Contemporary Mathematics

Anomalous Transport in Aperiodic Media

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ABSTRACT. Recent results on anomalous quantum transport are reviewed.

1. Introduction

Since the early 80's, there has been considerable mathematical research on almost-periodic Schrödinger operators with a particular focus on their spectral properties. With the discovery of quasicrystals in 1984, these models became also experimentally relevant. One of the most striking experimental features of these materials is the increase of their direct conductivity with increasing temperature. This is suprising because metals, other materials with nearly perfect crystaline ordering, exhibit a decrease of the conductivity with increasing temperature. Several theoretical explanation schemes have been proposed. The one discussed here supposes that the electronic wave packets are slowed down by destructive quantum interferences in the aperiodic crystal so that the quantum motion is anamolous and subdiffusive. This in turn leads to anomalies in Drude's formula for the conductivity which qualitatively predict its temperature increase [**SG, May**]. A rigorous analysis of this connection is discussed in Sec. 3. Within this framework, the task is then to show that one actually has anomalous quantum diffusion for particular aperiodic Schrödinger operators. In Sec. 5 we review recent improvements of a general lower bound on quantum motion due to Guarneri. One-dimensional models for which also upper bounds can be proven are presented in Sec. 6. This allows to construct $3D$-models with absolutely continuous (ac) spectrum and subdiffusive quantum transport, *cf.* Sec. 7.

2. Diffusion exponents

Given a positive measurable function f on $[b,\infty)$, we introduce an upper power growth exponents $\alpha^+ = \limsup_{T\to\infty} \log(f(T))/\log(T)$ and a lower exponent $\alpha^- = \liminf_{T\to\infty} \log(f(T))/\log(T)$ as well as an intermediate Mellin exponent α^m which is the infinimum of all $\gamma \in \mathbf{R}$ such that $\int_b^\infty dT\, f(T)/T^{1+\gamma}$ is finite. We then write $f(T) \underset{T\uparrow\infty}{\sim} T^{\alpha^*}$ where $*$ is $+$, $-$ or m. Similarly one defines $f(\epsilon) \underset{\epsilon\downarrow 0}{\sim} \epsilon^{\alpha^*}$.

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Let now $H : \ell^2(\Lambda) \to \ell^2(\Lambda)$ be a bounded self-adjoint Hamiltonian on a lattice Λ which can be either $\mathbf{Z}^D$ or $\mathbf{N}^D$. If $\vec{X}$ denotes the position operator on $\ell^2(\Lambda)$, we suppose that the current operator $\imath[\vec{X}, H]$ is also bounded. For a given $q \neq 0$, the diffusion exponent β_q^* is defined by

$$\int_{-T}^{T} \frac{dt}{2T} \, \langle 0| \, |\vec{X}(t) - \vec{X}|^q \, |0\rangle \underset{T\uparrow\infty}{\sim} T^{q\beta_q^*} \ , \tag{2.1}$$

where $\vec{X}(t) = e^{\imath Ht}\vec{X}e^{-\imath Ht}$ and $|0\rangle$ is the normalized state localized at the origin of Λ. Other initial states may give different exponents. It can be easily verified [**GSB3**] that the time mean $\int_{-T}^{T} dt/T$ can be replaced by a gaussian mean $\int_{\mathbf{R}} dt/T \exp(-t^2/T^2)$ without changing the exponents. Moreover, not substracting $\vec{X}$ will not change the exponents neither. Finally, let us note that it is possible to define diffusion exponents associated to a spectral interval Δ by introducing spectral projections in (2.1) [**SBB1**].

Further we consider covariant families of stochastic Hamiltonians $(H_\omega)_{\omega\in\Omega}$ where $(\Omega, T, \mathbf{Z}^D, \mathbf{P})$ is a compact space Ω furnished with an action T of $\mathbf{Z}^D$ and an invariant ergodic probability measure $\mathbf{P}$ such that the covariance relation w.r.t. the translations holds: $U(a)H_\omega U(A)^* = H_{T^a\omega}$. Their diffusion exponents $\overline{\beta}_q^*$ are defined by (2.1) with a supplementary mean over $\mathbf{P}$ on the l.h.s..

