Algebraic proof of the non-integrability of Hill's Problem

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Abstract

Hill's lunar problem appears in Celestial Mechanics as a limit of the Restricted Three-Body Problem. Besides, information on the former shows light on several other three-body problems. It contains no parameters and is globally far from any simple well–known problem. Strong numerical evidences of its lack of integrability have been given in the past. Here an algebraic proof of non–integrability is presented. Beyond the result in itself, the paper can also be considered as an example of the application of differential Galois theory to a significant problem.

Contents

1 Introduction

Hill's problem (*HP*), usually dubbed *lunar* as an homage to its earliest motivation, or *two-dimensional* in order to distinguish it from its own extension to $\mathbb{R}^3$, is a model originally based on the study of the Moon's motion under the joint influence of Earth and Sun ([5]); nowadays, the roles of these bodies are often played by others, since the same model turns out to be fairly accurate in some specific cases in which the body describing the problem has a negligible mass. More precisely, *HP* describes the motion of this first, massless object, under the gravitational action played by a second one (originally, the Earth) named the *secondary* and the perturbation practised by a third, large body or *primary* (originally, the Sun) which, although placed at an infinite distance from the other two, has a "suitably" infinite mass in order to assure both a parallel field and a finite though considerable influence on the first body. Among other nomenclatures regarding the mass hierarchy, we may mention the one labelling both massive bodies as *primaries* as opposed to the massless *secondary*.

Using Hamiltonian formulation, it is easy to complement the above approach with a far more accurate characterisation of the *HP* as a limit case of the *Restricted Three-Body Problem* (*RTBP*) after appropriate distance scalings; the limits are taken as μ, the mass of the secondary in the latter problem, tends to 0. The *RTBP* can in fact be written as an $O\left(\mu^{\frac{1}{3}}\right)$ perturbation of *HP* in a $O\left(\mu^{\frac{1}{3}}\right)$-sized neighbourhood of the secondary, in the initial variables of the *RTBP* in the configuration space. This said, it is worth noting that Hill's Problem *does not* depend on any parameter other than the energy: all global representations of the form

$$H = H_0\left(\mathbf{p}\right) + \varepsilon H_1\left(\mathbf{q}, \mathbf{p}\right) \tag{1}$$

are immediately discarded, and *HP* is therefore far enough, globally, from any known integrable system. The simplest expression known to date amounts to a two-degrees of freedom Hamiltonian which, expressed in terms of Levi-Civita variables, becomes the polynomial of degree six (10) shown in section 3.

Despite its being arguably the simplest paradigm of the *General Three-Body Problem* (see [16],[19]), *HP* still displays most of the numerical evidence inherent to chaotic dynamical systems. Hence, a tempting task would be to establish its non-integrability in a rigorous way but, at least in the present case, such an endeavour has proven more of a challenge than initially expected. Indeed, a simple use of Newton's polygon casted away any chance of obtaining a classical Hamiltonian through a polynomial symplectic change. Moreover, the unfeasibility of an expression such as (1) seemed to suppress every aspect of perturbation theory, including KAM, as a valid option, except for motions confined to a small neighbourhood of the secondary. Furthermore, monodromy groups for the normal variational equations yielded invariably resonant matrices, thus discarding the direct observance of Ziglin's theorem (see [21] for details).

Our next purpose was to overcome all such apparent obstructions through an algebraic approach, namely using Galois differential theory. This has been finally achieved using the *Morales–Ramis theorem* which connects two main notions: integrability of Hamiltonian systems and integrability of linear systems of equations. Thanks to this main result (stated as Theorem 2.12), a symplectic change and a series of minor operations, we afforded the proof avoiding burdensome calculations and strict dependence on numerical results.

As seen in the forthcoming sections, present work is conceived strictly as the confluence of already existing results. Accordingly, we rely entirely on the profusion of seminal texts assuring the validity of such results and deal with the proof in a purely expository manner, thus ignoring lesser intermediate steps and making no more forays than necessary into the topics of special functions, representation theory and algebraic geometry. Section 2 introduces the basic theoretical weaponry needed for our purpose, including all crucial theorems and definitions. As for section 3, it exposes the actual problem being dealt with and states the main results. Readers already acquainted with Hill's problem, Galois differential theory, Hamiltonian systems and canonical transformations are thereupon advised to skip everything in these two sections, except for the main statements in 3.2.

The next three sections are the main body of the proof. Its first part (corresponding to section 4) is based on the **computation of a particular solution of the** ***HP*** by means of a symplectic transformation; this solution and the sort of integral curve Γ it determines, are in turn useful for the second part, inscribed in section 5 and consisting on the **layout (and a fundamental matrix Ψ) of the variational equations of the HP along** Γ. The information we need about the matrix, included in 5.3, is actually less than computing the whole of Ψ explicitly, as we will see in section 6: the **study of the Galois differential group of the Picard-Vessiot extension for the aforementioned variational equations**. This will be the concluding part of our proof, using the relevant facts concerning Ψ to apply Morales–Ramis theorem.

As for the recent paper [10] devoted to the very same goal through different techniques, at the end of section 6 we extend on a comment regarding its authors' hypotheses.

2 Two notions of integrability

The idea underlying any apprehension of an **integrable** dynamical system

$$\frac{d}{dt}\mathbf{x} = f(t, \mathbf{x}), \tag{2}$$

is the ability to make assertions of a *global* sort about the system's evolution with respect to the time t. The outcome of such assertions is sometimes called a **solution**, although giving it (and, hence, those systems (2) bearing it) a strict definition has always proved a troublesome task, since each field of study has its own specialised notion of "solvability", seldom equivalent to the others'. This notwithstanding, and in spite of the lack of a total coincidence between them, the two specific definitions given in this section do have a common trait: the existence of functions, whether in implicit or parametric form, respectively.

2.1 Integrability of Hamiltonian systems

Let us restrict our attention to the case in which (2) has a very specific form, called *Hamiltonian* and shown in (3) below; further details may be found in [11], [12] and [18].

Integration by quadratures is defined as the computation of solutions of (2) through "algebraic" operations (including the inversion of functions) and "quadratures", that is, the computation of integrals of known functions. At a strictly theoretical level, this happens to be the adequate notion for integrability in the Hamiltonian realm; in other words: were function integration not an unsolved problem of its own, we would be plainly speaking of a solution in parametric form. *For practical purposes, though, the adequate definition actually uses implicit solutions*, as we claimed at the first paragraph of the present section.

A **symplectic manifold** is a differentiable manifold of even dimension $2n$ along with a nondegenerate closed 2-form Ω, called the **symplectic form**, whose nondegeneracy allows the definition of a musical isomorphism of vector bundles,

$$\flat : TM \to T^*M, \quad \flat X = \Omega(X).$$

These manifolds arise naturally as phase spaces of the class of differential systems we are now introducing.

A **Hamiltonian vector field** is a field X_H defined on the symplectic manifold M, such that $X_H = \flat^{-1} \cdot dH$ for some function H, usually called the **Hamiltonian**. The differential equation satisfied by the integral curves of a Hamiltonian vector field is called a **Hamiltonian system**; in virtue of *Darboux's theorem* (see, e.g., [12]), it may be written, in canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ (referred to as **positions** and **momenta**, respectively), in the following form

$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, n; \tag{3}$$

one usually calls these **Hamilton's equations** associated to Hamiltonian H. They may also be written as $\mathbf{z}' = X_H(\mathbf{z})$, noting $\mathbf{z} := (q_1, \ldots, q_n, p_1, \ldots, p_n)$. This context also allows the definition of **canonical transformations**, i.e. changes of the variables $\mathbf{z}, t$ under which the symplectic form remains invariant; *in other words, under which the Hamiltonian form of the equations is maintained.* Furthermore, the musical isomorphism $\flat$ allows the adjunction of a Poisson algebra structure on M, bearing Poisson brackets $\{f, g\} = \Omega(X_f, X_g)$ which in canonical coordinates may be expressed as $\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right)$. The following holds

Proposition 2.1 *f is a **first integral** (that is, a function constant over integral curves) of X_H if, and only if, $\{H, f\} = 0$ (i.e. H and f are **in involution**, or **commute**). In particular, H is always a first integral of X_H.*

Needless to say, given (2) as input, and whenever there exists a function $H \in \mathcal{C}^2$ such that (2) takes the form (3), we automatically call function H, field X_H and system (3) **Hamiltonian** and everything seen up to now applies to them.

The following result does not merely provide some Hamiltonians a description of their phase spaces; it also confers the whole area a precise notion of integrability.

Theorem 2.2 ***(Liouville–Arnold)*** *Let X_H be a real Hamiltonian with n degrees of freedom, defined on a symplectic manifold M. Assume there are n functionally independent first integrals $f_1 = H, f_2, \ldots, f_n$ in involution.*

Let $M_a = \{z \in M : f_i(z) = a_i, i = 1, \ldots, n\}$, $a = (a_1, \ldots, a_n)$, a non-critical level manifold of $f_1, \ldots, f_n$, i.e. such that $\operatorname{rank}(df_1, \ldots, df_n) \equiv n$ on M_a. Then,

1. *M_a is an invariant manifold of the flux of X_H, and if it is compact and connected then it is diffeomorphic to an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$;*

2. *in a neighbourhood of the torus M_a there is a canonical* ***action-angle*** *coordinate system,* $(I, \phi) = (I_1, \ldots, I_n, \phi_1, \ldots, \phi_n)$, *such that* $\phi_i \in \mathbb{T}$, *in which Hamilton's equations for* X_H *become*

$$I_i' = 0, \quad \phi_i = \omega_i, \qquad i = 1, \ldots, n,$$

the frequencies ω_i *depending solely on the actions,* $\omega_i = \omega_i (I_1, \ldots, I_n)$.

