Delocalization in random polymer models

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Abstract

A random polymer model is a one-dimensional Jacobi matrix randomly composed of two finite building blocks. If the two associated transfer matrices commute, the corresponding energy is called critical. Such critical energies appear in physical models, an example being the widely studied random dimer model. It is proven that the Lyapunov exponent vanishes quadratically at a generic critical energy and that the density of states is positive there. Large deviation estimates around these asymptotics allow to prove optimal lower bounds on quantum transport, showing that it is almost surely overdiffusive even though the models are known to have pure-point spectrum with exponentially localized eigenstates for almost every configuration of the polymers. Furthermore, the level spacing is shown to be regular at the critical energy.

1 Introduction

Until quite recently, common wisdom was that one-dimensional random Schrödinger operators are in a strong localization phase and that there is nothing else of any interest to be discovered. In 1990, Dunlap, Wu and Phillips [DWP] studied the random dimer model. It is similar to the conventional Bernoulli-Anderson model with a Hamiltonian given by the sum of the discrete Laplacian and a random potential taking only two values, except that these potential values now always come in neighboring pairs (dimers). For suitable values of the parameters, this model has so-called critical energies at which the two transfer matrices across the dimers commute. This leads to a vanishing of the Lyapunov exponent and a divergence of the localization length at these energies. They then argued and showed numerically that the second moment of the position operator X on the lattice grows superdiffusively under the dynamics like $\langle\psi|X^2(t)|\psi\rangle \approx C\,t^{3/2}$ for a localized initial state ψ and any typical dimer configuration. This is likely to be responsible for the high conductivities of certain organic polymer chains [PW] and quasi-$1D$ semiconductor superlattices [PTB].

This work considers Jacobi matrices H_ω ramdomly composed of two building blocks modeling two different polymers. The associated polymer transfer matrices are supposed to commute at an energy lying within the spectra of the periodic operators containing only one of the polymers. It is proven that for *almost every* polymer configuration ω and every $\alpha > 0$ there is a positive constant C_α such that

$$\int_0^T \frac{dt}{T} \, \langle 0|e^{\imath Ht}|X|^q e^{-\imath Ht}|0\rangle \;\geq\; C_\alpha \, T^{q-\frac{1}{2}-\alpha} \;, \tag{1}$$

where $|0\rangle$ is the state localized at the origin. This rigorously confirms the findings of [DWP]. Moreover, it is shown that for *every* configuration the l.h.s. is greater or equal than $C\,T^{q-1}$ for some $C > 0$. Note that (1) implies that the conductivity is infinite either at finite temperature or if the critical energy is at the Fermi level [SB].

This should be confronted with the fact that the spectrum of a random polymer model is almost surely pure-point with exponentially localized eigenfunctions. For the related Bernoulli-Anderson model, such *spectral* localization results were first proven in [CKM], later on in [SVW]. More recently, the random dimer model and the continuous Bernoulli-Anderson model were treated in [BG] and [DSS1] respectively. These works also established dynamical localization on energy intervals not containing a discrete set of special energies which includes the above critical energies. While [DSS1] considers continuum models, its approach can be carried over to prove spectral localization and dynamical localization away from the set of special energies for the polymer models studied here, see [DSS2] for the more general case of an arbitrary number of building blocks of bounded length.

The fact that spectral localization can in principle coexist with quantum transport (even almost ballistic; note that ballistic is impossible with pure point spectrum [Sim]) was demonstrated by an example in [RJLS] (see also [BT]). However, those examples were rather artificial and much research since then was devoted to the program of proving dynamical localization (i.e. boundedness in time of the left-hand side of (1)) in models of physical interest with previously established spectral localization, by means of upgrading the proof of pure point spectrum (see [GK] and references therein, also [BJ]). The success of this program may have raised doubts as to the validity of the distinction between spectral and dynamical localization in physically relevant contexts. This paper demonstrates that the distinction should indeed be made as it shows that exponential localization and quantum transport coexist also in physical models.

Let us sketch the heuristics of [DWP] leading to (1). It is known [Bov] and proven below that the Lyapunov exponent generically vanishes quadratically like $\gamma(E_c + \epsilon) = c\epsilon^2 + \mathcal{O}(\epsilon^3)$ in the vicinity of the critical energy E_c. The extension of the eigenstates in an ϵ-neighborhood of E_c is given by their localization length equal to the inverse of the Lyapunov exponent. Therefore the portion of the initial wave packet lying energetically in this ϵ-neighborhood spreads out ballistically up to time scales $T \approx \epsilon^{-2}$. Because it will be shown that the density of states is positive at E_c, this portion of states is proportional to ϵ. Consequently, the qth moment of the position operator should grow like T^q multiplied by this factor $\epsilon \approx T^{-1/2}$, showing that (1) should hold with high probability.

The main technical tools are adequate action-angle variables, also called modified Prüfer variables in the mathematical literature. Adapting techniques from [PF], one obtains perturbative expansions around the critical energy of both the density of states and the Lyapunov exponent. Large deviation estimates around these averaged quantities furnish sufficient information to complete the proof of (1). The methods of proof are calculatory, quantitative and optimal. For example, they allow to show how large the moments of the position operator have to be if the commutator of the transfer matrices is small, but does not vanish.

Acknowledgements: This paper is a heavily revised version of a preprint [JSS] which contained a first, but less direct proof of the deterministic lower bound stated in Theorem 1 below. The basic strategy (Lemma 1 and Section 6) of the present proof of the dynamical lower bound (Theorem 4) given the boundedness of transfermatrices (Theorem 7) was suggested by S. Tcheremchantsev. This technique, which will be published in full generality in [DT], is simpler than the Guarneri method [Gua] of proving lower bounds employed in [JSS] (and also applicable here) and allowed us to circumvent previous more intricate arguments. We greatly appreciate that S. Tcheremchantsev made his work available prior to publication. S. J. and H. S.-B. were supported by NSF grant DMS-0070755, H. S.-B. moreover by DFG grant SCHU 1358/1-1 and the SFB 288. G. S. was supported by NSF grant DMS-0070343. He would also like to acknowledge financial support of CNRS (France) and hospitality at Université Paris 7, where part of this work was done.

2 Model and main results

Let $\hat{t}_\pm = (\hat{t}_\pm(0), \dots, \hat{t}_\pm(L_\pm - 1))$ and $\hat{v}_\pm = (\hat{v}_\pm(0), \dots, \hat{v}_\pm(L_\pm - 1))$ be two pairs of finite sequences of real numbers, satisfying $\hat{t}_\pm(l) > 0$ for all $l = 0, \dots, L_\pm - 1$. These numbers are the hopping and potential terms of two different polymers. A family of random Jacobi matrices is now constructed by random juxtaposition of these polymers. More precisely, to any sequence $\omega = (\omega_l)_{l\in\mathbb{Z}}$ of signs $+$ and $-$ one associates sequences $t_\omega = (t_\omega(n))_{n\in\mathbb{Z}}$ and $v_\omega = (v_\omega(n))_{n\in\mathbb{Z}}$ by means of $t_\omega = (\dots, \hat{t}_{\omega_0}, \hat{t}_{\omega_1}, \dots)$ and $v_\omega = (\dots, \hat{v}_{\omega_0}, \hat{v}_{\omega_1}, \dots)$. An exact definition of the underlying probability space $(\Omega, \mathbf{P})$, which also requires to randomize the position of $\hat{v}_{\omega_0}(0)$ and $\hat{t}_{\omega_0}(0)$, is given in Section 4.1. The polymer Hamiltonian H_ω of the configuration ω is then defined by

$$(H_\omega\psi)(n) \;=\; -t_\omega(n+1)\psi(n+1) + v_\omega(n)\psi(n) - t_\omega(n)\psi(n-1)\;, \qquad \psi \in \ell^2(\mathbb{Z})\;, \tag{2}$$

and $(H_\omega)_{\omega\in\Omega}$ becomes a family of random operators if the signs are chosen with probabilities p_+ and $p_- = 1 - p_+$ respectively. The polymer transfer matrices $T^E_\pm$ at energy $E \in \mathbb{R}$ are introduced by

$$T^E_\pm \;=\; T_{\hat{v}_\pm(L_\pm-1)-E,\hat{t}_\pm(L_\pm-1)} \cdots T_{\hat{v}_\pm(0)-E,\hat{t}_\pm(0)}\;, \qquad \text{where} \qquad T_{v,t} \;=\; \frac{1}{t}\begin{pmatrix} v & -t^2 \\ 1 & 0 \end{pmatrix}\;. \tag{3}$$

The transfer matrices over several polymers are then

$$T^E_\omega(k,m) = T^E_{\omega_{k-1}} \cdot T^E_{\omega_{k-2}} \cdot \ldots \cdot T^E_{\omega_m} \;, \qquad k > m \;, \tag{4}$$

and $T^E_\omega(k,m) = T^E_\omega(m,k)^{-1}$ if $k < m$, $T^E_\omega(m,m) = \mathbf{1}$. The Lyapunov exponent at energy E, also called inverse localization length, is then almost surely defined by (some more details are given in Sections 3.4 and 4.1)

$$\gamma(E) \;=\; \lim_{k\to\infty} \frac{1}{k\,\langle L_\pm\rangle} \log\left(\left\|T^E_\omega(k,0)\right\|\right) \;, \tag{5}$$

where $\langle c_\pm\rangle = p_+c_+ + p_-c_-$ for any complex numbers $c_\pm$. Vanishing of the Lyapunov exponent is considered an indicator for possible delocalization. For a polymer chain, this happens in the following situation:

Definition 1 *An energy $E_c \in \mathbb{R}$ is called critical for the random family $(H_\omega)_{\omega\in\Omega}$ of polymer Hamiltonians if the polymer transfer matrices $T^{E_c}_\pm$ are elliptic (i.e. $|\mathrm{Tr}(T^{E_c}_\pm)| < 2$) or equal to $\pm\mathbf{1}$ and commute*

$$[T^{E_c}_-, T^{E_c}_+] \;=\; 0 \;. \tag{6}$$

Remark 1 The definition does *not* allow the critical energy to be in a spectral gap or at the band edges of one of the periodic operators (constructed from $(\hat{t}_+, \hat{v}_+)$ and $(\hat{t}_-, \hat{v}_-)$ respectively) except for points of band touching (where the transfer matrix is $\pm\mathbf{1}$).

