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Darboux Polynomials and First Integrals of Natural Polynomial Hamiltonian Systems

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Abstract

We show that for a natural polynomial Hamiltonian system the existence of a single Darboux polynomial (a partial polynomial first integral) is equivalent to the existence of an additional first integral functionally independent with the Hamiltonian function. Moreover, we show that, in a case when the degree of potential is odd, the system does not admit any proper Darboux polynomial, i.e., the only Darboux polynomials are first integrals.

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1 Introductions

Let us consider a system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{v}(\boldsymbol{x}), \qquad \boldsymbol{x} = (x_1, \ldots, x_n) \in U \subset \mathbb{C}^n, \tag{1}$$

where the right hand sides $\boldsymbol{v}(\boldsymbol{x}) = (v_i(\boldsymbol{x}), \ldots, v_n(\boldsymbol{x}))$ are smooth on a domain U in $\mathbb{C}^n$. Let $\boldsymbol{\varphi} : \mathbb{R} \supset I \mapsto U$ be a solution of (1) defined on a certain open non-empty interval I of the real axis. A continuous function $F : U \to \mathbb{R}$ is called a ***first integral*** of system (1) if it is constant along its solutions, i.e., if function $F \circ \boldsymbol{\varphi}$ is constant on its domain of definition for an arbitrary solution $\boldsymbol{\varphi}$ of (1). When F is differentiable, then it is a first integral of system (1) if

$$L_{\boldsymbol{v}}(F)(\boldsymbol{x}) := \sum_{i=1}^{n} v_i(\boldsymbol{x}) \partial_i F(\boldsymbol{x}) = 0, \qquad \partial_i \equiv \frac{\partial}{\partial x_i}, \tag{2}$$

where $L_{\boldsymbol{v}}$ is the Lie derivative along vector field $\boldsymbol{v}$. If F is a first integral of (1), then its constant value levels

$$M_c(F) = \{\boldsymbol{x} \in U \,|\quad F(\boldsymbol{x}) = c\,\}, \tag{3}$$

are invariant with respect to the flow generated by (1). It appears that together with the first integrals it is worth looking for partial first integrals. We say that F is a ***partial first integral*** if its zero level $M_0(F)$ is invariant with respect to the flow generated by (1). A differentiable function F is a partial first integral of system (1) if $L_{\boldsymbol{v}}(F)(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$ such that $F(\boldsymbol{x}) = 0$.

There are a lot of practical and theoretical reasons why we would like to know if a given system of differential equations possesses a first integral. However, in its most complete and general case this problem is intractable. We can simplify it a little bit, e.g., by imposing additional conditions on the smoothness of the right hand sides of (1) and smoothness of the first integrals. Nevertheless, even if we assume that the right hand sides of the investigated system are polynomial and we look for a polynomial first integral the problem is very hard to approach.

In this paper we investigate systems (1) of a special form with the polynomial right hand sides. Namely, we consider polynomial Hamiltonian systems defined on $\mathbb{C}^{2m}$. Let $\boldsymbol{q} = (q_1, \ldots, q_m)$ and $\boldsymbol{p} = (p_1, \ldots, p_m)$ denote the standard canonical coordinates and $H = H(\boldsymbol{q}, \boldsymbol{p})$ is a polynomial Hamiltonian function. We say that a first integral $F = F(\boldsymbol{q}, \boldsymbol{p})$ of the canonical Hamiltonian equations is an ***additional first integral*** if F and H are functionally independent.

It was observed [3; 6] that for most known polynomial Hamiltonian systems an additional first integral is a polynomial one and its degree with respect to $\boldsymbol{p}$

is small. However, there are also examples of polynomial Hamiltonian systems for which an additional first integral is more complicated.

Example 1 *For the Hamiltonian system given by*

$$H = \frac{1}{2}(p_1^2 + p_2^2) + 2q_2p_1p_2 - q_1$$

an additional first integral has the following form

$$F = p_2 \exp(p_1^2),$$

see [6].

Example 2 *For the Hamiltonian system given by*

$$H = \frac{1}{2}(p_1^2 + p_2^2) - f(p_1, p_2)(p_1q_1 - \alpha p_2q_2), \qquad \alpha \in \mathbb{C},$$

where $f(p_1, p_2)$ is an arbitrary polynomial, an additional first integral has the following form

$$F = p_1^{\alpha} p_2.$$

If $\alpha \notin \mathbb{Q}$ this first integral is not meromorphic, when $\alpha \in \mathbb{Q}$, then it is algebraic, when $\alpha \in -\mathbb{N}$, then it is rational, and finally when $\alpha \in \mathbb{N}$, then it is a polynomial of degree $\alpha + 1$ with respect to the momenta.