For a proof, see [1]. Trivially enough, part 2. yields the following

Corollary 2.3 *Let* H *be an* n*-degrees of freedom Hamiltonian having* n *functionally independent integrals* $f_1 = H, f_2, \ldots, f_n$ *in involution. Then, system (3) can be integrated by quadratures.*

We call H **integrable in the sense of Liouville-Arnold**, **completely integrable** or simply **integrable** (and extend this definition to X_H and (3)) if it verifies the hypotheses of Corollary 2.3, and henceforward take this as the definition of Hamiltonian integrability. It is worth noting that the vast majority of Hamiltonians *do not* respond to this definition; although this fact (and its acquaintance) falls beyond the scope and interests of this work, it does highlight the vital importance of the next result.

Lemma 2.4 *H given, let ϕ be a canonical change of variables. Then, H is integrable if and only if $K = H \circ \phi$ is. Hence,* integrability is preserved under symplectic transformations.

Proof. See [4, p. 94] for a slightly more general proposition.

This result, in fact one of the main tools there exist to prove (non-)integrability, allows us to crucially simplify the *HP* Hamiltonian in sections 4 and 5.

Remark. Although we restricted everything to real variables, Hamiltonian formulation may also be defined on the complex setting, thus allowing time and canonical variables to be complex, manifolds to be analytical, and functions and fields to be analytical or meromorphic. The only nuisance to more general purposes, though, is the fact that there is no complex analogue to the Liouville–Arnold theorem, except for a few special cases (see [12]); as will be said at the beginning of 2.3, this is not a problem in our case.

2.2 Integrability of linear differential systems

In the context of *linear* differential equations, the concept of *integrability* is conventionally limited to the possibility of solving the equation in terms of algebraic functions, integrals and exponentials of known functions or any finite combination of all three. This second notion is naturally inscribed in *differential Galois theory* as will be seen in definition 2.10 and theorem 2.11. Every single fact stated here is described in detail in reference [17].

Let K be a field. A **derivation** on K is an additive map $\partial : K \to K$ verifying the *Leibnitz rule*

$$\partial (ab) = \partial (a) b + a\partial (b), \quad a, b \in K.$$

A **differential field** is a pair (K, ∂_K) consisting of a field and a derivation on it. The **constants** are the elements of the subfield $\mathrm{Const}(K) := \ker \partial_K$ of K. We henceforth note $\partial = \partial_K$ unless necessary, and write a' (respectively, $a^{(n)}$) instead of $\partial (a)$ (resp. $\partial^n (a)$); we will also use this notation for elements of K^n, extending the derivation entrywise.

An **extension of differential fields**, usually noted $L \mid K$, is an inclusion $L \supset K$ such that $\partial_L|_K \equiv \partial_K$. All fields and extensions will be assumed to be differential from this point onward. We will also assume characteristic zero for every field considered. The set of all K**-automorphisms** of any differential extension $L \mid K$, (that is, field isomorphisms $\sigma : L \to L$ such that $\sigma|_K \equiv id_K$ and $\partial \circ \sigma \equiv \sigma \circ \partial$) is a group under map composition and will be denoted by $\mathrm{Aut}_K (L)$.

Given a linear homogeneous differential equation

$$y' = Ay, \qquad A \in \mathcal{M}_n (K), \tag{4}$$

where $\mathcal{M}_n (K)$ denotes the space of the $n \times n$ matrices with entries in K, and an extension $E \mid K$ containing a set V of solutions of (4), there is always a minimal subfield $L \subset E$ containing both K and the entries of the elements of V; we write $L = K(V)$ and say L is **generated over K as a differential field by the entries of elements of V**. $\mathrm{Aut}_K(L)$ preserves V and acts on it as a group of linear transformations over $\mathrm{Const}(K)$, and if $\mathrm{Const}(L) = \mathrm{Const}(K)$ the restriction of $\mathrm{Aut}_K(L)$ to V gives a faithful representation $\mathrm{Aut}_K(L) \to \mathrm{GL}(V)$. Furthermore, since the equation is linear and homogeneous, V is closed under addition and multiplication by scalars of $\mathrm{Const}(L)$; it is in fact a $\mathrm{Const}(L)$-vector space of dimension at most n. What gives V its relevance are those situations in which it is precisely defined as the maximal set of linearly independent solutions of (4), thus begetting the differential analogue of a Galois extension; such an analogue corresponds to the case $\dim_{\mathrm{Const}(L)}(V) = n$ and actually matches the situation in which no new constants are added to K.

Definition 2.5 *Let $y' = Ay$ be a matrix differential equation over K as in (4). A differential extension $L \mid K$ is a **Picard-Vessiot (P-V) extension of** K for (4) if*

1. $\mathrm{Const}(L) = \mathrm{Const}(K)$*;*
2. *there exists a fundamental matrix $\Phi \in \mathrm{GL}_n(L)$ for the equation; and*
3. *L is generated over K as a differential field by the entries of Φ.*

Given a *P-V* extension $L \mid K$ for (4) and an intermediate extension $L \supset L_1 \supset K$ then $L \mid L_1$ is also a *P-V* extension for some linear O.D.E. over L_1. We are calling $L \mid K$ a **Picard-Vessiot extension** if it is *P-V* for some linear O.D.E. over K; an intrinsic definition may indeed be made, regardless of the equation.

Everything said from now on needs no more considerations as to $\mathcal{C} = \mathrm{Const}\,(L)$ except for one extra assumption needed to assure existence of *P-V* extensions.

Theorem 2.6 *Let $y' = Ay$ be a matrix differential equation over a differential field K with algebraically closed field of constants $\mathcal{C}$. Then*

1. *(**Existence and uniqueness**) There exists a* P-V *extension $L \mid K$ for (4) over K, and any two such extensions are K-isomorphic;*
2. *For any $a \in L \setminus K$, there is a differential K-automorphism σ of L such that $\sigma(a) \neq a$.*

Definition 2.7 *If $L \mid K$ is a Picard-Vessiot extension for $y' = Ay$, then $\mathrm{Aut}_K(L)$ will be denoted $\mathrm{Gal}\,(L \mid K)$ and called the **Galois differential group of** L **over** K (or of the extension).*

Definition 2.8 *Let $\mathcal{C}$ be an algebraically closed field. A subgroup of $\mathrm{GL}_n(\mathcal{C})$, for some $n \in \mathbb{N}$, is a **linear algebraic group** over $\mathcal{C}$ provided its elements are matrices whose coefficients are zeros of polynomials in $\mathcal{C}$. A subgroup of a linear algebraic group is **closed** if it is a linear algebraic group itself.*

The Galois differential group of an equation (4) is a linear algebraic group; indeed, given a fundamental matrix $\Phi \in M_n(L)$, $\sigma(\Phi)$ is also a fundamental matrix and hence $\sigma(\Phi) = \Phi R(\sigma)$ with $R(\sigma) \in \mathrm{GL}_n(\mathcal{C})$, which yields an n-dimensional faithful representation

$$\rho : \mathrm{Gal}\,(L \mid K) \to \mathrm{GL}_n(\mathcal{C}), \qquad \sigma \mapsto R(\sigma); \tag{5}$$

this renders $\mathrm{Gal}\,(L \mid K)$ a linear group. For a proof of its being also closed, see [17, Theorem 1.27]. Furthermore, the *monodromy group* of an equation (4), attained through analytical continuation of solutions (see, e.g., [11]), is a (not necessarily closed) subgroup of the differential Galois group of the corresponding *P-V* extension. For a deeper study of the relationship between both groups see [17].

We now state the so-called *Fundamental Theorem of differential Galois theory.*

Theorem 2.9 *Let $L \mid K$ be a Picard-Vessiot extension with algebraically closed common field of constants $\mathcal{C}$. Furthermore, let $G = \mathrm{Gal}\,(L \mid K)$, $\mathcal{S}$ the set of closed subgroups of G and $\mathcal{L}$ the set of differential subfields of L. Define $\alpha : \mathcal{S} \to \mathcal{L}$, $\alpha(H) = L^H$, i.e. the subfield of L formed by H-invariant elements; let $\beta : \mathcal{L} \to \mathcal{S}$, $\beta(L_1) = \mathrm{Gal}\,(L \mid L_1)$, the subgroup of G of L_1-linear differential automorphisms of L. Then,*

1. *α i β are mutual inverses;*
2. *the following are equivalent,*
   (a) *$H \in \mathcal{S}$ is a normal subgroup of G;*
   (b) *$L_1 := \alpha(H) = L^H$ is a* P-V *extension of K;*

   and in such case $\mathrm{Gal}\,(L \mid L_1) = H$ and $\mathrm{Gal}\,(L_1 \mid K) \simeq G/H$
3. *Let G^0 be the identity component of G. Then, L^{G^0} is a* P-V *extension with Galois group G/G^0.*

As foretold at the start of this subsection, we are now introducing the strict definition of what is to be called an integrable linear differential equation; it is precisely one whose *P-V* extension falls into the following category.

Definition 2.10 *Let K be a differential field with an algebraically closed subfield of constants. An extension $L \mid K$ is called a **Liouville extension** of K if $\mathrm{Const}\,(L) = \mathrm{Const}\,(K)$ and there exists a tower of extensions*

$$K = L_0 \subset L_1 \subset \cdots \subset L_n = L$$

such that for $i = 1, \ldots, n$, $L_i = L_{i-1}(t_i)$ and one of the following holds: either

1. *$t_i' \in L_{i-1}$; we say t_i is an* ***integral*** *(of an element of L_{i-1}); or*

2. *$t_i \neq 0$ and $t_i'/t_i \in L_{i-1}$; in such case, t_i is an* ***exponential*** *(of an integral of an element of L_{i-1}); or*

3. *t_i is algebraic over L_{i-1}.*

If L is a Liouville extension of K and all t_i are integrals (resp. exponentials), we say L is an ***extension by integrals*** *(resp.* ***exponentials****) of K.*

What comes next, finally, is the fundamental characterisation of Liouville extensions.

