the integral equations

$$\begin{aligned}\hat{p}_s(x,k) &= e^{-|k|(x-1)}\hat{\varrho}_s(k) + \int_1^x dx_0\ e^{-|k|(x-x_0)}\hat{h}_s(x_0,k)\ ,\\ \hat{p}_u(x,k) &= -\int_x^\infty dx_0\ e^{|k|(x-x_0)}\hat{h}_u(x,k)\ ,\end{aligned}$$

where $\hat{\varrho}_s(k)$ is a function to be determined. From $\hat{p}_s$ and $\hat{p}_u$ we compute $(\hat{p},\hat{\eta})$. We get

$$\begin{aligned}\hat{p}(x,k) &= e^{-|k|(x-1)}\hat{\varrho}_s(k) + \int_1^x dx_0\ e^{-|k|(x-x_0)}\hat{h}_s(x_0,k) - \int_x^\infty dx_0\ e^{|k|(x-x_0)}\hat{h}_u(x_0,k)\ ,\\ \hat{\eta}(x,k) &= i\ \text{sign}(k)\left(e^{-|k|(x-1)}\hat{\varrho}_s(k) + \int_1^x dx_0\ e^{-|k|(x-x_0)}\hat{h}_s(x_0,k) + \int_x^\infty dx_0\ e^{|k|(x-x_0)}\hat{h}_u(x_0,k)\right)\ .\end{aligned}$$

We evaluate $\hat{\eta}$ at $x = 1$ and impose that this gives 0. This gives an expression for $\hat{\varrho}_s(k)$ in terms of $\int_1^\infty dx_0\ e^{|k|(1-x_0)}\hat{h}_+(x_0,k)$. Inserting back this expression in the integral equations for $\hat{p}$ and $\hat{\eta}$ gives the desired result, we omit the details.

Appendix D: Proof of Lemma 3.8

In order to prove Lemma 3.8, we bound each of the quadratic terms separately. For concision, we reformulate Lemma 3.8 with the help of the auxiliary functions w_α and $\bar{w}_\alpha$:

Definition D.1 *For all $(x,k) \in \Omega_+$, we define*

$$\begin{aligned}w_\alpha(x,k) &= \frac{1}{(1+|k|\sqrt{x})^\alpha}\ ,\\ \overline{w}_\beta(x,k) &= \left(\frac{x^{1/2}}{1+|k|x}\right)^\beta\ .\end{aligned}$$

Proposition D.2 *Let h_1 and h_2 satisfy Condition 3.7. Then for all $\nu \in (1/2,1)$, there exists a constant C such that for all $(x,k) \in \Omega_+$, we have*

$$\begin{aligned}|\hat{q}_{1,p}(x,k)| + |\hat{q}_{2,p}(x,k)| &\le C\ H\ x^{-1/2}w_\alpha(x,k)\ ,\\ |\hat{q}_1(x,k)| + |\hat{q}_2(x,k)| &\le C\ H\ x^{-1/2}|k|\ w_{\alpha+1}(x,k) + C\ H\ x^{-1/2}|k|w_{\alpha-1}(x,k)\bar{w}_2(x,k)\ ,\\ |D\hat{q}_1(x,k)| + |D\hat{q}_2(x,k)| &\le C\ H\ x^{\frac{\gamma-1}{2}}w_\alpha(x,k) + C\ H\ x^{-1/2}w_{\alpha-1}(x,k)\bar{w}_2(x,k)\ ,\\ |D\hat{q}_{1,p}(x,k)| + |D\hat{q}_{2,p}(x,k)| &\le C\ H\ x^{\frac{\gamma-1}{2}}w_\alpha(x,k) + C\ H\ x^{-1/2}w_{\alpha-1}(x,k)\bar{w}_2(x,k)\\ &\quad + C\ H\ w_{\alpha-1}\ x^{1/2-\nu}\ |k|^{-\nu}\ .\end{aligned}$$

Proof. See the following subsections. ■

D.1 Preliminary Bounds

We begin with a preparatory lemma.

Lemma D.3 *Let $\lambda > 0$. There exists a constant C such that for all $k \in \mathbf{R}$ and x, $x_0 \ge 1$ we have*

$$e^{-\lambda|k||x-x_0|}w_\alpha(x_0,k) \le C\ w_\alpha(x,k)\ . \tag{D.1}$$

Proof. Using the definition of w_α we find that

$$e^{-\lambda|k||x-x_0|}\frac{w_\alpha(x_0,k)}{w_\alpha(x,k)} \le C + C\ e^{-\lambda|k||x-x_0|}\frac{|k\sqrt{x}|^\alpha}{(1+|k\sqrt{x_0}|)^\alpha}\ .$$

For $x_0 \geq x/2$ we have that

$$e^{-\lambda|k||x-x_0|}\frac{|k\sqrt{x}|^\alpha}{1+|k\sqrt{x_0}|^\alpha} \leq C\ \frac{|k\sqrt{x}|^\alpha}{1+|k\sqrt{x/2}|^\alpha} \leq C\ ,$$

which proves (D.1) for this case. For $x_0 < x/2$ we have that $|x - x_0| > x/2$, and therefore

$$e^{-\lambda|k||x-x_0|}\frac{|k\sqrt{x}|^\alpha}{1+|k\sqrt{x_0}|^\alpha} \leq e^{-\lambda|k|x/2}|kx|^\alpha \leq C\ .$$

This achieves the proof. ■

The two following functions enter the definitions of q_i and $q_{i,p}$:

$$g_\pm(k, x_0, x) = e^{-|k|\ |x-x_0|} \pm e^{-|k|\ (x+x_0-2)}\ .$$

The following lemma summarizes the properties of these functions using Lemma D.3.

Lemma D.4 *There exists a constant C such that for all $k \in \mathbf{R}$, $x, x_0 \geq 1$, we have*

$$\begin{aligned}
&0 \leq g_\pm(k, x_0, x)\ w_\alpha(x_0, k) \leq C\ e^{-|k/2||x-x_0|}\ w_\alpha(x, k)\ , && \text{(D.2)}\\
&0 \leq g_\pm(k, x_0, x)\ w_\alpha(x_0, k) \leq C\ g_\pm(k/2, x_0, x)\ w_\alpha(x, k)\ , && \text{(D.3)}\\
&|\partial_k g_\pm(k, x_0, x)|\, w_\alpha(x_0, k) \leq \frac{C}{|k|}e^{-|k/2||x-x_0|}\ w_\alpha(x, k)\ , && \text{(D.4)}\\
&|\partial_k g_-(k, x_0, x)|\, w_\alpha(x_0, k) \leq \frac{C}{|k|}\left(e^{-|k/2||x-x_0|} - e^{-|k/2|(x+x_0-2)}\right)\ w_\alpha(x, k)\\
&\qquad\qquad + ||x-x_0| - (x + x_0 - 2)|\, e^{-|k/4|(|x-x_0|+(x+x_0-2))} w_\alpha(x, k)\ , && \text{(D.5)}\\
&|\partial_k g_+(k, x_0, x)|\, w_\alpha(x_0, k) \leq C\ \max\{(x_0 - 1), (x - 1)\}\ e^{-|k/2||x-x_0|}\ w_\alpha(x, k)\ . && \text{(D.6)}
\end{aligned}$$

Proof. There exists a constant C such that for all $\sigma \in \{1, -1\}$ and $0 \leq a \leq b$, we have

$$\begin{aligned}
0 \leq e^{-a} + \sigma e^{-b} &\leq 2e^{-a} \leq 2e^{-a/2}e^{-a/2}\ , && \text{(D.7)}\\
0 \leq e^{-a} + \sigma e^{-b} &\leq (e^{-a/2} + e^{-b/2})(e^{-a/2} + \sigma e^{-b/2}) \leq 2(e^{-a/2} + \sigma e^{-b/2})e^{-a/2}\ , && \text{(D.8)}\\
ae^{-a} + be^{-b} &\leq (e^{-a/2} + e^{-b/2})(ae^{-a/2} + be^{-b/2})\\
&\leq (e^{-a/2} + e^{-b/2})(e^{-a/4} + e^{-b/4})(ae^{-a/4} + be^{-b/4})\\
&\leq C\ (e^{-a/2} + e^{-b/2})e^{-a/4}\ , && \text{(D.9)}\\
ae^{-a} - be^{-b} &= (e^{-a/2} - e^{-b/2})(ae^{-a/2} + be^{-b/2}) + (a - b)e^{-(a+b)/2}\\
&\leq (e^{-a/2} - e^{-b/2})(e^{-a/4} + e^{-b/4})(ae^{-a/4} + be^{-b/4}) + (a - b)e^{-(a+b)/2}\\
&\leq C\ (e^{-a/2} - e^{-b/2})e^{-a/4} + |a - b|e^{-(a+b)/2}\\
&\leq C\ \left(e^{-a/2} - e^{-b/2} + |a - b|e^{-(a+b)/4}\right)e^{-a/4}\ . && \text{(D.10)}
\end{aligned}$$

Since $x + x_0 - 2 \geq |x - x_0|$, for $x,\ x_0 \geq 1$, we can use these inequalities with $a = |k||x - x_0|$ and $b = |k|(x + x_0 - 2)$. Inequality (D.2) follows from (D.7) using (D.1) with $\lambda = 1/2$. Inequality (D.3) follows from (D.8) using (D.1) with $\lambda = 1/2$. Inequality (D.4) follows from (D.9) using (D.1) with $\lambda = 1/4$. Finally inequality (D.5) follows from (D.10), using (D.1) with $\lambda = 1/4$. ■

Looking back at the bounds on h_1, h_2 of Condition 3.7, and using the bounds of Lemma D.4, we see that we will have to bound integrals of the form

$$\int dx_0\ x_0^{-3/2}\ e^{-|k||x-x_0|}\ , \tag{D.11}$$

$$\int dx_0\ (x_0 - 1)\ x_0^{-3/2}\ e^{-|k||x-x_0|}\ , \tag{D.12}$$

$$\int dx_0\ x_0^{-3/2}\ g_-(k, x_0, x)\ , \tag{D.13}$$

either on $[1, x]$ or on $[x, \infty)$. The three following lemmas deal with these three types of integrals.