Using DuHamel's formula in (2.1), it is straightforward to verify that $0 \leq \beta_q^* \leq 1$ for all $q > 0$ (similarly $0 \leq \overline{\beta}_q^* \leq 1$). If the l.h.s. of (2.1) is bounded in time (hence $\beta_q^* = 0$), one talks of strong localization; this is the case at high disorder and the band edges for the Anderson model. The quantum motion is called ballistic if $\beta_q^* = 1$ which is the case for Bloch electrons. $\beta_q^* = 1/2$ corresponds to the diffusive regime which is expected in the weak localization regime of the $3D$ Anderson model (low disorder and center of bands); it can be proven for the Anderson model with so-called free random variables, a generalized Wegner n-orbital model [**SBB1**]. For any other value of β_q^*, we shall speak of anomalous quantum diffusion. For $1D$-models $q \mapsto \beta_q^*$ is typically a strictly increasing curve, a phenomenon termed quantum intermittency in [**GM**].

3. Anomalous Drude Formula

It is implicitly contained in the Einstein relation between the direct conductivity and the diffusion coefficient that the conductivity is finite only if the motion of the electrons is diffusive. If this is not the case for the coherent quantum motion (*i.e.* $\beta_2 \neq 1/2$), one can superpose a collisional (Poisson) process similar to that in the classical Drude model and this forces the electron motion to be diffusive. These collisions are modelizing the interactions with phonons and other electrons as well as the static disorder of impurities if it is not already taken into account for in the one-particle Hamiltonian (e.g., if H is quasiperiodic without added disorder potential). Let τ be the relaxation time, notably the average time between collisions renormalized by their efficiency to diminish current. This time depends phenomenologically on the temperature, and the phonon and electron collision times diverge in the zero temperature limit. Then the (direction averaged) direct conductivity at inverse temperature β and chemical potential μ of a Fermi gas described by a covariant Hamiltonian $(H_\omega)_{\omega\in\Omega}$ is given by Kubo's formula:

$$\sigma(\beta,\mu,\tau) \;=\; \mathcal{T}\left(\imath[\vec{X},f_{\beta,\mu}(H_\omega)]\,\left(\frac{1}{\tau}-\mathcal{L}_{H_\omega}\right)^{-1}(\imath[\vec{X},H_\omega])\right)\;.$$

Here $f_{\beta,\mu}$ is the Fermi-Dirac function, $\mathcal{L}_H(A)=\imath[H,A]$ is the anti-self-adjoint Liouville (super-) operator (so that the appearing operator inverse exists for $\tau<\infty$), and finally $\mathcal{T}$ is the trace per unit volume on covariant operators defined by $\mathcal{T}(A_\omega)=\int d\mathbf{P}(\omega)\,\langle 0|A_\omega|0\rangle$. This Kubo formula takes fully into account the quantum interferences of the Hamiltonian H_ω up to times of order $1/tau$, but describes interactions with and dissipation to the rest of the solid phenomenologically.

THEOREM 3.1 (Anomalous Drude Formula [**SBB1, SBB2**]). *Let the temperature be fixed and finite. Then one has the following Mellin exponent behavior in the zero dissipation limit $\tau\to\infty$:*

$$\sigma(\beta,\mu,\tau)\;\underset{\tau\uparrow\infty}{\sim}\;\tau^{2\overline{\beta}_2^m-1}\;.$$

It would be desirable to prove that at zero temperature only the diffusion exponent at the Fermi energy enters this formula. Note first that for a ballistic quantum transport ($\overline{\beta}_2^m=1$), one recovers the usual Drude formula. In particular, the conductivity diverges in the zero dissipation limit. The same holds true in the super-diffusive regime ($\overline{\beta}_2^m>1/2$). Only in the diffusive regime $\overline{\beta}_2^m=1/2$, the conductivity at zero dissipation may be finite in accordance with the Einstein relation. In the subdiffusive regime, the conductivity vanishes in this limit. Let us remark that the relaxation time approximation and hence the above Kubo formula are not valid in the strong localization regime (impurity bands in doped semiconductors) where hopping conduction is predominant at low temperature.

Our belief is that the relaxation time approximation is valid in quasicrystals over a wide temperature range. If one supposes a temperature dependence $\tau(T)\sim T^{-\kappa}$ for some positive κ, then the anomalous Drude formula (fix β for simplicity) gives $\sigma(\beta,\mu,T^\kappa)\sim T^{\kappa(1-2\overline{\beta}_2^m)}$ so that the conductivity increases with increasing temperature whenever the quantum motion is subdiffusive ($\overline{\beta}_2^m<1/2$). This gives a qualitative explanation of the experimental observation discussed above.