These examples show that it is difficult to expect that the additional first integrals of a polynomial Hamiltonian system should be distinguished from the first integrals of an arbitrary polynomial system.

In mechanics there exists a well defined class of natural Hamiltonian systems for which Hamilton's function is a sum of the kinetic and potential energy. In this paper we consider systems for which the Hamiltonian has the following form

$$H = \frac{1}{2}\sum_{i=1}^{m} \mu_i p_i^2 + V(\boldsymbol{q}), \tag{4}$$

where $V(\boldsymbol{q})$ is a polynomial, and $\mu_i \in \mathbb{C}$ for $i = 1, \ldots, m$. To avoid trivialities we assume that $m > 1$ and $\deg V(\boldsymbol{q}) > 2$. For such systems canonical equations have the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q_i &= \mu_i p_i, \\ \frac{\mathrm{d}}{\mathrm{d}t} p_i &= -\frac{\partial V}{\partial q_i}, \qquad i = 1, \ldots, m. \end{aligned} \tag{5}$$

We did our best but we have not find even a single example of a system with the Hamiltonian of the form (4) admitting an additional first integral which is not a polynomial one. Thus, it seems reasonable to formulate the following questions:

(1) Does there exist a natural polynomial Hamiltonian system admitting a proper rational additional first integral?
(2) Does the existence of a proper rational additional first integral of a natural polynomial Hamiltonian system imply the existence of an additional polynomial first integral?
(3) Does there exist a natural polynomial Hamiltonian system admitting an analytic additional first integral which is not a polynomial one?
(4) Does the existence of an analytic additional first integral of a natural polynomial Hamiltonian system imply the existence of a polynomial one?
(5) Does there exist a natural polynomial Hamiltonian system admitting a meromorphic additional first integral which is not a rational one?
(6) Does the existence of a meromorphic additional first integral of a natural polynomial Hamiltonian system imply the existence of a rational one?

By a proper rational first integral we mean a rational first integral which is not a ratio of two polynomial first integrals. In the above questions we do not mention algebraic first integrals, because, as it is well known, the existence of an algebraic additional first integral implies the existence of a rational one, see e.g. [4].

In this paper we focused our attention on the first two questions. Roughly speaking, the main result of this paper is an affirmative answer to the second question.

In fact our result is stronger. Let us denote by L_H the Lie derivative along the Hamiltonian vector field defined by (5). A polynomial $G = G(\boldsymbol{q}, \boldsymbol{p}) \neq 0$ is called a ***Darboux polynomial*** of system (5) if

$$L_H G = \Lambda G \tag{6}$$

for a certain polynomial $\Lambda = \Lambda(\boldsymbol{q}, \boldsymbol{p})$, which is called ***cofactor*** of G. When $\Lambda = 0$, then the Darboux polynomial is a first integral. We say that G is a ***proper Darboux polynomial*** if $\Lambda \neq 0$. Let us assume that polynomials $G = G(\boldsymbol{q}, \boldsymbol{p})$ and $Q = Q(\boldsymbol{q}, \boldsymbol{p})$ are coprime, i.e., G and Q do not have common divisors different from constants. Then it is easy to show that if $F = G/Q$ is a proper rational first integral of system (5), then G and Q are proper Darboux polynomials with the same cofactor Λ.

The main results of this paper are formulated in the following theorems.

Theorem 1 *Assume that in Hamiltonian* (4) *potential* $V(\boldsymbol{q})$ *is of odd degree, then every Darboux polynomial of Hamiltonian system* (5) *is a first integral.*

Theorem 2 *Assume that in Hamiltonian* (4) *at least two* μ_i *are not zero and potential* $V(\boldsymbol{q})$ *and is of even degree. If Hamiltonian system* (5) *possesses a proper Darboux polynomial, then it admits an additional polynomial first integral.*

The statements of the above theorems are rather amazing. In fact, it is obvious that the conditions for the existence of a Darboux polynomial are much weaker than the conditions for the existence of a polynomial first integral. The first theorem says that it is not the case for natural polynomial Hamiltonian systems with a potential of odd degree. Moreover, let us notice that in the second theorem we do not exclude the existence of a proper Darboux polynomial (there are many examples where it exists). However, for the existence of a proper rational first integral we need ***two*** proper Darboux polynomials with the ***same*** cofactor. The second theorem says that one Darboux polynomial is enough to construct an additional polynomial first integral.

Proofs of the above theorems are purely algebraic and this is why we collect all necessary facts from (elementary) differential algebra in the next section. In Section 3 we give proofs of our theorems. In the last section we show some generalizations of our results and we give several remarks.