Lemma D.5 *There exists a constant C such that for all $(x,k)\in\Omega_+$, we have*

$$\int_x^\infty dx_0\ x_0^{-3/2}\ e^{-|k/2||x-x_0|} \le C\ x^{-1}\ \bar{w}_1(x,k)\ , \tag{D.14}$$

$$\int_1^x dx_0\ x_0^{-3/2}\ e^{-|k/2||x-x_0|} \le C\ \ x^{-1/2}\ (\bar{w}_2(x,k)+w_1(x,k))\ , \tag{D.15}$$

$$\int_1^\infty dx_0\ x_0^{-3/2}\ \ \min(x-1,x_0-1)e^{-|k/2|\max(x-1,x_0-1)} \le C\ x^{-1/2}\ \bar{w}_2(x,k)\ . \tag{D.16}$$

Proof. Define, for $t\ge 0$ the function I_1, I_2, I_3 and I_4 by the equations

$$\begin{aligned}
I_1(t) &= (1+t)\int_1^\infty ds\ s^{-3/2}\ e^{-t(s-1)}\ ,\\
I_2(t) &= (1+t)\int_{1/2}^1 ds\ s^{-3/2}\ e^{t(s-1)}\ ,\\
I_3(t) &= \frac{t}{t-1}\int_1^{(t+1)/2} dx_0\ x_0^{-3/2}\ ,\\
I_4(t) &= \left(\frac{t^2}{(t-1)^2\sqrt{t}}\int_1^t dx_0\ (x_0-1)x_0^{-3/2}\right)\ .
\end{aligned}$$

The functions I_1, I_2, I_3 and I_4 can be expressed in terms of elementary functions, in particular, they are bounded. We therefore have the bounds

$$\begin{aligned}
\int_x^\infty dx_0\ x_0^{-3/2}\ e^{-|k/2||x-x_0|} &= x^{-1/2}\int_1^\infty dx_0\ x_0^{-3/2}\ e^{-|k/2|x(x_0-1)} \le \frac{C\ x^{-1/2}}{1+|k|x}\ \sup_{x\ge 0} I_1(x)\ ,\\
\int_{(x+1)/2}^x dx_0\ x_0^{-3/2} e^{-|k/2||x-x_0|} &= x^{-1/2}\int_{\frac{x+1}{2x}}^1 dx_0\ x_0^{-3/2}\ e^{-|k/2|x(1-x_0)} \le \frac{C\ x^{-1/2}}{1+|k|x}\ \sup_{x\ge 0} I_2(x)\ ,\\
\int_1^{(x+1)/2} dx_0\ \frac{e^{-|k/2||x-x_0|}}{x_0^{3/2}} &\le e^{-|k/4|(x-1)}\frac{x-1}{x}\ \sup_{x\ge 1} I_3(x)\\
&\le \frac{C}{1+k^2x^2}\left((1+k^2x^2)\frac{x-1}{x}e^{-|k/4|(x-1)}\right)\\
&\le \frac{C}{1+k^2x^2}\left(1+k^2x(x-1)e^{-|k/4|(x-1)}\right)\\
&\le \frac{C}{1+k^2x^2}\left(1+\sup_{\xi\ge 0}(|k|\xi+\xi^2)e^{-\xi/4}\right)\\
&\le C\ \frac{1+|k|}{1+k^2x^2}\\
&\le C\ \left(x^{-1}\ \bar{w}_2(x,k)+x^{-1}\ \frac{1}{1+|k|x}\right)\ .
\end{aligned}$$

Inequality (D.14) follows at once by definition of $\bar{w}_1$, and inequality (D.15) follows since $(1+|k|x)^{-1}\le(1+|k|\sqrt{x})^{-1}$. Finally, to show (D.16), we consider separately the integrals over $[1,x]$ and $[x,\infty)$. On $[x,\infty)$, we use (D.14) to show that

$$\begin{aligned}
(x-1)\int_x^\infty dx_0\ x_0^{-3/2}\ e^{-|k/2|(x_0-1)} &\le (x-1)e^{-|k/4|(x-1)}\int_1^\infty dx_0\ x_0^{-3/2}e^{-|k/4|(x_0-1)}\\
&\le C\ \frac{x^{1/2}}{1+k^2x^2}(1+|k|x)\frac{x-1}{x}e^{-|k/4|(x-1)}\\
&\le C\ x^{-1/2}\ \bar{w}_2(x,k)\ ,
\end{aligned}$$

and on $[1,x]$, we have

$$e^{-|k/2|(x-1)}\int_1^x dx_0\ (x_0-1)\ x_0^{-3/2} = \frac{x^{1/2}}{1+k^2x^2}\left((1+k^2x^2)\frac{(x-1)^2}{x^2}e^{-|k/2|(x-1)}\right)I_4(x)$$

$$
\begin{aligned}
&\leq C\ \frac{x^{1/2}}{1+k^2x^2}\left(1+\sup_{\xi>0}\xi^2 e^{-\xi}\right)\sup_{x\geq 1} I_4(x) \\
&\leq C\ \frac{x^{1/2}}{1+k^2x^2} \leq C\ x^{-1/2}\ \bar{w}_2(x,k)\ .
\end{aligned}
$$

This achieves the proof. ■

Lemma D.6 *For all $\nu \in (1/2,1)$, there exists a constant C such that for all $(x,k)\in\Omega_+$, we have*

$$
\int_x^\infty dx_0\ e^{-|k/2||x-x_0|}\ (x_0-1)\ x_0^{-3/2} \leq C\ x^{1/2-\nu}\ |k|^{-\nu}\ . \tag{D.17}
$$

Proof. We have

$$
\begin{aligned}
\int_x^\infty dx_0\ e^{-|k/2||x-x_0|}\ (x_0-1)\ x_0^{-3/2} &\leq C\int_x^\infty dx_0\ e^{-|k/2||x-x_0|}\ x_0^{-1/2} \\
&\leq C\ |k|^{-\nu}\int_x^\infty dx_0\ \frac{|k(x-x_0)|^\nu e^{-|k/2||x-x_0|}}{x_0^{1/2}\ (x_0-x)^\nu} \\
&\leq C\ |k|^{-\nu}\ x^{1/2-\nu}\int_1^\infty ds\ s^{-1/2}(s-1)^{-\nu}\ .
\end{aligned}
$$

Since $\nu\in(1/2,1)$, the integral is convergent, and (D.17) follows. ■

Lemma D.7 *There exists a constant C such that for all $(x,k)\in\Omega_+$, we have*

$$
\int_1^\infty dx_0\ x_0^{-3/2} g_-(k/2,x_0,x) \leq C\ \ |k|\ x^{-1/2}\ \bar{w}_2(x,k)\ , \tag{D.18}
$$

$$
|k|\int_1^\infty dx_0\ x_0^{\frac{\gamma-1}{2}} g_-(k/2,x_0,x) \leq C\ x^{\frac{\gamma-1}{2}}\ . \tag{D.19}
$$

Proof. Define for $t\geq 0$ the function I_5, I_6 and I_7 by the equations

$$
\begin{aligned}
I_5(t) &= \frac{1+t^2}{t}e^{-t}\int_0^1 ds\ s^{-3/2}\left(e^{ts}-e^{-ts}\right)\ , \\
I_6(t) &= \frac{1+t^2}{t}\sinh(t)\int_1^\infty ds\ s^{-3/2}\ e^{-ts}\ , \\
I_7(x) &= \frac{x^{\frac{1-\gamma}{2}}}{x-1}\int_1^{(x+1)/2} dx_0\ x_0^{\frac{\gamma-1}{2}}\ .
\end{aligned}
$$

These functions can be expressed in terms of elementary functions, in particular, they are bounded. To prove (D.18), we split the integration in two parts. We first deal with $[x,\infty)$. Using (D.14), we have

$$
\begin{aligned}
\int_x^\infty dx_0\ x_0^{-3/2}\ g_-(k/2,x_0,x) &= e^{-|k/2|(x-1)}\sinh(|k/2|(x-1))\left(\int_x^\infty dx_0\ x_0^{-3/2}\ e^{-|k/2||x-x_0|}\right) \\
&\leq e^{-|k/2|(x-1)}\sinh(|k/2|(x-1))\ C\ x^{-1}\ \bar{w}_1(x,k) \\
&\leq C\frac{|k|}{1+|k|x}\bar{w}_1(x,k) \leq C|k|x^{-1/2}\bar{w}_2(x,k)\ .
\end{aligned}
$$

We now turn to $[1,x]$. We have

$$
\int_1^x dx_0\ \ x_0^{-3/2}\ g_-(k/2,x_0,x) \leq W_1(x,k)+W_2(x,k)\ ,
$$

where

$$
W_1(x,k) = e^{-|k/2|x}\int_1^x dx_0\ x_0^{-3/2}\left(e^{|k/2|x_0}-e^{-|k/2|x_0}\right) \leq C\ \frac{|k|x^{1/2}}{1+k^2x^2}I_5(kx/2)\ ,
$$

$$W_2(x,k) = 2e^{-|k/2|(x-1)} \sinh(|k/2|) \int_1^x dx_0\ x_0^{-3/2} e^{-|k/2|x_0} \leq C\ \frac{|k|}{1+k^2} e^{-|k/2|(x-1)} I_6(k/2)\ ,$$

so that we have

$$\int_1^x dx_0\ x_0^{-3/2}\ g_-(k/2,x_0,x) \leq C\ |k|\ x^{-1/2}\ \bar{w}_2(x,k)\ .$$

This completes the proof of (D.18). We now turn to (D.19). We split again the integral in two parts. We have

$$\begin{aligned}
|k| \int_{(x+1)/2}^{\infty} dx_0\ x_0^{\frac{\gamma-1}{2}}\ g_-(k/2,x_0,x) &\leq C\ x^{\frac{\gamma-1}{2}} \int_{-\infty}^{\infty} dx_0\ |k| e^{-|k/2||x-x_0|} \leq C\ x^{\frac{\gamma-1}{2}}\ , \\
|k| \int_1^{(x+1)/2} dx_0\ x_0^{\frac{\gamma-1}{2}}\ g_-(k/2,x_0,x) &\leq C\ |k|(x-1)e^{-|k/4|(x-1)}\ x^{\frac{\gamma-1}{2}}\ I_7(x) \leq C\ x^{\frac{\gamma-1}{2}}\ .
\end{aligned}$$

This completes the proof of the lemma. ■

D.2 Bound on $\hat{q}_1$ and $D\hat{q}_1$

Using (D.3) and (D.18), it is now an easy task to show that there exists a constant C such that for all $(x,k) \in \Omega_+$,

$$\begin{aligned}
|\hat{q}_1(x,k)| &\leq C\ H\ \left(\int_1^{\infty} dx_0\ x_0^{-3/2} g_-(k/2,x_0,x) \right) w_{\alpha-1}(x,k) \\
&\leq C\ H\ |k|\ x^{-1/2} w_{\alpha-1}(x,k) \bar{w}_2(x,k)\ .
\end{aligned}$$