Theorem 2.11 *Let L be a Picard-Vessiot extension of K with Galois differential group G. Then, the following are equivalent*

1. *L is a Liouville extension of K;*

2. *the identity component G^0 of G is solvable.*

Remark. If a Picard-Vessiot extension is defined solely by quadrature adjunction,

$$L = K\left(\int f_1, \int f_2, \dots, \int f_k\right),$$

where $f_1, f_2, \dots, f_k \in K$, the fact that these k new elements are either trivial or **transcendental** (i.e. no solution to any polynomial equation with its coefficients in K) renders its Galois group equal to $(\mathcal{C}_+)^s$, $s \leq k$. Here $\mathcal{C}_+$ denotes the additive group over the constant field of K. Indeed, $\mathrm{Gal}(L \mid K)$ acts on quadratures in an additive manner and the only algebraic subgroups of $\mathcal{C}_+$ are itself and the trivial group. See [17, Exercise 1.35 (1)], [7] for details.

2.3 The Morales-Ramis theorem

In order to use the result we are about to state, we must be able to rely on sections 2.1 and 2.2 despite working with an initially *real* Hamiltonian. Doing this on 2.2 is rather trivial; when it gets to the point of considering fields as vector spaces, whether we are considering $\mathbb{R}$ or $\mathbb{C}$ will not be relevant anymore. As for 2.1, as said in ending Remark, the task appears to be harder; a way to overcome this is working on complex Hamiltonian systems (whether holomorphic or meromorphic), such that if the coordinates are real then the system is also real, i.e. systems such that we obtain the real system as a subsystem, restricting time and canonical variables to real values. We may then apply Liouville-Arnold's theorem to the real system and subsequently perform the analytical continuation, that is, considering all variables (dependent and independent) as complex. Fortunately, this is overall the case for *HP*.

As mentioned in Section 1, the theorem bearing this subsection's name brings about the connection between the two notions of solvability listed in 2.1 and 2.2, namely when applied to a Hamiltonian X_H and the variational equation (*VL*) along an integral curve Γ of X_H, respectively. More precisely, if X_H is integrable *in the sense of Liouville-Arnold*, then the *VE* must also be integrable *in the sense of Definition 2.10 (and, therefore, Theorem 2.11)*, that is, it must yield a Liouville extension of differential fields, i.e., the identity component G^0 of the differential Galois group for the extension must be solvable. Accordingly, it relates the concept of integrability in the *linear* context to the one in the generally *non-linear* context of Hamiltonians.

The specific theorem is in fact more restrictive in its assertion about G^0, as well as wider than Liouville-Arnold in its scope regarding the analyticity properties of the first integrals. More precisely, the base field for the Picard-Vessiot extension (i.e. the one containing the coefficients of the variational equations) is the field $\mathcal{M}(\Gamma)$ of *meromorphic* functions defined on the integral curve of X_H.

Theorem 2.12 ***(Morales-Ramis)*** *Assume there exist n independent meromorphic first integrals in involution for X_H in a neighbourhood of an integral curve Γ. Then, the identity component of the Galois group of the variational equations along Γ is commutative.*

Proof. See [13, Corollary 8] (or [12, Theorem 4.1]) . □

Remark. We may also mention reference [14] extending the previous result to higher order variationals; for a detailed exposition when it was still a conjecture, see section 8.3 of [12].

3 Setting and main results

3.1 Hill's problem

First of all, let us review the steps towards the simplest existing Hamiltonian formulation of *HP*. Everything listed here comes detailed in [15] and, especially, [16].

In the context of the *(Planar Circular) Restricted Three-Body Problem* (*RTBP*), and after an adequate, non-binding choice of mass units, primaries P_1 and P_2 are assumed to move in circles around their common barycentre and carry masses $1-\mu$ and μ, respectively, $\mu \in \left(0, \frac{1}{2}\right]$ being the normalised mass unit. Their mean motion is $n = \sqrt{G} R^{-3/2}$, where G is the gravitational constant and $R = \operatorname{dist}(P_1, P_2)$. In a (so-called **synodical**) rotating reference frame (ξ, η) moving along with P_1, P_2 (i.e. whose angular velocity ω is equal to n), the primaries are correspondingly located in positions $\xi = -\mu R$ and $\xi = (1-\mu) R$. As the total mass M of our system is set equal to 1, the product $k = GM$ becomes G. In this frame, the equations for the *RTBP* are then

$$\left.\begin{aligned} \xi'' - 2\omega\eta' &= \Omega_\xi, \\ \eta'' + 2\omega\xi' &= \Omega_\eta, \end{aligned}\right\} \tag{6}$$

where

$$\Omega(\xi, \eta) = \frac{\omega^2}{2}\left(\xi^2 + \eta^2\right) + \frac{k(1-\mu)}{r_1} + \frac{k\mu}{r_2},$$

is the *centrifugal* plus the *gravitational potential* and

$$r_1^2 = (\xi - \mu R)^2 + \eta^2, \quad r_2^2 = (\xi - (1-\mu) R)^2 + \eta^2,$$

are the squared distances of the secondary to both primaries.

This fits Hamiltonian formulation, since (6) may in fact be expressed as

$$\frac{d}{dt}(\xi' - \omega\eta) = \omega\eta' + \Omega_\xi, \quad \frac{d}{dt}(\eta' + \omega\xi) = -\omega\xi' + \Omega_\eta,$$

thus allowing the completion of (ξ, η) to canonical coordinates,

$$q_1 = \xi, \qquad q_2 = \eta, \qquad p_1 = \xi' - \eta, \qquad p_2 = \eta' + \xi$$

in which to express the Hamiltonian

$$H_{RTBP}(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1 + q_2)^2 + \frac{1}{2}(p_2 - q_1)^2 - \Omega(q_1, q_2).$$

There is a function $C_J(q_1, q_2, p_1, p_2)$, called the **Jacobi constant**, which also happens to be a first integral of the *RTBP*. See [16] for more details.

Let us set the unit of length in order to obtain $R = 1$, and of time to set $G = 1$, so that $\omega = k = 1$. Translating the secondary to the origin and performing an adequate scaling,

$$(\bar{q}_1, \bar{q}_2) = \left(\mu^{-\frac{1}{3}}\left[q_1 + (1-\mu)\right], \mu^{-\frac{1}{3}} q_2\right),$$

expressing the right sides of both equations of the *RTBP* as fractional power series with respect to μ, and subsequently dividing both sides of these equations by $\mu^{\frac{1}{3}}$, we obtain

$$\left.\begin{aligned} \bar{q}_1'' &= -\frac{\bar{q}_1}{\left(\bar{q}_1^2 + \bar{q}_2^2\right)^{3/2}} + 2\bar{q}_2' + 3\bar{q}_1 + O\left(\mu^{\frac{1}{3}}\right), \\ \bar{q}_2'' &= -2\bar{q}_1' - \frac{\bar{q}_2}{\left(\bar{q}_1^2 + \bar{q}_2^2\right)^{3/2}} + O\left(\mu^{\frac{1}{3}}\right), \end{aligned}\right\} \tag{7}$$

thus establishing the *RTBP* as an $O\left(\mu^{\frac{1}{3}}\right)$ perturbation of

$$\left.\begin{aligned} \bar{q}_1'' &= -\frac{\bar{q}_1}{\left(\bar{q}_1^2 + \bar{q}_2^2\right)^{3/2}} + 2\bar{q}_2' + 3\bar{q}_1, \\ \bar{q}_2'' &= -\frac{\bar{q}_2}{\left(\bar{q}_1^2 + \bar{q}_2^2\right)^{3/2}} - 2\bar{q}_1', \end{aligned}\right\} \tag{8}$$

or, if preferred, the latter as the result of taking limits in (7) as $\mu \to 0$. In any case, (8) are the best known equations of *Hill's Problem*, or *HP*.

Same as in the *RTBP*, we may complete $(\bar{q}_1, \bar{q}_2)$ to canonical by setting, prior to the limit-taking in μ,

$$\bar{p}_1 = \mu^{-\frac{1}{3}} p_1, \quad \bar{p}_2 = \mu^{-\frac{1}{3}} \left(p_2 + (1-\mu)\right),$$

which yields

$$(q_1, q_2, p_1, p_2) = \left(\mu^{\frac{1}{3}} \bar{q}_1 - (1-\mu), \mu^{\frac{1}{3}} \bar{q}_2, \mu^{\frac{1}{3}} \bar{p}_1, \mu^{\frac{1}{3}} \bar{p}_2 - (1-\mu)\right),$$

as the inverse transformation. We now obtain the *HP Hamiltonian* for (8) after $\mu \to 0$

$$H_{HP}(\bar{\mathbf{q}}, \bar{\mathbf{p}}) := \frac{1}{2}\left(\bar{p}_1^2 + \bar{p}_2^2\right) - \frac{1}{\sqrt{\bar{q}_1^2 + \bar{q}_2^2}} + \left(\bar{p}_1 \bar{q}_2 - \bar{p}_2 \bar{q}_1\right) + \frac{1}{2}\left(\bar{q}_2^2 - 2\bar{q}_1^2\right). \tag{9}$$

The effect exerted on C_J by these transformations produces a corresponding first integral C_H for the *HP*, the *Hill-Jacobi constant*, satisfying $H_H = -C_H/2$.