Remark 2 The condition (6) contains 4 equations. Given a model, one can only vary the energy. Hence, in the space of polymer models existence of critical energies is a non-generic property. On the other hand, given an energy E_c, it is always possible to construct polymer models that have E_c as a critical energy.

Examples If $L_\pm = 1$, the model reduces to the Bernoulli-Anderson model and there are no critical energies. If $L_+ = 2$ and $L_- = 1$, an example can be constructed as follows: choose $t(l) = 1$ for all $l \in \mathbb{Z}$ and $\hat{v}_+ = (0,0)$ and $\hat{v}_- = (\lambda)$ and $|\lambda| < 2$, then $E_c = 0$ is the critical energy. The most prominent [DWP, Bov, BG] example is the random dimer model for which $L_+ = L_- = 2$ and $\hat{v}_+(0) = \hat{v}_+(1) = \lambda$ and $\hat{v}_-(0) = \hat{v}_-(1) = -\lambda$ ($\lambda \in \mathbb{R}$), and $t(l) = 1$ for all $l \in \mathbb{Z}$. This model has two critical energies $E_c = \lambda$ and $E_c = -\lambda$ as long as $\lambda < 1$. It was previously (non-rigorously) known that $\gamma(E_c + \epsilon) = \mathcal{O}(\epsilon^2)$ [Bov] for the random dimer model.

The definition of the critical energy assures that there exists a real invertible matrix M transforming $T^{E_c}_-$ and $T^{E_c}_+$ simultaneously into rotations by angles η_- and η_+ respectively:

$$M T^{E_c}_\pm M^{-1} \;=\; \begin{pmatrix} \cos(\eta_\pm) & -\sin(\eta_\pm) \\ \sin(\eta_\pm) & \cos(\eta_\pm) \end{pmatrix} \;. \tag{7}$$

Hence $\gamma(E_c) = 0$. Because $T^E_\pm$ are polynomials of degree $L_\pm$ in E, one can expand $T^{E_c+\epsilon}_\pm$ into powers of ϵ. Since $M T^{E_c}_\pm M^{-1}$ are rotations, this implies that $\|M T^{E_c+\epsilon}_\pm M^{-1}\| \le 1 + c|\epsilon|$ for $|\epsilon| \le \epsilon_0$ and one deduces the following:

Proposition 1 *For $\epsilon_0 > 0$ there exists a constant $C < \infty$ such that for all $|\epsilon| \leq \epsilon_0$ and $m, k \in \mathbb{Z}$,*

$$\left\| T_\omega^{E_c+\epsilon}(k,m) \right\| \;\leq\; C\, e^{C|\epsilon|\,|k-m|} \;. \tag{8}$$

In particular, $|\gamma(E_c+\epsilon)| \leq C'\,|\epsilon|$ for $C' > 0$.

Note that the bound in (8) does not depend on the configuration. To study a possible spreading of wave packets due to the divergence of the localization length, one best considers the moments of the associated probability distribution, notably the time-averaged moments of the position operator X on $\ell^2(\mathbb{Z})$:

$$M_{\omega,q}(T) \;=\; \int_0^\infty \frac{dt}{T}\, e^{-\frac{t}{T}} \,\langle 0|e^{\imath H_\omega t}|X|^q e^{-\imath H_\omega t}|0\rangle \;, \qquad q > 0 \;, \tag{9}$$

The exponential time average may be replaced by a Cesaro mean without changing the asymptotics (*e.g.* [GSB]). Proposition 1 will lead more or less directly to the following deterministic lower bound on transport.

Theorem 1 *There exists a constant C such that for every configuration ω and for $q \geq 0$*

$$M_{\omega,q}(T) \;\geq\; C\,T^{q-1} \;. \tag{10}$$

Remark 3 It is important that the initial condition in (9) is $|0\rangle$ and not an arbitrary state $\psi \in \ell^2(\mathbb{Z})$. In fact, ψ could be an eigenstate of H_ω and hence not lead to any diffusion.

In order to study the behavior of the Lyapunov exponent in the vicinity of the critical energy, that is, go beyond the trivial upper bound $|\gamma(E_c+\epsilon)| \leq C'\,|\epsilon|$, let us define the transmission and reflection coefficients $a_\pm^\epsilon$ and $b_\pm^\epsilon$ by

$$M T_\pm^{E_c+\epsilon} M^{-1}\, v \;=\; a_\pm^\epsilon v + b_\pm^\epsilon \overline{v} \;, \qquad v \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\imath \end{pmatrix} \;. \tag{11}$$

Both are polynomials in ϵ. As v is an eigenvector of all rotations, one has

$$a_\pm^0 \;=\; e^{\imath\eta_\pm} \;, \qquad b_\pm^0 \;=\; 0 \;. \tag{12}$$

Furthermore let us set $e^{\imath\eta_\pm^\epsilon} = a_\pm^\epsilon/|a_\pm^\epsilon|$ so that $\eta_\pm^0 = \eta_\pm$. Averages will always be denoted as $\langle c_\pm\rangle = p_+c_+ + p_-c_-$.

Theorem 2 *Suppose that $\langle e^{2\imath\eta_\pm}\rangle \neq 1$ and $\langle e^{4\imath\eta_\pm}\rangle \neq 1$. Then the Lyapunov exponent of a random polymer chain satisfies*

$$\gamma(E_c+\epsilon) \;=\; \frac{2\,p_+p_-}{\langle L_\pm\rangle}\, \frac{|b_+^\epsilon\,\sin(\eta_-^\epsilon) - b_-^\epsilon\,\sin(\eta_+^\epsilon)|^2}{|1-\langle e^{2\imath\eta_\pm^\epsilon}\rangle|^2} + \mathcal{O}\left(|b_\pm^\epsilon|^3\right) \;. \tag{13}$$

Definition 2 *A critical energy $E_c \in \mathbb{R}$ of a polymer Hamiltonian is said to be of order r if $|b^\epsilon_\pm| = \mathcal{O}(\epsilon^r)$, but not $|b^\epsilon_\pm| = \mathcal{O}(\epsilon^{r+1})$ for both polymers.*

Remark 4 If E_c is a critical energy of order r, then $\gamma(E_c+\epsilon) = C\,\epsilon^{2r} + \mathcal{O}(\epsilon^{2r+1})$ for some non-negative constant C. Since $b^\epsilon_\pm = \mathcal{O}(\epsilon)$, the order of every critical energy is as least 1, i.e. the Lyapunov exponent vanishes at least quadratically. Generically the order of a critical energy is 1 (this is the case in the dimer model). In the latter case, more explicit formulas for the coefficient in (13) invoking only the values of $T^{E_c}_\pm$ and $\partial_E T^{E_c}_\pm$ at the critical energy can easily be written out. A comparison of (13) with a random phase approximation is made in Section 4.7. Finally, let us note that (13) also proves positivity of the Lyapunov exponent close to E_c whenever the numerator does not vanish.

Remark 5 The conditions $\langle e^{2\imath\eta_\pm}\rangle \neq 1$ and $\langle e^{4\imath\eta_\pm}\rangle \neq 1$ in Theorems 2 and 3 below are linked to anomalies studied in [CK]. For the dimer model, the condition $\langle e^{4\imath\eta_\pm}\rangle \neq 1$ is verified if and only if $\lambda \neq 1/\sqrt{2}$. This particular value already appeared in [BG].

Theorem 2 shows how the localization length diverges at the critical energy. The next result concerns the asymptotics of the integrated density of states $\mathcal{N}$ (denoted IDS, its definition is recalled in Section 3.4 below). Let $\mathcal{N}_\pm$ and $\mathcal{N}'_\pm$ denote the absolutely continuous IDS and their densities associated to the models with $L_\pm$-periodic models composed of only one of the polymers. By definition of a critical energy, $\mathcal{N}'_\pm(E_c) > 0$.

Theorem 3 *Suppose that $\langle e^{2\imath\eta_\pm}\rangle \neq 1$. Then the* IDS *of a random polymer chain satisfies*

$$\mathcal{N}(E_c+\epsilon) \;=\; \frac{\langle L_\pm \mathcal{N}_\pm(E_c)\rangle}{\langle L_\pm\rangle} + \epsilon\,\frac{\langle L_\pm \mathcal{N}'_\pm(E_c)\rangle}{\langle L_\pm\rangle} + \mathcal{O}(\epsilon^2)\;. \tag{14}$$

The theorem states that $\mathcal{N}$ is linearly increasing at E_c so that there are many states in the vicinity of a critical energy. The spreading of these states is quantitatively nicely characterized by the diffusion exponents, namely the power law growth exponents of the moments $M_{\omega,q}(T)$ defined in (9) above:

$$\beta^\pm_{\omega,q} \;=\; \lim_{T\to\infty}{}^{\pm} \frac{\log(M_{\omega,q}(T))}{\log(T^q)}\;. \tag{15}$$

Here $\lim^\pm$ denote the superior and inferior limit respectively. The main result is the following:

Theorem 4 *Suppose that $|\langle e^{2\imath\eta_\pm}\rangle| < 1$. Then* $\mathbf{P}$*-almost surely*

$$\beta^\pm_{\omega,q} \;\geq\; 1-\frac{1}{2q}\;. \tag{16}$$

It is interesting to compare Theorems 4 and 1. The latter implies for all configurations a weaker lower bound in (16) of the form $1-\frac{1}{q}$. One can construct configurations ω with slower transport than in (16). Therefore - seemingly paradoxically - typical random configurations do not lead the slowest possible transport for this model.

Finally it is worth mentioning a large deviation result here. The IDS and Lyapunov exponent are both averaged quantities describing the behavior at the infinite volume limit. Given their asymptotics $\mathcal{N}(E_c+\epsilon)=\mathcal{N}(E_c)+\mathcal{N}'(E_c)\epsilon+\mathcal{O}(\epsilon^2)$ and $\gamma(E_c+\epsilon)=C\epsilon^2+\mathcal{O}(\epsilon^3)$ (here $C=0$ if the order of E_c is $p>1$), one therefore expects that typically (w.r.t. $\mathbf{P}$) the following holds for the finite (but sufficiently large) size Hamiltonian $H_{\omega,N}$ found by restricting H_ω to $\ell^2(\{0,\dots,N-1\})$ (with Dirichlet boundary conditions): $H_{\omega,N}$ has $cN^{1/2}$ equally spaced eigenstates in the interval $[E_c-N^{-1/2},E_c+N^{-1/2}]$ which are all spread out over the whole sample. Here we give an upper bound on the probability of the set of atypical configurations for which the average *metal-like* behavior of the eigenvalue spacing does not hold. This result shows on which scales there is strong level repulsion.