Similarly, using (D.3), (D.5), (D.16), (D.18) and (D.19), we show that there exists a constant C such that for all $(x,k) \in \Omega_+$,

$$\begin{aligned}
|D\hat{q}_1(x,k)| \leq\ & C\ H\ \frac{w_{\alpha-1}(x,k)}{|k|} \int_1^{\infty} dx_0\ x_0^{-3/2}\ g_-(k/2,x_0,x) \\
&+ C\ H\ w_{\alpha}(x,k) \int_1^{\infty} dx_0\ x_0^{\frac{\gamma-1}{2}}\ |k| g_-(k/2,x_0,x) \\
&+ C\ H\ w_{\alpha-1}(x,k) \int_1^{\infty} dx_0\ x_0^{-3/2}\ \min(x-1,x_0-1) e^{-|k/2| \max(x-1,x_0-1)} \\
\leq\ & C\ H\ x^{\frac{\gamma-1}{2}}\ w_{\alpha}(x,k) + C\ H\ x^{-1/2}\ w_{\alpha-1}(x,k) \bar{w}_2(x,k)\ .
\end{aligned}$$

D.3 Bound on $\hat{q}_2$ and $D\hat{q}_2$

Using (D.2), (D.14) and (D.15), we show that there exists a constant C such that for all $(x,k) \in \Omega_+$,

$$\begin{aligned}
|\hat{q}_2(x,k)| &\leq C\ H\ \left(\int_1^{\infty} dx_0\ x_0^{-3/2}\ e^{-|k/2||x-x_0|} \right) |k| w_{\alpha}(x,k) \\
&\leq C\ H\ x^{-1/2}\ (\bar{w}_2(x,k) + w_1(x,k)) |k|\ w_{\alpha}(x,k) \\
&\leq C\ H\ x^{-1/2} |k| (w_{\alpha+1} + w_{\alpha} \bar{w}_2)(x,k)\ .
\end{aligned}$$

Similarly, using (D.3), (D.4), (D.14) and (D.15), we show that there exists a constant C such that for all $(x,k) \in \Omega_+$,

$$\begin{aligned}
|D\hat{q}_2(x,k)| &\leq C\ H\ \left(\int_1^{\infty} dx_0\ x_0^{-3/2}\ e^{-|k/2||x-x_0|} \right) w_{\alpha-1}(x,k) \\
&\leq C\ H\ x^{-1/2}\ (w_{\alpha} + w_{\alpha-1} \bar{w}_2)(x,k)\ ,
\end{aligned}$$

as claimed.

D.4 Bound on $\hat{q}_{1,p}$ and $D\hat{q}_{1,p}$

Using (D.3), (D.14) and (D.18), we show that there exists a constant C such that for all $(x,k) \in \Omega_+$,

$$\begin{aligned}
|\hat{q}_{1,p}(x,k)| &\le C\, w_{\alpha-1}(x,k) \left(\int_1^x dx_0\, x_0^{-3/2}\, g_-(k/2,x_0,x) + \int_x^\infty dx_0\, x_0^{-3/2} e^{-|k/2||x-x_0|} \right) \\
&\le C\, w_{\alpha-1}(x,k)\, x^{-1/2} \left(|k|\bar{w}_2(x,k) + w_1(x,k)\right) \le C\, x^{-1/2}\, w_\alpha(x,k)\ .
\end{aligned}$$

To bound $|D\hat{q}_{1,p}(x,k)|$, we split the integration in the definition of $\hat{q}_{1,p}$ in two parts, $[1,x]$ and $[x,\infty)$. On $[1,x]$, the integrand is the same as the one of $\hat{q}_1$, so that we can repeat the arguments of Subsection D.2. All we have to do is to bound the integral over $[x,\infty)$. For all $\nu \in (1/2,1)$, there exists a constant C such that for all $(x,k) \in \Omega_+$,

$$\begin{aligned}
|D\hat{q}_{1,p}(x,k)| &\le C\, H\, x^{\frac{\gamma-1}{2}}\, w_\alpha(x,k) + C\, H\, x^{-1/2}\, w_{\alpha-1}(x,k)\bar{w}_2(x,k) \\
&\quad + C\, H \int_x^\infty dx_0\, |\partial_k g_+(k,x_0,x)|\, w_{\alpha-1}(x_0,k)\, x_0^{-3/2} \\
&\quad + C\, H \int_x^\infty dx_0\, x_0^{\frac{\gamma-1}{2}}\, |k|\, e^{-|k||x-x_0|}\, w_\alpha(x_0,k) \\
&\le C\, H\, x^{\frac{\gamma-1}{2}}\, w_\alpha(x,k) + C\, H\, x^{-1/2}\, w_{\alpha-1}(x,k)\bar{w}_2(x,k) \\
&\quad + C\, H\, w_{\alpha-1}(x,k) \int_x^\infty dx_0\, e^{-|k/2||x-x_0|}(x_0-1)\, x_0^{-3/2} \\
&\le C\, H\, x^{\frac{\gamma-1}{2}}\, w_\alpha(x,k) + C\, H\, x^{-1/2}\, w_{\alpha-1}(x,k)\bar{w}_2(x,k) \\
&\quad + C\, H\, x^{1/2-\nu}\, |k|^{-\nu}\, w_{\alpha-1}(x,k)
\end{aligned}$$

as claimed.

D.5 Bound on $\hat{q}_{2,p}$ and $D\hat{q}_{2,p}$

Using (D.2), (D.14) and (D.15), we show that there exists a constant C such that for all $(x,k) \in \Omega_+$,

$$\begin{aligned}
|\hat{q}_{2,p}(k,t)| &\le C\, H\, |k| \int_1^\infty dx_0\, x_0^{-3/2}\ e^{-|k||x-x_0|} w_\alpha(x_0,k) \\
&\le w_\alpha(x,k)\, |k|\, x^{-1/2} \left(w_1(x,k) + \bar{w}_1(x,k)\right) \\
&\le C\, x^{-1}\, w_\alpha(x,k)\ .
\end{aligned}$$

Similarly, using (D.3), (D.4), (D.14) and (D.15), we show that there exists a constant C such that for all $(x,k) \in \Omega_+$,

$$\begin{aligned}
|D\hat{q}_{2,p}(x,k)| &\le C\, H\ \left(\int_1^\infty dx_0\, x_0^{-3/2}\, e^{-|k/2||x-x_0|} \right) w_\alpha(x,k) \\
&\le C\, H\, x^{-1/2}\, (w_\alpha + w_{\alpha-1}\bar{w}_2)(x,k)
\end{aligned}$$

as claimed.

Appendix E: Proof of Lemma 3.10

The proof of Lemma 3.10 relies on a certain number of preliminary estimates. They are given in the following section. Throughout this section, we will omit the argument 'k' in $\Lambda_\pm$ to simplify notation.

E.1 Preliminary Bounds

We begin by defining the following function

$$g_\gamma(k) = \left(\frac{|k|}{1+|k|} \right)^\gamma\ .$$

The following lemma summarizes some of the properties of the functions g_γ, $\Lambda_\pm$ and

$$w_\alpha(x,k) = \frac{1}{(1+|k|\sqrt{x})^\alpha} \ .$$

Lemma E.1 *Let $p \geq 1$, $\alpha \geq 1$, $0 \leq \gamma \leq 1$ and $\beta \geq 0$. There exists a constant C such that for all $x \geq x_0 \geq 1$, $k \in \mathbf{R}$, we have*

$$|k|^\gamma \leq |\Lambda_-|^{\gamma/2}(1+|\Lambda_-|)^{\gamma/2} \ , \tag{E.1}$$

$$|k|^\gamma \leq g_{\gamma/2}(\Lambda_-)(1+|\Lambda_-|)^\gamma \ , \tag{E.2}$$

$$x^{\gamma/2}|k|^\gamma \frac{x-1}{x} e^{\Lambda_-(x-1)/p} \leq C \ , \tag{E.3}$$

$$x^{\beta/2} g_\beta(k) e^{\Lambda_-(x-1)/p} \leq C \ , \tag{E.4}$$

$$\frac{w_\alpha(x_0,k)}{w_\alpha(x,k)} e^{\Lambda_-(x-x_0)/p} \leq C \ . \tag{E.5}$$

Proof. Inequalities (E.1) and (E.2) are elementary. For inequality (E.3), we use that $\frac{x-1}{x} \leq \left(\frac{x-1}{x}\right)^\gamma$ to get

$$|k|^\gamma \frac{x-1}{x} e^{\Lambda_-(x-1)/p} \leq \frac{1}{x^\gamma} \sup_{\xi\geq 0} \xi^{\gamma/2}(x-1+\xi)^{\gamma/2} e^{-\xi/p} \leq \frac{1}{x^{\gamma/2}} \ .$$

For inequality (E.4), we use that $g_\beta(k) \leq g_{\beta/2}(\Lambda_-) \leq 2$ to get

$$\begin{aligned} x^{\beta/2} g_\beta(k) e^{\Lambda_-(x-1)/p} &\leq x^{\beta/2} g_{\beta/2}(\Lambda_-) e^{-|\Lambda_-|(x-1)/p} \\ &\leq \left(g_1(\Lambda_-) + |\Lambda_-|(x-1)\right)^{\beta/2} e^{-|\Lambda_-|(x-1)/p} \\ &\leq \sup_{\xi \geq 0} \left(1+\xi\right)^{\beta/2} e^{-\xi/p} \leq C \ . \end{aligned}$$

Finally, for inequality (E.5), we consider first $x_0 \geq \frac{x+1}{2}$, then $\frac{w_\alpha(x_0,k)}{w_\alpha(x,k)} \leq \frac{w_\alpha(k,(x+1)/2)}{w_\alpha(x,k)} \leq C$. On the other hand, when $x_0 \leq \frac{x+1}{2}$, then

$$\frac{w_\alpha(x_0,k)}{w_\alpha(x,k)} e^{\Lambda_-(x-x_0)/p} \leq \frac{w_\alpha(1,k)}{w_\alpha(x,k)} e^{\Lambda_-(x-1)/(2p)} \leq 1 + x^{\alpha/2} g_\alpha(k) e^{\Lambda_-(x-1)/(2p)} \leq C \ .$$

This achieves the proof. ■

This lemma will help us estimate the derivatives.