We may now perform the Levi-Civita regularisation step. Defining

$$L(\hat{\mathbf{q}}) := \begin{pmatrix} \hat{q}_1 & -\hat{q}_2 \\ \hat{q}_2 & \hat{q}_1 \end{pmatrix}, \quad \hat{r}^2 := \hat{q}_1^2 + \hat{q}_2^2 = r = \sqrt{q_1^2 + q_2^2},$$

the change (and its completion to conjugate momenta) is

$$\bar{\mathbf{q}} = L(\hat{\mathbf{q}})\hat{\mathbf{q}}, \qquad \bar{\mathbf{p}} = \frac{2}{\hat{r}^2} L(\hat{\mathbf{q}})\hat{\mathbf{p}},$$

which, along with time transformation $d\tau = \frac{4}{r} dt$, calls for a formulation of the problem in the *extended* phase space (see [18]) introducing canonical variables $q_0 = t$, $p_0 = -H$. We have

$$dt = g(\hat{q}_0, \hat{q}_1, \hat{q}_2; \hat{p}_0, \hat{p}_1, \hat{p}_2) d\tau = \frac{r}{4} dt,$$

where $\hat{q}_0 = t$, $\hat{p}_0 = -\bar{H}$, and thus ([18])

$$\begin{aligned} \hat{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) &= g(\hat{q}_0, \hat{\mathbf{q}}; \hat{p}_0, \hat{\mathbf{p}}) \cdot \left[H(\hat{q}, \hat{p}) + p_0\right] \\ &= \frac{\hat{p}_1^2 + \hat{p}_2^2}{2} + \frac{\hat{r}^2}{2}(\hat{p}_1 \hat{q}_2 - \hat{p}_2 \hat{q}_1) - \frac{\hat{r}^2}{4}\left(\hat{q}_2^4 - 4\hat{q}_2^2 \hat{q}_1^2 + \hat{q}_1^4\right) + \frac{p_0 \hat{r}^2}{4} - \frac{1}{4}. \end{aligned}$$

In order to suppress explicit dependence on the energy, we add 1/4 to $\hat{H}$ and set $c = \frac{p_0}{2}$; generalised canonical transformation $(\hat{\mathbf{q}}; \hat{\mathbf{p}}) = \left(2c^{\frac{1}{4}} \mathbf{Q}; 2c^{\frac{3}{4}} \mathbf{P}\right)$ then yields the Hamiltonian

$$H(\mathbf{Q}, \mathbf{P}) = 2c^{3/2}\left[Q^2 + P^2 - 4Q^2\left(P_2 Q_1 - P_1 Q_2\right) - Q^2(Q_1^4 - 4Q_1^2 Q_2^2 + Q_2^4)\right],$$

and it only takes a scaling $\mathcal{H} = \frac{c^{-\frac{3}{2}}}{4} H$ to discard c; we are doing this only for the $c > 0$ case; for $c \leq 0$ there are slightly different transformations and final Hamiltonian (see [16] for details). Anyhow, the final expression is the sum

$$\mathcal{H}(\mathbf{Q}, \mathbf{P}) = H_2 + H_4 + H_6, \tag{10}$$

of homogeneous polynomials of degrees 2, 4 and 6, respectively:

$$\begin{aligned} H_2(\mathbf{Q}, \mathbf{P}) &= \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{2}(Q_1^2 + Q_2^2), \\ H_4(\mathbf{Q}, \mathbf{P}) &= -2(Q_1^2 + Q_2^2)(P_2 Q_1 - P_1 Q_2), \\ H_6(\mathbf{Q}, \mathbf{P}) &= -4(Q_1^2 + Q_2^2)(Q_1^4 - 4Q_1^2 Q_2^2 + Q_2^4). \end{aligned}$$

Our main statements and proofs (that is, the rest of section 3 and sections 4 through 6) will stem from Hamiltonian (10).

3.2 Statement of the main results

A first statement will restrict our study to a particular solution of HP contained in a (complex) linear submanifold of the phase space; we call it an *invariant plane* solution.

Lemma 3.1 *$X_{\mathcal{H}}$ has a particular solution (depending on the energy level h) of the form*

$$Q_1(t) = \frac{\phi(t)}{\sqrt{2}}, \quad Q_2(t) = -\frac{\phi(t)}{\sqrt{2}}i, \quad P_1(t) = \frac{\phi(t)}{\sqrt{2}}i, \quad P_2(t) = \frac{\phi(t)}{\sqrt{2}}, \tag{11}$$

such that for all values $0 < h < \frac{1}{6\sqrt{3}}$, $\phi^2(t)$ is elliptic with two simple *poles in each period parallelogram.*

Using this and some properties of the specific elliptic function involved in $\phi(t)$, we then obtain

Lemma 3.2 *The variational equations (VE) of $X_{\mathcal{H}}$ along the invariant plane solution (11) have a fundamental matrix of the form*

$$\Psi(t) = \begin{pmatrix} \Phi_N(t) & \Phi_N(t)\int_0^t V(\tau)d\tau \\ 0 & \Phi_N(t) \end{pmatrix},$$

where

$$\Phi_N(t) = \begin{pmatrix} \xi_1(t) & \xi_2(t) \\ i\xi_1'(t) & i\xi_2'(t) \end{pmatrix}$$

is a fundamental matrix of the normal *variational equations; furthermore,*

$$\xi_1(t) = \sqrt{-\phi^2(t) + 4\phi^4(t) + 2h},$$

where h is the energy level, ξ_2 is a linear combination of elliptic functions and nontrivial elliptic integrals of first and second classes, and $\int_0^t V(\tau)d\tau$ is a 2×2 matrix function containing logarithmic terms in its diagonal.

This allows a careful study of the P-V extension for the (VE) yielding the following

Theorem 3.3 *The identity component of the Galois differential group of (VE) is non-commutative.*

This proved, it only takes straight observance of Theorem 2.12 and Corollary 2.3 to obtain the result central to the work.

Corollary 3.4 *Hill's problem does not admit a meromorphic integral of motion independent of its Hamiltonian.* □

4 Proof of Lemma 3.1

4.1 Change of variables

Matrix

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

provides for a symplectic change of variables,

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = A\begin{pmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{pmatrix}, \qquad \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = (A^{-1})^T\begin{pmatrix} \bar{P}_1 \\ \bar{P}_2 \end{pmatrix},$$

which in turn transforms Hamiltonian (10) into

$$H = i(\bar{Q}_1\bar{Q}_2 - \bar{P}_1\bar{P}_2) - 4i(3\bar{Q}_1^4 - 2\bar{Q}_1^2\bar{Q}_2^2 + 3\bar{Q}_2^4)\bar{Q}_1\bar{Q}_2 - 4\bar{Q}_1\bar{Q}_2(\bar{Q}_1\bar{P}_1 - \bar{Q}_2\bar{P}_2);$$

the corresponding differential system is now

$$\left.\begin{aligned} \bar{Q}_1' &= -i\bar{P}_2 - 4\bar{Q}_1^2\bar{Q}_2, \\ \bar{Q}_2' &= -i\bar{P}_1 + 4\bar{Q}_1\bar{Q}_2^2, \\ \bar{P}_1' &= \quad i\left[-1 - 4i(2\bar{P}_1\bar{Q}_1 - \bar{P}_2\bar{Q}_2) - 24\bar{Q}_1^2\bar{Q}_2^2 + 60\bar{Q}_1^4 + 12\bar{Q}_2^4\right]\bar{Q}_2, \\ \bar{P}_2' &= \quad i\left[-1 - 4i(\bar{P}_1\bar{Q}_1 - 2\bar{P}_2\bar{Q}_2) - 24\bar{Q}_1^2\bar{Q}_2^2 + 12\bar{Q}_1^4 + 60\bar{Q}_2^4\right]\bar{Q}_1 \end{aligned}\right\} \tag{12}$$

and its phase space displays two invariant planes

$$\pi_1 : \left\{\bar{Q}_2 = \bar{P}_1 = 0\right\}, \quad \pi_2 : \left\{\bar{Q}_1 = \bar{P}_2 = 0\right\}.$$

In any of them, all nontrivial information of (12) reduces to a hyperelliptic equation,

$$\phi'' = -\phi + 12\phi^5, \tag{13}$$

which through multiplication by ϕ' and subsequent integration becomes

$$(\phi')^2 = -\phi^2 + 4\phi^6 + 2h. \tag{14}$$

Defining $w = \phi^2,\ z = 2\phi\phi'$, we arrive to the system

$$w' = z, \quad z' = 4\left(-w + 8w^3 + h\right), \tag{15}$$

whose Hamiltonian (at level zero energy) is

$$\mathcal{K}(w, z) = \frac{1}{2}z^2 + 2w^2 - 8w^4 - 4hw.$$

Remark. The fact that in these invariant planes everything becomes simpler has a clear mechanical meaning. Some difficulties appear in (10) due to the presence of H_4, which mixes positions and momenta. It corresponds to the Coriolis term coming from the rotating frame. The present choice of variables singles out (complex) planes in which this term becomes zero.

