MEASURE ZERO SPECTRUM OF A CLASS OF SCHRÖDINGER OPERATORS

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Abstract. We study the measure of the spectrum of a class of one-dimensional discrete Schrödinger operators $H_{v,\omega}$ with quasi-periodic potential $v(\omega)$ generated by any primitive substitution. It is known that the spectrum of $H_{v,\omega}$ is singular continuous ([8]). We will give a more exact result that the spectrum of $H_{v,\omega}$ is a Cantor set of Lebesgue measure zero.

1. Introduction

Since the discovery of quasi-crystal [17], the spectral properties of one-dimensional discrete Schrödinger operator with quasi-periodic potential have attracted considerable attention during the past two decades. In this paper, we study such kind of Schrödinger operator with potential generated by primitive substitution, i.e.,

$$(H_{v,\omega}\phi)_n = -\phi_{n+1} - \phi_{n-1} + v(\omega_n)\phi_n, \quad n \in \mathbb{Z}, \phi \in l^2(\mathbb{Z}), \tag{1}$$

where v is an injective map from a finite alphabet $\mathcal{A}$ to $\mathbb{R}$, $\omega \in \mathcal{A}^{\mathbb{Z}}$ is a two-side infinite sequence. Moreover, we require that ω is generated by primitive substitution over alphabet $\mathcal{A}$.

Before setting the generation, we introduce some terms.

Free Monoid $\mathcal{A}^*$. Let $\mathcal{A} = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ be an alphabet (we usually fix $\alpha_1 = a$). A word over $\mathcal{A}$ is a finite sequence $x = x_1x_2\cdots x_n$, $x_i \in \mathcal{A}, 1 \le i \le n$. The length n of the word x is also denoted by $|x|$. The set of words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$ (we suppose the empty word belong to $\mathcal{A}^*$). The concatenation of two words is the operation, denoted multiplicatively, consisting in putting these words end to end. Endowed with this operation, $\mathcal{A}^*$ is a free monoid.

Substitution. A substitution over the alphabet $\mathcal{A}$ is a morphism ξ from $\mathcal{A}$ to $\mathcal{A}^*$. A substitution ξ is said to be primitive if, there exists an integer k, such that for any letter α in $\mathcal{A}$ the word $\xi^k(\alpha)$ contains all letters in $\mathcal{A}$.

Id a be the first letter (prefix) of $\xi(a)$. Then $\lim_{n\to\infty} \xi^n(a) = \xi^\infty =: z^+$ exists and is called the *substitution sequence* of ξ. z^+ is a fixed point of ξ, i.e., there exists $l > 0$ such that $\xi^l(z^+) = z^+$. The fixed point of ξ may not be unique.

Strictly ergodic substitution dynamical system. Let ξ be a primitive substitution over alphabet $\mathcal{A}$. Choose a fixed point of ξ (for example the substitution sequence z^+). Taking any $z \in A^{\mathbb{Z}}$ satisfying $z_n = z_n^+$ for $n \in \mathbb{N}$ and defining Ω_ξ as the set of accumulation points of $\{T^n z : n \in \mathbb{N}\}$, where T is the left shift on $\mathcal{A}^{\mathbb{Z}}$. As claimed in [7, 8], (Ω_ξ, T) is uniquely ergodic (i.e., there exists only one T-invariant measure)and minimal (i.e., the set of translates of every $\omega \in \Omega_\xi$ is dense in Ω_ξ), so it is strictly ergodic. The proof is

This work is partly supported by the Special Funds for Major State Basic Research Projects of China and NSFC 19901025. The authors thank Prof. Zhi-Ying Wen for his help.

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essentially contained in [15]. We note that Ω_ξ is independent of the selection of fixed point.

Generation of ω. We call any $\omega \in \Omega_\xi$ a sequence generated by primitive substitution ξ. By minimality of (Ω_ξ, T), the spectrum of $H_{v,\omega}$ are equal for all $\omega \in \Omega_\xi$, and we denote the set by Σ_ξ.

It was proved by Hof, Knill and Simon [8] that, for any primitive substitution ξ, and any $\omega \in \Omega_\xi$, Σ_ξ is singular continuous (note that a singular spectrum may have positive Lebesgue measure, but Σ_ξ was conjectured to have Lebesgue measure zero). They applied a result proved by Hof [7] that, for any $E \in \mathbb{C}$, the Lyapunov exponent $\gamma_{v,\omega}(E)$ exists and are equal for any $\omega \in \Omega_\xi$. It is known that, $\gamma_{v,\omega}(E) > 0$ for $E \in \mathbb{C}\backslash\Sigma_\xi$, and $\gamma_{v,\omega}(E) \geq 0$ for any $E \in \Sigma_\xi$. We will prove that $\gamma_{v,\omega}(E) = 0$ for any $E \in \Sigma_\xi$, then by a theorem of Kotani [10], Σ_ξ is a set of Lebesgue measure zero.