4. Spectral dimensions

Here we introduce dimensional properties of the (spectral) measures which will be linked to the diffusion exponents below. Given a Borel measure μ on $\mathbf{R}$, its local dimension $d_\mu^*(E)$ at $E\in\mathbf{R}$ are defined by $\mu([E-\epsilon,E+\epsilon])\underset{\epsilon\downarrow 0}{\sim}\epsilon^{d_\mu^*(E)}$. The Hausdorff dimension $\dim_{\mathrm{H}}(\mu)$ of μ and its packing dimension $\dim_{\mathrm{P}}(\mu)$ are defined to be the μ-essential supremum of respectively the lower dimensions $d_\mu^-(E)$ and the upper ones $d_\mu^+(E)$. For an ac measure, these dimension are equal to 1, for a pure-point measure equal to 0, and for any other value in $(0,1)$, μ has a singular continuous component. If $\dim_{\mathrm{H}}(\mu)<\dim_{\mathrm{P}}(\mu)$, then μ has bad scaling properties. For equivalent definitions and further properties we refer to [**Pes, GSB2**]. A finer characterization of μ is given by its multifractal dimensions ($q\neq 1$):

$$\int d\mu(E)\,\mu([E-\epsilon,E+\epsilon])^{q-1}\;\underset{\epsilon\downarrow 0}{\sim}\;\epsilon^{(q-1)D_\mu^*(q)}\;.$$

The functions $q\mapsto D_\mu^*(q)$ are non-increasing and moreover $q\mapsto(q-1)D_\mu^*(q)$ is concave. Other than $\dim_{\mathrm{H}}(\mu)$ and $\dim_{\mathrm{P}}(\mu)$, the multifractal dimensions depend on

μ in its measure class [**SBB1**]. Under the hypothesis that $D^-_\mu(q) = D^+_\mu(q)$ and $D_\mu(q) < \infty$ for some $q < 0$, one can directly implement large deviation theory in order to study the fluctuations of the local dimensions [**GSB3**]. Let $f_\mu(\alpha)$ be the Legendre transform of the function $q \mapsto (q-1)D^*_\mu(q)$. Set $I_\alpha(\epsilon) = \{E \in \mathbf{R} \,|\, \mu([E-\epsilon, E+\epsilon]) < \epsilon^\alpha\}$. Then there exist constants c, C such that for any $\alpha \geq f_\mu(\alpha)$:

$$c\,\epsilon^{\alpha - f_\mu(\alpha)} \;\leq\; \mu(I_\alpha(\epsilon)) \;\leq\; C\,\epsilon^{\alpha - f_\mu(\alpha)} \;. \tag{4.1}$$

A less explicit, but for our purposes sufficient large deviation bound which moreover does not depend on the hypothesis $D^-_\mu(q) = D^+_\mu(q)$ can be found in [**BGT**]: Let μ have a compact support, then for any $q \in [0,1]$ there exists $\alpha = \alpha(\epsilon, q)$ such that

$$\mu(I_\alpha(\epsilon)) \;\geq\; \epsilon^{(1-q)\alpha}\,\frac{-1}{\log(\epsilon)} \int d\mu(E)\, \mu([E-\epsilon, E+\epsilon])^{q-1} \;. \tag{4.2}$$

In all the definitions and bounds above, the indicator function on $[E-\epsilon, E+\epsilon]$ may be replaced by a gaussian of width ϵ and centered at E.

5. General lower bounds on quantum diffusion

THEOREM 5.1. *Let H be a bounded tight-binding Hamiltonian on $\Lambda = \mathbf{Z}^D$ or $\mathbf{N}^D$. Then for any $q > 0$:*

$$\text{(i) } D\beta^-_q \;\geq\; \dim_H(\mu)\;, \quad \text{(ii) } D\beta^+_q \;\geq\; \dim_P(\mu)\;, \quad \text{(iii) } D\beta^\pm_q \;\geq\; D^\pm_\mu\left(\frac{D}{D+q}\right)\;.$$

Part (i) is the original Guarneri Inequality [**G**], in a form given by Last who removed the hypothesis that the spectral measure be exactly scaling citeLas. Combes showed that it also holds for continuous Schrödinger operators [**Com**]. The inequality (ii) first appeared in [**GSB2**] and (iii) in [**GSB3**], but under the stronger hypothesis that $D^+_\mu(q) = D^-_\mu(q)$. The above form of (iii) is due to [**BGT**]. It is particularly interesting for $1D$-models for which anomalous diffusion goes along with multifractal properties of the spectral measures (see [**GM, Man**] and Sec. 6). General results about the various dimensions of μ show that the inequality (iii) implies both (i) and (ii). The bounds (i) and (ii) transpose to covariant Hamiltonians [**L, SBB1, GSB2**].