4.2 Solution of the new equation

The solution to system (15), or equivalently to equation

$$(w')^2 = -4w^2 + 16w^4 + 8hw,$$

is the inverse of an elliptic integral of the first class:

$$t = \pm \int_0^{w(t)} \frac{dy}{\sqrt{-4y^2 + 16y^4 + 8hy}} + K_1, \quad K_1 \in \mathbb{C},$$

translation $t \mapsto t - K_1$ being the next obvious step. It is a known fact (see [20]) that given a polynomial of degree four without repeated factors,

$$p_4(x) = a_4x^4 + 4a_3x^3 + 6a_2x^2 + 4a_1x + a_0,$$

and defining constants (called *invariants*)

$$g_2 = a_4a_0 - 4a_3a_1 + 3a_2^2, \qquad g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_4a_1^2 - a_3^2a_0,$$

then the solution for

$$t = \int_a^{w(t)} \frac{1}{\sqrt{p_4(x)}}dx,$$

is the following:

$$w(t) = a + \frac{\sqrt{p_4(a)}\wp'(t; g_2, g_3) + \frac{1}{2}p_4'(a)\left[\wp(t; g_2, g_3) - \frac{1}{24}p_4''(a)\right] + \frac{1}{24}p_4(a)p_4'''(a)}{2\left[\wp(t; g_2, g_3) - \frac{1}{24}p_4''(a)\right]^2 - \frac{1}{48}p_4(a)p_4^{(4)}(a)},$$

where $\wp(t; g_2, g_3)$ is the *Weierstrass elliptic function.* In our specific case, this becomes

$$w(t) = \frac{6h}{F(t)}, \quad z(t) = -\frac{18h\wp'(t; g_2, g_3)}{F^2(t)},$$

where $F(t) := 3\wp(t; g_2, g_3) + 1$. In particular,

$$\phi_1(t) = \sqrt{\frac{6h}{F(t)}}, \quad \phi_2(t) = -\phi_1(t),$$

are solutions to original equation (13). Furthermore, a simple calculation proves $h^* = 1/(6\sqrt{3})$ to be a separatrix value in which $\phi_1^2(t) = \phi_2^2(t)$ breaks down into combinations of hyperbolic functions. In order to step into the next subsection, we are therefore assuming $0 < h < h^*$.

4.3 Singularities of $\phi^2(t)$

We are now proving that, for the above range of h, $w(t)$ has two simple poles in each period parallelogram. The sides of that parallelogram will be denoted as $2\omega_1, 2\omega_2$, as usual. In virtue of [3, p. 96], expression $1/(\wp(t)-\wp(t^*))$ (in our case, $\wp(t^*)=-1/3$) has exactly two simple poles in $t^*, -t^*$ (mod $2\omega_1, 2\omega_2$), with respective residues $1/\wp'(t^*)$ and $-1/\wp'(t^*)$. Therefore, all *double* poles, if any, of $1/(\wp(t)-\wp(t^*))$, expanding around $t=t^*$, are precisely those t^* such that $\wp'(t^*)=0$. We have

$$(\wp'(t;g_2,g_3))^2 = 4(\wp(t;g_2,g_3))^3 - g_2\wp(t;g_2,g_3) - g_3 = 4\wp^3 - \frac{4}{3}\wp - \frac{8}{27} + 64h^2,$$

and every pole (whether double or not) must satisfy $\wp(t^*)=-1/3$; it is obvious that $\wp=-1/3$ is not a root of $4\wp^3-\frac{4}{3}\wp-\frac{8}{27}+64h^2$ unless $h=0$. This ends the proof. □

5 Proof of Lemma 3.2

5.1 Layout of the system

Reordering the vector of dependent canonical variables as $(\bar Q_1, \bar P_2, \bar Q_2, \bar P_1)^T$ and restricting the resulting matrix to invariant plane π_1, we obtain the matrix for the linear variational equations as

$$M = \begin{pmatrix} A_1 & B_1 \\ 0 & A_1 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 0 & -i \\ i(-1+60\bar Q_1^4) & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} -4\bar Q_1^2 & 0 \\ -8\bar Q_1 \bar P_2 & 4\bar Q_1^2 \end{pmatrix},$$

and, restricted to our particular solution

$$\bar Q_1 = \phi, \quad \bar Q_2 = 0, \quad \bar P_1 = 0, \quad \bar P_2 = i\phi',$$

we may express the *variational equations* (*VE*)

$$\begin{pmatrix} \bar\xi' \\ \bar\eta' \\ \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} 0 & -i & -4w & 0 \\ i(-1+60w^2) & 0 & -4iz & 4w \\ 0 & 0 & 0 & -i \\ 0 & 0 & i(-1+60w^2) & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \eta_1 \\ \eta_2 \end{pmatrix}, \tag{16}$$

and their lower right block, the *normal variational equations* (*NVE*)

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i(-1+60w^2) & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{17}$$

that is,

$$\xi''(t) = (-1+60w^2(t))\xi(t). \tag{18}$$

Since the next step is to obtain a fundamental matrix for (17), an obvious shortcut is taking w as new independent variable and thus defining $\Xi(w), H(w)$ such that $\xi = \Xi\circ w$ and $\eta = H\circ w$. We then have

$$\frac{d^2\Xi}{dw^2} = 4\left(\frac{w-8w^3-h}{wf(w,h)}\right)\frac{d\Xi}{dw} + \frac{-1+60w^2}{wf(w,h)}\Xi, \tag{19}$$

also expressible in matrix form

$$\begin{pmatrix} \frac{d}{dw}\Xi \\ \frac{d}{dw}H \end{pmatrix} = \frac{1}{\sqrt{wf(w,h)}} \begin{pmatrix} 0 & -i \\ i(-1+60w^2) & 0 \end{pmatrix} \begin{pmatrix} \Xi \\ H \end{pmatrix}, \tag{20}$$

where

$$f = f(w,h) = 4(4w^3 - w + 2h).$$

5.2 Fundamental matrix of the *VE*

We are now interested in the fundamental matrix of (16). Let us start from the block notation

$$\Psi = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{21}$$

P, Q, R, S being 2×2 matrices with their entries in some differential field to be described in section 6. We can assume $\Psi(0) = Id_4$, which, along with the triangular form of (16), assures $R \equiv 0$. In particular, the matrix form of the *NVE* (i.e., of (17) through (20)) can be written as $S' = A_1 S$. Let us now proceed to integrate these normal equations. More precisely, we are going to explicit all necessary information about the fundamental matrix $\Phi_N(t)$ of (17) with initial condition $\Phi_N(0) = Id_2$.

Using well-known properties of $\wp'$ and $\wp''$, it is easy to prove that $\Xi_1(w) = \sqrt{f(w,h)}$ is a solution of (19), and therefore

$$\xi_1(t) = \Xi_1(w(t)) = \frac{\wp'(t; g_2, g_3)}{(3\wp(t; g_2, g_3) + 1)^{\frac{3}{2}}},$$

is a solution of (18). A first solution of the *NVE* (17) is then

$$\begin{pmatrix} \xi_1(t) \\ \eta_1(t) \end{pmatrix} = C_1 \begin{pmatrix} \sqrt{-4w(t) + 16w^3(t) + 8h} \\ -2i\sqrt{w(t)}\left(-1 + 12w^2(t)\right) \end{pmatrix}, \quad C_1 \in \mathbb{C}.$$

We now recall *d'Alembert's method* ([6, p. 160], [8, p. 122]) in order to obtain a second solution of (18) independent of ξ_1. This solution is

$$\xi_2(t) = \xi_1(t) \int_0^t \{\xi_1(\tau)\}^{-2} \, d\tau; \tag{22}$$

See subsection 5.3.1 for further details. After recovering our former independent variable t through composition we have a fundamental matrix for the *NVE*, that is, the block S in (21),

$$\Phi_N(t) = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ i\xi_1' & i\xi_2' \end{pmatrix}.$$

In particular, this also provides $P(t) \equiv S(t)$ since they are both fundamental matrices for the same initial value problem. We now compute the block Q in (21); the standing equations (in vector form) are

$$\begin{pmatrix} \bar{\xi}' \\ \bar{\eta}' \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i(-1 + 60w^2) & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi} \\ \bar{\eta} \end{pmatrix} + \begin{pmatrix} -4w & 0 \\ -4iz & 4w \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{23}$$

where $(\xi, \eta)^T$ are the solutions to the normal equations. $Q(t)$ is obtained by applying the **variation of constants** formula ([6], [8]) to (23)

$$Q(t) = \Phi_N(t) \int_0^t V(\tau) d\tau, \tag{24}$$

where

$$C(t) = \begin{pmatrix} -4w(t) & 0 \\ -4iz(t) & 4w(t) \end{pmatrix}, \qquad V(t) = \Phi_N^{-1}(t) C(t) \Phi_N(t).$$

In other words, the fundamental matrix of (16) has the form

$$\Psi(t) = \begin{pmatrix} \Phi_N(t) & \Phi_N(t) \int_0^t V(\tau) d\tau \\ 0 & \Phi_N(t) \end{pmatrix}. \tag{25}$$

Remark. In view of (24), computing Ψ *explicitly* would now only take the computation of four integrals. *The path we are taking, however, is a different one,* although we are keeping in mind all of this notation and the final expression (25).

5.3 Relevant facts concerning $\Psi(t)$

As said in Section 1 and in the above remark, we are not coping with the symbolical calculus needed to obtain (24) explicitly. Instead, our next aim is to prove only two specific properties of the fundamental matrix Ψ of (16), namely the existence of elliptic integrals and logarithmic terms in its coefficients. The two consecutive steps of *transcendence* forced by these two new objects will provide the rest of our proof.

