On complex singularity analysis for linear $q-$difference-differential equations

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Abstract

We investigate the existence of global holomorphic solutions of linear $q-$difference-differential equations, with meromorphic coefficients, having complex singularities along $q-$spirals. We also study the rate of growth of the constructed solutions near these singular points.

Key words: Cauchy problem, functional differential equation, $q-$difference equation, entire function with $q-$exponential growth, transseries expansion, singularity analysis. 2000 MSC: 35C10, 35C20.

1 Introduction

We study linear $q-$difference-differential equations of the form

$$\partial_z^S X(t,z) = \sum_{k=0}^{S-1} b_k(t,z)(\partial_z^k X)(q^{p_{0,k}}t, zq^{-m_{1,k}}) \tag{1}$$

where q is a complex number such $|q| > 1$, $S, p_{0,k}, m_{1,k} \geq 1$ are positive integers and $b_k(t,z)$ are meromorphic functions in t and polynomial in z on $\mathbb{C}^* \times \mathbb{C}$. As in [7], we will consider the case when each deviation $(t,z) \mapsto (q^{p_{0,k}}t, zq^{-m_{1,k}})$ is a *volume shrinking map* meaning that the modulus of the determinant of its jacobian matrix is smaller than one, which is equivalent to the fact that $p_{0,k} < m_{1,k}$.

Our goal is to investigate a singularity analysis of holomorphic solutions of (1) using the approach of formal series expansions which belong to the class of so-called *transseries* (cf. [4]). This point of view has shown a great success in the context of differential and difference equations in the complex domain. In particular, the formation of complex singularities and local behaviour of the holomorphic solutions near the singular points along Stokes directions for systems of non linear difference and differential equations with irregular singularities have been studied using the concept of formal transseries, see [2], [5], [6].

We are looking for formal series expansions solutions of (1) of the form

$$\hat{X}(t,z) = \sum_{l,\beta \geq 0} X_{l,\beta}(t)(e_q(t))^l \frac{z^\beta}{\beta!}. \tag{2}$$

The function $e_q(t)$ is a meromorphic function defined on a domain $\mathcal{S} \subset \mathbb{C}^*$ invariant with respect to the dilation $t \mapsto qt$ having poles along a part of a union of $q-$spirals (as defined in [3], [9]) denoted by Θ. The coefficients $X_{l,\beta}(t)$ are bounded holomorphic functions on $\mathcal{S}$.

In the framework of differential equations, the transseries are constructed in the form $\hat{y}(t) = \sum_{l\geq 0} \hat{x}_l(t)e^{l/t}$ where $\hat{x}_l(t)$ are formal power series. These transseries yield actual holomorphic solutions to differential equations of the form $y(t) = \sum_{l\geq 0} x_l(t)e^{l/t}$, where $x_l(t)$ are the $k-$sums of $\hat{x}_l(t)$ in some direction d, see [5], [6]. In our context, the function $e_q(t)$ will play the same role as the exponential function $e^{1/t}$ and the coefficients $X_{l,\beta}(t)$ will play the role of the $k-$sums $x_l(t)$.

In a first part of the paper, we construct the function $e_q(t)$ as a particular solution of a $q-$difference equation on $\mathbb{C}^*$ with rational coefficient.

In a second part, we study a Cauchy problem in a space of entire functions with $q-$exponential growth (Theorem 1).

In the third part, we construct formal transseries solutions (2) of (1) and we give sufficient conditions under which these solutions define holomorphic functions on the open set $(\mathcal{S} \setminus \Theta) \times \mathbb{C}$ using a majorazing series technique reducing the problem to the study of the Cauchy problem of the second part. Finally, using the fact that $e_q(t)$ has meromorphic singularities on Θ and classical growth properties of $q-$Gevrey entire functions, we get that these solutions have actually *sub-exponential* rate of growth for t near the points of Θ and z in $\mathbb{C}$ (Theorem 2).

2 A $q-$difference equation

In this section, we study a particular solution of the following $q-$difference equation

$$e_q(qt) = (1 + u(1/t))e_q(t) \tag{3}$$

where q is a complex number such that $|q| > 1$ and u is a rational function. Let $u = P/Q$ with P, Q be two polynomials. We make the following assumptions on u.

(H1). Assume that $P(0) = 0$, $Q(0) \neq 0$ and if $\deg(P)$ denotes the degree of the polynomial P, we have $1 \leq \deg(P) \leq \deg(Q)$ and $\deg(P + Q) \geq 1$.

(H2). We denote by $d_1, \ldots, d_{\deg(Q)}$ the roots of Q and $n_1, \ldots, n_s$ the roots of $P+Q$. By **(H1)**, we get that $n_j \neq 0$, $1 \leq j \leq s$. For $1 \leq j \leq s$, let $\delta_j > 0$ be positive real numbers, let $B(1/n_j, \delta_j)$ be the open disc centered at $1/n_j$ with radius δ_j and $\bar{B}(1/n_j, \delta_j)$ its closure. We assume that there exist $\delta_j > 0$, $1 \leq j \leq s$, $\eta > 0$ such that

$$\mathrm{dist}(d_k \bar{B}(1/n_j, \delta_j)q^{\mathbb{Z}}, 1) := \min_{H\in\mathbb{Z}, x_j\in\bar{B}(1/n_j,\delta_j)} |d_k x_j q^H - 1| \geq \eta,$$

for all $1 \leq k \leq \deg(Q)$, $1 \leq j \leq s$. Moreover, we assume that $\delta_j > 0$ can be chosen small enough such that $B(1/n_j, \delta_j)$ and $\bar{B}(1/n_j, \delta_j)$ belong to $\mathbb{C}^*$ for $1 \leq j \leq s$. In the following, we denote by $\mathcal{S}$ the open set $\mathcal{S} = \cup_{j=1}^{s} q^{\mathbb{Z}} B(1/n_j, \delta_j)$ and $\bar{\mathcal{S}}$ the set $\bar{\mathcal{S}} = \cup_{j=1}^{s} q^{\mathbb{Z}} \bar{B}(1/n_j, \delta_j)$.