Lemma E.2 *There exists a constant C such that for all $(x,k) \in \Omega_+$, we have*

$$\left| D\int_1^x dx_0 \ e^{\Lambda_-(x-x_0)} \hat{q}(x_0,k) \right| \leq C \ \frac{1+|k|}{|k|} \int_1^x dx_0 \ e^{\Lambda_-/2(x-x_0)} \Big(\frac{|\hat{q}(x_0,k)|}{1+|k|} + g_1(k)|D\hat{q}(x_0,k)| \Big) \ ,$$

$$\left| D\int_x^\infty dx_0 \ e^{\Lambda_+(x-x_0)} \hat{q}(x_0,k) \right| \leq C \ \int_x^\infty dx_0 \ e^{\Lambda_+/2(x-x_0)} \Big(|\hat{q}(x_0,k)| + |D\hat{q}(x_0,k)| \Big) \ .$$

Proof. We have

$$\begin{aligned} \left| D\int_1^x dx_0 \ e^{\Lambda_-(x-x_0)} \hat{q}(x_0,k) \right| &\leq \left| \int_1^x dx_0 \ e^{\Lambda_-(x-x_0)} \Big(\Lambda_-(x-x_0) \frac{2\Lambda_+}{k\Delta(k)} \hat{q}(x_0,k) + D\hat{q}(x_0,k) \Big) \right| \\ &\leq \frac{1}{|k|} \int_1^x dx_0 \ e^{\Lambda_-/2(x-x_0)} \left(\Big(\sup_{\xi\geq 0} \xi e^{-\xi} \Big) |\hat{q}(x_0,k)| + |k||D\hat{q}(x_0,k)| \right) \ , \\ \left| D\int_x^\infty dx_0 \ e^{\Lambda_+(x-x_0)} \hat{q}(x_0,k) \right| &\leq \int_x^\infty dx_0 \ e^{\Lambda_+(x-x_0)} \Big(\Lambda_+(x_0-x)|\hat{q}(x_0,k)| + |D\hat{q}(x_0,k)| \Big) \\ &\leq \int_x^\infty dx_0 \ e^{\Lambda_+(x-x_0)/2} \Big(\Big(\sup_{\xi\geq 0} \xi e^{-\xi} \Big) |\hat{q}(x_0,k)| + |D\hat{q}(x_0,k)| \Big) \ . \end{aligned}$$

This achieves the proof. ■

The following functions enter the definition of N_i.

Definition E.3 *For all* $(x,k) \in \Omega_+$, *we define the following functions.*

$$\hat{N}_{1,1}(x,k) = \frac{ik}{\Delta(k)} \int_1^x dx_0 \; e^{\Lambda_-(x-x_0)} \hat{q}(x_0,k) \;,$$
$$\hat{N}_{1,2}(x,k) = \frac{ik}{\Delta(k)} \int_x^\infty dx_0 \; e^{\Lambda_+(x-x_0)} \hat{q}(x_0,k) \;,$$
$$\hat{N}_{2,1}(x,k) = -\frac{\Lambda_-}{\Delta(k)} \int_1^x dx_0 \; e^{\Lambda_-(x-x_0)} \hat{q}(x_0,k) \;,$$
$$\hat{N}_{2,2}(x,k) = -\frac{\Lambda_+}{\Delta(k)} \int_x^\infty dx_0 \; e^{\Lambda_+(x-x_0)} \hat{q}(x_0,k) \;.$$

The following lemma summarizes properties of the derivatives of $\hat{N}_{i,j}$.

Proposition E.4 *There exists a constant* C *such that for all* $(x,k) \in \Omega_+$, *we have*

$$\left|D\hat{N}_{1,1}(x,k)\right| \leq C \int_1^x dx_0 \; e^{\Lambda_-/2(x-x_0)} \Big(|\hat{q}(x_0,k)| + g_1(k)|D\hat{q}(x_0,k)| \Big) \;,$$
$$\left|D\hat{N}_{2,1}(x,k)\right| \leq C \; g_1(k) \int_1^x dx_0 \; e^{\Lambda_-/2(x-x_0)} \Big(|\hat{q}(x_0,k)| + g_1(k)|D\hat{q}(x_0,k)| \Big) \;,$$
$$\left|D\hat{N}_{1,2}(x,k)\right| \leq C \int_x^\infty dx_0 \; e^{\Lambda_+/2(x-x_0)} \Big(|\hat{q}(x_0,k)| + g_1(k)|D\hat{q}(x_0,k)| \Big) \;,$$
$$\left|D\hat{N}_{2,2}(x,k)\right| \leq C \int_x^\infty dx_0 \; e^{\Lambda_+/2(x-x_0)} \Big(|\hat{q}(x_0,k)| + |D\hat{q}(x_0,k)| \Big) \;.$$

proof of prop. E.4. Use that $\left|D\frac{k}{\Delta(k)}\right| \leq \frac{1}{1+|k|^3}$, $\left|D\frac{\Lambda_\pm}{\Delta(k)}\right| \leq \frac{2|k|}{1+|k|^3}$, together with Lemma E.2. ∎

The following functions will help us estimate the functions $N_{i,j}$ and their derivatives with respect to k.

Definition E.5 *For* $(x,k) \in \Omega_+$, *we define the following functions*

$$\bar{w}_\alpha = \left(\frac{x^{1/2}}{1+|k|x} \right)^\alpha \;,$$
$$E_{\alpha,\beta,\gamma,\delta,p}(x,k) = g_\beta(k)|k|^\gamma \int_1^x dx_0 \; e^{\Lambda_-(x-x_0)/p} \; x_0^{-\delta} w_\alpha(x_0,k) \;,$$
$$F_{\alpha,\beta,p}(x,k) = g_\beta(k) \int_1^x dx_0 \; e^{\Lambda_-(x-x_0)/p} \; |k| \; x_0^{-1/2} \; \bar{w}_2(x_0,k) \; w_{\alpha-1}(x_0,k) \;,$$
$$G_{\alpha,\beta,p}(x,k) = \int_x^\infty dx_0 \; e^{\Lambda_+(x-x_0)/p} \; x_0^{-\beta} \; w_\alpha(x_0,k) \;,$$
$$H_{\alpha,\gamma,p}(x,k) = \frac{(|k|x)^\gamma}{\sqrt{x}} \int_x^\infty dx_0 \; e^{\Lambda_+(x-x_0)/p} \; x_0^{-1/2} \; \bar{w}_2(x_0,k) \; w_{\alpha-1}(x_0,k) \;.$$

We summarize the properties of these functions in the following proposition.

Proposition E.6 *Let* $\beta \geq 0$, $0 \leq \gamma \leq 1$, $0 \leq \delta \leq 1$, $\alpha \geq 0$, $\xi \in \mathbf{R}$ *and* $p \geq 1$ *with* $\beta+\gamma \geq 0$. *Define*

$$\tilde{\beta} = \begin{cases} \beta - \frac{1}{2} & \text{if} \quad \beta \geq \frac{1}{2} \\ \beta - 1 & \text{if} \quad \beta < \frac{1}{2} \end{cases} \;.$$

Then there exists a constant C *such that for all* $(x,k) \in \Omega_+$, *we have*

$$E_{\alpha,\beta,\gamma\delta,p}(x,k) \leq C \; w_\alpha(x,k) \; J_{\beta,\gamma,\delta}(x), \tag{E.6}$$
$$F_{\alpha,\beta,p}(x,k) \leq C \; w_\alpha(x,k) \; x^{-\min(1/2,\tilde{\beta}/2)} \;, \tag{E.7}$$
$$G_{\alpha,\xi,p}(x,k) \leq C w_\alpha(x,k) \; w_1(1,k) \; x^{-\xi} \;, \tag{E.8}$$
$$H_{\alpha,\gamma,p}(x,k) \leq C \; w_{\alpha-1}(x,k) \;, \tag{E.9}$$

where

$$J_{\beta,\gamma,\delta}(x) = x^{1-\delta-min(1,(\beta+\gamma)/2)} + \delta_{1,\delta} \; \ln(1+x) \; x^{-(\beta+\gamma)/2} \;.$$

proof of prop. E.6. We begin with the proof of (E.6). We have

$$\frac{E_{\alpha,\beta,\gamma,\delta,p}(x,k)}{w_\alpha(x,k)} \leq C\ T_1(x,k) + C\ T_2(x,k)\ ,$$

where

$$\begin{aligned} T_1(x,k) &= g_\beta(k)|k|^\gamma e^{\Lambda_-(x-1)/(2p+2)} \int_1^{(x+1)/2} dx_0\ x_0^{-\delta} \\ T_2(x,k) &= g_\beta(k)|k|^\gamma \frac{1}{x^\delta} \int_{(x+1)/2}^x dx_0\ e^{\Lambda_-(x-x_0)/(p+1)}\ . \end{aligned}$$

For T_1, we have

$$T_1(x,k) \leq C\ J_{\beta,\gamma,\delta}(x) \left(\sup_{x\geq 1} \frac{1}{x^{1-\delta} + \delta_{1,\delta} \ln(1+x)} \frac{x}{x-1} \int_1^x dx_0\ x_0^{-\delta} \right) \leq C\ J_{\beta,\gamma,\delta}(x)\ .$$

For T_2, since $\beta + \gamma \geq 0$, we have

$$\begin{aligned} T_2(x,k) &\leq \frac{C}{x^\delta} g_\beta(k)|k|^\gamma \frac{1}{|\Lambda_-|} \left(1 - e^{-|\Lambda_-|(x-1)}\right) \\ &\leq \frac{C}{x^{\delta-1}} g_{\beta/2}(\Lambda_-) \frac{|k|^\gamma}{1+|\Lambda_-|t} \\ &\leq \frac{C}{x^{\delta-1}} g_{(\beta+\gamma)/2}(\Lambda_-) \frac{1+|\Lambda_-|^\gamma}{1+|\Lambda_-|t} \\ &\leq C\ J_{\beta,\gamma,\delta}(x)\ \left(\frac{|\Lambda_-|t}{1+|\Lambda_-|}\right)^{\min(1,(\beta+\gamma)/2)} \frac{1+|\Lambda_-|^\gamma}{1+|\Lambda_-|t} \\ &\leq C\ J_{\beta,\gamma,\delta}(x)\ \left(\frac{|\Lambda_-|t}{1+|\Lambda_-|t}\right)^{\min(1,(\beta+\gamma)/2)} \frac{1+|\Lambda_-|^\gamma}{1+|\Lambda_-|} \\ &\leq C\ J_{\beta,\gamma,\delta}(x)\ . \end{aligned}$$

For inequality (E.7), we have

$$\frac{F_{\alpha,\beta,p}(x,k)}{w_{\alpha-1}(x,k)} \leq C\ T_3(x,k) + C\ T_4(x,k)\ ,$$

where

$$\begin{aligned} T_3(x,k) &= g_\beta(k) e^{\Lambda_-(x-1)/(2p)} \int_1^{(x+1)/2} dx_0\ \frac{|k|\sqrt{x_0}}{1+k^2 x_0^2}\ , \\ T_4(x,k) &= g_\beta(k) \frac{|k|\sqrt{x}}{1+k^2x^2} \int_{(x+1)/2}^x dx_0\ e^{\Lambda_-(x-x_0)}\ . \end{aligned}$$

Now take $\beta < 1/2$, then

$$T_3(x,k) \leq C\ x^{-\beta/2} \frac{e^{\Lambda_-(x-1)/(3p)}}{1+|k|} \int_1^{(x+1)/2} dx_0\ x_0^{-1/2} \leq C\ x^{(1-\beta)/2}\ w_1(x,k)\ ,$$

whereas for $\beta \geq 1/2$,

$$\begin{aligned} T_3(x,k) &\leq g_{\beta-1/2}(k)\ \frac{e^{\Lambda_-(x-1)/(2p)}}{1+|k|}\ \sup_{k\geq 0} \sqrt{1+|k|} \int_0^\infty d\xi\ \frac{\sqrt{\xi+|k|}}{1+(\xi+|k|)^2} \\ &\leq C\ x^{1/4-\beta/2}\ w_1(x,k)\ . \end{aligned}$$