Theorem 5 *For every $\alpha>0$ there exist $c>0$ and $C<\infty$ such that for all $N\in\mathbb{N}$ there are sets $\Omega_N(\alpha)\subset\Omega$ satisfying*

$$\mathbf{P}(\Omega_N(\alpha)) \;=\; \mathcal{O}(e^{-cN^\alpha})\;,$$

such that for every configuration ω in the complementary set $\Omega_N(\alpha)^c=\Omega\backslash\Omega_N(\alpha)$ the following statement holds: the interval $[E_c-N^{-1/2-\alpha},E_c+N^{-1/2-\alpha}]$ contains of the order of $N^{1/2-\alpha}$ eigenvalues of $H_{\omega,N}$ which are equally spaced and have eigenfunctions spread out over the whole sample, namely adjacent eigenvalues E and E' satisfy

$$\frac{1}{C\,N}\;\leq\;|E-E'|\;\leq\;\frac{C}{N}\;, \tag{17}$$

and for all normalized eigenfunctions ψ of $H_{\omega,N}$ it holds that

$$\frac{1}{C\,N}\;\leq\;|\psi(k-1)|^2+|\psi(k)|^2\;\leq\;\frac{C}{N} \tag{18}$$

for $0\leq k\leq N-1$, where $\psi(-1)=\psi(N)=0$.

Outside of the interval $[E_c-N^{-1/2-\alpha},E_c+N^{-1/2-\alpha}]$ we expect Poisson statistics. It seems unknown what the level statistics is like on the boundaries of this interval, but it is possibly not of the Wigner-type.

Theorems 2 and 3 are proved through the perturbation analysis of polymer phase shifts and action multipliers (essentially, appropriately modified Prüfer variables). The key for the proof of Theorems 4, and 5 is Theorem 7 which states that with high probability norms of the transfer matrices $T^E_\omega(k,m)$, $1\leq m\leq k\leq N$ for energies in the interval $[E_c-N^{-1/2-\alpha},E_c+N^{-1/2-\alpha}]$ are uniformly bounded. This theorem is proved in Section 5 by establishing large-deviation estimates for random Weyl-type sums defined in terms of polymer phase shifts.

Let us conclude with a brief remark about the one-dimensional Anderson model in the weak coupling limit, namely $H_{\lambda,\omega} = H_0 + \lambda V_\omega$ where H_0 is a periodic operator, V_ω the usual Anderson potential and λ a (small) coupling constant. Pastur and Figotin [PF] showed (in the case where H_0 is the discrete Laplacian) that away from band-center and band edges of the periodic operator, the Lyapunov exponent grows quadratically in λ. The large deviation results and dynamical lower bounds presented here transpose in order to show that almost surely

$$\sup_{T>0} M_{\omega,\lambda,q}(T) \;\geq\; C_\alpha\, \lambda^{-2q+\alpha} \;.$$

Hence the presented techniques allow to study in a very detailed way the metal-insulator transition driven by either the disorder strength or the sample size. This transition appears at a single energy, the critical energy, in the polymer models studied here.

3 Brief review of basic formulas

3.1 Transfer matrices

Let $(t(n))_{n\in\mathbb{Z}}$ be a sequence of positive numbers and $(v(n))_{n\in\mathbb{Z}}$ a sequence of real numbers. As in (2) they define a Jacobi matrix H acting on $\ell^2(\mathbb{Z})$. Given an initial angle $\theta^0 \in \mathbb{R}$ and a complex energy $z \in \mathbb{C}$, let us construct the formal solution $(u^z(n))_{n\in\mathbb{Z}}$ by

$$-t(n+1)u^z(n+1) + v(n)u^z(n) - t(n)u^z(n-1) \;=\; zu^z(n) \;, \tag{19}$$

and the initial conditions

$$\begin{pmatrix} t(0)\,u^z(0) \\ u^z(-1) \end{pmatrix} \;=\; \begin{pmatrix} \cos(\theta^0) \\ \sin(\theta^0) \end{pmatrix} \;.$$

Using the definition (3) of the single site transfer matrices $T_{v,t}$, the transfer matrix from site k to n is introduced by

$$\mathcal{T}^z(n,k) \;=\; \prod_{l=n-1}^{k} T_{v(l)-z,t(l)} \;.$$

It allows to rewrite the (formal) eigenfunction equation (19) as

$$\begin{pmatrix} t(n)\,u^z(n) \\ u^z(n-1) \end{pmatrix} \;=\; \mathcal{T}^z(n,k) \begin{pmatrix} t(k)\,u^z(k) \\ u^z(k-1) \end{pmatrix} \;. \tag{20}$$

Note that the transfer matrices satisfy the transitivity relation $\mathcal{T}^z(n,k) = \mathcal{T}^z(n,m)\mathcal{T}^z(m,k)$. A direct inductive argument then shows that, for $\zeta \in \mathbb{C}$,

$$\mathcal{T}^{z+\zeta}(n,k) \;=\; \mathcal{T}^{z}(n,k) - \zeta \sum_{l=k}^{n-1} \mathcal{T}^{z+\zeta}(n,l+1)\, \frac{1}{t(l)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{T}^{z}(l,k) \;. \tag{21}$$

Taking the norm of (21), estimating the r.h.s. and consequently taking the supremum over $0 \le k \le n \le m$ leads to the following perturbative result which in a slightly different form is given in [Sim2] (see also [DT].)

Lemma 1 *Suppose*

$$\sup_{0\le k\le n\le m} \|\mathcal{T}^{z}(n,k)\| \;\le\; C \qquad D \;=\; \sup_{0\le l\le m-1} \frac{1}{|t_l|} \;.$$

Then, as long as $CD|\zeta|m < 1$*,*

$$\sup_{0\le k\le n\le m} \|\mathcal{T}^{z+\zeta}(n,k)\| \;\le\; \frac{C}{1 - CD|\zeta|m} \;.$$

3.2 Free Prüfer variables

Let now $E \in \mathbb{R}$ and u^E be given by (19). The free Prüfer phases $\theta^{0,E}(n)$ and amplitudes $R^{0,E}(n) > 0$ are now defined by

$$R^{0,E}(n) \begin{pmatrix} \cos(\theta^{0,E}(n)) \\ \sin(\theta^{0,E}(n)) \end{pmatrix} \;=\; \begin{pmatrix} t(n)u^E(n) \\ u^E(n-1) \end{pmatrix} \;, \tag{22}$$

the above initial conditions as well as

$$-\frac{\pi}{2} < \theta^{0,E}(n+1) - \theta^{0,E}(n) < \frac{3\pi}{2} \;.$$

Note that the θ^0-dependence of the Prüfer variables is suppressed.

Lemma 2

$$R^{0,E}(n)^2\, \partial_E\, \theta^{0,E}(n) \;=\; \begin{cases} \sum_{l=0}^{n-1} u^E(l)^2 & \textit{if} \quad n > 0 \;, \\[2ex] -\sum_{l=n}^{-1} u^E(l)^2 & \textit{if} \quad n < 0 \;. \end{cases} \tag{23}$$

Proof: From the recurrence relation (19) and the definition of $\theta^{0,E}(n)$ one gets

$$\cot(\theta^{0,E}(n)) \;=\; -t^2(n-1)\tan(\theta^{0,E}(n-1)) + v(n-1) - E \;.$$

Differentiation leads to

$$\partial_E \theta^{0,E}(n) \;=\; \frac{t^2(n-1)\sin^2(\theta^{0,E}(n))}{\cos^2(\theta^{0,E}(n-1))}\, \partial_E \theta^{0,E}(n-1) + \sin^2(\theta^{0,E}(n)) \;.$$

Multiplying with $R^{0,E}(n)^2$ and using the definition of $R^{0,E}(n)$ and $\theta^{0,E}(n)$ gives

$$R^{0,E}(n)^2\partial_E\theta^{0,E}(n) \;=\; R^{0,E}(n-1)^2\partial_E\theta^{0,E}(n-1)+u^E(n-1)^2\;. \tag{24}$$

The above deduction of (24) has used that $u^E(n-1)\neq 0$. If $u^E(n-1)=0$, then one may deduce (24) in a similar way from

$$\tan(\theta^{0,E}(n)) \;=\; \frac{\cot(\theta^{0,E}(n-1))}{-t^2(n-1)+(v(n-1)-E)\cot(\theta^{0,E}(n-1))}\;.$$

The lemma now follows by iterating (24). □

Note in particular that (23) implies that $\partial_E\theta^{0,E}(n)$ is strictly positive for $n\geq 2$ and strictly negative for $n\leq -2$. Furthermore, it follows from elementary considerations for transfer matrices that there are constants C_1 and C_2 such that

$$0\;<\;C_1\;\leq\;\left|\partial_E\,\theta^{0,E}(n)\right|\;\leq\;C_2\;<\;\infty\;, \tag{25}$$

where C_1 and C_2 can be uniformly bounded away from 0 and ∞ as long as $|n|\geq 2$ and the quantities $|n|$, E and $\max_{|k|\leq|n|}\{|v(k)|,t(k),1/t(k)\}$ remain bounded (where only the lower bound requires $|n|\geq 2$).

Let Π_N be the projection on $\ell^2(\{0,\ldots,N-1\})$ and denote the associated finite-size Jacobi matrix by $H_N=\Pi_N H\Pi_N$. As H_N has Dirichlet boundary conditions, let us choose $u^E(-1)=0$ and $t(0)u^E(0)=1$ as initial conditions in the recurrence relation (19). This corresponds to an initial Prüfer phase $\theta^0=0$. The formal solution u^E then gives an eigenvector (and E is an eigenvalue of H_N) if and only if $t(N)u^E(N)=R^{0,E}(N)\cos(\theta^{0,E}(N))=0$, that is $\theta^{0,E}(N)=\frac{\pi}{2}$ mod π (note herefore that $u^E(0)\neq 0$ for any eigenvector of H_N).