Remark: Similar $q-$difference equations like (3) giving rise to $q-$analogs of exponential functions and Jacobi theta functions have been studied in [8] and similar sets $\mathcal{S}$ which are invariant under the dilation $t \mapsto qt$ have been considered in the theory of resummation of formal series along $q-$spirals, see [3], [9].

Lemma 1 *a) For all $n \geq 0$, there exists $M_n > 0$ such that, for all $t \in \bar{\mathcal{S}}$,*

$$|u(1/(q^n t))| \leq M_n.$$

b) Let $h \in \mathbb{Z}$. Then, there exist $M_h > 0$ and $n_{0,h} > 0$ depending on h such that

$$|u(1/(q^n t))| \leq M_h (\frac{1}{|q|^n})^{\mathrm{val}(P)},$$

for all $n \geq n_{0,h}$, for all $t \in \cup_{j=1}^{s} q^h \bar{B}(1/n_j, \delta_j)$, where $\mathrm{val}(P)$ denotes the valuation of the polynomial P.

Proof a) Let $n \geq 0$ be a fixed integer. If $Q(X) = a(d_1 - X) \cdots (d_{\deg(Q)} - X)$ where $a \in \mathbb{C}^*$, we have

$$u(\frac{1}{q^n t}) = \frac{P(\frac{1}{q^n t})}{a(d_1 - \frac{1}{q^n t}) \cdots (d_{\deg(Q)} - \frac{1}{q^n t})}.$$

If $t = q^h x_j \in \bar{\mathcal{S}}$, $1 \leq j \leq s$, $h \in \mathbb{Z}$, $x_j \in \bar{B}(1/n_j, \delta_j)$, due to **(H2)** we have

$$|d_k - \frac{1}{q^{n+h} x_j}| = |\frac{d_k x_j q^{n+h} - 1}{q^{n+h} x_j}| \geq \frac{\eta}{|q|^{n+h}|x_j|} > 0, \tag{4}$$

for $1 \leq k \leq \deg(Q)$. In particular, $u(1/q^n t)$ is well defined for all $t \in \bar{\mathcal{S}}$.

There exist $M > 0$ and $A > 0$ such that for all $z \in \mathbb{C}$, $|z| \geq A$ implies $|P(z)| \leq M|z|^{\deg(P)}$. In particular, there exists $h_{0,n} < 0$ depending on n such that for all $h \leq h_{0,n}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$, we have $1/(|q|^{n+h}|x_j|) \geq A$ which implies

$$|P(\frac{1}{q^{n+h} x_j})| \leq M \frac{1}{(|q|^{n+h}|x_j|)^{\deg(P)}}. \tag{5}$$

From (4) and (5), we deduce

$$|u(1/(q^{n+h} x_j))| \leq \frac{M}{|a|\eta^{\deg(Q)}} \frac{(|q|^{n+h}|x_j|)^{\deg(Q)}}{(|q|^{n+h}|x_j|)^{\deg(P)}}.$$

From **(H1)**, the sequence $|q|^{h(\deg(Q) - \deg(P))}$ is bounded as $h \to -\infty$. So that there exists $M_{0,n} > 0$ depending on n, there exists $h_{0,n} < 0$ depending on n such that for all $h \leq h_{0,n}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$,

$$|u(1/(q^{n+h} x_j))| \leq M_{0,n}. \tag{6}$$

There exist $M > 0$ and $\delta > 0$ such that for all $z \in \mathbb{C}$, $|z| < \delta$ implies $|P(z)| \leq M|z|^{\mathrm{val}(P)}$. In particular, there exists $h_{1,n} > 0$ depending on n such that for all $h \geq h_{1,n}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$, we have $1/(|q|^{n+h}|x_j|) \leq \delta$ which implies

$$|P(\frac{1}{q^{n+h} x_j})| \leq M \frac{1}{(|q|^{n+h}|x_j|)^{\mathrm{val}(P)}}. \tag{7}$$

From **(H1)** we have $d_k \neq 0$, $1 \leq k \leq \deg(Q)$ and from the fact that $\lim_{h \to +\infty} 1/(q^{n+h} x_j) = 0$, we deduce that there exist $\eta_1 > 0$, $h_{1,n} > 0$ such that for all $h \geq h_{1,n}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$, $1 \leq k \leq \deg(Q)$,

$$|d_k - \frac{1}{q^{n+h} x_j}| \geq \eta_1 \tag{8}$$

From (8) and (7), we deduce

$$|u(1/(q^{n+h}x_j))| \leq \frac{M}{|a|(\eta_1)^{\deg(Q)}(|q|^{n+h}|x_j|)^{\mathrm{val}(P)}}$$

From **(H1)**, we have $\mathrm{val}(P) \geq 1$, so that $1/|q|^{h\mathrm{val}(P)}$ tends to 0 as $h \to +\infty$. We deduce that there exists $M_{1,n} > 0$ depending on n, there exists $h_{1,n} > 0$ depending on n such that for all $h \geq h_{1,n}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$,

$$|u(1/(q^{n+h}x_j))| \leq M_{1,n}. \tag{9}$$

From (4), $u(1/(q^{n+h}x_j))$ is well defined for all $h_{0,n} \leq h \leq h_{1,n}$, all $x_j \in \bar{B}(1/n_j, \delta_j)$. We deduce that there exists $M_{2,n} > 0$ depending on n such that for all $h_{0,n} \leq h \leq h_{1,n}$, all $x_j \in \bar{B}(1/n_j, \delta_j)$,

$$|u(1/(q^{n+h}x_j))| \leq M_{2,n}. \tag{10}$$

Finally, from (6), (9), (10), we get $M_n = \sup(M_{0,n}, M_{1,n}, M_{2,n})$ depending on n such that for all $h \in \mathbb{Z}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$, $1 \leq j \leq s$,

$$|u(1/(q^{n+h}x_j))| \leq M_{3,n}.$$

The part a) of Lemma 1 follows.