Combining these inequalities gives

$$T_3(x,k) \leq C\ x^{-\min(1/2,\tilde{\beta}/2)}\ w_1(x,k)\ .$$

For T_4, one has

$$T_4(x,k) \leq Cg_\beta(k)\frac{|k|\ x^{1/2}\ (x-1)}{1+k^2x^2}\frac{1-e^{\Lambda_-(x-1)/4p}}{|\Lambda_-|(x-1)} \leq Cg_\beta(k)\frac{|k|\ x^{1/2}}{1+k^2x^2}\frac{x-1}{1+|\Lambda_-|(x-1)}\ ,$$

and now, we use that

$$\begin{aligned} g_\beta(k)\frac{|k|\ x^{1/2}}{1+k^2x^2}\frac{x-1}{1+|\Lambda_-|(x-1)} &\leq g_\beta(k)\frac{|k|x^{3/2}}{1+k^2x^2}\frac{1}{1+|k|\sqrt{x}} \\ &\leq x^{-\min(1/2,\beta-1/2)}\left(\frac{|k|x}{1+|k|}\right)^{\min(\beta,1)}\frac{1}{1+|k|x}w_1(x,k) \\ &\leq x^{-\min(1/2,\beta-1/2)}w_1(x,k) \leq x^{-\min(1/2,\tilde{\beta}/2)}w_1(x,k)\ . \end{aligned}$$

For the inequality (E.8), since $x_0 \geq x$, we have

$$x^\xi G_{\alpha,\xi,p}(x,k) \leq w_\alpha(x,k)\int_x^\infty dx_0\ e^{\Lambda_+(x-x_0)}\frac{x^\xi}{x_0^\xi}\ .$$

By a change of variable, we have

$$\begin{aligned} \int_x^\infty dx_0\ e^{\Lambda_+(x-x_0)}\frac{x^\xi}{x_0^\xi} &\leq \frac{1}{\Lambda_+}\int_0^\infty dy\ \left(1+\frac{y}{\Lambda_+ x}\right)^{-\xi} e^{-y} \\ &\leq \frac{1}{\Lambda_+}\max\left\{1, \int_0^\infty dy\ (1+y)^{-\xi}\ e^{-y}\right\} \leq C\ w_1(1,k)\ . \end{aligned}$$

We conclude the proof with inequality (E.7). We have

$$\begin{aligned} H_{\alpha,\gamma,p}(x,k) &\leq w_{\alpha-1}(x,k)\frac{(|k|x)^\gamma}{1+k^2x^2}\frac{1}{\sqrt{x}}\int_x^\infty dx_0\ e^{\Lambda_+(x-x_0)}\sqrt{x_0} \\ &\leq w_{\alpha-1}(x,k)\left(x^{-1/2}G_{0,-1/2,1}(x,k)\right) \leq w_{\alpha-1}(x,k)\ . \end{aligned}$$

This achieves the proof. ■
Finally, this lemma completes information we gave in Section C.3.

Lemma E.7 *Let*

$$\begin{aligned} P_1(x,k) &= -(\Lambda_+(k)+|k|)\left[\frac{ik}{\Delta(k)(\Lambda_-(k)+|k|)}e^{\Lambda_-(k)(x-1)} - i\ \mathrm{sign}(k)e^{-|k|(x-1)}\right]\ , \\ P_2(x,k) &= (\Lambda_+(k)+|k|)\left[\frac{\Lambda_-(k)}{\Delta(k)(\Lambda_-(k)+|k|)}e^{\Lambda_-(k)(x-1)} + e^{-|k|(x-1)}\right]\ . \end{aligned}$$

Then there exists a constant C such that for all $(x,k) \in \Omega_+$, we have

$$\begin{aligned} |P_1(x,k)| &\leq C\ e^{\Lambda_-(k)(x-1)/2}\ , \\ |P_2(x,k)| &\leq C\ e^{\Lambda_-(k)(x-1)/2}\ , \\ |DP_1(x,k)| &\leq C\ \sqrt{x}e^{\Lambda_-(k)(x-1)/2} + C\ xe^{-|k|/2(x-1)}\ , \\ |DP_2(x,k)| &\leq C\ e^{\Lambda_-(k)(x-1)/2} + C\ xe^{-|k|/2(x-1)}\ . \end{aligned}$$

Proof. This is elementary. ■

E.2 Proof of Lemma 3.10

Using Definition E.3, we have $|\hat{N}_i| \leq |\hat{N}_{i,1}| + |\hat{N}_{i,2}| + |P_i\tau|$. The following proposition bounds the contribution of $\hat{N}_{i,j}$ to $\hat{N}_i$.

Proposition E.8 *For all $\gamma_2 \in [0,1)$, there exists a constant C such that for all $(x,k) \in \Omega_+$, we have*

$$\begin{aligned}
\left|\hat{N}_{1,1}(x,k)\right| &\leq C\ Q\ E_{\alpha,1,1,(1-\gamma)/2,2}(x,k) + C\ Q\ F_{\alpha,1,2}(x,k)\ , \\
\left|D\hat{N}_{1,1}(x,k)\right| &\leq C\ Q\ g_{-\gamma}(k)\left(E_{\alpha,\gamma,1,(1-\gamma)/2,2}(x,k) + F_{\alpha,\gamma,2}(x,k)\right)\ , \\
\left|\hat{N}_{2,1}(x,k)\right| &\leq C\ Q\ E_{\alpha,2,0,1,2}(x,k) + C\ Q\ F_{\alpha,2,2}(x,k)\ , \\
\left|D\hat{N}_{2,1}(x,k)\right| &\leq C\ Q\ E_{\alpha,1,1,(1-\gamma)/2,2}(x,k) + C\ Q\ F_{\alpha,1,2}(x,k)\ , \\
\left|\hat{N}_{1,2}(x,k)\right| &\leq C\ Q\ G_{\alpha,1/2,1}(x,k)\ , \\
\left|D\hat{N}_{1,2}(x,k)\right| &\leq C\ Q\ g_{-\gamma}(k) G_{\alpha,(1-\gamma)/2,1}(x,k)\ , \\
\left|\hat{N}_{2,2}(x,k)\right| &\leq C\ Q\ G_{\alpha,1/2,1}(x,k)\ , \\
\left|D\hat{N}_{2,2}(x,k)\right| &\leq C\ Q\ G_{\alpha,(1-\gamma)/2,1}(x,k) + C\ Q\ \frac{\sqrt{x}}{|kx|^{\gamma_2} + |kx|}\left(H_{\alpha,\gamma_2,1}(x,k) + H_{\alpha,1,1}(x,k)\right)\ .
\end{aligned}$$

Proof. From Condition 3.9, there exists a constant Q such that for all $(x,k) \in \Omega_+$, we have

$$\begin{aligned}
|\hat{q}(x,k)| &\leq Q\ x^{-1}\ w_\alpha(x,k) + Q\ |k|\ x^{1/2}\ \bar{w}_2(x,k) w_\alpha(x,k)\ , \\
|\hat{q}(x,k)| &\leq Q\ x^{-1/2}\ w_\alpha(x,k)\ , \\
|\hat{q}(x,k)| &\leq Q\ |k|\ x^{(\gamma-1)/2}\ w_{\alpha+1}(x,k) + Q\ |k|\ x^{-1/2}\ \bar{w}_2(x,k) w_{\alpha-1}(x,k)\ , \\
|D\hat{q}(x,k)| &\leq Q\ x^{(\gamma-1)/2}\ w_\alpha(x,k) + Q\ x^{-1/2} \bar{w}_2(x,k) w_{\alpha-1}(x,k)\ , \\
|\hat{q}(x,k)| + g_1(k)|D\hat{q}(x,k)| &\leq Q\ |k|\ x^{(\gamma-1)/2}\ w_\alpha(x,k) + Q\ |k|\ x^{-1/2}\ \bar{w}_2(x,k) w_{\alpha-1}(x,k)\ , \\
|\hat{q}(x,k)| + g_1(k)|D\hat{q}(x,k)| &\leq Q\ x^{(\gamma-1)/2}\ w_\alpha(x,k)\ , \\
|\hat{q}(x,k)| + |D\hat{q}(x,k)| &\leq Q\ x^{(\gamma-1)/2} w_\alpha(x,k) + Q\ x^{-1/2}\ \bar{w}_2(x,k) w_{\alpha-1}(x,k)\ .
\end{aligned}$$

To achieve the proof, replace E, F, G and H by their definitions in the estimations of the proposition, and check (use Proposition E.4 for the derivatives) that the integrand appears in the list of estimations on $\hat{q}$ above. We omit the details. ■

We conclude the proof with the contribution of $P_i\ \tau$ to $\hat{N}_i$. This lemma provides bounds on τ.

Lemma E.9 *There exists a constant C such that for all $k \in \mathbf{R}$,*

$$\begin{aligned}
|\tau(k)| &\leq C\ Q\ |k|\ w_{\alpha+2}(1,k)\ , \\
|D\tau(k)| &\leq C\ Q\ w_{\alpha+1}(1,k)\ .
\end{aligned}$$

Proof. The following bounds follow from Condition 3.9

$$\begin{aligned}
|\hat{q}(x,k)| &\leq Q\ |k|\ \sqrt{x}\ w_{\alpha+1}(1,k)\ , \\
|\hat{q}(x,k)| + |D\hat{q}(x,k)| &\leq Q\ \sqrt{x}\ w_\alpha(1,k)\ .
\end{aligned}$$

By the definition of $G_{\alpha,\beta,\gamma}$ and Lemma (E.2), it follows that

$$\begin{aligned}
|\tau(k)| &\leq C\ Q\ |k|\ G_{\alpha+1,-1/2,1}(1,k) \leq C\ Q\ |k|\ w_{\alpha+2}(1,k)\ , \\
|D\tau(k)| &\leq C\ Q\ G_{\alpha,-1/2,1}(1,k) \leq C\ Q\ w_{\alpha+1}(1,k)\ .
\end{aligned}$$

This achieves the proof. ■

Remark E.10 *It is now an easy task to check that the functions $P_i\ \tau$, $\hat{N}_{i,1}$ and $\hat{N}_{i,2}$ satisfy the estimations of Lemma 3.10 for $\hat{N}_i$. This completes the proof of Lemma 3.10.*

Appendix F: Bounds on Nonlinear Terms

The proof of Proposition 4.4 relies on some preliminary estimates. They are given in the next subsection.

F.1 Preliminary Bounds

The following lemma will help us estimate the convolutions appearing in $q_0(\mathbf{v})$, $h_1(\mathbf{v})$ and $h_2(\mathbf{v})$.