2 Basic notions of differential algebra

When the algebraic methods are used to study a polynomial or rational differential equation, it is convenient to use tools of differential algebra. It gives a natural language for the problem, simplifies exposition and proofs. In this section we fix notation and we collect basic facts from differential algebra. We follow the excellent lectures of A. Nowicki [1] [5] simplifying only some assumptions. All propositions formulated in this section are very simple. We left their proofs to the readers, and we refer them to [5] and references therein.

2.1 Notation

By $\mathbb{N}_0$ we denote the set of non-negative integers. If k is a field (we always assume that $k=\mathbb{R}$ or $k=\mathbb{C}$) then $k[x_1,\ldots,x_n]$ and $k(x_1,\ldots,x_n)$ denote the ring of polynomials and the field of rational functions, respectively. We use also an abbreviated notation, i.e., $k[\boldsymbol{x}]=k[x_1,\ldots,x_n]$ and $k(\boldsymbol{x})=k(x_1,\ldots,x_n)$.

[1] The whole book can be found at `http://www.mat.uni.torun.pl/~anow/polder.html`

If $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, then we denote

$$\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad |\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i.$$

A non-zero element $\boldsymbol{\gamma} \in \mathbb{N}^n$ we called a ***direction***, and we denote

$$\langle \boldsymbol{\gamma}, \boldsymbol{\alpha} \rangle = \sum_{i=1}^n \gamma_i \alpha_i, \qquad \text{for} \quad \boldsymbol{\alpha} \in \mathbb{N}_0^n.$$

A non-zero polynomial $F \in k[\boldsymbol{x}]$ is called a ***$\boldsymbol{\gamma}$-form of $\boldsymbol{\gamma}$-degree*** r if it has the form

$$F = \sum_{\langle \boldsymbol{\gamma}, \boldsymbol{\alpha} \rangle = r} F_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}, \qquad F_{\boldsymbol{\alpha}} \in k.$$

In such a case we also say that F is a ***$\boldsymbol{\gamma}$-homogeneous polynomial of $\boldsymbol{\gamma}$-degree*** r. The zero polynomial is a $\boldsymbol{\gamma}$-form of an arbitrary degree.

Proposition 1 *If F is a non-zero polynomial in $k[\boldsymbol{x}]$, then the following conditions are equivalent:*

(1) F is a $\boldsymbol{\gamma}$-form of $\boldsymbol{\gamma}$-degree r.
(2) $F(t^{\gamma_1} x_1, \ldots, t^{\gamma_n} x_n) = t^r F(x_1, \ldots, x_n)$.
(3) $\gamma_1 x_1 \partial_1 F + \cdots + \gamma_n x_n \partial_n F = rF$.

The introduced $\boldsymbol{\gamma}$-homogeneity converts $k[\boldsymbol{x}]$ into a ***$\boldsymbol{\gamma}$-graded ring***. Namely, we have

$$k[\boldsymbol{x}] = \bigoplus_{r \in \mathbb{N}_0} k_r^{\boldsymbol{\gamma}}[\boldsymbol{x}],$$

where $k_r^{\boldsymbol{\gamma}}[\boldsymbol{x}]$ denotes the group all $\boldsymbol{\gamma}$-forms of $\boldsymbol{\gamma}$-degree r, and

$$k_r^{\boldsymbol{\gamma}}[\boldsymbol{x}] k_s^{\boldsymbol{\gamma}}[\boldsymbol{x}] \subseteq k_{r+s}^{\boldsymbol{\gamma}}[\boldsymbol{x}],$$

for all $r, s \in \mathbb{N}_0$. If there is no ambiguity, we write $k_r[\boldsymbol{x}]$ instead of $k_r^{\boldsymbol{\gamma}}[\boldsymbol{x}]$. Thus, every polynomial $F \in k[\boldsymbol{x}]$ has the following ***$\boldsymbol{\gamma}$-decomposition***

$$F = \sum_{s \in S} F_s,$$

where F_s is a $\boldsymbol{\gamma}$-form of $\boldsymbol{\gamma}$-degree s, and S is a finite set of integers. By $\boldsymbol{\gamma}$-deg(F) we denote the ***$\boldsymbol{\gamma}$-degree*** of F, that is, for $F \neq 0$, the maximal s such that $F_s \neq 0$ in the above $\boldsymbol{\gamma}$-decomposition of F; for $F = 0$ we put $\boldsymbol{\gamma}$-deg$(F) = -\infty$. If $s = \boldsymbol{\gamma}$-deg(F) then we denote $F_s = F^+$.

If $F, G \in k[\boldsymbol{x}]$ and FG is a non-zero $\boldsymbol{\gamma}$-form, then F and G are $\boldsymbol{\gamma}$-forms.