b) Let $h \in \mathbb{Z}$. From **(Q1)** we have $d_k \neq 0$, $1 \leq k \leq \deg(Q)$ and from the fact that $\lim_{n\to+\infty} 1/(q^{n+h}x_j) = 0$, there exists $\eta_1 > 0$, there exists $n_{0,h} \geq 0$ (depending on h) such that for all $n \geq n_{0,h}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$, $1 \leq j \leq s$,

$$|d_k - \frac{1}{q^{n+h}x_j}| \geq \eta_1 \tag{11}$$

On the other hand, there exist $M > 0$ and $\delta > 0$ such that for all $z \in \mathbb{C}$, $|z| < \delta$ implies $|P(z)| \leq M|z|^{\mathrm{val}(P)}$. In particular, there exists $n_{0,h} \geq 0$ (depending on h) such that for all $n \geq n_{0,h}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$ implies $1/(|q|^{n+h}|x_j|) \leq \delta$, which implies

$$|P(\frac{1}{q^{n+h}x_j})| \leq M\frac{1}{(|q|^{n+h}|x_j|)^{\mathrm{val}(P)}} \leq M\frac{1}{(|q|^{n+h}m_j)^{\mathrm{val}(P)}}. \tag{12}$$

where $m_j > 0$ depends on $\bar{B}(1/n_j, \delta_j)$.

From (11) and (12), we deduce that

$$|u(1/(q^{n+h}x_j))| \leq \frac{M}{|a|(\eta_1)^{\deg(Q)}(|q|^{n+h}m_j)^{\mathrm{val}(P)}}$$

for all $n \geq n_{0,h}$, for all $x_j \in \bar{B}(1/n_j, \delta_j)$, $1 \leq j \leq s$. The part b) of Lemma 1 follows. □

Proposition 1 *a) The infinite product* $\prod_{n=0}^{+\infty}(1 + u(1/(q^n t)))$ *converges uniformly on the compact sets* $\cup_{j=1}^{s} q^h \bar{B}(1/n_j, \delta_j)$ *for any* $h \in \mathbb{Z}$.
b) We denote by $\Theta = \cup_{j=1}^{s} \frac{1}{n_j} q^{-\mathbb{N}}$. *Then, the function*

$$e_q(t) = \frac{1}{\prod_{n=0}^{+\infty}(1 + u(1/(q^n t)))},$$

is a solution of (3) which is holomorphic on the open set $\mathcal{S}\backslash\Theta$ *and has meromorphic singularities on* Θ.

Proof a) Let $h \in \mathbb{Z}$. From Lemma 1) b), there exist $M_h > 0$ and $n_{0,h} > 0$ depending on h such that

$$|u(1/(q^n t))| \leq M_h (\frac{1}{|q|^n})^{\mathrm{val}(P)},$$

for all $n \geq n_{0,h}$, for all $t \in \cup_{j=1}^{s} q^h \bar{B}(1/n_j, \delta_j)$. We deduce that there exist $M_{1,h} > 0$ and $n_{1,h} > 0$ such that

$$|\mathrm{Log}(1 + u(\frac{1}{q^n t}))| \leq M_{1,h} (\frac{1}{|q|^n})^{\mathrm{val}(P)}$$

for all $n \geq n_{1,h}$, for all $t \in \cup_{j=1}^{s} q^h \bar{B}(1/n_j, \delta_j)$. From **(H1)**, we have $\mathrm{val}(P) \geq 1$. So that the series

$$\sum_{n \geq n_{1,h}} \mathrm{Log}(1 + u(\frac{1}{q^n t}))$$

converge uniformly on $\cup_{j=1}^{s} q^h \bar{B}(1/n_j, \delta_j)$. The part a) of Proposition 1 follows.

b) From Proposition 1 a), the infinite product $\prod_{n=0}^{+\infty}(1 + u(1/(q^n t)))$ defines a holomorphic function on $\mathcal{S}$. Moreover, by construction for a given $n \geq 0$, the solutions of $1 + u(1/(q^n t)) = 0$ are of the form $t = q^{-n}/n_j$, $1 \leq j \leq s$. Moreover, these complex numbers are poles for the function $1/(1 + u(1/(q^n t)))$ because u is a rational function. So that the function $e_q(t) = 1/\prod_{n=0}^{+\infty}(1 + u(1/(q^n t)))$ is holomorphic on $\mathcal{S}$ outside the points q^{-n}/n_j, $1 \leq j \leq s$, $n \geq 0$, and is meromophic at these points. As in [8], page 72, one verifies that $e_q(t)$ is formally a solution of (3) by plugging the expression $1/\prod_{n=0}^{+\infty}(1 + u(1/(q^n t)))$ in (3). The part b) of Proposition 1 follows. □

3 A Cauchy problem in a class of entire functions with $q-$exponential growth

Definition 1 *Let q, T, X be real numbers such that $q > 1$, $T, X > 0$. We define a vector space $G^q(T, X)$ which is a subspace of the vector space of entire functions on $\mathbb{C}^2$ denoted by $A(\mathbb{C}^2)$. An entire function*

$$U(t,x) = \sum_{l,\beta \geq 0} u_{l,\beta} t^l \frac{x^\beta}{\beta!}$$

belongs to $G^q(T, X)$, if the series

$$\sum_{l,\beta \geq 0} \frac{|u_{l,\beta}|}{q^{P(l,\beta)} \beta!} T^l X^\beta,$$

converges, where $P(l, \beta)$ denotes the polynomial

$$P(l, \beta) = l\beta - \beta^2 - l^2$$

and we also define a weighted L^1 norm on $G^q(T, X)$ as

$$||U(t,x)||_{T,X} = \sum_{l,\beta \geq 0} \frac{|u_{l,\beta}|}{q^{P(l,\beta)} \beta!} T^l X^\beta.$$

One can easily show that $(G^q(T, X), ||.||_{T,X})$ is a Banach space.