Lemma F.1 *Let $\alpha \geq 0$, $0 \leq \gamma < 1$, $k, \xi \in \mathbf{R}$ and $\lambda \geq 0$. Define the function $E_{\alpha,\gamma}$ by*

$$E_{\alpha,\gamma}(k,\xi,\lambda) = \int dk' \, \frac{e^{-\lambda|k'|/2}}{(1+|k-k'|)^{\alpha}} \frac{1}{|\xi-k'|^{\gamma}} \, . \tag{F.1}$$

Then there exists a constant C such that for all $(x,k) \in \Omega_+$, $\lambda \geq 0$ and for all $\xi \in \mathbf{R}$,

$$\sup_{\xi \in \mathbf{R}} E_{\alpha,\gamma}(k\sqrt{x},\xi,\lambda) \leq C \, w_{\alpha}(x,k) \lambda^{\gamma-1} \max\left\{1, \frac{1}{\lambda^{\alpha-1+\gamma}}\right\} \, , \tag{F.2}$$

where $w_{\alpha}(x,k) = (1+|k|\sqrt{x})^{-\alpha}$.

Proof. We observe first that

$$\begin{aligned}\sup_{\xi \in \mathbf{R}} \int dk' \, \frac{e^{-\lambda|k'|/2}}{|\xi-k'|^{\gamma}} &\leq \lambda^{\gamma-1} \sup_{z \in \mathbf{R}} \int dk' \, \frac{e^{-|k'|/2}}{|z-k'|^{\gamma}} \\ &\leq C \, \lambda^{\gamma-1} \, .\end{aligned}$$

Next, we split the integral in the definition of $E_{\alpha,\gamma}$ in two parts. For the integral over $(-\infty, k\sqrt{x}/2]$ we have

$$\begin{aligned}\sup_{\xi \in \mathbf{R}} \int_{-\infty}^{k\sqrt{x}/2} dk' \, \frac{e^{-\lambda|k'|/2}}{(1+|k\sqrt{x}-k'|)^{\alpha}} \frac{1}{|\xi-k'|^{\gamma}} &\leq \frac{1}{(1+|k|\sqrt{x}/2)^{\alpha}} \sup_{\xi \in \mathbf{R}} \int_{-\infty}^{\infty} dk' \, \frac{e^{-\lambda|k'|/4}}{|\xi-k'|^{\gamma}} \\ &\leq \lambda^{\gamma-1} C \, w_{\alpha}(x,k) \, .\end{aligned}$$

For the integral over $(k\sqrt{x}/2, \infty)$ we consider separately $k\sqrt{x} < 1$ and $k\sqrt{x} \geq 1$. If $k\sqrt{x} < 1$ then

$$\begin{aligned}\sup_{\xi \in \mathbf{R}} \int_{k\sqrt{x}/2}^{\infty} dk' \, \frac{e^{-\lambda|k'/2|}}{(1+|k\sqrt{x}-k'|)^{\alpha}} \frac{1}{|\xi-k'|^{\gamma}} &\leq \sup_{\xi \in \mathbf{R}} \int_{k\sqrt{x}/2}^{\infty} dk' \, \frac{e^{-\lambda|k'|/2}}{|\xi-k'|^{\gamma}} \\ &\leq C \, \lambda^{\gamma-1} \leq C \, \lambda^{\gamma-1} \, w_{\alpha}(x,k) \, ,\end{aligned}$$

which proves (F.2) for this case. If $k\sqrt{x} \geq 1$, we have

$$\begin{aligned}\sup_{\xi \in \mathbf{R}} \int_{k\sqrt{x}/2}^{\infty} dk' \, \frac{e^{-\lambda|k'/2|}}{(1+|k\sqrt{x}-k'|)^{\alpha}} \frac{1}{|\xi-k'|^{\gamma}} &\leq e^{-\lambda k\sqrt{x}/4} \sup_{\xi \in \mathbf{R}} \int_{-\infty}^{\infty} dk' \, \frac{1}{(1+|k\sqrt{x}-k'|)^{\alpha}} \frac{1}{|\xi-k'|^{\gamma}} \\ &\leq C \, e^{-\lambda k\sqrt{x}/4} \\ &\leq C w_{\alpha}(x,k) \sup_{k\sqrt{x} \geq 1} \left(e^{-\lambda k\sqrt{x}/4} (\lambda k\sqrt{x})^{\alpha}\right) \sup_{k\sqrt{x} \geq 1} \left(\frac{1+k\sqrt{x}}{\lambda k\sqrt{x}}\right)^{\alpha} \\ &\leq C w_{\alpha}(x,k) \frac{1}{\lambda^{\alpha}} \sup_{\xi \geq 1} \left(e^{-\xi/4} \xi^{\alpha}\right) \sup_{\xi \geq 1} \left(\frac{1+\xi}{\xi}\right)^{\alpha} \\ &\leq C \, w_{\alpha}(x,k) \lambda^{\gamma-1} \max\left\{1, \frac{1}{\lambda^{\alpha-1+\gamma}}\right\} \, .\end{aligned}$$

This completes the proof of (F.2). ■

The following proposition provides bounds for convolution products and uses the preceding lemma.

Proposition F.2 *Let α,$\alpha' > 1$, $\delta > -1$, $0 \leq \gamma < 1$. There exists a constant C such that for all $(x,k) \in \Omega_+$, we have the bounds*

$$((g_{-\gamma} w_{\alpha'}) * w_{\alpha})(x,k) \leq C \, x^{\frac{\gamma-1}{2}} \, w_{\min\{\alpha,\alpha'\}}(x,k) \, , \tag{F.3}$$

$$((g_{-\gamma} w_{\alpha}) * e_{0,\delta})(x,k) \leq C \, x^{\gamma-1} \, w_{\alpha}(x,k) \, , \tag{F.4}$$

$$(w_{\alpha} * e_{-\gamma,\delta})(x,k) \leq C \, x^{\gamma-1} \, w_{\alpha}(x,k) \, , \tag{F.5}$$

where

$$w_\alpha(x,k) = \frac{1}{(1+|k|\sqrt{x})^\alpha} ,$$
$$g_\gamma(k) = \left(\frac{|k|}{1+|k|}\right)^\gamma ,$$
$$e_{-\gamma,\delta}(x,k) = |k|^{-\gamma}\, e^{-|k|(x+\delta)/2} .$$

Proof. w_α, and $e_{\gamma,\delta}$ are even functions of k, and we can therefore assume without loss of generality that $k \geq 0$. A change of variables shows that (F.3) follows from

$$\int_{-\infty}^{\infty} dk' \, \frac{x^{-\gamma/2}+|k'|^{-\gamma}}{(1+|k-k'|)^\alpha \,(1+|k'|)^{\alpha'}} \leq \frac{C}{1+|k|^{\min\{\alpha,\alpha'\}}} . \tag{F.6}$$

Indeed, we have

$$\int_{-\infty}^{\infty} dk' \, \frac{x^{-\gamma/2}+|k'|^{-\gamma}}{(1+|k-k'|)^\alpha \,(1+|k'|)^{\alpha'}} \leq \int_{-\infty}^{\infty} dk' \, \frac{1+|k'|^{-\gamma}}{(1+|k-k'|)^\alpha \,(1+|k'|)^{\alpha'}} .$$

We split the integrals in two parts. For the integral over $(-\infty, k/2)$ we have

$$\int_{-\infty}^{k/2} dk' \, \frac{1+|k'|^{-\gamma}}{(1+|k-k'|)^\alpha \,(1+|k'|)^{\alpha'}} \leq \frac{1}{(1+k/2)^\alpha} \int_{-\infty}^{\infty} dk' \, \frac{1+|k'|^{-\gamma}}{(1+|k'|)^{\alpha'}} \leq \frac{C}{1+k^\alpha} ,$$

and for the integral over $(k/2, \infty)$ we have

$$\int_{k/2}^{\infty} dk' \, \frac{1+|k'|^{-\gamma}}{(1+|k-k'|)^\alpha \,(1+|k'|)^{\alpha'}} \leq \frac{1}{(1+k/2)^{\alpha'}} \int_{-\infty}^{\infty} dk' \, \frac{1+|k'|^{-\gamma}}{1+|k-k'|^\alpha} \leq \frac{C}{1+k^{\alpha'}} ,$$

ant the bound (F.6) follows. For inequality (F.5), by a change of variables, we have from (F.1)

$$\left(w_\alpha * e_{-\gamma,\delta}\right)(x,k) \leq x^{\frac{\gamma-1}{2}} E_{\alpha,\gamma}(k\sqrt{x}, 0, \sqrt{x}(1+\delta/x)) .$$

Therefore, by Lemma F.1,

$$\begin{aligned}
\left(w_\alpha * e_{-\gamma,\delta}\right)(x,k) &\leq C\, x^{\gamma-1} w_\alpha(x,k) \frac{1}{(1+\delta/x)^{1-\gamma}} \max\left\{1, \frac{1}{(\sqrt{x})^{\alpha-1+\gamma}(1+\delta/x)^{\alpha-1+\gamma}}\right\} \\
&\leq C\, x^{\gamma-1}\, w_\alpha(x,k)\, \frac{1}{(1+\delta)^{1-\gamma}} \max\left\{1, \frac{1}{(1+\delta)^{\alpha-1+\gamma}}\right\} , \\
&\leq C\, x^{\gamma-1}\, w_\alpha(x,k) ,
\end{aligned}$$

as claimed. For inequality (F.4) we use $g_{-\gamma}(k) \leq 1+|k|^{-\gamma}$ and find

$$\left((g_{-\gamma} w_\alpha) * e_{0,\delta}\right)(x,k) \leq x^{-1/2} E_{\alpha,0}(k\sqrt{x}, 0, \sqrt{x}(1+\delta/x) + x^{\frac{\gamma-1}{2}} E_{\alpha,\gamma}(k\sqrt{x}, k\sqrt{x}, \sqrt{x}(1+\delta/x)) .$$

Therefore by Lemma F.1

$$\begin{aligned}
\left((g_{-\gamma} w_\alpha) * e_{0,\delta}\right)(x,k) &\leq C\, x^{\gamma-1} w_\alpha(x,k) \max\left\{1, \frac{1}{(1+\delta)^\alpha}\right\} , \\
&\leq C\, x^{\gamma-1}\, w_\alpha(x,k) ,
\end{aligned}$$

as claimed. ■

The following lemma is very similar to Lemma C.2.