Let us fix two $\boldsymbol{\gamma}$-forms F and G of $\boldsymbol{\gamma}$-degree p and q, respectively. In the ring $k[X, Y]$ we define $\boldsymbol{\lambda}$-gradation putting $\boldsymbol{\lambda} = (p, q)$. For a polynomial $W \in k[X, Y]$ we define a polynomial $\widetilde{W} \in k[\boldsymbol{x}]$ as $\widetilde{W} = W[F, G]$.

Proposition 2 *Let $P \in k[X,Y]$, and*

$$P = \sum_{s\in S} P_s,$$

where S is a finite set of integers, be its $\boldsymbol{\lambda}$-decomposition. If $\widetilde{P} = 0$, then $\widetilde{P}_s = 0$ for $s \in S$.

PROOF. For each $i, j \in \mathbb{N}_0$ polynomials $F^i G^j$ are $\boldsymbol{\gamma}$-forms in $k[\boldsymbol{x}]$ of $\boldsymbol{\gamma}$-degree $ip + jq$. Thus, a $\boldsymbol{\gamma}$-homogeneous component of $\widetilde{P}$ of $\boldsymbol{\gamma}$-degree s is just $\widetilde{P}_s$. $\square$

We say that rational functions $F_i \in k(\boldsymbol{x})$, $i = 1, \ldots, m$ are ***functionally independent*** if their differentials $\mathrm{d}F_i$ are linearly independent over k, i.e., if matrix

$$\begin{bmatrix} \partial_1 F_1 & \ldots & \partial_n F_1 \\ \cdots & \cdots & \cdots \\ \partial_1 F_m & \ldots & \partial_n F_m \end{bmatrix}$$

has the maximal rank.

We say that rational functions $F_i \in k[\boldsymbol{x}]$, $i = 1, \ldots, m$ are ***algebraically dependent over*** k if there exists a non-zero polynomial $W \in k[X_1, \ldots, X_m]$ such that $W(F_1(\boldsymbol{x}), \ldots, F_m(\boldsymbol{x})) = 0$.

Later, we use the following lemma, see e.g. Proposition 1.16 in [1].

Lemma 1 *In $\mathbb{C}(\boldsymbol{x})$ functional and algebraic independences are equivalent.*

2.2 Derivations.

Let us fix n polynomials $v_i \in k[\boldsymbol{x}]$, $i = 1, \ldots, n$, and denote $\boldsymbol{v} = (v_1, \ldots, v_n)$. Then, let $\boldsymbol{d}_{\boldsymbol{v}} : k[\boldsymbol{x}] \mapsto k[\boldsymbol{x}]$ be a map defined in the following way

$$\boldsymbol{d}_{\boldsymbol{v}}(F) = \sum_{i=1}^{n} v_i \partial_i F, \qquad \text{for} \quad F \in k[\boldsymbol{x}]. \tag{7}$$

Obviously it is k-linear and satisfies the Leibnitz rule, i.e., we have

$$\boldsymbol{d}_{\boldsymbol{v}}(FG) = \boldsymbol{d}_{\boldsymbol{v}}(F)G + F\boldsymbol{d}_{\boldsymbol{v}}(G),$$

for all $F, G \in k[\boldsymbol{x}]$. Such maps are called derivations. For a more general definition see [5]. In the ring $k[\boldsymbol{x}]$ every derivation is of this form. For an element $R \in k(\boldsymbol{x})$ given by $R = F/G$, where F and G are coprime polynomials, we put

$$\boldsymbol{d_v}(F) = \frac{\boldsymbol{d_v}(F)G - F\boldsymbol{d_v}(G)}{R^2}.$$

In this way we extend uniquely $\boldsymbol{d_v}$ into $k(\boldsymbol{x})$. If it is clear from the context, we write $\boldsymbol{d}$ instead of $\boldsymbol{d_v}$. The kernel of $\boldsymbol{d}$ in $k[\boldsymbol{x}]$, which we denote

$$k[\boldsymbol{x}]^{\boldsymbol{d}} = \{F \in k[\boldsymbol{x}] \,|\, \boldsymbol{d}(F) = 0\},$$

is a subring of $k[\boldsymbol{x}]$. It is called the ***ring of constants of derivation*** $\boldsymbol{d}$. Similarly, the kernel of $\boldsymbol{d}$ in $k(\boldsymbol{x})$, which we denote

$$k(\boldsymbol{x})^{\boldsymbol{d}} = \{R \in k(\boldsymbol{x}) \,|\, \boldsymbol{d}(R) = 0\},$$

is a subfield of $k(\boldsymbol{x})$, and it is called the ***field of constants of derivation*** $\boldsymbol{d}$.