Remark: Let $U(t,x)$ be in $G^q(T_0,X_0)$ for given $T_0,X_0>0$. Then, $U(t,x)$ also belong to the spaces $G^q(T,X)$ for all $X\leq X_0$ and $T\leq T_0$. Moreover, the maps $X\mapsto ||U(t,x)||_{T,X}$ and $T\mapsto ||U(t,x)||_{T,X}$ are increasing functions from $[0,X_0]$ (resp. $[0,T_0]$) into $\mathbb{R}_+$.

We define the integration operator $\partial_x^{-1}: A(\mathbb{C}^2)\to A(\mathbb{C}^2)$ as

$$\partial_x^{-1}U(t,x):=\int_0^x U(t,\zeta)d\zeta.$$

Definition 2 *For any polynomial $a(t,x)=\sum_{\alpha,\mu\geq 0} a_{\alpha,\mu}t^\alpha x^\mu\in\mathbb{C}[t,x]$, we denote by $|a|(t,x)$ the polynomial $|a|(t,x)=\sum_{\alpha,\mu\geq 0}|a_{\alpha,\mu}|t^\alpha x^\mu$.*

Definition 3 *Let $h_2,m_1,m_2\geq 0$ be non negative integers. Let the set*

$$\mathcal{P}_{h_2,m_1,m_2}=\{(\alpha,\mu)\in\mathbb{N}^2 \quad : \quad h_2+\mu\geq 2\alpha+m_1 \quad , \quad \alpha+m_2\geq 2(h_2+\mu)\}$$

assumed not to be empty. A polynomial $a(t,x)\in\mathbb{C}[t,x]$ is said of $\mathcal{P}_{h_2,m_1,m_2}-$type if it is of the form

$$a(t,x)=\sum_{(\alpha,\mu)\in\mathcal{P}_{h_2,m_1,m_2}} a_{\alpha,\mu}t^\alpha x^\mu.$$

Remark: If $(\alpha,\mu)\in\mathcal{P}_{h_2,m_1,m_2}$ then we get $0\leq 3\alpha\leq m_2-2m_1$ and $0\leq 3\mu\leq -3h_2+2m_2-m_1$. In particular, the set $\mathcal{P}_{h_2,m_1,m_2}$ is bounded in $\mathbb{N}^2$.

Lemma 2 *Let $h_2,m_1,m_2\geq 0$ be non negative integers. Assume that the set $\mathcal{P}_{h_2,m_1,m_2}$ is not empty. Then, there exists a constant $C>0$ (depending on h_2,m_1,m_2) such that for all polynomials $a(t,x)$ of $\mathcal{P}_{h_2,m_1,m_2}-$type, one has*

$$||a(t,x)(\partial_x^{-h_2}U)(q^{m_1}t,q^{-m_2}x)||_{T,X}\leq C|a|(T,X)X^{h_2}||U(t,x)||_{T,X}$$

for all $U(t,x)\in G^q(T,X)$.

Proof For $U(t,x)\in G^q(T,X)$ we have

$$a_{\alpha,\mu}t^\alpha x^\mu(\partial_x^{-h_2}U)(q^{m_1}t,q^{-m_2}x)=a_{\alpha,\mu}\sum_{l,\beta}\frac{\beta!}{(\beta-\mu)!}u_{l-\alpha,\beta-h_2-\mu}q^{m_1(l-\alpha)}q^{-m_2(\beta-\mu)}t^l\frac{x^\beta}{\beta!}.$$

By definition, we have

$$\begin{aligned}&||a_{\alpha,\mu}t^\alpha x^\mu(\partial_x^{-h_2}U)(q^{m_1}t,q^{-m_2}x)||_{T,X}\\&\qquad=|a_{\alpha,\mu}|T^\alpha X^{\mu+h_2}\sum_{l,\beta}\frac{|u_{l-\alpha,\beta-h_2-\mu}|(\beta!/(\beta-\mu)!)q^{m_1(l-\alpha)}q^{-m_2(\beta-\mu)}}{q^{P(l,\beta)}\beta!}T^{l-\alpha}X^{\beta-h_2-\mu}\end{aligned}$$

So that

$$\begin{aligned}&||a_{\alpha,\mu}t^\alpha x^\mu(\partial_x^{-h_2}U)(q^{m_1}t,q^{-m_2}x)||_{T,X}\\&\qquad=|a_{\alpha,\mu}|T^\alpha X^{\mu+h_2}\sum_{l,\beta}A(l,\beta)\frac{|u_{l-\alpha,\beta-h_2-\mu}|}{q^{P(l-\alpha,\beta-h_2-\mu)}(\beta-h_2-\mu)!}T^{l-\alpha}X^{\beta-h_2-\mu}\end{aligned}$$

where

$$A(l,\beta)=\frac{(\beta-h_2-\mu)!}{(\beta-\mu)!q^{(P(l,\beta)-P(l-\alpha,\beta-h_2-\mu)-m_1(l-\alpha)+m_2(\beta-\mu))}}.$$