Lemma F.3 *Let* $h : \Omega_+ \to \mathbf{R}$. *Assume that there exist constants* H *and* $\alpha \geq 0$ *such that for all* $(x,k) \in \Omega_+$,

$$|h(x,k)| \leq H\, w_\alpha(x,k) ,$$
$$|Dh(x,k)| \leq H\, \sqrt{x}\, w_\alpha(x,k) .$$

Then there exists a constant C such that for all $(x,k) \in \Omega_+$, we have

$$\Big|h(x,k) - e^{-k^2 x} h(x,0)\Big| \le C\, H\, |k|\sqrt{x}\, w_{\alpha+1}(x,k)\;, \tag{F.7}$$

in particular, if h is odd in k, we have

$$|h(k)| \le C\, H\, |k|\sqrt{x}\, w_{\alpha+1}(x,k)\;. \tag{F.8}$$

Proof. As is easily seen, (F.8) for an odd function follows from (F.7), because an odd function with bounded derivative vanishes at $k=0$. To prove (F.7), we first make the following observations. There exists a constant C such that for all $(x,k) \in \Omega_+$, $0 \le |\xi| \le |k|$,

$$\begin{aligned} \Big|\partial_\xi e^{-(k-\xi)^2 x}\Big| &\le 2|k-\xi|\; x\, e^{-(k-\xi)^2 x} \le C\,\sqrt{x}\, e^{-(k-\xi)^2 x/2}\;, \\ \frac{(1+|k|\sqrt{x})^\alpha}{(1+|\xi|\sqrt{x})^\alpha} e^{-(k-\xi)^2 x/2} &\le C\, e^{-(k-\xi)^2 x/4}\;. \end{aligned}$$

Using this gives

$$\begin{aligned} \Big|h(x,k) - e^{-k^2\,x} h(x,0)\Big| &\le \left|\int_0^k d\xi\; \partial_\xi \left(e^{-(k-\xi)^2 x} h(x,\xi)\right)\right| \\ &\le C\, H\, \sqrt{x} \int_0^{|k|} d\xi\; \frac{e^{-(|k|-\xi)^2 x/2}}{(1+|\xi|\sqrt{x})^\alpha} \\ &\le C\, H\, \sqrt{x}\, w_\alpha(x,k) \int_0^{|k|} d\xi\; e^{-(|k|-\xi)^2 x/4} \\ &\le C\, H\, w_\alpha(x,k) \int_0^{|k|\sqrt{x}} d\xi\; e^{-\xi^2/4} \\ &\le C\, H\, w_\alpha(x,k) \frac{|k|\sqrt{x}}{1+|k|\sqrt{x}} \\ &\le C\, H\, |k|\sqrt{x}\, w_{\alpha+1}(x,k)\;. \end{aligned}$$

This achieves the proof. ■

The next lemma gives miscellaneous properties we will use below.

Lemma F.4 *Let $\alpha \ge 1$ and $f, g : \Omega_+ \to \mathbf{R}$. There exists a constant C such that for all $(x,k) \in \Omega_+$, we have for the multiplication operator (by k) K and the derivative D*

$$\begin{aligned} K(f*g) &= Kf*g + f*Kg\,, \\ DK(f) &= f + KD(f)\,, \\ |Kw_\alpha(x,k)| &\le \frac{C}{x^{1/2}} w_{\alpha-1}(x,k)\;. \end{aligned}$$

Proof. This is elementary. ■

F.2 Proof of Proposition 4.4

The proof consists only in applying Proposition F.2. In order to simplify the notation we omit in the following subsections the arguments x and k, whenever no confusion is possible. We begin with two lemmas which give bounds on $(u,v) \in \mathcal{B}$ and on the function $\mathbf{u}_E$ of Lemma 1.3.

Lemma F.5 *Let $\mathbf{v} = (u,v) \in \mathcal{B}$, then there exists a constant C such that for all $0 \le \alpha' \le \alpha$, and for all $(x,k) \in \Omega_+$, we have*

$$\begin{aligned} |\hat{u}(x,k)| &\le \|\mathbf{v}\|\, w_{\alpha'}(x,k)\;, \\ |D\hat{u}(x,k)| &\le \|\mathbf{v}\|\, x^{\frac{1-\gamma_1}{2}}\; g_{-\gamma_1}(k)\; w_{\alpha'}(x,k)\;, \\ |\hat{v}(x,k)| &\le \|\mathbf{v}\|\, x^{-1/2}\; w_{\alpha'}(x,k)\;, \\ |KD\hat{v}(x,k)| &\le C\; \|\mathbf{v}\|\; x^{-1/2}\; w_{\alpha'-1}(x,k)\;. \end{aligned} \tag{F.9}$$

Proof. Only inequality (F.9) requires a proof, the others follow directly from the definition of $\mathcal{B}$. For (F.9), we note that

$$|KD\hat{v}(x,k)| \leq w_{\alpha-1}(x,k)\frac{|k\sqrt{x}|}{|kx|^{\gamma_2}+|kx|} \leq w_{\alpha'-1}(x,k)x^{-1/2} \sup_{\xi\in\mathbf{R}} \frac{|\xi|}{|\xi|^{\gamma_2}+|\xi|} \leq C\, x^{-1/2}\, w_{\alpha'-1}(x,k)\ .$$

This achieves the proof. ■

Lemma F.6 *There exists a constant C such that for all $(x,k)\in\Omega_+$, the function $\mathbf{u}_E=(u_E,v_E)$ of Lemma 1.3 satisfies*

$$|\hat{u}_E(x,k)| \leq C\,\varepsilon\, e_{0,\delta}(x,k)\ , \tag{F.10}$$
$$|\hat{v}_E(x,k)| \leq C\,\varepsilon\, e_{0,\delta}(x,k)\ , \tag{F.11}$$
$$|K\ \hat{u}_E(x,k)| \leq C\,\frac{\varepsilon}{x}\, e_{0,\delta}(x,k)\ , \tag{F.12}$$
$$|K\ \hat{v}_E(x,k)| \leq C\,\frac{\varepsilon}{x}\, e_{0,\delta}(x,k)\ , \tag{F.13}$$
$$|D\ \hat{u}_E(x,k)| \leq C\,\varepsilon\left(x\, e_{0,\delta}(x,k)+(1+\delta)^{1-\gamma_E}\, e_{-\gamma_E,\delta}(x,k)\right)\ , \tag{F.14}$$
$$|D\ K\ \hat{v}_E(x,k)| \leq C\,\varepsilon\, e_{0,\delta}(x,k)\ . \tag{F.15}$$

Proof. Using the notation of Lemma 1.3, we have

$$\begin{aligned}\hat{u}_E(x,k) &= f(k)\, e^{-|k|(x-1)}\ ,\\ \hat{v}_E(x,k) &= -i\ \operatorname{sign}(k)\ f(k)\, e^{-|k|(x-1)},\end{aligned}$$

with $f\in\mathcal{A}_{\alpha,\gamma_E,\delta}$, so that inequalities (F.10), (F.11) and (F.14) follow at once. To prove inequalities (F.12) and (F.13), we use

$$|k|e^{-|k|(x+\delta)/2} \leq \frac{C}{x+\delta} = \frac{C}{x}\frac{1}{1+\delta/x} \leq \frac{C}{x}\ .$$

Finally, to prove inequality (F.15), we use

$$|DK\hat{v}_E(x,k)| \leq |\hat{u}_E(x,k)| + |KD\hat{u}_E(x,k)|\ .$$

By inequality (F.10), we have $|\hat{u}_E(x,k)| \leq C\,\varepsilon\, e_{0,\delta}(x,k)$, and for $|KD\hat{u}_E(x,k)|$, we have

$$\begin{aligned}|KD\hat{u}_E(x,k)| &\leq \varepsilon\, e^{-|k|(x+\delta)/2}\left(|k|\ x\ e^{-|k|(x+\delta)/2}+(|k|(1+\delta))^{1-\gamma_E}e^{-|k|(1+\delta)/2}\right)\\ &\leq C\,\varepsilon\, e^{-|k|(x+\delta)/2}\left(\frac{x}{x+\delta}+\frac{1}{(1+\delta)^{\gamma_E}}\right)\\ &\leq C\,\varepsilon\, e_{0,\delta}(x,k)\ .\end{aligned}$$

This achieves the proof. ■

F.2.1 Bound on $\hat{q}_0$ and $D\hat{q}_0$

We first show that

$$|\hat{q}_0| \leq C\ (\varepsilon+\|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{-1}\ w_\alpha\ . \tag{F.16}$$

We have $|\hat{q}_0| \leq |\hat{u}*\hat{v}|+|\hat{u}_E*\hat{v}|+|\hat{v}_E*\hat{u}|$, and since

$$\begin{aligned}|(\hat{u}*\hat{v})| &\leq C\ \|\mathbf{v}\|^2\ x^{-1/2}\ (w_\alpha * w_\alpha) \leq C\ \|\mathbf{v}\|^2\ x^{-1}\ w_\alpha\ ,\\ |(\hat{u}_E*\hat{v})|+|(\hat{u}*\hat{v}_E)| &\leq C\ \|\mathbf{v}\|\ \varepsilon\ (1+x^{-1/2})\ (e_{0,\delta}*w_\alpha) \leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-1}\ w_\alpha\ ,\end{aligned}$$

this completes the proof of (F.16). Next, we show that

$$|D\hat{q}_0| \leq C\ (\varepsilon+\|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{\frac{\gamma_1-1}{2}}\ w_\alpha\ . \tag{F.17}$$

We have $|D\hat{q}_0| \leq |D\hat{u} * \hat{v}| + |D\hat{u}_E * \hat{v}| + |D\hat{u} * \hat{v}_E|$, and since

$$\begin{aligned}
|(D\hat{u} * \hat{v})| &\leq C\ \|\mathbf{v}\|^2\ x^{-\gamma_1/2}(g_{-\gamma_1} w_\alpha * w_\alpha) \leq C\ \|\mathbf{v}\|^2\ x^{-1/2}\ w_\alpha\ , \\
|(D\hat{u}_E * \hat{v})| &\leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-1/2}\ \ (x\ (e_{0,\delta} * w_\alpha) + (e_{-\gamma_E,\delta} * w_\alpha)) \\
&\leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-1/2}\ \left(1 + \frac{1}{(x+\delta)^{1-\gamma_E}}\right) w_\alpha \leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-1/2}\ w_\alpha\ , \\
|(D\hat{u} * \hat{v}_E)| &\leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{\frac{1-\gamma_1}{2}}\ (g_{-\gamma_1} w_\alpha * e_{0,\delta}) \leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{\frac{\gamma_1 - 1}{2}}\ w_\alpha\ ,
\end{aligned}$$

this completes the proof of (F.17). Finally, u, u_E are even, and v, v_E are odd, so that $\hat{q}_0$ is odd. From (F.16) and (F.17) and Lemma F.3 we conclude that

$$|\hat{q}_0(x,k)| \leq C\ (\varepsilon + \|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{\frac{\gamma_1-1}{2}} |k|\ w_{\alpha+1}(x,k)\ . \tag{F.18}$$

This achieves the proof of (4.4) and (4.5).