For E sufficiently close to $-\infty$, one checks iteratively in n that $u(n)>0$ for all $n=0,\ldots,N-1$. Moreover, $\lim_{E\to-\infty}u^E(n)/u^E(n-1)=\infty$ for such n. Hence $\lim_{E\to-\infty}\theta^{0,E}(N)=0$. As $\theta^{0,E}(N)$ is monotone increasing in E, it follows that the jth eigenvalue E_j of H_N (counted from below $E_1<E_2<\ldots<E_N$) satisfies

$$\theta^{0,E_j}(N)\;=\;\frac{\pi}{2}+\pi(j-1)\;,\qquad\theta^0\;=\;0\;. \tag{26}$$

This oscillation theorem implies immediately:

$$\left|\frac{1}{\pi}\,\theta^{0,E}(N)\;-\;\#\;\{\text{negative eigenvalues of }\;(H_N-E)\,\}\right|\;\leq\;\frac{1}{2}\;. \tag{27}$$

3.3 Modified Prüfer variables

Let us fix $M \in SL(2,\mathbb{R})$. Set $e_\theta = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$. Define a smooth function $m : \mathbb{R} \to \mathbb{R}$ with $m(\theta+\pi) = m(\theta)+\pi$ and $0 < C_1 \leq m' \leq C_2 < \infty$, by

$$r(\theta) e_{m(\theta)} = M e_\theta, \qquad r(\theta) > 0 \;, \qquad m(0) \in [-\pi,\pi) \;.$$

Then the M-modified Prüfer variables $(R^E(n), \theta^E(n)) \in \mathbb{R}_+ \times \mathbb{R}$ for initial condition $\theta^E(0) = \theta = m(\theta^0)$ are given by

$$\theta^E(n) \;=\; m(\theta^{0,E}(n)) \;, \tag{28}$$

and

$$\begin{pmatrix} R^E(n)\cos(\theta^E(n)) \\ R^E(n)\sin(\theta^E(n)) \end{pmatrix} \;=\; M \begin{pmatrix} t(n)\, u^E(n) \\ u^E(n-1) \end{pmatrix} \;, \tag{29}$$

where the dependence on the the initial phase is again suppressed. Bounds of the form (25) also hold for an M-modified Prüfer phase $\theta^E(n)$ because $\theta^E(n) = m(\theta^{0,E}(n))$ leads to $(\min m')|\partial_E \theta^{0,E}| \leq |\partial_E \theta^E| \leq (\max m')|\partial_E \theta^{0,E}|$. Furthermore, as $|\theta^E(n) - \theta^{0,E}(n)| \leq 2\pi$, (27) implies that for the choice $\theta = m(0)$

$$\left| \frac{1}{\pi}\, \theta^E(N) \;-\; \#\left\{\text{negative eigenvalues of } \quad (H_N - E)\right\} \right| \;\leq\; \frac{5}{2} \;. \tag{30}$$

The goal to have in mind when choosing M is to make the M-modified transfer matrices as simple as possible so that the M-modified Prüfer variables are easy to calculate. Whenever E is in the spectrum, the most simple matrix to obtain is a rotation. Anything close to it can then be treated by perturbation theory. This is the strategy followed for the random polymer model below where M is chosen as in (7)

Example: Let us consider an L-periodic Jacobi matrix H. If $E \in \mathbb{R}$ is in the interior of the spectrum of H, there exists a matrix M (depending on E, of course) such that $MT^E(L)M^{-1} = R_\eta$ where R_η is the rotation by an angle $\eta = \eta(E)$ obtained in accordance with the definition (28). The M-modified Prüfer variables are then simply given by $(R^E(kL), \theta^E(kL)) = (1, k\eta)$ and the IDS is $\mathcal{N}(E) = \eta(E)/(L\pi)$.

3.4 Covariant Jacobi matrices

Let $(\Omega, T, \mathbb{Z}, \mathbf{P})$ be a compact space Ω, endowed with a $\mathbb{Z}$-action T and a T-invariant and ergodic probability measure $\mathbf{P}$. For a function $f \in L^1(\Omega, \mathbf{P})$, let us denote $\mathbf{E}(f(\omega)) = \int d\mathbf{P}(\omega)\, f(\omega)$. A strongly continuous family $(H_\omega)_{\omega\in\Omega}$ of two-sided tridiagonal, self-adjoint matrices on $\ell^2(\mathbb{Z})$ is called covariant if the covariance relation $UH_\omega U^* = H_{T\omega}$ holds where U is the translation on $\ell^2(\mathbb{Z})$. H_ω is characterized by two sequences $(t_\omega(n))_{n\in\mathbb{Z}}$ and $(v_\omega(n))_{n\in\mathbb{Z}}$ such that (2) holds.

The IDS at energy $E \in \mathbb{R}$ of the family $(H_\omega)_{\omega\in\Omega}$ can $\mathbf{P}$-almost surely be defined by [PF]

$$\mathcal{N}(E) \;=\; \lim_{N\to\infty}\, \frac{1}{N}\, \mathrm{Tr}(\chi_{(-\infty,E]}(\Pi_N H_\omega \Pi_N)) \;, \tag{31}$$

while the Lyapunov exponent $\gamma(E)$ for $E \in \mathbb{R}$ is $\mathbf{P}$-almost surely given by the formula

$$\gamma(E) \;=\; \lim_{N\to\infty} \frac{1}{N} \log\left(\|\mathcal{T}^E_\omega(N,0)\|\right) \;,$$

where the transfer matrix $\mathcal{T}^E_\omega(N,0)$ from site 0 to N is defined as in Section 3.1. Both the IDS and the Lyapunov exponent are self-averaging quantities, notably an average over $\mathbf{P}$ may be introduced before taking the limit without changing the result [PF].

For each H_ω let $(R^E_\omega(n), \theta^E_\omega(n))$ denote the associated M-modified Prüfer variables with some initial condition, then according to (30)

$$\mathcal{N}(E) \;=\; \lim_{N\to\infty}\, \frac{1}{\pi}\, \frac{1}{N}\, \mathbf{E}\left(\theta^E_\omega(N)\right) \;. \tag{32}$$

While it is readily seen that $\gamma(E) \geq \lim^+_{N\to\infty} \frac{1}{N}\mathbf{E}(\log(R^E_\omega(N))$, one may in general not get equality here as demonstrated by a counterexample in Section 4.1. This is due to the dependence of $R^E_\omega(N)$ on the initial phase θ. The next lemma solves this problem by (continuously) averaging over θ.

Lemma 3 *For $E \in \mathbb{R}$ and any continuous (i.e. non-atomic) measure ν on $\mathbb{R}P(1) = [0,\pi)$*

$$\gamma(E) \;=\; \lim_{N\to\infty}\, \frac{1}{N} \int d\nu(\theta)\; \mathbf{E}\left(\log(R^E_\omega(N))\right) \;. \tag{33}$$

Proof: As $\|\mathcal{T}^E_\omega(N,0)e_{m^{-1}(\theta)}\| \;=\; \|Me_{m^{-1}(\theta)}\|\|M^{-1}e_{\theta^E_\omega(N)}\|R^E_\omega(N)$, a change of variables and elementary estimates show that it is sufficient to show that $\gamma(E)$ is equal to

$$\lim_{N\to\infty} \frac{1}{N} \int d\nu(\theta)\; \mathbf{E}\left(\log(\|\mathcal{T}^E_\omega(N,0)e_\theta\|)\right)$$

for any continuous probability measure ν. This is easy to see if $\gamma(E) = 0$, thus we now assume that $\gamma(E) > 0$. Suppose the contrary, that is there exists a ν such that

$$\lim_{N\to\infty} \frac{1}{N} \int d\nu(\theta)\; \mathbf{E}\left(\log(\|\mathcal{T}^E_\omega(N,0)e_\theta\|)\right) \;<\; \gamma(E) \;.$$

By Fatou's lemma this implies that $\int d\nu(\theta)\mathbf{E}(\lim_{N\to\infty} \frac{1}{N}\log(\|\mathcal{T}^E_\omega(N,0)e_\theta\|)) \;<\; \gamma(E)$. Because for a.e. ω the limit inside of the expectation is equal to either $\gamma(E)$ or $-\gamma(E)$ by Oseledec's Theorem, there has to exist a set $\mathcal{E} \subset [0,\pi) \times \Omega$ of positive $\nu \otimes \mathbf{P}$-measure such that $\lim_{N\to\infty} \frac{1}{N}\log(\|\mathcal{T}^E_\omega(N,0)e_\theta\|)$ is equal to $-\gamma(E)$ for all $(\theta,\omega) \in \mathcal{E}$. Hence there exists an ω such that the set $\{\theta \in [0,\pi) \,|\, E \text{ eigenvalue of } H_\omega(\theta)\}$ has positive ν-measure where $H_\omega(\theta)$ is the half-line operator with θ-boundary condition. As ν is continuous, this set has to contain at least two distinct points. This is in contradiction to the fact that the difference equation $H_\omega u = Eu$ has, up to constant multiples, at most one square-summable solution at $+\infty$. $\square$

4 Asymptotics of IDS and Lyapunov exponent

Generalizing the strategy suggested by Pastur and Figotin [PF], this chapter is devoted to the calculation of the asymptotics for the IDS and the Lyapunov exponent near the critical energy of a random polymer model, that is the proof of Theorems 2 and 3. The techniques of [CS] allow to treat also the case of strongly mixing (instead of random) configurations of polymer chains giving similar formulas, containing a correction factor given by the Fourier transform of the correlation function. No further details are given here concerning this generalization.