By construction, we have

$$\begin{aligned}&P(l,\beta)-P(l-\alpha,\beta-h_2-\mu)-m_1(l-\alpha)+m_2(\beta-\mu)\\&=l(h_2+\mu-2\alpha-m_1)+\beta(\alpha-2h_2-2\mu+m_2)-\alpha h_2-\alpha\mu+h_2^2+2h_2\mu+\mu^2+\alpha^2+m_1\alpha-m_2\mu.\end{aligned}$$

So that there exists a constant $C>0$ (depending on h_2,m_1,m_2) such that if $(\alpha,\mu)\in\mathcal{P}_{h_2,m_1,m_2}$ then $A(l,\beta)\leq C$, for all l,β. This gives the lemma. $\square$

Lemma 3 *Let $\delta_1,\delta_2>0$ be positive real numbers and let $U(t,x)$ be in $G^q(T_0,X_0)$. Then, there exist $X,T>0$ small enough (depending on δ_1,δ_2) such that the function $U(\delta_1 t,\frac{x}{\delta_2})$ belongs to $G^q(T,X)$.*

Proof Let $U(t,x)\in G^q(T_0,X_0)$. We have

$$U(\delta_1 t,\frac{x}{\delta_2})=\sum_{l,\beta\geq 0}u_{l,\beta}{\delta_1}^l{\delta_2}^{-\beta}t^l\frac{x^\beta}{\beta!}.$$

By definition, we have

$$||U(\delta_1 t,\frac{x}{\delta_2})||_{T,X}=\sum_{l,\beta}\frac{|u_{l,\beta}|}{q^{P(l,\beta)}\beta!}(\delta_1 T)^l(X/\delta_2)^\beta.$$

Due to the fact that $U(t,x)\in G^q(T_0,X_0)$, this latter series converges if $X/\delta_2\leq X_0$ and $\delta_1 T\leq T_0$. So that, if $X/\delta_2\leq X_0$ and $\delta_1 T\leq T_0$, then $U(\delta_1 t,\frac{x}{\delta_2})\in G^q(T,X)$. $\square$

Lemma 4 *Let $\alpha,\mu\in\mathbb{N}$ and let $U(t,x)\in G^q(T_0,X_0)$. Then, there exist $T,X>0$ (depending on α,μ) small enough such that the function $t^\alpha x^\mu U(t,x)$ belongs to $G^q(T,X)$.*

Proof Let $U(t,x)\in G^q(T_0,X_0)$. From the proof of Lemma 2, we get a constant $C>0$ such that

$$||t^\alpha x^\mu U(t,x)||_{T,X}\leq C(q^{2\alpha}T)^\alpha(q^{2\mu}X)^\mu q^{\alpha\mu}\sum_{l,\beta}\frac{|u_{l-\alpha,\beta-\mu}|}{q^{P(l-\alpha,\beta-\mu)}(\beta-\mu)!}(q^{2\alpha}T)^{l-\alpha}(q^{2\mu}X)^{\beta-\mu}.$$

Due to the fact that $U(t,x)\in G^q(T_0,X_0)$, the series on the right hand side of the latter inequality is convergent, which yields $t^\alpha x^\mu U(t,x)\in G^q(T,X)$, if $q^{2\mu}X\leq X_0$ and $q^{2\alpha}T\leq T_0$. $\square$

Let $\mathcal{D}$ be the linear operator from $A(\mathbb{C}^2)$ into $A(\mathbb{C}^2)$ defined as

$$\mathcal{D}(u(t,x))=\partial_x^S u(t,x)-\sum_{k=0}^{S-1}a_k(t,x)(\partial_x^k u)(q^{m_{0,k}}t,xq^{-m_{1,k}}),$$

where $S\geq 1$, $m_{0,k},m_{1,k}\geq 0$ are integers and $a_k(t,x)\in\mathbb{C}[t,x]$ for $0\leq k\leq S-1$. We make the following assumption.

Assumption (A):
i) The set $\mathcal{P}_{S-k,m_{0,k},m_{1,k}}$ is not empty,
ii) The polynomial $a_k(t,x)$ is of $\mathcal{P}_{S-k,m_{0,k},m_{1,k}}-$type,

for all $0 \leq k \leq S-1$.

We consider the following operator from $A(\mathbb{C}^2)$ into $A(\mathbb{C}^2)$,

$$\mathcal{D} \circ \partial_x^{-S} = \mathrm{id} - \mathcal{A},$$

where

$$\mathcal{A}(u(t,x)) = \sum_{k=0}^{S-1} a_k(t,x)(\partial_x^{-(S-k)}u)(q^{m_{0,k}}t, xq^{-m_{1,k}}),$$

for all $u(t,x) \in A(\mathbb{C}^2)$, and id is the identity operator. As a consequence of Lemma 2 we get the following lemma.

Lemma 5 *There exist real numbers $X, T > 0$ small enough (that depend on the coefficients of $a_k(t,x)$ for $0 \leq k \leq S-1$) such that $\mathcal{A}$ is a bounded linear operator, from $(G^q(T,X), ||.||_{T,X})$ into itself. Moreover the estimates hold,*

$$||\mathcal{A}u(t,x)||_{T,X} \leq (1/2)||u(t,x)||_{T,X},$$

for all $u(t,x) \in G^q(T,X)$.

Corollary 1 *There exist real numbers $X, T > 0$ small enough (that depend on the coefficients of $a_k(t,x)$ for $0 \leq k \leq S-1$) such that $\mathcal{D} \circ \partial_x^{-S}$ is a bounded invertible operator from $(G^q(T,X), ||.||_{T,X})$ into itself. In particular, there exists a constant $C > 0$ such that*

$$||(\mathcal{D} \circ \partial_x^{-S})^{-1} b(t,x)||_{T,X} \leq C||b(t,x)||_{T,X},$$

for all $b(t,x) \in G^q(T,X)$.