As we see, having a system of polynomial equations (1) with right hand sides $\boldsymbol{v}$, we associate with this system a derivation $\boldsymbol{d} = \boldsymbol{d_v}$ which is, in this context, nothing else but the Lie derivative along $\boldsymbol{v}$ considered as a differential operator. Thus, $k[\boldsymbol{x}]^{\boldsymbol{d}}$ and $k(\boldsymbol{x})^{\boldsymbol{d}}$ are sets of the polynomial and rational first integrals, respectively.

Now, the notion of Darboux polynomial is defined in the following way. We say that non-zero polynomial $F \in k[\boldsymbol{x}]$ is a ***Darboux polynomial*** of derivation $\boldsymbol{d}$ if $\boldsymbol{d}(F) = \Lambda F$ for a certain $\Lambda \in k[\boldsymbol{x}]$. Polynomial Λ is called a ***cofactor*** of Darboux polynomial F.

The following proposition explains the importance of the Darboux polynomials.

Proposition 3 *Let F and G be non-zero coprime polynomials in $k[\boldsymbol{x}]$. Then*

$$\boldsymbol{d}(F/G) = 0,$$

if and only if

$$\boldsymbol{d}(F) = PF \quad \text{and} \quad \boldsymbol{d}(G) = PG,$$

for some $P \in k[\boldsymbol{x}]$.

Two important properties of Darboux polynomials are described in the following propositions.

Proposition 4 *If $\boldsymbol{d}(F_i) = \Lambda_i F_i$ for $i = 1, 2$, then $\boldsymbol{d}(F_1F_2) = (\Lambda_1 + \Lambda_2)(F_1F_2)$.*

Proposition 5 *If $F \in k[\boldsymbol{x}]$ is a Darboux polynomial of derivation $\boldsymbol{d}$, then all factors of F are also Darboux polynomials of derivation $\boldsymbol{d}$.*

Now, we fix a direction $\boldsymbol{\gamma}$. We say that derivation $\boldsymbol{d}$ is ***$\boldsymbol{\gamma}$-homogeneous of $\boldsymbol{\gamma}$-degree*** s if

$$\boldsymbol{d}(k_r^{\boldsymbol{\gamma}}[\boldsymbol{x}]) \subseteq k_{r+s}^{\boldsymbol{\gamma}}[\boldsymbol{x}],$$

for an arbitrary $r \in \mathbb{Z}$. For checking if a given derivation is $\boldsymbol{\gamma}$-homogeneous it is convenient to apply the following proposition.

Proposition 6 *Derivation $\boldsymbol{d}$ is $\boldsymbol{\gamma}$-homogeneous of $\boldsymbol{\gamma}$-degree s if and only if $\boldsymbol{d}(x_i)$ is a $\boldsymbol{\gamma}$-form of $\boldsymbol{\gamma}$-degree $s + \gamma_i$, for $i = 1, \ldots, n$.*

Homogeneity of a derivation is important when we look for its Darboux polynomials.

Proposition 7 *If $F \in k[\boldsymbol{x}]$ is a Darboux polynomial of a $\boldsymbol{\gamma}$-homogeneous derivation $\boldsymbol{d}$ of $\boldsymbol{\gamma}$-degree s, then a cofactor $\Lambda \in k[\boldsymbol{x}]$ is $\boldsymbol{\gamma}$-form of $\boldsymbol{\gamma}$-degree s, and all $\boldsymbol{\gamma}$-components of F are also Darboux polynomials with the common cofactor Λ.*

For a polynomial derivation $\boldsymbol{d}_{\boldsymbol{v}}$ and a given $\boldsymbol{\gamma}$-gradation we can always represent $\boldsymbol{d}_{\boldsymbol{v}}$ as a finite sum of $\boldsymbol{\gamma}$ homogeneous derivations. Namely, for an integer l we denote

$$\boldsymbol{v}^{(\boldsymbol{\gamma},l)} = (v_1^{(\boldsymbol{\gamma},l)}, \ldots, v_n^{(\boldsymbol{\gamma},l)}),$$

where $v_i^{(\boldsymbol{\gamma},l)}$ is the $\boldsymbol{\gamma}$-homogeneous component of v_i of $\boldsymbol{\gamma}$-degree $l + \gamma_i$. Then by Proposition 6 $\boldsymbol{d}_{\boldsymbol{v}^{(\gamma,l)}}$ is a $\boldsymbol{\gamma}$-homogeneous derivation of $\boldsymbol{\gamma}$-degree l. By $\boldsymbol{d}_{\boldsymbol{v}}^{+}$ we denote a $\boldsymbol{\gamma}$-homogeneous derivation $\boldsymbol{d}_{\boldsymbol{v}^{(\gamma,l)}}$ corresponding to the maximal value of l. By $\boldsymbol{\gamma}$-degree of $\boldsymbol{d}_{\boldsymbol{v}}$ we understand $\boldsymbol{\gamma}$-degree of $\boldsymbol{d}_{\boldsymbol{v}}^{+}$