4.1 Random polymer chains

For sake of completeness, let us briefly indicate how to construct $(\Omega, T, \mathbb{Z}, \mathbf{P})$ for the random polymer Hamiltonians defined in Chapter 2. Let Ω_0 be the Tychonov space of two-sided sequences of signs. Set $\Omega_\pm = \{\omega \in \Omega_0 \,|\, \omega_0 = \pm\} \times \{0, \ldots, L_\pm - 1\}$ and $\Omega = \Omega_+ \cup \Omega_-$. Now $T : \Omega \to \Omega$ is defined by

$$T(\omega, l) \;=\; \begin{cases} (\omega, l+1) & \text{if } \; l < L_{\omega_0} - 1 \;, \\[2ex] (T_0\omega, 0) & \text{if } \; l = L_{\omega_0} - 1 \;, \end{cases}$$

where T_0 is the left shift on Ω_0. Now for any set $A_\pm \subset \Omega_0$ of codes all having $\omega_0 = \pm$, one sets for all $l \in \{0 \ldots L_\pm - 1\}$

$$\mathbf{P}(\{(\omega, l) \in \Omega \,|\, \omega \in A_\pm\}) \;=\; \frac{\mathbf{P}_0(A_\pm)}{\langle L_\pm \rangle} \;,$$

where $\mathbf{P}_0$ is the Bernoulli measure on Ω_0. It can then be verified that $\mathbf{P}$ is invariant and ergodic (the latter by mimicking the proof for $(\Omega_0, T_0, \mathbf{P}_0)$). Random hopping terms and potential are then given by $t_{(\omega,l)} = (\ldots, \hat{t}_{\omega_0}, \hat{t}_{\omega_1}, \ldots)$ and $v_{(\omega,l)} = (\ldots, \hat{v}_{\omega_0}, \hat{v}_{\omega_1}, \ldots)$ with choice of origin $t_{(\omega,l)}(0) = \hat{t}_{\omega_0}(l)$ and $v_{(\omega,l)}(0) = \hat{v}_{\omega_0}(l)$. This leads to the covariant family $(H_{(\omega,l)})_{(\omega,l)\in\Omega}$ of Jacobi matrices. It is this family which is refered to as (H_ω) in Section 2 and, in particular, in Theorems 2 to 5.

According to Section 3.4 the Lyapunov exponent satisfies

$$\gamma(E) \;=\; \lim_{N\to\infty} \frac{1}{N}\, \mathbf{E}\left(\log\left(\|\mathcal{T}^E_{(\omega,l)}(N,0)\|\right)\right) \;=\; \lim_{N\to\infty} \frac{1}{N} \log(\|\mathcal{T}^E_{(\omega,l)}(N,0)\|) \;, \tag{34}$$

for $\mathbf{P}$-a.e. $(\omega, l) \in \Omega$. Here $\mathbf{E} = \int d\mathbf{P}$. On the other hand, there is also a Lyapunov exponent associated with random products of the unimodular matrices $T^E_\pm$:

$$\gamma_0(E) \;=\; \lim_{k\to\infty} \frac{1}{k}\, \mathbf{E}_0\left(\log\left(\|T^E_\omega(k,0)\|\right)\right) \;=\; \lim_{k\to\infty} \frac{1}{k} \log\left(\|T^E_\omega(k,0)\|\right) \;, \tag{35}$$

for $\mathbf{P}_0$-a.e. $\omega \in \Omega_0$ and $\mathbf{E}_0 = \int d\mathbf{P}_0$. To compare $\gamma(E)$ and $\gamma_0(E)$, let $\tilde{\Omega}_0$ be the full measure set of those $\omega \in \Omega_0$ such that (35) holds and also $\sum_{l=0}^{k-1} L_{\omega_l}/k \to \langle L_\pm \rangle$ as $k \to \infty$. For $\omega \in \tilde{\Omega}_0$ it is easily seen that $\lim_{N\to\infty} \frac{1}{N} \log(\|\mathcal{T}^E_{(\omega,0)}(N,0)\|) = \gamma_0(E)/\langle L_\pm\rangle$. Since $\mathbf{P}\{(\omega,0) \mid \omega \in \tilde{\Omega}_0\} = 1/\langle L_\pm\rangle > 0$, one concludes from (34) that

$$\gamma(E) \;=\; \frac{1}{\langle L_\pm\rangle}\,\gamma_0(E)\;. \tag{36}$$

While γ_0 is not defined through a covariant operator, it follows by the same argument as in Lemma 3 that for any continuous measure ν on $[0,\pi)$

$$\gamma_0(E) \;=\; \lim_{k\to\infty} \frac{1}{k} \int d\nu(\theta)\mathbf{E}_0 \log\left(\|MT^E_\omega(k,0)M^{-1}e_\theta\|\right)\;. \tag{37}$$

Counterexample: The continuity condition on ν in Lemma 3 cannot be weakened as shows the following example. Consider the polymer model with $L_\pm = 3$, $t(l) = 1$ for all $l \in \mathbb{Z}$ and $\hat{v}_+ = (\frac{1}{2}, 2, 0)$ and $\hat{v}_- = (-\frac{1}{2}, -2, 0)$, and choose $M = \mathbf{1}$. For $E = 0$ it is easily seen that $T^0_\pm e_{\pi/2} = \mp\frac{1}{2} e_{\pi/2}$ and thus $\frac{1}{k}\log\|T^0_\omega(k,0)e_{\pi/2}\| = -\frac{1}{2}$ for all ω and k, while $\gamma_0(0) = \frac{1}{2}$. Hence a measure having an atom at $\theta = \frac{\pi}{2}$ will not satisfy (37). This also provides a counterexample to Lemma 3 with $(\Omega, T, \mathbb{Z}, \mathbf{P})$ as above. For this one uses that the event $\{\omega \,|\, v_\omega(0) = \hat{v}_{\omega_0}(0)\}$ has probability 1/3 in Ω.

4.2 Polymer phase shifts

For M given by (7), let the polymer action multipliers $\rho^\epsilon_\pm(\theta)$ and the polymer phase shifts $\mathcal{S}_{\epsilon,\pm}(\theta)$ be the M-modified Prüfer amplitude and phase for the $L_\pm$-periodic polymers with initial phase θ at 0 and evaluated at $L_\pm$ (i.e. over a single polymer $(\hat{t}_+, \hat{v}_+)$ and $(\hat{t}_-, \hat{v}_-)$, respectively). By definition of the modified Prüfer variables, this means

$$\rho^\epsilon_\pm(\theta) e_{\mathcal{S}_{\epsilon,\pm}(\theta)} \;=\; MT^{E_c+\epsilon}_\pm M^{-1} e_\theta \tag{38}$$

for all $\theta \in \mathbb{R}$. The iterated polymer phase shifts are then denoted by

$$\mathcal{S}^{l+1}_{\epsilon,\omega}(\theta) \;=\; \mathcal{S}_{\epsilon,\omega_l}(\mathcal{S}^l_{\epsilon,\omega}(\theta))\;, \qquad \mathcal{S}^0_{\epsilon,\omega}(\theta) \;=\; \theta\;.$$

From (7) it follows that (independent of θ) $\rho^0_\pm(\theta) = 1$ and $\eta_\pm = \mathcal{S}_{0,\pm}(\theta) - \theta$, at least up to a multiple of 2π which is hereby fixed. The former readily implies that $\gamma(E_c) = 0$. To study the Lyapunov exponent in a vicinity of E_c, iterate (38) in order to deduce

$$\log\left(\left\|MT^{E_c+\epsilon}_\omega(N,0)M^{-1}e_\theta\right\|\right) \;=\; \sum_{l=0}^{N-1} \log\left(\rho^\epsilon_{\omega_l}(\mathcal{S}^l_{\epsilon,\omega}(\theta))\right)\;, \tag{39}$$

which combined with (36) and (37) gives

$$\gamma(E_c+\epsilon) \;=\; \frac{1}{\langle L_\pm\rangle}\lim_{N\to\infty}\frac{1}{N}\sum_{l=0}^{N-1}\int d\nu(\theta)\,\mathbf{E}_0\left(\log(\rho^\epsilon_{\omega_l}(\mathcal{S}^l_{\epsilon,\omega}(\theta)))\right)\;. \tag{40}$$

To also express the IDS in terms of the polymer phase shifts, let $(n_{(\omega,l),k})_{k\in\mathbb{Z}}$ be the sequence of lower polymer nodes for a given $(\omega,l)\in\Omega$, i.e. the integers determined by $v_{(\omega,l)}(n_{(\omega,l),k})=\hat{v}_{\omega_k}(0)$, for any choice of $\hat{v}$. For $N\in\mathbb{N}$, let $n_{(\omega,l),k}$ be the polymer node closest to N. Since $\mathcal{S}^k_{\epsilon,\omega}(\theta)-\theta$ is a rotation number for a matrix which arises from $H_{(\omega,l),N}$ by a perturbation of rank bounded by $C\max\{L_-,L_+\}$, it follows that $|\theta^{E_c+\epsilon}_{(\omega,l)}(N)-(\mathcal{S}^k_{\epsilon,\omega}(\theta)-\theta)|\leq C\max\{L_-,L_+\}$ uniformly in θ. Thus it follows from (32) that $\mathcal{N}(E_c+\epsilon)=\lim_{N\to\infty}\frac{1}{\pi N}(\mathcal{S}^k_{\epsilon,\omega}(\theta)-\theta)$ almost surely and in expectation. Since $k/N\to 1/\langle L_\pm\rangle$ almost surely as $N\to\infty$, this implies that

$$\begin{aligned}\mathcal{N}(E_c+\epsilon) \;&=\; \frac{1}{\pi\langle L_\pm\rangle}\lim_{k\to\infty}\frac{1}{k}\,\mathbf{E}_0(\mathcal{S}^k_{\epsilon,\omega}(\theta)-\theta)\\ &=\; \frac{1}{\pi\langle L_\pm\rangle}\lim_{k\to\infty}\frac{1}{k}\sum_{l=0}^{k-1}\mathbf{E}_0\left(\mathcal{S}_{\epsilon,\omega_l}(\mathcal{S}^l_{\epsilon,\omega}(\theta))-\mathcal{S}^l_{\epsilon,\omega}(\theta)\right)\;.\end{aligned} \tag{41}$$

4.3 Calculation of phase shifts and action multipliers

The aim of this paragraph is to calculate the polymer phase shifts and action multipliers needed in (41) and (40) in terms of the transmission and reflection coefficients defined in (11). Because $\det(MT^{E_c+\epsilon}_\pm M^{-1})=1$, these coefficients satisfy

$$|a^\epsilon_\pm|^2-|b^\epsilon_\pm|^2\;=\;1\;.$$

A further short calculation shows that

$$\rho^\epsilon_\pm(\theta)^2\;=\;1+2\,\Re e\left(a^\epsilon_\pm b^\epsilon_\pm\,e^{2\imath\theta}\right)+2|b^\epsilon_\pm|^2\;, \tag{42}$$

and

$$e^{\imath(\mathcal{S}_{\epsilon,\pm}(\theta)\,-\,\theta)}\;=\;\frac{a^\epsilon_\pm+\overline{b^\epsilon_\pm}e^{-2\imath\theta}}{\left|a^\epsilon_\pm+\overline{b^\epsilon_\pm}e^{-2\imath\theta}\right|}\;.$$