The main result of this section is the following.

Theorem 1 *Consider a partial $q-$difference differential equation*

$$\partial_x^S u(t,x) = \sum_{k=0}^{S-1} a_k(t,x)(\partial_x^k u)(q^{m_{0,k}}t, xq^{-m_{1,k}}), \tag{13}$$

where $S \geq 1$, $m_{0,k}, m_{1,k} \geq 0$ are integers and $a_k(t,x) \in \mathbb{C}[t,x]$ for $0 \leq k \leq S-1$.

We make the hypothesis that the assumption **(A)** *is fulfiled.*

We impose the initial conditions : For all $0 \leq j \leq S-1$,

$$(\partial_x^j u)(t,0) = \phi_j(t), \tag{14}$$

where $\phi_j(t) \in G^q(T_0, X_0)$, for given $T_0, X_0 > 0$.

Then, there exist X, T small enough such that the problem (13), (14) has a unique solution $u(t,x)$ in $G^q(T,X)$.

Proof Every function $u(t,x) \in A(\mathbb{C}^2)$ can be written in the form

$$u(t,x) = \partial_x^{-S} v(t,x) + w(t,x),$$

where

$$w(t,x) = \sum_{j=0}^{S-1} \varphi_j(t) \frac{x^j}{j!}.$$

From the initial conditions (14) and Lemma 4 let us assume that $w(t,x)$ belongs to $G^q(T,X)$ for $X,T>0$ small enough. Then, the function $u(t,x)$ is a solution of (13), (14) if and only if $v(t,x)$ satisfies the equation

$$\mathcal{D} \circ \partial_x^{-S} v(t,x) = -\mathcal{D} w(t,x). \tag{15}$$

From the fact that $w(t,x) \in G^q(T,X)$ is a polynomial in x of degree less than $S-1$, we deduce from the lemma 3,4 that $-\mathcal{D}w(t,x)$ also belongs to $G^q(T,X)$, for all $X,T>0$ small enough. From the corollary 1, we deduce that for X,T small enough, there exists a unique solution $v(t,x)$ of (15) which belongs to $G^q(T,X)$.

It remains to show that $u(t,x)$ belongs to $G^q(T,X)$ for X,T small enough. Indeed, from Lemma 2, the function $\partial_x^{-S} v(q^S t, \frac{x}{q^{2S}})$ belong to $G^q(T,X)$ and from Lemma 3, the function $w(q^S t, xq^{-2S})$ belong to $G^q(T,X)$ for T,X small enough. So that the function $\nu(t,x) = u(q^S t, xq^{-2S}) = \partial_x^{-S} v(q^S t, \frac{x}{q^{2S}}) + w(q^S t, xq^{-2S})$ belongs to $G^q(T,X)$ for T,X small enough. Finally, again by Lemma 3, we deduce that $u(t,x) = \nu(\frac{t}{q^S}, q^{2S}x)$ belongs to $G^q(T,X)$ for X,T small enough. □

4 Transseries solutions of partial $q-$difference differential equations

We consider a partial $q-$difference differential equation

$$\partial_z^S X(t,z) = \sum_{k=0}^{S-1} b_k(t,z)(\partial_z^k X)(q^{p_{0,k}} t, zq^{-m_{1,k}}) \tag{16}$$

where q is a complex number such that $|q|>1$ and $p_{0,\kappa}$, $m_{1,\kappa}$, S are integers larger than 1. The functions $b_k(t,z)$, $0 \le k \le S-1$ are of the form

$$b_k(t,z) = \sum_{l,\beta \ge 0} b_{l,\beta,k}(t)(e_q(t))^l z^\beta = \sum_{(l,\beta)\in I_k} b_{l,\beta,k}(t)(e_q(t))^l z^\beta$$

where $b_{l,\beta,k}(t)$ are bounded holomophic functions on the open set $\mathcal{S}$, continuous on $\bar{\mathcal{S}}$, considered in the section 2, such that $b_{l,\beta,k}(t) = 0$, for all $t \in \bar{\mathcal{S}}$, for $(l,\beta) \notin I_k$, where I_k, $0 \le k \le S-1$, are bounded subsets of $\mathbb{N}^2$.

We are looking for formal *transseries* solutions $X(t,z)$ in the powers of $e_q(t)$ and z, of (16) of the form

$$X(t,z) = \sum_{l,\beta \ge 0} X_{l,\beta}(t)(e_q(t))^l \frac{z^\beta}{\beta!} \tag{17}$$

where $X_{l,\beta}(t)$ are holomorphic functions on the open set $\mathcal{S}$.

By inserting the formal transseries (17) in (16), identifying the powers of $e_q(t)$ and z, and using the relation

$$e_q(q^{p_{0,k}}t) = \prod_{h=0}^{p_{0,k}-1} (1 + u(\frac{1}{q^h t}))e_q(t)$$

for the function $e_q(t)$ deduce from (3), we get the following lemma.

Lemma 6 *For given holomorphic functions on $\mathcal{S}$, $X_{l,j}(t)$, $l \geq 0$, $0 \leq j \leq S-1$, consider the functions $X_{l,\beta}(t)$, $l \geq 0$, $\beta \geq S$ defined by the following recursion relation*

$$(18) \qquad \frac{X_{l,\beta+S}(t)}{\beta!} = \sum_{k=0}^{S-1} \sum_{l_1+l_2=l} \sum_{\beta_1+\beta_2=\beta} \frac{b_{l_1,\beta_1,k}(t) X_{l_2,\beta_2+k}(q^{p_{0,k}}t)}{\beta_2!} (\prod_{h=0}^{p_{0,k}-1} (1 + u(\frac{1}{q^h t})))^{l_2} q^{-m_{1,k}\beta_2}$$

for all $t \in \mathcal{S}$, for all $\beta \geq 0$. Then the formal transseries $X(t,z)$ in (17) is a solution of (16).