Note that in one dimension, ac spectrum ($\dim_{\rm H}(\mu) = 1$) implies ballistic quantum mution. For $D = 2$, diffusive motion and ac spectrum may coexist; for $D = 3$ the motion may even be subdiffusive with $\beta \approx 1/3$ in presence of ac spectrum. For higher dimensional models, slow quantum transport may hence be heavily influenced by properties of the generalized eigenfunctions. How such infromation can be incorporated in the lower bound was shown in [**KPG, KL**].

The main idea of proof is as follows: a given continuity of a chosen part of the spectral measure (characterized by a spectral dimension) forces the corresponding portion of the wave packet out of a ball of a certain size, due to a refined RAGE theorem and control its size by (4.1). Maximising over the free parameters gives the lower bound (iii). More precisely, for a given $R \in \mathbf{R}$:

$$\langle 0|\,|\vec{X}|^q(t)|0\rangle \;\geq\; R^q \sum_{|n|>R} |\langle n|e^{-\imath Ht}|0\rangle|^2 \;=\; R^q \left(1 - \sum_{|n|\leq R} |\langle n|e^{-\imath Ht}|0\rangle|^2\right) \;.$$

Now we decompose $|0\rangle = |\psi\rangle + |\psi^c\rangle$. Then

$$\sum_{|n|\leq R} |\langle n|e^{-\imath Ht}|0\rangle|^2 \;\leq\; 1 - \|\psi\|^2 + \sum_{|n|\leq R} |\langle n|e^{-\imath Ht}|\psi\rangle|^2 + \text{crossed terms} \;.$$

With little effort the crossed terms in ψ and ψ^c can be treated similarly to the third one, so that we will neglect them here. Hence we get

$$\langle 0|\,|\vec{X}|^q(t)|0\rangle \;\geq\; R^q \left(\|\psi\|^2 - \sum_{|n|\leq R} |\langle n|e^{-\imath Ht}|\psi\rangle|^2 + \text{crossed terms} \right) \;. \tag{5.1}$$

Now for a given time T we choose $|\psi\rangle = \chi(H)|0\rangle$ where χ is the indicator function on $I_\alpha(1/T)$ w.r.t. the spectral measure μ of $|0\rangle$. According to the large deviation bound (4.1), $\|\psi\|^2 = \mu(I(T)) \geq c\,T^{f_\mu(\alpha)-\alpha}$. We now have to choose $R = R(T)$ sufficiently small such that the second term in the l.h.s. of (5.1) is of comparable size, but smaller than this. To bound this second term, a very conveniant way of dealing with oscillation of $e^{-\imath Ht}$ is to take a gaussion mean of (5.1) up to time T [**L**]. If P_n are the generalized eigenfunctions (that is $\langle 0|g(H)|n\rangle = \int d\mu(E)P_n(E)g(E)$ and $\int d\mu(E)|P_n(E)|^2 = 1$), this gives by the Cauchy-Schwarz inequality

$$\int \frac{dt}{T}\, e^{-t^2/T^2} |\langle n|e^{-\imath Ht}|\psi\rangle|^2 \;\leq\; \int_{I(T)} d\mu(E)\, P_n(E)^2 \int_{I(T)} d\mu(E')\, e^{-(E-E')^2T^2} \;\leq\; T^{-\alpha}.$$

Now we need to choose $R(T)$ such that $T^{f_\mu(\alpha)-\alpha} \approx R(T)^D T^{-\alpha}$. This is satisfied for $R(T) \approx T^{f_\mu(\alpha)/D}$. Hence the gaussian time mean of $\langle 0|\,|\vec{X}|^q(t)|0\rangle$ is bounded below by $T^{(1+q/D)f_\mu(\alpha)-\alpha}$. Maximising the exponent over α and using the fact that the Legendre transform of f_μ is linked to the multifractal dimensions allows to conclude immediately. Alternatively, one can use the bound (4.2).