Now using the phase $\eta^\epsilon_\pm$ of $a^\epsilon_\pm$,

$$a^\epsilon_\pm\;=\;e^{\imath\eta^\epsilon_\pm}+\mathcal{O}(|b^\epsilon_\pm|^2)\;. \tag{43}$$

This leads to the following expansions:

$$\log(\rho^\epsilon_\pm(\theta)^2)\;=\;2\,\Re e\left(a^\epsilon_\pm b^\epsilon_\pm e^{2\imath\theta}\right)+|b^\epsilon_\pm|^2-\Re e\left((a^\epsilon_\pm b^\epsilon_\pm)^2e^{4\imath\theta}\right)+\mathcal{O}\left(|b^\epsilon_\pm|^3\right)\;, \tag{44}$$

and

$$e^{2\imath(\mathcal{S}_{\epsilon,\pm}(\theta)-\theta)} \;=\; e^{2\imath\eta^\epsilon_\pm} + \overline{b^\epsilon_\pm} e^{\imath\eta^\epsilon_\pm}\, e^{-2\imath\theta} - b^\epsilon_\pm e^{3\imath\eta^\epsilon_\pm}\, e^{2\imath\theta} + \mathcal{O}\left(|b^\epsilon_\pm|^2\right)\;. \tag{45}$$

4.4 Oscillatory sums

Proposition 2 *Let $c_\pm \in \mathbb{C}$, $j = 1, 2$, and set*

$$I^j_N(\theta,\epsilon) \;=\; \mathbf{E}_0(I^j_{\omega,N}(\theta,\epsilon))\;, \qquad I^j_{\omega,N}(\theta,\epsilon) \;=\; \sum_{l=0}^{N-1} c_{\omega_l}\, e^{2\imath j \mathcal{S}^l_{\epsilon,\omega}(\theta)}\;.$$

Let ϵ be sufficiently small. If $\langle e^{2\imath j\eta_\pm}\rangle \neq 1$, then $I^j_N(\theta,\epsilon) = \mathcal{O}(N|b^\epsilon_\pm|, 1)$. If $\langle e^{2\imath j\eta_\pm}\rangle \neq 1$ for both $j = 1, 2$,

$$I^1_N(\theta,\epsilon) \;=\; N\,\langle c_\pm\rangle\, \frac{\langle \overline{b^\epsilon_\pm} e^{\imath\eta^\epsilon_\pm}\rangle}{1-\langle e^{2\imath\eta^\epsilon_\pm}\rangle} + \mathcal{O}(N|b^\epsilon_\pm|^2,\,1)\;.$$

Proof: Since $\mathcal{S}^{l+1}_{\epsilon,\omega}(\theta) = \mathcal{S}_{\epsilon,\omega_l}(\mathcal{S}^l_{\epsilon,\omega}(\theta))$ and $\mathcal{S}^l_{\epsilon,\omega}(\theta)$ is independent of ω_l, one gets

$$I^1_N(\theta,\epsilon) = \langle e^{2\imath\eta^\epsilon_\pm}\rangle\, I^1_{N-1}(\theta,\epsilon) + \langle c_\pm\rangle\, e^{2\imath\theta} + \langle c_\pm\rangle \sum_{l=1}^{N-1} \mathbf{E}_0\left(e^{2\imath \mathcal{S}_{\epsilon,\omega_l}(\mathcal{S}^l_{\epsilon,\omega}(\theta))} - e^{2\imath(\eta^\epsilon_{\omega_l}+\mathcal{S}^l_{\epsilon,\omega}(\theta))}\right). \tag{46}$$

Equation (45) shows that $e^{2\imath\mathcal{S}_{\epsilon,\omega_l}(\theta)} - e^{2\imath(\eta^\epsilon_{\omega_l}+\theta)} = \mathcal{O}(|b^\epsilon_\pm|)$. As $I^1_N(\theta,\epsilon) = I^1_{N-1}(\theta,\epsilon) + \mathcal{O}(1)$ and $\langle e^{2\imath\eta^\epsilon_\pm}\rangle \neq 1$ by hypothesis, one can solve for $I^1_N(\theta,\epsilon)$ which directly implies that $I^1_N(\theta,\epsilon) = \mathcal{O}(N|b^\epsilon_\pm|\,1)$. Along the same lines, $I^2_N(\theta,\epsilon) = \mathcal{O}(N|b^\epsilon_\pm|,\,1)$. Now insert the expansion (45) in (46). Due to the above, the oscillatory terms in that formula are then of order $\mathcal{O}(N|b^\epsilon_\pm|^2)$. Thus only the non-oscillatory term on the r.h.s. of (45) gives a contribution to the leading order. □

4.5 Asymptotics of the IDS

Proof of Theorem 3: Formula (45) leads to $\mathcal{S}_{\epsilon,\pm}(\theta)-\theta = \eta_\pm + \epsilon d_\pm - \epsilon\,\Im m(c_\pm e^{2\imath\theta}) + \mathcal{O}(\epsilon^2)$, where $d_\pm = (\partial_\epsilon \eta^\epsilon_\pm)\big|_{\epsilon=0}$ and $c_\pm = (\partial_\epsilon b^\epsilon_\pm)\big|_{\epsilon=0} e^{\imath\eta_\pm}$. Inserting this in (41) yields

$$\mathcal{N}(E_c+\epsilon) = \frac{1}{\pi\langle L_\pm\rangle}\left(\langle\eta_\pm\rangle + \epsilon\,\langle d_\pm\rangle - \epsilon \lim_{k\to\infty}\frac{1}{k}\,\Im m\;\mathbf{E}_0\left(\sum_{l=0}^{k-1} c_{\omega_l} e^{2\imath S^l_{\epsilon,\omega}(\theta)}\right) + \mathcal{O}(\epsilon^2)\right)\;.$$

By Proposition 2, the expectation of the oscillatory sum is of order $\mathcal{O}(k|b^\epsilon_\pm|, 1)$ and thus

$$\mathcal{N}(E_c+\epsilon) = \frac{1}{\pi\langle L_\pm\rangle}\left(\langle\eta_\pm\rangle + \epsilon\langle d_\pm\rangle + \mathcal{O}(\epsilon^2)\right)\;. \tag{47}$$

Setting $p_+ = 1$ and $p_+ = 0$ yields that in particular $\mathcal{N}_\pm(E_c+\epsilon) = \frac{1}{\pi L_\pm}(\eta_\pm + \epsilon d_\pm + \mathcal{O}(\epsilon^2))$, allowing to identify $\mathcal{N}_\pm(E_c) = \eta_\pm/\pi L_\pm$ and $\mathcal{N}'_\pm(E_c) = d_\pm/\pi L_\pm$. Using this to insert for $\eta_\pm$ and $d_\pm$ in (47) completes the proof. □

4.6 Asymptotics of the Lyapunov exponent

Proof of Theorem 2: Replacing (44) into (40) shows that $2\langle L_\pm\rangle\,\gamma(E_c+\epsilon)$ is, up to corrections of order $\mathcal{O}(|b^\epsilon_\pm|^3)$, equal to the ν-average of

$$\langle|b^\epsilon_\pm|^2\rangle+2\,\Re e\left(\langle a^\epsilon_\pm b^\epsilon_\pm\rangle\lim_{N\to\infty}\frac{1}{N}\sum_{l=0}^{N-1}\mathbf{E}_0\left(e^{2\imath\mathcal{S}^l_{\epsilon,\omega}(\theta)}\right)\right)-\Re e\left(\langle(a^\epsilon_\pm b^\epsilon_\pm)^2\rangle\lim_{N\to\infty}\frac{1}{N}\sum_{l=0}^{N-1}\mathbf{E}_0\left(e^{4\imath\mathcal{S}^l_{\epsilon,\omega}(\theta)}\right)\right).$$

By Proposition 2, the first oscillatory sum has a contribution of the order $\mathcal{O}(|b^\epsilon_\pm|^2)$ (which is given there) while the second oscillatory sum is of order $\mathcal{O}(|b^\epsilon_\pm|^3)$ and can hence be neglected. Therefore one obtains

$$\gamma(E_c+\epsilon)\;=\;\frac{1}{\langle L_\pm\rangle}\left[\frac{1}{2}\,\langle|b^\epsilon_\pm|^2\rangle+\Re e\left(\frac{\langle b^\epsilon_\pm e^{\imath\eta^\epsilon_\pm}\rangle\,\langle\overline{b^\epsilon_\pm}e^{\imath\eta^\epsilon_\pm}\rangle}{1-\langle e^{2\imath\eta^\epsilon_\pm}\rangle}\right)\right]+\mathcal{O}\left(|b^\epsilon_\pm|^3\right)\;. \tag{48}$$

It can be directly verified that the given leading order term vanishes if either $p_+=0$ or $p_+=1$, which also follows from the fact that in this case H_ω is a periodic Jacobi matrix, whose Lyapunov exponent vanishes in the interior of its spectral bands. Next rewrite (48) as a fraction with common denominator $|1-\langle e^{2\imath\eta^\epsilon_\pm}\rangle|^2$. Since $p_-=1-p_+$, the numerator is a polynomial of degree at most 3 in p_+ vanishing at $p_+=0$ and $p_+=1$. Elementary but lengthy algebra shows that moreover its third derivative vanishes identically. Calculating the first order derivative allows to conclude. □

4.7 Comments

A random phase approximation consists in supposing that the incoming phases $\mathcal{S}^l_{\epsilon,\omega}(\theta)$ in each summand of (40) and (41) is completely random, that is distributed according to the Lebesgue measure. It can easily be checked that one actually obtains the correct answers for the derivatives of both the IDS and the Lyapunov exponent at the critical energy within this approximation. However, the lowest order non-vanishing term in the Lyapunov exponent is $\mathcal{O}(|b^\epsilon_\pm|^2)$ and the random phase approximation gives together with the expansion (44) that $\gamma(E_c+\epsilon)\approx\langle|b^\epsilon_\pm|^2\rangle/(2\,\langle L_\pm\rangle)$, namely only the first term in (48). As we shall argue now, the second contribution is due to the presence of correlations (or memory) in the family of discrete time random dynamical systems $(\mathcal{S}_{\epsilon,\pm},\mathbb{R}P(1),\Omega,\mathbf{P})_{\epsilon\in\mathbb{R}}$.