In the next result we give sufficient conditions for which there exist holomorphic solutions $X(t,z)$ of (16) on the product $(\mathcal{S} \setminus \Theta) \times \mathbb{C}$. We also study the rate of growth of these solutions near the singular set $\Theta \times \mathbb{C}$.

From Lemma 1 a), to each integer $p_{0,k} \geq 1$ one can attach an integer $m_{0,k} \geq 1$ such that

$$\sup_{t \in \bar{\mathcal{S}}} \left| \prod_{h=0}^{p_{0,k}-1} (1 + u(\frac{1}{q^h t})) \right| \leq |q|^{m_{0,k}}.$$

Theorem 2 *Consider the partial $q-$difference differential equation (16). We make the following assumption.*

Assumption (B)*:*
i) The set $\mathcal{P}_{S-k,m_{0,k},m_{1,k}}$, defined in section 3, is not empty,
ii) $I_k = \mathcal{P}_{S-k,m_{0,k},m_{1,k}}$,
for all $0 \leq k \leq S-1$.

Let $X_{l,j}(t)$, $l \geq 0$, $0 \leq j \leq S-1$, be bounded holomorphic functions on $\mathcal{S}$, continuous on $\bar{\mathcal{S}}$, satisfying the following estimates. There exist $C, K > 0$ such that

$$(19) \qquad \sup_{t \in \bar{\mathcal{S}}} |X_{l,j}(t)| \leq C K^l |q|^{-l^2}$$

for all $l \geq 0$, for all $0 \leq j \leq S-1$.

Then, the formal transseries $X(t,z) = \sum_{l,\beta \geq 0} X_{l,\beta}(t)(e_q(t))^l \frac{z^\beta}{\beta!}$ constructed in Lemma 6, solution of (16), defines a holomorphic function on the open set $\mathcal{M} = (\mathcal{S} \setminus \Theta) \times \mathbb{C}$.

Moreover, the function $X(t,z)$ has the following rate of growth near the set $\Theta \times \mathbb{C}$. There exist constants $C > 0, X > 0$ such that for all $1 \leq j \leq s$, $n \geq 0$, there exists a constant $C_{j,n} > 0$ such that

$$(20) \qquad \sup_{|z|<r} |X(t,z)| \leq C \exp(\frac{1}{2}(C_{j,n}(\log|t - q^{-n}/n_j|)^2 + (\log(\frac{2r}{X}))^2))$$

for all $t \neq q^{-n}/n_j$ in a small neighborhood of q^{-n}/n_j in $\mathbb{C}^*$ and all $r > 0$. This implies in particular that $X(t,z)$ has at most sub-exponential growth near the singular sets $\{q^{-n}/n_j\} \times \mathbb{C}$, $1 \leq j \leq s$, $n \geq 0$.

Proof From Lemma 1, we get a constant $m_{0,k} > 0$ such that

$$\sup_{t \in \bar{\mathcal{S}}} \left| \prod_{h=0}^{p_{0,k}-1} (1 + u(\frac{1}{q^h t})) \right| \leq |q|^{m_{0,k}}.$$

Let $X_{l,j}(t)$, $l \geq 0$, $0 \leq j \leq S-1$, be bounded holomorphic functions on $\mathcal{S}$, continuous on $\bar{\mathcal{S}}$ satisfying the estimates (19) and let $X_{l,\beta}(t)$, $l, \beta \geq 0$ be the functions constructed from the recursion relation (18). In the following, we put

$$w_{l,\beta} = \sup_{t \in \bar{\mathcal{S}}} |X_{l,\beta}(t)| \quad , \quad |b|_{l,\beta,k} = \sup_{t \in \bar{\mathcal{S}}} |b_{l,\beta,k}(t)|,$$

for all $l, \beta \geq 0$ and $0 \leq k \leq S-1$. By construction, we have that $|b|_{l,\beta,k} = 0$ if $(l,\beta) \notin I_k$, for $0 \leq k \leq S-1$.

From the fact that the set $\mathcal{S}$ is invariant under the dilations $t \mapsto q^{p_{0,k}} t$, we get the following inequalities for the numbers $w_{l,\beta}$,

$$\frac{w_{l,\beta+S}}{\beta!} \leq \sum_{k=0}^{S-1} \sum_{l_1+l_2=l} \sum_{\beta_1+\beta_2=\beta} \frac{|b|_{l_1,\beta_1,k} w_{l_2,\beta_2+k}}{\beta_2!} |q|^{m_{0,k} l_2} |q|^{-m_{1,k}\beta_2} \tag{21}$$

for all $l, \beta \geq 0$.

In the following, we put

$$\phi_j(t) = \sum_{l \geq 0} w_{l,j} t^l \quad , \quad |b|_k(t,x) = \sum_{l,\beta \geq 0} |b|_{l,\beta,k} t^l x^\beta = \sum_{(l,\beta) \in I_k} |b|_{l,\beta,k} t^l x^\beta$$

for all $0 \leq j, k \leq S-1$.