6. Model specific upper bounds on quantum diffusion

The link between diffusion exponents and spectral dimensions is most likely to be tight for Jacobi matrices on $\Lambda = \mathbf{N}$ ($1D$ next neighbor hopping models) where the generalized eigenfunctions are given by the orthogonal polynomials and hence entirely determined by the spectral measure. This fact is fully exploited in subordinacy theory [**JL**], but here we only use the generalized Herbert-Jones-Thouless formula [**ST**]:

$$\lim_{n\to\infty} \frac{1}{n} \log(|P_n(z)|) \;=\; \int d\mathcal{N}(E) \log(|E-z|) - \log(\text{cap}(\text{supp}(\mu))) \;. \tag{6.1}$$

Here $z \in \mathbf{C}\backslash\text{supp}(\mu)$, $\mathcal{N}$ is the density of states and $\text{cap}(J)$ denotes the logarithmic capacity of a set $J \subset \mathbf{C}$. This formula holds (and $\mathcal{N}$ exists) for a large class of Jacobi matrices called asymptotically regular in [**ST**].

Here we consider Jacobi matrices associated to fractal measures of the Cantor type. Let I^1_0 and I^1_1 be two disjoint closed intervals contained in a closed interval I^0. Let $S : \mathbf{R} \to \mathbf{R}$ be such that its restrictions $S^{-1}_{0,1} : I^1_{0,1} \to I^0$ are analytic bijections satisfying $S(I^0 \setminus (I^1_0 \cup I^1_1)) \cap I^0 = \emptyset$. The inverse images $S^{-N}(I^0)$ consist of 2^N intervals of generation N. A coding $\sigma = (\sigma_i)_{i\geq 1}$ with $\sigma_i \in \{0,1\}$ permits to describe them through $I^N_\sigma = S^{-1}_{\sigma_1} \circ \ldots \circ S^{-1}_{\sigma_N}(I^0)$. The codes of length N are denoted by Σ^N. Finally S is supposed to be expansive, namely we suppose that there exist positive constants $a < 1$ and b such that the lenghts of I^N_σ decrease faster than ba^N uniformly in σ. Then the set $J = \bigcap_{N\geq 1} S^{-N}(I^0)$ is a self-similar fractal invariant under S. The usual Cantor sets are obtained by chossing $S_{0,1}^{-1}$ to be linear, here particular focus will be on real Jula sets accociated to $S(E)=E^2-\lambda$, $\lambda>2$. For the following, we could choose μ to be an arbitrary Gibbs measure of the dynamical system (J,S), but for simplicity we only consider the equal weight Bernoulli measure satisfying $\mu(I_\sigma^N)=2^{-N}$. It is known [**Pes**] that $q\mapsto D_\mu^+(q)=D_\mu^-(q)$ is then real analytic. Now the orthogonal polynomials P_n in $L^2(\mathbf{R},\mu)$ form a three term recurrence relation, the coefficients of which give our Hamiltonian H acting on $\ell^2(\mathbf{N})$. It can be verified to be asymptotically regular [**GSB1**]. If J is a Julia set, H is called a Julia matrix. They are known to be limit-periodic for λ sufficiently large [**BBM**], but it is an open problem to show that the Jacobi matrices associated to general self-similar measures of the above type are almost periodic.

THEOREM 6.1. [**BaSB**] *For Julia matrices* $\beta_q^+\leq D_\mu(1-q)$ *for all* $0\leq q\leq 2$.

Numerics and non-rigorous arguments [**Man**] show that this is actually an equality for all $q>0$. Bounds for other models such as any Bernoulli measure on linear Cantor sets can also be obtained [**GSB1**], but they cannot be expressed only in terms of the multifractal dimensions because the asymptotic properties of the generalized eigenfunctions entering via the Thouless formula do not cancel out as for Julia matrices. Theoretical arguments and numerics show that a similar formula $\overline{\beta}_q=D_{\mathcal{N}}(1-q)$ holds for phase-averaged exponents of $1D$ quasiperiodic models [**Pie**]. No proof is known at the time being.

Let us develop the key element of the proof developed in [**GSB1**]. At a given time, one has to show asymptotic fall off of $|\langle n|e^{-\imath Ht}|0\rangle|$ in n. Herefore we approximate $e^{-\imath Ht}|0\rangle$ up to time T fairly well by $|\psi(t)\rangle=\sum_\sigma e^{-\imath E_\sigma^N t}\chi_{I_\sigma^N}(H)|0\rangle$ where $E_\sigma^N\in I_\sigma^N$ as long as the length of (most of) the intervals $|I_\sigma^N|\approx e^{-N\Lambda}$ are of order $1/T$ (here Λ is the Lyapunov exponent. If one then moreover chooses the E_σ^N's to be part of a regular lattice of spacing $1/T$, then

$$\int_{-T}^{T}\frac{dt}{2T}\,|\langle n|e^{-\imath Ht}|0\rangle|^2\;\leq\;\text{correction terms}+\sum_{\sigma\in\Sigma^N}|\langle n|\chi_{I_\sigma^N}(H)|0\rangle|^2\;.$$