F.2.2 Bound on $\hat{h}_1$ and $D\hat{h}_1$

We first show that

$$\left|\hat{h}_1\right| \leq C\ (\varepsilon + \|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{-3/2}\ w_{\alpha-1}\ . \tag{F.19}$$

We have $\left|\hat{h}_1\right| \leq |\hat{v} * K\hat{u}| + |\hat{v} * K\hat{u}_E| + |\hat{v}_E * K\hat{u}|$, and since

$$\begin{aligned}
|(\hat{v} * K\hat{u})| &\leq C\ \|\mathbf{v}\|^2\ x^{-1}\ (w_\alpha * w_{\alpha-1}) \leq C\ \|\mathbf{v}\|^2\ x^{-3/2}\ w_{\alpha-1}\ , \\
|(\hat{v} * K\hat{u}_E)| &\leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-3/2}\ (w_\alpha * e_{0,\delta}) \\
&\leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-5/2}\ w_{\alpha-1} \leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-3/2}\ w_{\alpha-1}\ , \\
|(\hat{v}_E * K\hat{u})| &\leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-1/2}\ (e_{0,\delta} * w_{\alpha-1}) \leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-3/2}\ w_{\alpha-1}\ ,
\end{aligned}$$

this completes the proof of (F.19). Next, we show that

$$\left|D\hat{h}_1\right| \leq C\ (\varepsilon + \|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{\frac{\gamma_1-1}{2}}\ |k|\ w_\alpha\ . \tag{F.20}$$

We can show easily using Lemma F.4 that

$$iD\hat{h}_1 = -\hat{q}_0 - KD\hat{q}_0 + (K\hat{v} * D\hat{u} + K\hat{v} * D\hat{u}_E + K\hat{v}_E * D\hat{u})\ \ ,$$

and therefore we have

$$\left|D\hat{h}_1\right| \leq |\hat{q}_0| + |KD\hat{q}_0| + |\widetilde{h}|\ ,$$

where

$$\tilde{h} = K\hat{v} * D\hat{u} + K\hat{v} * D\hat{u}_E + K\hat{v}_E * D\hat{u}\ .$$

By (F.17) and (F.18), we have

$$\begin{aligned}
|\hat{q}_0| &\leq C\ (\varepsilon + \|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{\frac{\gamma_1-1}{2}}\ |k|\ w_{\alpha+1} \leq C\ (\varepsilon + \|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{\frac{\gamma_1-1}{2}}\ |k|\ w_\alpha\ , \\
|KD\hat{q}_0| &\leq C\ (\varepsilon + \|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{\frac{\gamma_1-1}{2}}\ |k|\ w_\alpha\ .
\end{aligned}$$

By Lemma F.3, to prove (F.20), it is sufficient to prove that

$$|\tilde{h}| \leq C\ (\varepsilon + \|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{-1}\ w_{\alpha-1}\ \ , \tag{F.21}$$

$$|D\widetilde{h}| \leq C\ (\varepsilon + \|\mathbf{v}\|)\ \|\mathbf{v}\|\ x^{\frac{\gamma_1-1}{2}}\ w_{\alpha-1}\ . \tag{F.22}$$

We prove first (F.21). We have $|\tilde{h}| \leq |K\hat{v} * D\hat{u}| + |K\hat{v} * D\hat{u}_E| + |K\hat{v}_E * D\hat{u}|$, and since

$$\begin{aligned}
|(K\hat{v} * D\hat{u})| &\leq C\ \|\mathbf{v}\|^2\ x^{-1/2-\gamma/2}(g_{-\gamma_1} w_{\alpha-1} * w_\alpha) \leq C\ \|\mathbf{v}\|^2\ x^{-1}\ w_{\alpha-1}\ , \\
|(K\hat{v} * D\hat{u}_E)| &\leq C\ \|\mathbf{v}\|\ \varepsilon\ x^{-1}\ \ (x\ (e_{0,\delta} * w_{\alpha-1}) + (e_{-\gamma_E,\delta} * w_{\alpha-1}))
\end{aligned}$$

$$
\begin{aligned}
&\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-1}\,\left(1+\frac{1}{(x+\delta)^{1-\gamma_E}}\right)w_{\alpha-1} \leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-1}\,w_{\alpha-1}\ , \\
|(K\hat{v}_E * D\hat{u})| &\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-\frac{1+\gamma_1}{2}}\,(g_{-\gamma_1}w_\alpha * e_{0,\delta}) \\
&\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{\frac{\gamma_1-3}{2}}w_\alpha \leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-1}\,w_{\alpha-1}\ ,
\end{aligned}
$$

this completes the proof of (F.21). To prove (F.22), we use

$$
\begin{aligned}
D\tilde{h} &= DK\hat{v} * D\hat{u} + DK\hat{v} * D\hat{u}_E + DK\hat{v}_E * D\hat{u} \\
&= (\hat{v} * D\hat{u} + \hat{v} * D\hat{u}_E) + (KD\hat{v} * D\hat{u} + KD\hat{v} * D\hat{u}_E) + DK\hat{v}_E * D\hat{u}\ ,
\end{aligned}
$$

and therefore

$$
|D\tilde{h}| \leq |\hat{v} * D\hat{u}| + |\hat{v} * D\hat{u}_E| + |KD\hat{v} * D\hat{u}| + |KD\hat{v} * D\hat{u}_E| + |DK\hat{v}_E * D\hat{u}|\ .
$$

The functions $\hat{v} * D\hat{u}$ and $\hat{v} * D\hat{u}_E$ have already been bounded before. Namely,

$$
|\hat{v} * D\hat{u}| + |\hat{v} * D\hat{u}_E| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\|\,x^{-1/2}\,w_\alpha\ .
$$

And finally, we have

$$
\begin{aligned}
|(KD\hat{v} * D\hat{u})| &\leq C\,\|\mathbf{v}\|^2\,x^{-\gamma/2}(g_{-\gamma_1}w_\alpha * w_{\alpha-1}) \leq C\,\|\mathbf{v}\|^2\,x^{-1/2}\,w_{\alpha-1}\ , \\
|(KD\hat{v} * D\hat{u}_E)| &\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-1/2}\,\left(x\,(e_{0,\delta} * w_{\alpha-1}) + (e_{-\gamma_E,\delta} * w_{\alpha-1})\right) \\
&\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-1/2}\,\left(1+\frac{1}{(x+\delta)^{1-\gamma_E}}\right)w_{\alpha-1} \leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-1/2}\,w_{\alpha-1}\ , \\
|(DK\hat{v}_E * D\hat{u})| &\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{\frac{1-\gamma_1}{2}}\,(g_{-\gamma_1}w_\alpha * e_{0,\delta}) \leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{\frac{\gamma_1-1}{2}}\,w_\alpha\ .
\end{aligned}
$$

This completes the proof of (F.22), so that the proof of (4.6) and (4.7) is completed.

F.2.3 Bound on $\hat{h}_2$ and $D\hat{h}_2$

We first show that

$$
\left|\hat{h}_2\right| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\|\,x^{-3/2}\,|k|\,w_\alpha\ .
$$

Since $\left|\hat{h}_2\right| \leq |k|\,(|\hat{v} * \hat{v}| + 2\,|\hat{v} * \hat{v}_E|)$, we have

$$
\begin{aligned}
|(\hat{v} * \hat{v})| &\leq C\,\|\mathbf{v}\|^2\,x^{-1}\,(w_\alpha * w_\alpha) \leq C\,\|\mathbf{v}\|^2\,x^{-3/2}\,w_\alpha\ , \\
|(\hat{v} * \hat{v}_E)| &\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-1/2}(e_{0,\delta} * w_\alpha) \leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-3/2}\,w_\alpha\ .
\end{aligned}
$$

Next, we show that

$$
\left|D\hat{h}_2\right| \leq C\,(\varepsilon + \|\mathbf{v}\|)\,\|\mathbf{v}\|\,x^{-3/2}\,w_{\alpha-1}\ .
$$

Since

$$
\hat{h}_2 = \frac{i}{\pi}\left(K\hat{v} * \hat{v} + K\hat{v} * \hat{v}_E + \hat{v} * K\hat{v}_E\right)\ ,
$$

we have

$$
\begin{aligned}
iD\hat{h}_2 &= -\frac{1}{\pi}\left(DK\hat{v} * \hat{v} + DK\hat{v} * \hat{v}_E + \hat{v} * DK\hat{v}_E\right) \\
&= \frac{1}{\pi}\left(\hat{v} * \hat{v} + \hat{v} * \hat{v}_E - KD\hat{v} * \hat{v} - KD\hat{v} * \hat{v}_E - \hat{v} * DK\hat{v}_E\right)\ .
\end{aligned}
$$

Since

$$
\begin{aligned}
|(\hat{v} * \hat{v})| + |(KD\hat{v} * \hat{v})| &\leq C\,\|\mathbf{v}\|^2\,x^{-1}\,(w_{\alpha-1} * w_\alpha) \\
&\leq C\,\|\mathbf{v}\|^2\,x^{-3/2}\,w_{\alpha-1}\ , \\
|(\hat{v} * \hat{v}_E)| + |(KD\hat{v} * \hat{v}_E)| + |(\hat{v} * DK\hat{v}_E)| &\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-1/2}(e_{0,\delta} * w_{\alpha-1}) \\
&\leq C\,\|\mathbf{v}\|\,\varepsilon\,x^{-3/2}\,w_{\alpha-1}\ ,
\end{aligned}
$$

this ends the proof of (4.8) and (4.9) and the proof of Proposition 4.4.

References

[Bricmont et al.] Bricmont, J., Kupiainen, A. & Lin, G., Renormalization Group and Asymptotics of Solutions of Nonlinear Parabolic Equations. Comm. Pure Appl. Math. **47**, no. 6, 893–922 (1994)

[Gallay] Gallay, Th.: A Center–Stable Manifold Theorem for Differential Equations in Banach Spaces. Comm. Math. Phys. **152**, no. 2, 249–268 (1993)

[Batchelor] Batchelor, G. K., An Introduction to Fluid Dynamics. Cambridge University Press (1999)

[Smith] Smith, D. R., Estimates at Infinity for Stationary Solutions of the Navier–Stokes Equations in Two Dimensions. Arch. Rational Mech. Anal. **20** 341–372 (1965)

[Galdi] Galdi, G. P., An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I+II. Springer Tracts in Natural Philosophy, **38–39**. Springer-Verlag, New York, (1994)

[Amick] Amick, C. J., On the Asymptotic Form of Navier-Stokes Flow Past a Body in the Plane. J. Differential Equations **91**, no. 1, 149–167 (1991)

[Wayne] Wayne, E. C., Invariant Manifolds for Parabolic Partial Differential Equations on Unbounded Domains. Arch. Rational Mech. Anal. **138**, no. 3, 279–306 (1997)