5.3.1 Elliptic integrals in Φ_N

Let K be the field of all elliptic functions of the complex plane. We know a solution of the *NVE* (17),

$$\xi_1(t) = \sqrt{-w(t) + 4w^3(t) + 2h},$$

and can obtain a second one using (22) and the chain rule. Let us define $\alpha_1, \alpha_2, \alpha_3$ as the values of w for which $f(w,h) = 0$, the functions

$$\beta(w,h) := \arcsin\left(\sqrt{\frac{w(\alpha_3-\alpha_1)}{\alpha_3(w-\alpha_1)}}\right), \qquad k(h) := \sqrt{\frac{\alpha_3(\alpha_1-\alpha_2)}{\alpha_2(\alpha_1-\alpha_3)}},$$

(both attaining complex, nonzero values if $h \in (0,h^*)$ and therefore $w(t) \neq 0$) and let

$$E(\beta|k) := \int_0^\beta (1-k^2\sin^2\theta)^{-\frac{1}{2}}d\theta, \qquad F(\beta|k) := \int_0^\beta (1-k^2\sin^2\theta)^{\frac{1}{2}}d\theta.$$

the *elliptic integrals of first and second class*, respectively (see [3], [9], [20]), We then obtain a fundamental matrix for the *NVE* (20),

$$\overline{\Phi}_N(w) = \begin{pmatrix} \Xi_1(w) & \Xi_2(w) \\ H_1(w) & H_2(w) \end{pmatrix} = \begin{pmatrix} \sqrt{f(w,h)} & g_1\{f_1E(\beta|k)+f_2F(\beta|k)+g_2\} \\ 2i\sqrt{w}(-1+12w^2) & i\frac{d}{dw}(g_1\{f_1E(\beta|k)+f_2F(\beta|k)+g_2\}) \end{pmatrix},$$

for some $f_1 = f_1(h), f_2 = f_2(h), g_1 = g_1(w,h), g_2 = g_2(w,h)$, the first three nonvanishing if $h \in (0,h^*)$, and the last two linked to w by *algebraic* equations. In particular, this yields our fundamental matrix $\Phi_N(t) = \overline{\Phi}_N(w(t))$ for (17).

It is to be noted that $E(\beta|k)$ and $F(\beta|k)$ *are transcendental over* K; more precisely, nontrivial elliptic integrals of the first and second classes cannot be expressed in terms of elliptic functions under any relation of algebraic dependence (see [3]). This, as will be stated in the next section, is the first nontrivial contribution to the identity component of the Galois group.

5.3.2 Logarithms in Ψ

Let us prove the existence of terms with nonzero residue in the diagonal of matrix $V(t)$. We know

$$\Phi_N(t) = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ i\xi_1' & i\xi_2' \end{pmatrix},$$

and this, being the fundamental matrix of a Hamiltonian linear system, is a symplectic matrix itself (see, e.g., [11]). Its inverse is therefore

$$\Phi_N^{-1}(t) = \begin{pmatrix} i\xi_2' & -\xi_2 \\ -i\xi_1' & \xi_1 \end{pmatrix},$$

and the integrand in (24) becomes

$$V(t) = 4i\begin{pmatrix} -w(\xi_2\xi_1'+\xi_1\xi_2')+w'\xi_1\xi_2 & -\xi_2(2\xi_2'w-\xi_2w') \\ (2w\xi_1'-w'\xi_1)\xi_1 w(\xi_1\xi_2'+\xi_1'\xi_2)-w'\xi_1\xi_2 & \end{pmatrix} =: 4i\begin{pmatrix} u(t) & v_1(t) \\ v_2(t) & -u(t) \end{pmatrix}.$$

For every $h \in (0,h^*)$, and taking profit of what was proved in 4.3, we expand these four entries around a simple pole t^* of $w(t)$; expressing only the first term in each power series, we have

$$\begin{aligned} w(t) &= C_0(t-t^*)^{-1} + O(1), \\ \xi_1(t) &= 2C_0^{\frac{3}{2}}(t-t^*)^{-\frac{3}{2}} + O\left((t-t^*)^{-\frac{1}{2}}\right), \\ \xi_2(t) &= \frac{1}{8}C_0^{-\frac{3}{2}}(t-t^*)^{\frac{5}{2}} + O\left((t-t^*)^{\frac{7}{2}}\right), \end{aligned}$$

for some $C_0 = C_0(h) \in \mathbb{C}$; therefore,

$$\begin{aligned} u(t) &= -\frac{1}{2}C_0(t-t^*)^{-1} + O(1), \\ v_1(t) &= -\frac{3}{32C_0^2}(t-t^*)^3 + O\left((t-t^*)^4\right), \\ v_2(t) &= -8C_0^4(t-t^*)^{-5} + O\left((t-t^*)^{-4}\right). \end{aligned}$$

Hence, and except for the only value of h forcing $C_0 = 0$ (i.e. $h = 0$), we have a nonzero residue in $u(t)$, which results in the aforementioned logarithmic terms in the diagonal of

$$\int_0^t V(\tau)d\tau = \begin{pmatrix} \int_0^t u(\tau)d\tau & \int_0^t v_1(\tau)d\tau \\ \int_0^t v_2(\tau)d\tau & -\int_0^t u(\tau)d\tau \end{pmatrix}. \quad \square$$

Whence appears a class of functions, namely logarithms, which *cannot be linked with elliptic functions nor elliptic integrals of the first and second classes under relations of algebraic dependence* (see [3] once more). This, as will be stated shortly, results in a new nontrivial, transcendental extension of fields of functions. Both this and the previous extension will ultimately render the component identity of the Galois group non-commutative.

6 Proof of Theorem 3.3

Let us interpret our results in terms of field extensions. First of all, we note that using coordinates $(x, y) = (\phi, \phi')$ all solutions of the equation (14) roam in the hyperelliptic curve

$$\Gamma_h := \left\{(x, y) \in \mathbb{C}^2 : y^2 = -x^2 + 4x^6 + 2h\right\}.$$

Denote by VE_{Γ_h} the expression of VE (16) and $\widehat{G} := \mathrm{Gal}(EV_{\Gamma_h})$; let $\widehat{G}^0$ be the identity component of the latter. The previous transformation $w = x^2, z = 2xy$ induces a finite branched covering

$$\Gamma_h \to \Lambda_h,$$

where Λ_h is the elliptic curve defined by

$$\Lambda_h := \{\frac{1}{2}z^2 + 2w^2 - 8w^4 - 4hw = 0\}$$

and the group $\widehat{G}^0$ does not change; this is a consequence [13, Theorem 5] (see also [12, Theorem 2.5]), according to which the identity component of the Galois group remains invariant under covering maps of this sort. We may thus keep with the abuse in notation of calling $\widehat{G}$ and $\widehat{G}^0$ the Galois group of VE_{Γ_h} and its identity component, respectively, *now in variable* w.

Keeping K $(= \mathcal{M}(\Lambda_h))$ as the field of all elliptic functions, let us explicit the Picard-Vessiot extension over K for VE_{Γ_h}

1. First of all, let us define the extension
$$K \subset K_1 := K(\xi_1, \xi_1'),$$
based on the adjunction of the first solution ξ_1 of (17) and its derivative, which is an algebraic (in fact, quadratic) one. The identity component of the Galois group of this extension is, therefore, trivial.

2. Second of all, adjoining to this new field the solution ξ_2 from (22) we obtain the extension
$$K_1 \subset L_1 := K_1(\xi_2, \xi_2') = K(\xi_1, \xi_1', \xi_2, \xi_2'),$$
which is nontrivial (for it introduces objects different from elliptic functions, see subsection 5.3.1), and *transcendental*, since it is defined exclusively by an adjunction of quadratures.

3. Third of all, adjoining the matrix integral from (24) to L_1, we have
$$L_1 \subset L_2 := L_1\left(\int_0^t u, \int_0^t v_1, \int_0^t v_2\right),$$
which is also given by quadratures (three in this case), and thus transcendental. It is also nontrivial due to the logarithm in $\int_0^t u(\tau)d\tau$ found in subsection 5.3.2.

So far, the Picard-Vessiot extension $L_2 \mid K$ of the VE has been decomposed as a tower of P-V extensions

$$K \subset K_1 \subset L_1 \subset L_2.$$

Let $\widehat{G} := \mathrm{Gal}(L_2 \mid K)$. The fact that each of above extensions results from adjoining either algebraic elements or quadratures renders $L_2 \mid K$ a *Liouville* extension, and thus $\widehat{G}^0$ a *solvable* group. Our aim is to prove that the (stronger) hypothesis demanded by theorem 2.12 is not fulfilled, i.e. $\widehat{G}^0$ *is not commutative.*

Step 0. Let us recall some facts and introduce major conventions prior to the proof. Given an arbitrary P-V extension $L \mid K$ with Galois group G:

1. We are identifying elements σ of G with the corresponding matrices $R(\sigma)$ defining representation ρ in (5). In other words, we will be dealing indistinctively with the *linear algebraic* group G and the *matrix* group $\rho(G)$.

2. Given any $m \in \mathbb{N}$, and using the propagation of morphism axioms of any $\sigma \in G$ to elements of $\mathcal{M}_m(K)$,

$$(\sigma(a_{i,j}))_{1\le i,j\le m}(\sigma(b_{i,j}))_{1\le i,j\le m} = \left(\sum_{r=1}^{m}\sigma(a_{i,r})\sigma(b_{r,j})\right)_{1\le i,j\le m} = \left(\sigma\left(\sum_{r=1}^{m}a_{i,r}b_{r,j}\right)\right)_{1\le i,j\le m},$$

we will engage in as many abuses of notation as necessary when extending σ entrywise to any $m \times m$ matrix, regardless of m.

Step 1. Since $L_2 \mid K_1$ is transcendental and $K_1 \mid K$ is algebraic, we may assume the base field of the tower to be K_1, for

$$\widehat{G}^0 \cong \mathrm{Gal}(L_2 \mid K_1);$$

indeed, all of the contributions derived from transcendental elements stay in the identity component, and the last part of Theorem 2.9 asserts

$$\widehat{G}/\widehat{G}^0 \cong \mathrm{Gal}(K_1 \mid K).$$

This restricts our study to $\mathrm{Gal}(L_2 \mid K_1)$, besides proving it connected and thus equal to its connected component; in a further abuse of notation, we may call it $\widehat{G}^0$ again.

Step 2. Let us prove that the elements $R(\sigma)$ of the Galois group $\widehat{G}^0 = \mathrm{Gal}(L_2 \mid K_1)$ are unipotent matrices of the following kind:

$$\widehat{G}^0 = \left\{ \begin{pmatrix} 1 & \mu & A_1 & A_2 \\ 0 & 1 & A_3 & A_4 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu \in S_0;\ A_1, A_2, A_3, A_4 \in T_0 \right\} \tag{26}$$

for some subsets $S_0, T_0 \subset \mathbb{C}$ such that $S_0 \neq \{0\}$.