Proposition 8 *Assume that $F \in k[\boldsymbol{x}]$ is a Darboux polynomial of a derivation $\boldsymbol{d}_{\boldsymbol{v}}$ and $\Lambda \in k[\boldsymbol{x}]$ is a corresponding cofactor. Then for an arbitrary $\boldsymbol{\gamma}$-gradation we have $\boldsymbol{d}_{\boldsymbol{v}}^{+}(F^{+}) = P^{+}F^{+}$.*

Proposition 9 *If $\boldsymbol{d}_{\boldsymbol{v}}(F) = \Lambda F$, then $\boldsymbol{\gamma}$-$\deg \Lambda \leq r$ where r is $\boldsymbol{\gamma}$-degree of $\boldsymbol{d}_{\boldsymbol{v}}$.*

3 Proofs

To translate the language of Hamiltonian systems into the framework of differential algebra, we introduce the following conventions. By $k[\boldsymbol{q}, \boldsymbol{p}]$ we denote the ring of polynomials of $n = 2m$ variables

$$\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p}) = (q_1, \ldots q_m, p_1, \ldots, p_m).$$

For a polynomial Hamiltonian $H \in k[\boldsymbol{q}, \boldsymbol{p}]$ we define a derivation $\boldsymbol{d}_H$ in the following way

$$\boldsymbol{d}_H(q_i) = \frac{\partial H}{\partial p_i}, \qquad \boldsymbol{d}_H(p_i) = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \dots, m.$$

Thus, for Hamilton's function (4) the explicit form of $\boldsymbol{d}_H$ is following

$$\boldsymbol{d}_H(q_i) = \mu_i p_i, \qquad \boldsymbol{d}_H(p_i) = -\frac{\partial V}{\partial q_i}, \qquad i = 1, \dots, m. \tag{8}$$

Natural Hamiltonian systems have a very important property of the time reversibility. We can define this property in the following way. Let $\tau : k[\boldsymbol{q}, \boldsymbol{p}] \mapsto k[\boldsymbol{q}, \boldsymbol{p}]$ denote an automorphism of $k[\boldsymbol{q}, \boldsymbol{p}]$ given by:

$$\tau(q_i) = q_i, \qquad \tau(p_i) = -p_i, \qquad i = 1, \dots, m.$$

The following proposition is easy to check.

Proposition 10 *Derivation* (8) *has the following property*

$$\tau^{-1} \circ \boldsymbol{d}_H \circ \tau = -\boldsymbol{d}_H.$$

For a given Hamilton's function (4) we fix a $\boldsymbol{\gamma}$-gradation defined by the following direction

$$\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_m, \gamma_{m+1}, \dots, \gamma_{2m}) = (2, \dots, 2, r, \dots, r),$$

where $r = \boldsymbol{\gamma}\text{-}\deg V$. We prove the following simple but important lemma.

Lemma 2 *Assume that potential $V \in k[\boldsymbol{q}]$ is homogeneous of degree $r > 2$. Then derivation $\boldsymbol{d}_H$ given by* (8) *is $\boldsymbol{\gamma}$-homogeneous of degree $s = r - 2$.*

PROOF. Variables q_i and p_i are $\boldsymbol{\gamma}$-forms of degree 2 and r, respectively. Thus,

$$\boldsymbol{\gamma}\text{-}\deg(\boldsymbol{d}_H(q_i)) = \boldsymbol{\gamma}\text{-}\deg(p_i) = r = s + \gamma_i,$$

as $\gamma_i = 2$ for $i = 1, \dots, m$. Moreover, we have

$$\boldsymbol{\gamma}\text{-}\deg((\boldsymbol{d}_H(p_i)) = \boldsymbol{\gamma}\text{-}\deg\left(-\frac{\partial V}{\partial q_i}\right) = 2(r-1) = (r-2) + r = s + \gamma_{i+m},$$

as $\gamma_{i+m} = r$ for $i = 1, \dots, m$. Now, the proof follows from Proposition 6. $\square$

It is easy to show that

$$H^+ = \frac{1}{2}\sum_{i=1}^{m} \mu_i p_i^2 + V^+(\boldsymbol{q}),$$

where $V^+(\boldsymbol{q}) \neq 0$ is the homogeneous component of $V(\boldsymbol{q}) \neq 0$ of the highest degree r. An important consequence of the previous lemma is following.