We consider now the auxiliary $q-$difference differential equation

$$\partial_x^S V(t,x) = \sum_{k=0}^{S-1} |b|_k(t,x) (\partial_x^k V)(|q|^{m_{0,k}} t, x|q|^{-m_{1,k}}) \tag{22}$$

with initial conditions

$$(\partial_x^j V)(t,0) = \phi_j(t) \quad , \quad 0 \leq j \leq S-1. \tag{23}$$

From the estimates (19), we get in particular that the series $\phi_j(t)$ belong to the space $G^{|q|}(T_0, X_0)$ for $T_0 > 0$ small enough and $X_0 > 0$. From the assumption **(B)**, the set $\mathcal{P}_{S-k,m_{0,k},m_{1,k}}$ is not empty and the polynomial $|b|_k(t,x)$ is of $\mathcal{P}_{S-k,m_{0,k},m_{1,k}}-$type, for all $0 \leq k \leq S-1$. We deduce that the assumption **(A)** of Theorem 1 is fulfilled for the Cauchy problem (22), (23). From Theorem 1, we deduce that the problem (22), (23) has a unique formal series solution

$$V(t,x) = \sum_{l,\beta \geq 0} v_{l,\beta} t^l \frac{x^\beta}{\beta!}$$

which satisfies $V(t,x) \in G^{|q|}(T,X)$ for $T, X > 0$ small enough.

By inserting this latter expansion in (22), (23), we get the following equalities for the numbers $v_{l,\beta}$,

$$\frac{v_{l,\beta+S}}{\beta!} = \sum_{k=0}^{S-1} \sum_{l_1+l_2=l} \sum_{\beta_1+\beta_2=\beta} \frac{|b|_{l_1,\beta_1,k} v_{l_2,\beta_2+k}}{\beta_2!} |q|^{m_{0,k} l_2} |q|^{-m_{1,k}\beta_2}. \tag{24}$$

for all $l, \beta \geq 0$, with initial data

$$v_{l,j} = w_{l,j} \quad , \quad l \geq 0 \quad , \quad 0 \leq j \leq S-1. \tag{25}$$

From the relations (21), (24) and (25), we get that

$$w_{l,\beta} \leq v_{l,\beta} \tag{26}$$

for all $l, \beta \geq 0$. In particular, from the fact that $V(t,x) \in G^{|q|}(T,X)$ for $T, X > 0$ small enough and the inequality $l\beta \leq l^2/2 + \beta^2/2$, we get that there exist $C, T, X > 0$ such that

$$|w_{l,\beta}| \leq C|q|^{P(l,\beta)}\beta!(\frac{1}{T})^l(\frac{1}{X})^\beta \leq C|q|^{-l^2/2-\beta^2/2}\beta!(\frac{1}{T})^l(\frac{1}{X})^\beta \tag{27}$$

for all $l, \beta \geq 0$.

From (27), we deduce that the formal series in the powers of τ and z

$$W(t,\tau,z) = \sum_{l,\beta \geq 0} X_{l,\beta}(t)\tau^l \frac{z^\beta}{\beta!}$$

satisfies the estimates for $\tau \in \mathbb{C}$, $z \in \mathbb{C}$, $|z| < r$, $r > 0$,

$$\sup_{|z|<r} \sup_{t \in \bar{\mathcal{S}}} |W(t,\tau,z)| \leq \sum_{l,\beta \geq 0} |w_{l,\beta}||\tau|^l \frac{r^\beta}{\beta!} \leq C(\sum_{l \geq 0} |q|^{-l^2/2}(\frac{|\tau|}{T})^l)(\sum_{\beta \geq 0} |q|^{-\beta^2/2}(\frac{r}{X})^\beta)$$

for all $\tau \in \mathbb{C}$, $z \in \mathbb{C}$, $|z| < r$, for all $r > 0$. So that $W(t,\tau,z)$ defines a holomorphic function on $\mathcal{S} \times \mathbb{C}^2$. Moreover, from Lemma 2.2 in [8], we get the following estimates. There exists $C' > 0$ such that

$$\sup_{|z|<r} \sup_{t \in \mathcal{S}} |W(t,\tau,z)| \leq C' \exp(\frac{1}{2}((\log(2|\tau|/T))^2 + (\log(2r/X))^2)) \tag{28}$$

for all $\tau \in \mathbb{C}^*$, all $r > 0$.

We deduce that the function $X(t,z) = W(t, e_q(t), z)$, solution of (16), defines a holomorphic function on $\mathcal{M} = (\mathcal{S} \setminus \Theta) \times \mathbb{C}$. Moreover, from the estimates (28), there exist $C', T, X > 0$ such that

$$\sup_{|z|<r} |X(t,z)| \leq C' \exp(\frac{1}{2}((\log(2|e_q(t)|/T))^2 + (\log(2r/X))^2)) \tag{29}$$

for all $t \in \mathcal{S} \setminus \Theta$ and all $r > 0$.

From Proposition 1, the function $e_q(t)$ is meromorphic at the point q^{-n}/n_j for all $1 \leq j \leq s$ and $n \geq 0$. So that for all $1 \leq j \leq s$ and $n \geq 0$, there exist two constants $m_{j,n}, M_{j,n} > 0$ and a positive integer $d_{j,n} \geq 1$ such that

$$m_{j,n}/|t - q^{-n}/n_j|^{d_{j,n}} \leq |e_q(t)| \leq M_{j,n}/|t - q^{-n}/n_j|^{d_{j,n}},$$

for all $t \neq q^{-n}/n_j$ in a small neighborhood of q^{-n}/n_j. In particular, we get that

$$0 \leq \log(\frac{2|e_q(t)|}{T}) \leq \log(\frac{2M_{j,n}}{T|t - q^{-n}/n_j|^{d_{j,n}}}) \tag{30}$$

for all $t \neq q^{-n}/n_j$ in a small neighborhood of q^{-n}/n_j. Finally, the estimates (29) and (30) yield the estimates (20). □

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