Indeed, expressing any such $R(\sigma)$ in block notation,

$$R(\sigma) = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

let us write the equation $\sigma(\Psi) = \Psi R(\sigma)$ explicitly

$$\begin{aligned}
\sigma(\Psi) &= \begin{pmatrix} \Phi_N(t) & \Phi_N(t)\int_0^t V(\tau)d\tau \\ 0 & \Phi_N(t) \end{pmatrix}\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \\
&= \begin{pmatrix} \Phi_N M_1 + (\Phi_N \int_0^t V)M_3 & \Phi_N M_2 + (\Phi_N \int_0^t V)M_4 \\ \Phi_N M_3 & \Phi_N M_4 \end{pmatrix} \qquad (27)\\
&= \begin{pmatrix} \sigma(\Phi_N) & \sigma(\Phi_N \int_0^t V) \\ \sigma(0) & \sigma(\Phi_N) \end{pmatrix}. \qquad (28)
\end{aligned}$$

The last two equations imply $M_1 = M_4$ and $M_3 = 0$. We are now working on $\mathrm{Gal}(L_2 \mid K_1)$, and the first column of $\Phi_N(t)$ is $(\xi_1(t), i\xi'(t))^T$, whose entries are already in K_1; thus, morphism σ must leave it fixed: that is, defining

$$M_1 = M_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$\sigma\begin{pmatrix} \xi_1 & \xi_2 \\ i\xi_1' & i\xi_2' \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_1 b + \xi_2 d \\ i\xi_1' & i\xi_1' b + i\xi_2' d \end{pmatrix},$$

so the first column in M_1 and M_4 must be $(a,c)^T = (1,0)^T$. Their second column $(b,d)^T$ must then be of the form $(\mu, 1)^T$ for some $\mu \in \mathbb{C}$, since $\sigma(\Phi_N(t)) = \Phi_N(t)M_1$ must be symplectic and thus $\det(\sigma(\Phi_N(t))) = 1$. This altogether forces the given expression for the diagonal blocks in (26).

The actual domain of definition S_0 for μ will be seen in the next step, but we can already assert μ is not identically zero. For, if it were, then the action of $\widehat{G}^0$ would leave $\xi_2, \xi_2' \in L_2$ fixed; this, the definition of L_2, L_1 and the third part of Theorem 2.6 would in turn imply $\xi_2, \xi_2' \in K_1$, i.e. we would have elliptic integrals in an algebraic extension of the field of elliptic functions; as said at the end of subsection 5.3.1, this is absurd. Consequently, $S_0 \neq \{0\}$.

STEP 3. Let us prove $S_0 = \mathbb{C}$. Indeed, the action of σ on $\mathrm{diag}\,(\Phi_{\mathrm{N}}, \Phi_{\mathrm{N}})$ is of the form

$$\tilde{G} = \left\{ \begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu \in S_0 \right\}, \tag{29}$$

which is a representation of the additive group $\mathbb{C}_+$ over the field of constants. $\mathbb{C}_+$ has only two *algebraic* subgroups, namely itself and $\{0\}$; step 2 already discarded the first case, so we are left with $S_0 = \mathbb{C}$.

The form of (29) clearly embodies our need for the *whole* fundamental matrix Ψ; in other words, solving the *NVE* is not enough to prove Theorem 3.3. Indeed, the theorems due to Ziglin and Morales-Ramis are of no use up to this step, since $\tilde{G}$ is a *commutative* group of unipotent (and thus *resonant*) matrices.

STEP 4. We are now giving a new provisional form to our group. We already know $\sigma\Phi_N = \Phi_N M_1$; let us first study the action of any $\sigma \in \widehat{G}^0$ over the four entries of $\int V$. Applying the identity $\partial \circ \sigma \equiv \sigma \circ \partial$ on $\int V$ and integrating the resulting equation, we obtain

$$\sigma \int \Phi_N^{-1} C \Phi_N = \int \sigma \left(\Phi_N^{-1} C \Phi_N\right) + M, \tag{30}$$

for some

$$M = \begin{pmatrix} \delta & \gamma \\ \beta & \kappa \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}).$$

Besides, using $\sigma C = C$ we have

$$\sigma(\Phi_N^{-1} C \Phi_N) = (\sigma\Phi_N)^{-1} C (\sigma\Phi_N) = M_1^{-1} (\Phi_N^{-1} C \Phi_N) M_1, \tag{31}$$

which translates (30) into

$$\sigma \int \Phi_N^{-1} C \Phi_N = M_1^{-1} \left(\int \Phi_N^{-1} C \Phi_N \right) M_1 + M, \tag{32}$$

that is, the following separate actions of σ on the entries of $\int(\Phi_N^{-1} C \Phi_N)$:

$$\int u \quad \mapsto \quad \int u - \mu \int v_2 + \delta, \tag{33}$$

$$\int v_1 \quad \mapsto \quad 2\mu \int u - \mu^2 \int v_2 + \int v_1 + \gamma, \tag{34}$$

$$\int v_2 \quad \mapsto \quad \int v_2 + \beta, \tag{35}$$

$$-\int u \quad \mapsto \quad \mu \int v_2 - \int u + \kappa, \tag{36}$$

the first and fourth of which readily imply $\delta = -\kappa$.

On the other hand, (31) allows us to write (27) in the equivalent form

$$\sigma(\Psi) = \begin{pmatrix} \sigma\Phi_N & (\sigma\Phi_N)\left[M_1^{-1} M_2 + \int \sigma(\Phi_N^{-1} C \Phi_N)\right] \\ 0 & \sigma\Phi_N \end{pmatrix},$$

morphism axioms (and (28)) now rendering the latter's upper right block equal to $(\sigma\Phi_N)\,(\sigma\!\int \Phi_N^{-1} C \Phi_N)$, and thus forcing the following to hold,

$$\sigma \left(\int \Phi_N^{-1} C \Phi_N \right) = M_1^{-1} M_2 + \int \sigma(\Phi_N^{-1} C \Phi_N),$$

which along with (32) yields $M_1^{-1} M_2 = M$. This gives us the explicit form for the upper 2×2 block in the generic expression (26) for $R(\sigma)$:

$$M_2 = M_1 M = \begin{pmatrix} -\kappa + \mu\beta & \gamma + \mu\kappa \\ \beta & \kappa \end{pmatrix}.$$

In particular,

$$\widehat{G}^0 = \left\{ \begin{pmatrix} 1 & \mu & -\kappa+\mu\beta & \gamma+\mu\kappa \\ 0 & 1 & \beta & \kappa \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu \in \mathbb{C}, \kappa \in S_1, \beta \in S_2, \gamma \in S_3 \right\} \tag{37}$$

for some subsets $S_1, S_2, S_3 \subset \mathbb{C}$.

Step 5. Further specification of the domains of definition of κ, β, γ will finish our proof. We already know $S_0 = \mathbb{C}$ is the domain for μ. Given any element $a_{\mu,\kappa,\beta,\gamma} \in \widehat{G}^0$, we have

$$\begin{aligned} a_{\mu,\kappa,\beta,\gamma} &= \begin{pmatrix} 1 & \mu & -\kappa+\mu\beta & \gamma+\mu\kappa \\ 0 & 1 & \beta & \kappa \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\kappa & 0 \\ 0 & 1 & 0 & \kappa \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \\ &=: U_\mu V_\kappa W_\beta X_\gamma, \end{aligned}$$

Assume, for the moment, $S_1 = S_2 = S_3 = \mathbb{C}$. Defining the subgroups of $\widehat{G}^0$ generated by U_μ and $V_\kappa W_\beta X_\gamma$,

$$G = \left\{ \begin{pmatrix} 1 & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu \in \mathbb{C} \right\}, \qquad H = \left\{ \begin{pmatrix} 1 & 0 & -\kappa & \gamma \\ 0 & 1 & \beta & \kappa \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \kappa, \beta, \gamma \in \mathbb{C} \right\},$$

it is easy to prove that G is a representation of the additive group $\mathbb{C}_+$, and H, unlike G, is a normal subgroup of $\widehat{G}^0$. The facts $G \cap H = Id_4$ and $\widehat{G}^0 = GH$ therefore prove $\widehat{G}^0$ to be the *semidirect product* ([7]) of $G \simeq \mathbb{C}_+$ and H.