The lengthy analysis of the correction terms is based on the study of the fluctuations of the band lengths of generation N, eventually leading to the multifractal dimensions [**BaSB**]. Let us focus on the argument for the main term [**GSB1**]. One has $S^{-1}(I_\sigma^{N-1})=I_\sigma^N\cup I_{\sigma'}^N$ (for suitable σ') and $I_\sigma^N\subset I_+^1$ and $I_{\sigma'}^N\subset I_-^1$ (or vica versa). Introducing $\mathcal{S}|\phi\rangle=|\phi\circ\mathcal{S}\rangle$ on $L^2(\mathbf{R},\mu)$, we therefore obtain $\chi_{I_\sigma^N}(H)=\chi_{I_+^1}(H)\mathcal{S}\chi_{I_\sigma^N}(H)$ so that

$$\langle n|\chi_{I_\sigma^N}(H)|0\rangle\;=\;\sum_{k,l}\langle n|\chi_{I_+^1}(H)|k\rangle\;\langle k|\mathcal{S}|l\rangle\;\langle l|\chi_{I_\sigma^{N-1}}(H)|0\rangle\;.\tag{6.2}$$

Now a Combes-Thomas argument shows that $\langle n|\chi_{I_+^1}(H)|k\rangle$ decreases exponentially in $|n-k|$. Supposing that S is analytic, one can develop $P_l(S(E))$ using Cauchy's formula with a contour Γ_R of radius R bigger than the spectral radius R_c around the center E_c of the spectrum. Hence by orthogonality of the P_m's:

$$\begin{aligned}\langle k|\mathcal{S}|l\rangle &= \sum_{m\geq k}\int_{\Gamma_R}\frac{dz}{2\pi\imath}\,\frac{P_l(S(z))}{(z-E_c)^{m+1}}\int d\mu(E)\,P_k(E)\,(E-E_c)^m\\ &\leq \frac{(R_c/R)^k}{1-R_c/R}\,\max_{z\in\Gamma_R}|P_l(S(z))|\;.\end{aligned}$$

Now $P_l(S(z))$ is bounded by the Thouless formula. If S is a polynomial of degree 2, this gives for large R the bound $\langle k|\mathcal{S}|l\rangle \leq \min\{c\,R^{-k+2l},1\}$ (as R is arbitrary, this actually implies the known renormalization property $P_l(S(z)) = P_{2l}(z)$ [**BBM**]). Replacing this into (6.2), a recurrence argument over n shows that $\langle n|\chi_{I_\sigma^N}(H)|0\rangle$ decreases exponentially in n for $n \geq 2^N \approx T^{\log(2)/\Lambda} = T^{\dim_{\mathrm{H}}(\mu)}$. This is the upper bound by the fractal dimension.

7. Subdiffusive transport in higher dimensional models

A simple way to construct higher dimensional models with controllable transport exponents is to take a direct sum $H^\eta = \sum_{j=1}^D \eta_j\; \mathbf{1}\otimes\ldots\otimes H_j\otimes\ldots\otimes\mathbf{1}$ acting on $\ell^2(\mathbf{N}^D)$ where $\eta = (\eta_1,\ldots,\eta_D)\in\mathbf{R}_+^D$ and the H_j's are Julia matrices. In fact elementary inequalities imply that $\beta_q^+(H^\eta) = \max_{j=1\ldots D}\beta_q^+(H_j)$ so that the transport is subdiffusive if it is so for each of the H_j's. The spectral properties can be analysed using results from [**HT**]:

THEOREM 7.1. [**BeSB**] *Let the spectral measure of H^η w.r.t. $|0\rangle\otimes\ldots\otimes|0\rangle$ be μ^η. Then for almost all $\eta\in\mathbf{R}_+^D$, $\dim_{\mathrm{H}}(\mu^\eta) = \min\{1,\sum_{j=1}^D \dim_{\mathrm{H}}(\mu_j)\}$ and, if $\sum_{j=1}^D\dim_{\mathrm{H}}(\mu_j) > 1$, then μ^η is absolutely continuous.*

Hence choosing $\dim_{\mathrm{H}}(\mu) = D_\mu(1)\approx 1/3$, the theorem guarantees the existence of a $3D$ model with ac spectrum for which the diffusion exponent $\beta_{0^+} = 1/3$ is only slightly bigger than imposed by the Guarneri Inequality.

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