Lemma 3 *Let* $\deg V \geq 2$ *and* $F \in k[\boldsymbol{q}, \boldsymbol{p}]$ *be a Darboux polynomial of* $\boldsymbol{d}_H$ *with cofactor* Λ. *Then* $\Lambda \in k[\boldsymbol{q}]$.

PROOF. We know that $\boldsymbol{\gamma}$-degree of $\boldsymbol{d}_H$ is $s = r - 2$. Thus, by Proposition 9 $\boldsymbol{\gamma}\text{-}\deg \Lambda \leq s = r - 2$. But $\boldsymbol{\gamma}\text{-}\deg p_i = r > s$ for $i = 1, \ldots, m$. Hence, Λ does not depend on $\boldsymbol{p}$. $\square$

Now, the proof of Theorem 1 is elementary.

PROOF of Theorem 1. Let us assume the opposite, i.e., let F be a Darboux polynomial of $\boldsymbol{d}_H$, with non-zero cofactor Λ. Then, by Proposition 8, we have $\boldsymbol{d}_{H^+}F^+ = \Lambda^+ F^+$. Hence, by Proposition 7, Λ^+ is a non-zero $\boldsymbol{\gamma}$-form and $\boldsymbol{\gamma}\text{-}\deg \Lambda^+ = r - 2$, i.e., for an odd r it is of odd degree. But, by Lemma 3, Λ^+ depends only on $\boldsymbol{q}$, so its $\boldsymbol{\gamma}$-degree is necessarily even. We have the contradiction. Thus, we must have $\Lambda = 0$. $\square$

When the degree of the potential is even, then derivation $\boldsymbol{d}_H$ can possess a proper Darboux polynomial. However, in such a case we have also a polynomial first integral.

Lemma 4 *Assume that the degree of* V *is even and that* $\boldsymbol{d}_H$ *has a proper Darboux polynomial* F *with cofactor* Λ. *Then* $\tau(F)$ *is also a proper Darboux polynomial with cofactor* $-\Lambda$, *and* $G = \tau(F)F$ *is a polynomial first integral of* $\boldsymbol{d}_H$.

PROOF. If $\boldsymbol{d}_H(F) = \Lambda F$, then we have

$$\tau(-\boldsymbol{d}_H(F)) = \tau(-\Lambda F) = -\tau(\Lambda)\tau(F).$$

But, from Proposition 10, it follows that

$$\tau(-\boldsymbol{d}_H(F)) = \tau(\tau^{-1} \circ \boldsymbol{d}_H \circ \tau(F)) = \boldsymbol{d}_H(\tau(F)).$$

Hence $\boldsymbol{d}_H(\tau(F)) = -\tau(\Lambda)\tau(F)$. However, by Lemma 3, cofactor Λ does not depend on $\boldsymbol{p}$, so $\tau(\Lambda) = \Lambda$. As a result we have

$$\boldsymbol{d}_H(\tau(F)) = -\Lambda\tau(F).$$

Now, the fact that $G = \tau(F)F$ is a first integral of $\boldsymbol{d}_H$ follows from Proposition 4. $\square$

At this point, the question is if the first integral constructed in the above Lemma is functionally independent of H. To show this we need to use the fact that the Hamiltonian (4) is irreducible.

Lemma 5 *If $m \geq 2$, $V \neq 0$, and among numbers μ_i, $i = 1, \ldots, m$, at least two are non-zero, then polynomial $H \in k[\boldsymbol{q}, \boldsymbol{p}]$ given by (4) is not reducible.*

PROOF. Without loss of generality we can assume that $\mu_1 \neq 0$ and $\mu_2 \neq 0$. Assume that H is reducible. As H is a polynomial of second degree with respect to $\boldsymbol{p}$, its factors are polynomials of degree one with respect to $\boldsymbol{p}$. So, we can write

$$2H = \sum_{i=1}^{m} \mu_i p_i^2 + 2V(\boldsymbol{q}) = \left(\sum_{i=1}^{m} \alpha_i p_i + W_1 \right) \left(\sum_{i=1}^{m} \beta_i p_i + W_2 \right), \tag{9}$$

where $\alpha_i, \beta_i \in \mathbb{C}$, for $i = 1, \ldots, m$, $W_1, W_2 \in k[\boldsymbol{q}]$, and $W_1 W_2 = 2V$. Comparing the left and right hand sides in the above equality we find, among others, that

$$\mu_1 = \alpha_1 \beta_1, \qquad \mu_2 = \alpha_2 \beta_2, \qquad \alpha_1 \beta_2 + \alpha_2 \beta_1 = 0, \tag{10}$$

$$\alpha_1 W_2 + \beta_1 W_1 = 0, \qquad \alpha_2 W_2 + \beta_2 W_1 = 0, \tag{11}$$