It is a result of Furstenberg [Fur] (his hypothesis can be checked here) that for each $\epsilon\neq0$ (small enough) there exists a unique invariant measure ν_ϵ on $\mathbb{R}P(1)$ satisfying

$$\int d\nu_\epsilon(\theta)\,f(\theta)\;=\;\int d\nu_\epsilon(\theta)\,\langle f(\mathcal{S}_{\epsilon,\pm}(\theta))\rangle\;,\qquad f\in C(\mathbb{R}P(1))\;.$$

For $\epsilon=0$, one invariant measure is given by the Lebesgue measure (it is unique if $\eta_+-\eta_-$ is irrational). For finite ϵ, iteration of the invariance property and the Proposition 2 implies

$$\int d\nu_\epsilon(\theta)\,e^{2\imath\theta}\;=\;\int d\nu_\epsilon(\theta)\,\mathbf{E}\left(\frac{1}{N}\sum_{l=0}^{N-1}e^{2\imath\mathcal{S}^l_{\epsilon,\omega}(\theta)}\right)\;=\;\frac{\langle\overline{b^\epsilon_\pm}e^{\imath\eta^\epsilon_\pm}\rangle}{1-\langle e^{2\imath\eta^\epsilon_\pm}\rangle}+\mathcal{O}(|b^\epsilon_\pm|^2)\;.$$

Similarly $\int d\nu_\epsilon(\theta)\, e^{4\imath\theta} = \mathcal{O}(|b^\epsilon_\pm|)$. These facts express the deviations of the invariant measure from the Lebesgue measure and hence from the random phase approximation. Moreover, for small enough $\epsilon \neq 0$ the invariant measure ν_ϵ is known to be Hölder continuous [BL, p. 161] so that one can use it in (40). Hence

$$\gamma(E_c+\epsilon) \;=\; \frac{1}{2}\,\frac{1}{\langle L_\pm\rangle}\int d\nu_\epsilon(\theta)\;\left\langle \log(\rho^\epsilon_\pm(\theta)^2)\right\rangle \;.$$

Developing $\log(\rho^\epsilon_\pm(\theta)^2)$ as in (44) then also leads an alternative proof of (48) and the second contribution in (48) is indeed due to the correlations as claimed above. Finally let us point out that higher order terms in ϵ can readily be calculated, under adequate (weak) hypothesis.

5 Large deviation estimates

Using elementary estimates on the boundary terms M and M^{-1} in (39), as well as (12), and the expansions (43) and (44), one obtains that for all $0 \leq m \leq k \leq N$

$$\log\left(\left\|T^{E_c+\delta}_\omega(k,m)\right\|^2\right) \;=\; 2\delta \sup_{\theta\in[0,\pi)} \Re e \sum_{l=m}^{k-1} c_{\omega_l}\, e^{2\imath\mathcal{S}^l_{\delta,\omega}(\theta)} + \mathcal{O}(N\delta^2,1)\;, \tag{49}$$

where $c_\pm = e^{\imath\eta_\pm}(\partial_\delta b^\delta_\pm)|_{\delta=0}$. If the order of the critical energy is 1, then $c_\pm = \mathcal{O}(1)$. In order to prove the delocalization results, it is necessary to show that the l.h.s. of (49) is of order 1 as long as $\mathcal{O}(N\delta^2) = 1$. Therefore one needs to show that sums like $I^1_{\omega,k}(\theta,\delta)$ defined in Proposition 2 are with high probability of order $\sqrt{N}$ for $\mathcal{O}(N\delta^2) = 1$ and $|k| \leq N$. These random Weyl sums can be thought of as a discrete time (variable N) correlated random walk in the complex plain, the correlation being due to the presence of the dynamics $\mathcal{S}_{\delta,\pm}$. For the present purposes, it is sufficient to show that this sum actually behaves as a random walk on adequate time scales. Hence let us introduce, for every δ, θ,

$$\Omega^0_N(\alpha,\delta,\theta) \;=\; \left\{\omega\in\Omega_0 \;\middle|\; \exists\, k\leq N \text{ such that } |I^1_{\omega,k}(\theta,\delta)| \geq N^{\alpha+1/2}\right\}\;. \tag{50}$$

Theorem 6 *If $|\langle e^{2\imath\eta_\pm}\rangle| < 1$ and $\alpha > 0$, there exist constants C_1 and C_2 such that for all θ, N and δ with $N\delta^2 \leq 1$:*

$$\mathbf{P}_0(\Omega^0_N(\alpha,\delta,\theta)) \;\leq\; C_1\, e^{-C_2 N^\alpha}\;. \tag{51}$$

The proof of this estimate will be given in Section 5.1. First, let us deduce the following consequence:

Theorem 7 *Let $|\langle e^{2i\eta_\pm}\rangle| < 1$ and $\alpha > 0$. Then there are $c, c' > 0$, $C < \infty$ such that for every $N \in \mathbb{N}$, there exists a set $\Omega_N(\alpha) \subset \Omega$ satisfying*

$$\mathbf{P}(\Omega_N(\alpha)) \;=\; \mathcal{O}(e^{-cN^\alpha})\;,$$

and such that for every configuration (ω, l) in the complementary set $\Omega_N(\alpha)^c = \Omega\backslash\Omega_N(\alpha)$, one has

$$\left\| \mathcal{T}_{(\omega,l)}^{E_c+\delta+\imath\kappa}(k,m) \right\| \;\leq\; C \;,$$

for all $0 \leq m \leq k \leq N$ and all $|\delta| \leq N^{-\alpha-1/2}$, $|\kappa| \leq c'/N$.

Proof: In order to estimate the norms of the transfer matrices using the Weyl sums, note that for any 2×2 matrix A,

$$\|A\| \;=\; \sup_{\theta\in[0,\pi)} \|Ae_\theta\| \;\leq\; \sqrt{2} \max_{\theta=0,\frac{\pi}{2}} \|Ae_\theta\| \;. \tag{52}$$

Set $\Omega_N^0(\alpha,\delta) = \Omega_N^0(\alpha,\delta,0) \cup \Omega_N^0(\alpha,\delta,\frac{\pi}{2})$. Then, combining Theorem 6 with (49) and (52) as well as the fact that $T(k,m) = T(k,0)T(m,0)^{-1}$, one deduces that for all $\omega \in \Omega_N^0(\alpha,\delta)^c$ with $|\delta| < N^{-\alpha-1/2}$, norms of the transfer-matrices $T_\omega^{E_c+\delta}(k,m)$ are uniformly bounded by a constant, not dependent on δ.

Now let $\Omega_N(\alpha,\delta) = \{(\omega,l) \in \Omega \mid \omega \in \Omega_N^0(\alpha,\delta)\}$. It follows that

$$\mathbf{P}(\Omega_N(\alpha,\delta)) \;\leq\; \frac{L_+ + L_-}{\langle L_\pm\rangle}\, \mathbf{P}_0(\Omega_N^0(\alpha,\delta)) \;\leq\; C_5 e^{-C_4 N^\alpha}.$$

Elementary estimates (based on uniform bounds on norms of transfer matrices over blocks of length no more than $\max L_\pm$) imply that

$$\|\mathcal{T}_{(\omega,l)}^{E_c+\delta}(k,m)\| \;\leq\; C' \;,$$

for all $(\omega,l) \in \Omega_N(\alpha,\delta)^c$, $|\delta| \leq N^{-\alpha-1/2}$ and $0 \leq m \leq k \leq N$ (in fact this holds for m,k up to the N-th polymer node). Set $\epsilon = N^{-\alpha-1/2}$. The theorem then follows from Lemma 1, by taking $\Omega_N(\alpha) = \bigcup_{k=-N}^{N} \Omega_N(\alpha, k\epsilon/N)$. □

5.1 Correlation bounds: Proof of Theorem 6

Lemma 4 *Let $\kappa = |\langle e^{2\imath\eta_\pm}\rangle| < 1$. Then there exists a centered complex random variable $X(\omega)$ depending on $\omega_1, \ldots, \omega_r$ such that*

$$e^{2\imath\mathcal{S}_{\delta,\omega}^r(\theta)} \;=\; X(\omega)e^{2\imath\theta} + \mathcal{O}(r\delta,\kappa^r) \;.$$

Moreover, $|X(\omega)|$ is uniformly bounded by 2.

Proof: Let us set $\kappa_r(\omega) = \exp(2\imath\sum_{m=1}^r \eta_{\omega_m})$. Note that $|\mathbf{E}_0(\kappa_r(\omega))| = \kappa^r$. Iteration of $e^{2\imath\mathcal{S}_{\delta,\pm}(\theta)} = e^{2\imath(\eta_\pm+\theta)} + \mathcal{O}(\delta)$ and centering the random variable $\kappa_r(\omega)$ shows

$$e^{2\imath\mathcal{S}_{\delta,\omega}^r(\theta)} \;=\; (\kappa_r(\omega) - \langle e^{2\imath\eta_\pm}\rangle^r)e^{2\imath\theta} + \mathcal{O}(\kappa^r,\delta r) \;,$$

as claimed. □

Proof of Theorem 6: Let r be the smallest integer larger than $\log(N^{-\alpha-1/2})/\log(\kappa)$. Applying Lemma 4 to each term (except the first r terms) of the sum $I^1_{\omega,k}(\theta,\delta)$ shows that

$$I^1_{\omega,k}(\theta,\delta) \;=\; \sum_{l=0}^{k-1} c_{\omega_{l+r+1}}\, X(T_0^l\,\omega)\, e^{2\imath \mathcal{S}^l_{\delta,\omega}(\theta)} + \mathcal{O}(rk\delta,\, k\kappa^r,\, r)\;. \tag{53}$$

(Recall here that $T_0^l\omega$ is the l-fold shift of ω). Under the hypothesis of the theorem and because of the choice of r, the error term in (53) is $\mathcal{O}(N^{1/2}\log N)$. Thus it is sufficient to prove probabilistic estimates of the appearing sum, which will be denoted by $Z_k(\omega,\delta)$.