Consider, besides, the three subgroups of H formed by matrices of the form $V_\kappa, W_\beta, X_\gamma$, respectively; they are all normal subgroups of H and representations of $\mathbb{C}_+$, and their pairwise intersections are $\{Id_4\}$. Therefore, writing $\times$ as the direct product and $\ltimes$ as the semidirect product, we have

$$\widehat{G}^0 = G \ltimes H \simeq \mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+ \times \mathbb{C}_+).$$

So far we have assumed $S_1 = S_2 = S_3 = \mathbb{C}$; were that false for any of them, say, S_i, it would still have to be the underlying set of a *linear algebraic* subgroup of the additive group $\mathbb{C}_+$, since each of κ, β, γ comes from one quadrature; indeed, if we consider L_1 as our base group, we have $\mathrm{Gal}(L_2 \mid L_1) = H$ and $\mu = 0$ in formulae (33)–(36), which in turn yield *additive* actions on $\int V$:

$$\int u \mapsto \int u - \kappa, \quad \int v_1 \mapsto \int v_1 + \gamma, \quad \int v_2 \mapsto \int v_2 + \beta. \tag{38}$$

Parameters κ, β, γ thus belong to an algebraic subgroup of $\mathbb{C}_+$ (i.e., $\mathbb{C}$ or $\{0\}$), so we end up with

$$S_i \in \{\{0\}, \mathbb{C}\}, \quad i = 1, 2, 3.$$

However, κ *is not* identically zero. If it were, (38) would then prove $\int u$ invariant under any morphism $\sigma \in \widehat{G}^0$; this and the logarithmic term in $\int u$ are obviously in contradiction with part 3 of theorem 2.6. Furthermore, γ is also *not* identically zero. Otherwise the product in $\widehat{G}^0$ would not be defined. Therefore,

$$S_1 = S_3 = \mathbb{C}, \qquad S_2 \in \{\{0\}, \mathbb{C}\}. \tag{39}$$

Resetting K_1 as our base field in order to obtain the remaining parameter μ, and using both the factorisation $a_{\mu,\kappa,\beta,\gamma} = U_\mu V_\kappa W_\beta X_\gamma$ and the isomorphism provided by the second part of Fundamental Theorem 2.9,

$$\mathrm{Gal}(L_1 \mid K_1) \simeq \mathrm{Gal}(L_2 \mid K_1)/\mathrm{Gal}(L_2 \mid L_1),$$

we actually have (in this case) a splitting of $\mathrm{Gal}(L_2 \mid K_1)$ as the semidirect product

$$\widehat{G}^0 = G \ltimes H = G \ltimes \left(\widehat{G}^0/G\right) \simeq \mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+ \times S_2).$$

Both this and condition (39) force $\widehat{G}^0$ to be isomorphic to one of the following:

$$\mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+ \times \mathbb{C}_+) \quad \text{or} \quad \mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+),$$

non-commutative, in any case. □

Remarks.

1. *An alternative approach to step 4.*

Recall the **unipotent radical** of any given linear algebraic group G as being the (unique) largest closed, connected, normal subgroup formed by unipotent matrices in G. We know, thanks to [2, p.27], that the unipotent radical of the symplectic group $Sp(2, \mathbb{C}$ may be expressed, in an suitable basis $\{e_1, e_2, e_3, e_4\}$, as follows:

$$G_u = \left\{ \begin{pmatrix} 1 & \mu & \kappa + \mu\beta & \gamma \\ 0 & 1 & \beta & \kappa \\ 0 & 0 & 1 & -\mu \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mu, \kappa, \beta, \gamma \in \mathbb{C} \right\}.$$

The coordinates being still canonical, let us first transform the fundamental matrix (e_1, e_2, e_3, e_4) into $(e_1, e_2, \tilde{e}_3, e_4)$, where $e_3 = -\tilde{e}_3$. The fact that these new coordinates are not canonical will not affect our result: the symplectic manifold, the symplectic bundle and the Galois group do not change, and submatrix $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$ is still symplectic; the only relevant fact is that they are not Darboux coordinates anymore.

Subsequent changes $\beta \mapsto -\beta$ and $\gamma \mapsto \gamma + \mu\kappa$, in this order, make the representation become

$$G_u \cong \left\{ \begin{pmatrix} A & AB \\ 0 & A \end{pmatrix} : A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} -\kappa & \gamma \\ \beta & \kappa \end{pmatrix},\ \mu, \kappa, \beta, \gamma \in \mathbb{C} \right\},$$

that is, exactly in the form (37).

Let us return to $\widehat{G}^0$. The fact that this group is a connected, normal and unipotent subgroup of the symplectic group assures $\widehat{G}^0 \subset G_u$. This is just what was proven in step 4.

2. *An alternative ending to Step 5.*

In the general expression of $\widehat{G}^0$, as we know, the domain of definition for μ, κ, γ is all of $\mathbb{C}$, and the one for β is either $\mathbb{C}$ once again or $\{0\}$; given any $a_i := a_{\mu_i,\kappa_i,\beta_i,\gamma_i} \in \widehat{G}^0,\ i = 1, 2$, their commutator is

$$\begin{aligned} a_1 a_2 a_1^{-1} a_2^{-1} &= a_{\mu_1,\kappa_1,\beta_1,\gamma_1} a_{\mu_2,\kappa_2,\beta_2,\gamma_2} a^{-1}_{\mu_2,\kappa_2,\beta_2,\gamma_2} a^{-1}_{\mu_1,\kappa_1,\beta_1,\gamma_1} \\ &= \begin{pmatrix} 1 & 0 & \mu_1\beta_2 - \beta_1\mu_2 & 2(\mu_1\kappa_2 - \mu_2\kappa_1) - \mu_1\beta_2(\mu_1 + 2\mu_2) + \beta_1\mu_2(\mu_2 + 2\mu_1) \\ 0 & 1 & 0 & \beta_1\mu_2 - \mu_1\beta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

It is now a simple exercise to prove this is not identically equal to Id_4, which also proves $\widehat{G}^0$ non-commutative.

3. *Some comments on preliminary methodology and checks.*

Reliance on numerics not only provided significant preliminary information prior to the actual proof; it also shed some light into the ensuing algebraic framework, namely in the relationship between the monodromy group and the presumably larger one $\widehat{G}^0 = \mathrm{Gal}(L_2 \mid K_1)$ containing it. We first considered system (13), along with the related variational equations as given in (16), from a numerical point of view. Clearly, for $h \in (0, h^*)$ (13) has both a real and a purely imaginary period (with ϕ' real in both cases). It is enough to take $(\phi(0), \phi'(0)) = (0, \sqrt{2h})$ as initial conditions and then real or imaginary times, respectively.

Let M_1 and M_2 be the monodromy matrices along the real and the imaginary periods, respectively. These matrices have the common structure

$$M = \begin{pmatrix} 1 & p & q & 0 \\ 0 & 1 & 0 & -q \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which, of course, turns out to be a particular case of (37). In the real period case $p = ia$, $q = b(1 - i)$ has been found, and in the imaginary period case $p = c$, $q = d(1 + i)$, with $a, b, c, d \in \mathbb{R}$, all of them positive. The computed values of a, b, c, d are displayed, as a function of h, in Figure 1. Let p_1, q_1 and p_2, q_2 the p, q entries in M_1 and M_2. These matrices commute if and only if $\Delta := p_1 q_2 - q_1 p_2 = (i - 1)(ad + bc) = 0$. From the positive character of

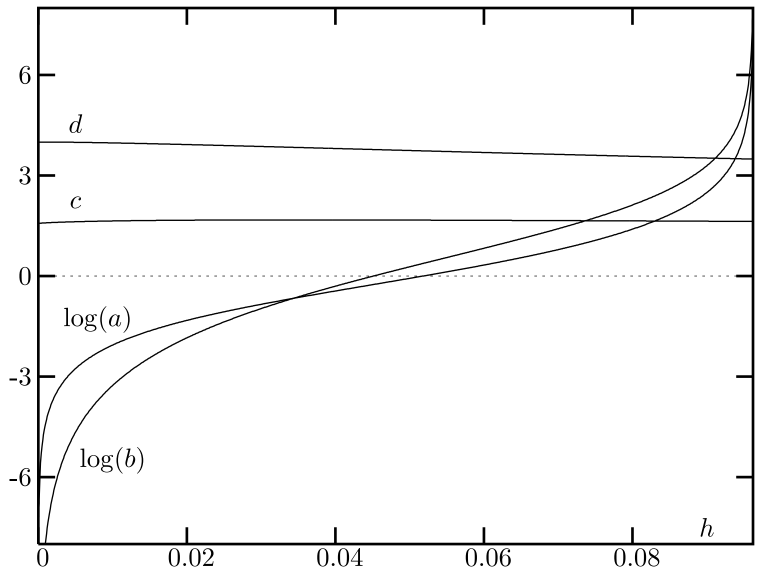

Figure 1: Values of parameters a, b, c, d (first two in logarithmic scale) appearing in the monodromy matrices for real and purely imaginary period. See the text for details.

a, b, c, d it follows $\Delta \neq 0$ for all $h \in (0, h^*)$. Furthermore, the coefficient of $(i-1)$ in Δ is far away from zero, except for small h, a domain amenable to perturbative computations.

The group generated by M_1 and M_2, which is a subgroup of the monodromy one and, hence, a subgroup of the Galois group, has the same structure than G_u with $\beta = 0$. This is in favour of the second of the options presented for $\widehat{G}^0$, i.e., $\widehat{G}^0 \simeq \mathbb{C}_+ \ltimes (\mathbb{C}_+ \times \mathbb{C}_+)$.

A last remark

In reference [10], the authors start from (9) expressed in polar canonical coordinates and *without the aforementioned changes allowing* $k = \omega = R = 1$, that is,

$$H_H = \frac{1}{2}\left(p_\rho^2 + \frac{p_\theta^2}{\rho^2}\right) - \frac{k}{\rho} - \omega p_\theta + \omega^2 \frac{\rho^2}{2}(1 - 3\cos^2\theta).$$

The first step is to construct the more general Hamiltonian

$$\tilde{H}_{b,\omega} = H_0 + \omega^2 H_2,$$

assuming ω^2 is sufficiently small, where H_0 is the Hamiltonian of Kepler's classical problem in a reference frame rotating with angular velocity b, and $\tilde{H}_{\omega,\omega} = H_H$. The strategy followed henceforth is based in proving there is no first integral Φ at a time independent of $\tilde{H}_{b,\omega}$ and analytical with respect to ω in an open neighbourhood of $\omega = 0$.

The authors presumably afford their non-integrability proof restricting it to first integrals which are analytical with respect to the conjugate variables *and the parameter* ω; in other words, their proof does not deny, for instance, the existence of additional first integrals meromorphic *with respect to phase variables* and satisfying the Liouville-Arnold hypotheses. That denial, which discards any restriction concerning ω, comes precisely from our proof.

(**Pregunta 4c**: No tinc clar que s'hagin usat TOTES les fòrmules que porten número. Cal verificar!)

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