From equations (10) it follows that

$$\alpha_1^2 \mu_2 + \alpha_2^2 \mu_1 = 0. \tag{12}$$

Then, multiplying the first equation (11) by $\alpha_1 \mu_2$, and the second one by $\alpha_2 \mu_1$ and adding them together we obtain

$$(\alpha_1^2 \mu_2 + \alpha_2^2 \mu_1) W_2 + 2\mu_1 \mu_2 W_1 = 0.$$

Hence, taking into account (12) we have

$$2\mu_1 \mu_2 W_1 = 0.$$

But, $\mu_1 \neq 0$, $\mu_2 \neq 0$, and $W_1 \neq 0$. The contradiction finishes the proof. $\square$

Lemma 6 *Let us assume that Hamiltonian $H \in k[\boldsymbol{q}, \boldsymbol{p}]$ satisfies the following conditions:*

(1) $V \neq 0$, is a homogeneous potential of even degree;
(2) $m \geq 2$ and among numbers μ_i, $i = 1, \ldots, m$ at least two are non-zero;
(3) there exists a proper Darboux polynomial F of $\boldsymbol{d}_H$.

Then the first integrals H and $G = \tau(F)F$ of $\boldsymbol{d}_H$ are algebraically independent.

PROOF. Let us assume that H and G are algebraically dependent and let $W \in k[X,Y]$ be the minimal polynomial, i.e. $\widetilde{W} = W[H,G] = 0$. From Proposition 2 it follows that we can assume that W is $\boldsymbol{\lambda}$-homogeneous. As W is minimal, it is irreducible and this is why it can be written in the following form $W = X^i + YW_1(X,Y)$ where $i > 0$ is an integer and $W_1 \neq 0$ is an polynomial in $k[X,Y]$. Hence, we have

$$GW_1(H,G) = -H^i.$$

Thus irreducible factors of G divide H. But it is impossible as, under the assumptions we made, by Lemma 5, H is irreducible. A contradiction finishes the proof. □

Now, we are ready to prove Theorem 2.

PROOF of Theorem 2. Let F be a proper Darboux polynomial of $\boldsymbol{d}_H$ with cofactor Λ, so $G = \tau(F)F$ is a first integral of $\boldsymbol{d}_H$. Then, by Proposition 8, we have $\boldsymbol{d}_{H^+}F^+ = \Lambda^+F^+$. Hence, by Lemma 4, $G^+ = \tau(F^+)F^+$ is a polynomial first integral of $\boldsymbol{d}_{H^+}$. Moreover, by Lemma 5, G^+ and H^+ are algebraically, and thus functionally independent. But this implies that G and H are functionally independent.

4 Remarks and comments

There is an important consequence of Theorem 1. Namely, if a system with Hamilton's function given by (4) and the potential of odd degree possesses an additional first integral then it possesses an additional irreducible first integral. In fact, for such system, by Theorem 1, all factors of the a polynomial first integral are first integrals.

In a case of the potential of even degree, an additional first integral can be reducible or irreducible. As an example, let us consider a Hamiltonian system with two degrees of freedom with a homogeneous potential of degree 4

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2). \tag{13}$$

The list of integrable systems of this form with a first integral of degree at most 4 in momenta is given in [2]. We have the following possibilities.

(1) When $V = q_1^4$, then the additional first integral $F = p_2$ is irreducible.

(2) When $V=(q_1^2+q_2^2)^2$, then the additional first integral $F=q_1p_2-q_2p_1$ is irreducible.
(3) If $V=Cq_1^2+q_2^4$, then the additional reducible first integral $F=p_2^2+2q_2^2=(\mathrm{i}p_2+\sqrt{2}q_2^2)(-\mathrm{i}p_2+\sqrt{2}q_2^2)$. Obviously in this case $G_1=\mathrm{i}p_2+\sqrt{2}q_2^2$ and $G_2=\tau(G_1)$ are proper Darboux polynomials.
(4) For $V=4q_1^4/3+q_1^2q_2^2+q_2^4/12$ the additional first integral $F=p_2(p_1q_2-p_2q_1)+q_2^2(2q_1^3+q_1q_2^2)/3$ is irreducible.
(5) When $V=4q_1^4/3+q_1^2q_2^2+q_2^4/6$ the additional first integral F of degree 4 with respect to momenta is reducible and can be written in the form $F=\tau(G)G$ where

$$G=3\sqrt{6}p_2^2+12\mathrm{i}p_2q_1q_2+q_2^2\left[-6\mathrm{i}p_1+\sqrt{6}(2q_1^2+q_2^2)\right].$$

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