In order to decouple the correlations, divide $\{0,\ldots,k-1\}$ in $2R$ pieces $I_0,\ldots,I_{2R-1}$ of equal length $[k^\alpha]$, where $R=[k/(2[k^\alpha])]$, i.e. $I_s=\{s[k^\alpha],(s+1)[k^\alpha]-1\}$, $s=0,\ldots,2R-1$. Here $[x]$ denotes the largest integer smaller or equal to x. This excludes ck^α terms which in the following can be absorbed in the error. Set for $j=0,1$:

$$Z^j_R(\omega,\delta) \;=\; \sum_{s=0}^{R-1} Y_{2s+j}(\omega,\delta)\;, \qquad Y_s(\omega,\delta) \;=\; \sum_{l\in I_s} c_{\omega_{l+r+1}}\, X(T_0^l\omega)\, e^{2\imath \mathcal{S}^l_{\delta,\omega}(\theta)}\;.$$

Thus

$$Z_k(\omega,\delta) \;=\; Z^0_R(\omega,\delta) + Z^1_R(\omega,\delta) + \mathcal{O}(k^\alpha)\;. \tag{54}$$

The random variable $Y_s(\omega,\delta)$ satisfies uniformly $|Y_s(\omega,\delta)|\le c_1\,k^\alpha$. If $\mathbf{E}_s$ denotes the averaging procedure (conditional expectation) over all random variables ω_l for $l\ge s$, then Lemma 4 implies $\mathbf{E}_{s[k^\alpha]+1}(Y_s(\omega,\delta))=0$.

In the following estimates, real and imaginary parts of $Z^j_R(\omega,\delta)$ are treated separately, but in exactly the same way; hence one may suppose that $Z^j_R(\omega,\delta)$ and all the summands therein are real. For $\lambda>0$ and $\beta>0$, the Tchebychev and Cauchy-Schwarz inequalities imply

$$\mathbf{P}_0(\{\omega\in\Omega_0\,|\,Z_k(\omega,\delta)>\lambda\}) \;\le\; e^{-\beta\lambda}\,\mathbf{E}_0(e^{\beta Z_k(\omega,\delta)}) \;\le\; e^{-\beta\lambda+C\beta k^\alpha}\,\max_{j=0,1}\,\mathbf{E}_0(e^{2\beta Z^j_R(\omega,\delta)})\;.$$

Now if $-1\le Y\le 1$, by convexity $2\,e^{\beta Y}\le(1-Y)e^{-\beta}+(1+Y)e^{\beta}$. Thus if Y is a real centered random variable,

$$\mathbf{E}(e^{\beta Y(\omega)}) \;\le\; (e^{-\beta}+e^{\beta})/2 \;\le\; e^{\beta^2/2}\;. \tag{55}$$

one may assume that $[k^\alpha]>r-1$ and $k\ge N^{\frac{1+\alpha}{2}}$ (otherwise it is trivially true that $|I^1_{\omega,k}(\theta,\delta)|<N^{\alpha+1/2}$). Thus $Z^j_{R-1}(\omega,\delta)$ does not depend on the ω_l with $l\ge(2(R-1)+j)[k^\alpha]-1$ and a rescaled version of (55) can be iteratively applied to the conditional expectations, leading to

$$
\begin{aligned}
\mathbf{E}_0(e^{2\beta Z^j_R(\omega,\delta)}) \;&\leq\; \mathbf{E}_0\left(\mathbf{E}_{(2(R-1)+j)[k^\alpha]-1}(e^{2\beta Y_{2(R-1)+j}(\omega,\delta)})\, e^{2\beta Z^j_{R-1}(\omega,\delta)}\right)\\
&\leq\; e^{(2c_1k^\alpha\beta)^2/2}\mathbf{E}_0\left(e^{2\beta Z^j_{R-1}(\omega,\delta)}\right)\\
&\leq\; e^{c_2\beta^2k^{1+\alpha}}\;.
\end{aligned}
$$

Choosing $\beta = \lambda/(2c_2k^{1+\alpha})$ and proceeding similarly for $\{\omega \in \Omega_0 \,|\, Z_k(\omega,\delta) < -\lambda\}$ thus shows (after recombining real and imaginary parts)

$$
\mathbf{P}_0(\{\omega \in \Omega_0 \,|\, |Z_k(\omega,\delta)| > \lambda\}) \;\leq\; 4\, e^{-\lambda^2/(4c_2\, k^{1+\alpha})+\frac{C\lambda}{2c_2k}}\;.
$$

Using this estimate for $\lambda = N^{\alpha+1/2}$ and renormalizing the constants in order to compensate for the error terms in (53) as well as for summation over k concludes the proof. □

5.2 Eigenvalue distribution in the metalic phase

Proof of Theorem 5: Let us fix a configuration $(\omega, l) \in \Omega_N(\alpha)$ (see Theorem 7) and suppress its index. Using (20) and the fact that the norm of a transfer matrix is equal to the norm of its inverse yields $\|\mathcal{T}^{E_c+\delta}(k,0)\|^{-1} \leq R^{0,E_c+\delta}(k) \leq \|\mathcal{T}^{E_c+\delta}(k,0)\|$. Theorem 7 therefore guarantees the existence of a constant C such that for $0 \leq k \leq N$ and for $-N^{-\alpha-1/2} < \delta < N^{-\alpha-1/2}$,

$$
\frac{1}{C} \;\leq\; R^{0,E_c+\delta}(k)^2 \;\leq\; C\;, \qquad \frac{1}{C} \;\leq\; |u^{E_c+\delta}(k)|^2 + |u^{E_c+\delta}(k-1)|^2 \;\leq\; C\;.
$$

This readily yields (18). Now by Lemma 2,

$$
\frac{N}{C} \;\leq\; \partial_E\, \theta^{0,E_c+\delta}(N) \;\leq\; N\,C\;.
$$

Upon integration, one deduces

$$
\frac{N}{C}\,|E-E'| \;\leq\; |\theta^{0,E}(N) - \theta^{0,E'}(N)| \;\leq\; N\,C\,|E-E'|\;,
$$

for all $E, E' \in [E_c - N^{-\alpha-1/2}, E_c + N^{-\alpha-1/2}]$. The oscillation theorem discussed in Section 3.2 now gives (17). □

6 Lower bound on dynamics

The deterministic part of the argument presented in this section follows [DT]. Let us return to the simplified notation from Section 2 and write ω instead of (ω, l) since based on the results of Section 5 the value of l will not influence the considerations. Let us begin with some preliminaries and introduce the Green's function

$$
G^z_\omega(n) \;=\; \left\langle n \left| \frac{1}{H_\omega - z} \right| 0 \right\rangle\;.
$$

Note that

$$-t_\omega(n+1)\,G^z_\omega(n+1)+(v_\omega(n)-E_c-z)\,G^z_\omega(n)-t_\omega(n)\,G^z_\omega(n-1)\;=\;\delta_{n,0}\;. \tag{56}$$

Using transfer matrices, one now has for $n\leq 0$,

$$\begin{pmatrix} t_\omega(n)\,G^z_\omega(n) \\ G^z_\omega(n-1)\end{pmatrix}\;=\;\mathcal{T}^z_\omega(n,0)\begin{pmatrix} t_\omega(0)\,G^z_\omega(0) \\ G^z_\omega(-1)\end{pmatrix}\;, \tag{57}$$

while for $n\geq 1$,

$$\begin{pmatrix} t_\omega(n)\,G^z_\omega(n) \\ G^z_\omega(n-1)\end{pmatrix}\;=\;\mathcal{T}^z_\omega(n,1)\begin{pmatrix} t_\omega(1)\,G^z_\omega(1) \\ G^z_\omega(0)\end{pmatrix}\;. \tag{58}$$

The following identity is well-known:

$$M_{\omega,q}(T)\;=\;\frac{1}{\pi}\,\frac{1}{T}\sum_{n\in\mathbb{Z}}|n|^q\int_{\mathbb{R}}dE\;|G^z_\omega(n)|^2\;,\qquad z\;=\;E+\frac{\imath}{T}\;. \tag{59}$$

Proof of Theorem 4. For given $\alpha>0$ let $c,c'>0$ and $C<\infty$ be the constants form Theorem 7 and choose $N=[c'\,T]$ and $\epsilon=N^{-1/2-\alpha}$. By Theorem 7 there exists $\Omega_N(\alpha)\subset\Omega$ with $\mathbf{P}(\Omega_N(\alpha))=\mathcal{O}(e^{-cN^\alpha})$ and such that for $\omega\in\Omega_N(\alpha)^c$ one has $\|\mathcal{T}^{E_c+\delta+\imath/T}_\omega(n,1)\|\leq C$ for all $|\delta|\leq N^{-\alpha-1/2}$ and $n\leq N$.

For such ω, because of the uniform bounds on the matrix elements $t_\omega(n)$ and $v_\omega(n)$, for $n=0$ one of the three terms on the l.h.s. of (56) has to be large. Suppose first that $|G^z_\omega(0)|^2+|G^z_\omega(1)|^2\geq C_6>0$, then it follows from (58) and $\|(\mathcal{T}^z_\omega(n,1))^{-1}\|=\|\mathcal{T}^z_\omega(n,1)\|$ that

$$\max\left\{|G^z_\omega(n)|^2,|G^z_\omega(n-1)|^2\right\}\;\geq\;\frac{C_7}{\|\mathcal{T}^z_\omega(n,1)\|^2}\;.$$

According to the above, as long as $\delta\in[-\epsilon,\epsilon]$ the transfer matrices are bounded from above by C as long as $n\leq[c_1\,T]$, in which case at least every second $|G^z_\omega(n)|^2$ is bigger than C_7/C^2. Replacing this into (59),

$$M_{\omega,q}(T)\;\geq\;\frac{1}{2\pi T}\sum_{0\leq n\leq[c_1\,T]}n^q\int_{[-\epsilon,\epsilon]}d\delta\;\frac{C_7}{C^2}\;\geq\;C_8\,T^q\epsilon\;=\;C_8\,T^{q-\frac{1}{2}-\alpha}\;,$$

for some constant $C_8>0$. If, on the other hand $|G^z_\omega(-1)|^2\geq C_6>0$, then one gets this estimate in the same way, but based on (57) instead of (58). This uses the fact that the analysis of Section 5 can also be carried out for $\mathcal{T}^{E_c+\delta}_\omega(n,0)$ with negative n. A Borel-Cantelli lemma shows that a.s. $\beta^-_{\omega,q}\geq 1-(\frac{1}{2}-\alpha)/q$. Since $\alpha>0$ is arbitrary, this finishes the proof. □

Proof of Theorem 1. Follow the above argument by using the deterministic Proposition 1. □

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