Let us recall two papers that had ever estimated the related Lyapunov exponent. It was proved by Bellissard *et al.* [2] that for ω generated by a class of circle map, which is the usually called sturmian sequence (some stumian sequences can be generated by invertible substitution over two-letter alphabet), $\gamma_{v,\omega}(E) = 0$ for any $E \in \sigma(H_{v,\omega})$. For non-periodic ω generated by more general primitive substitution, Bovier and Ghez [3] provided a way to prove zero Lyapunov exponent for E in the spectrum, if the substitution satisfies two hypotheses they gave. What we want to do in this paper is to remove the two hypotheses, so for any non-periodic ω generated by primitive substitution, $\sigma(H_{v,\omega})$ is a set of Lebesgue measure zero. Moreover, it was proved by Sütö [19] such that Σ_ξ contains no isolated point (we will give another proof in Theorem 1), so it is also a Cantor set.

We note that absence of isolated point does not mean that Σ_ξ contains no point spectrum of $H_{v,\omega}$ for any $\omega \in \Omega_\xi$. Uniform absence of point spectrum for some strictly ergodic potentials is closely studied by Damanik and Lenz recently (see [6] and reference there in), he had proved that, for any sturmian sequence ω, $\sigma(H_{v,\omega})$ contains no point spectrum (i.e. eigenvalue).

Our work is a continuation of Casdagli [5], Sütö [18], Bellissard *et al.* [2], Bovier and Ghez [3], in sense of applying the so called trace polynomial.

Casdagli [5] proved that pseudo-spectrum B_∞ of Schrödinger operator for some potential generated by Fibonacci substitution is a Cantor set. B_∞ constructs a bridge to apply the trace map theory.

Sütö [18] proved that $\sigma(H_{v,\omega})$ is a Cantor set for some v and a ω generated by Fibonacci substitution. He established a relation between B_∞ and $\sigma(H_{v,\omega})$ by method of periodical approach, which can also be extended to any primitive substitution by choosing a suitable way of generation.

Bellissard *et al.* [2] proved that $\sigma(H_{v,\omega})$ is a Cantor set of zero Lebesgue measure, for any $\omega \in \Omega_\tau$ with τ an arbitrary invertible substitution over two letter alphabet. They introduced some result on random Schrödinger operator ([12] and [10]), and discussed the set Ω_τ instead of a single sequence ω.

Bovier and Ghez [3] gave a general result under two hypotheses:

Theorem A1. *Let ξ be a primitive substitution with non-periodic substitution sequence on a finite alphabet $\mathcal{A}$. Let v be a non-constant map from $\mathcal{A}$ to $\mathbb{R}$, ω a two-side infinite sequence generated by ξ, and $H_{v,\omega}$ be the Schrödinger operator defined in* (1). *Suppose there exists a trace map whose induced substitution ϕ, defined on an alphabet $\mathcal{B}$, is semi-primitive. Assume further that there exists $k < \infty$ such that $\xi^k(a)$ contains the word $\beta\beta$ for some $\beta \in \mathcal{B}$. Then the spectrum of $H_{v,\omega}$ is singular and supported on a set of zero Lebesgue measure.*

Where they applied leading term of trace polynomial to estimate trace, introduced the semi-primitivity of induced substitution, and applied some methods of Casdagli, Sütö and Bellissard *et al.* In this paper we will follow these methods.

Note that in [3], the generation of a two-side infinite sequence from a primitive substitution, say ξ, is different from the generation we give above, which caused some trouble and is modified in [4]. And it may still happen that the generated sequence ω' they given in [4] is not in the set Ω_ξ. In fact, it can be deduced from our proof that the spectrum for this kind of potential is still Lebesgue measure zero, but we do not know whether it is equal to Σ_ξ.

They checked the hypotheses of semi-primitivity and existence of square word for some substitutions, and they found that all the examples they had checked satisfy the two hypotheses, except that the respective induced substitution of Rudin-Shapiro substitution is not semi-primitive. But we found that they had made some errors on it. The trace map they chose is got from [9], while some trace polynomial in the trace map has no leading term (a term in a polynomial with a degree strictly larger than any other terms in the polynomial) for $\mathcal{A}$-degree they defined themselves, this may cause trouble in their estimating. Moreover, we would like to note that since trace map of a substitution over three or more letter alphabet is not unique, absence of semi-primitivity for the induced substitution of one trace map associated a primitive substitution (say ξ) does not imply that of any other trace map associated ξ. The usually called trace map will be called polynomial map associated substitution, since it may be not unique for a given substitution.

To assure the existence of leading term, we will focus our attention on a class of special trace polynomials, which is called natural trace polynomial in this paper. Even the natural trace polynomial may be not unique for word over three or more letter alphabet. So it may need complex and tedious calculation to justify if there are semi-primitive induced substitution for a given primitive substitution. In fact we had made out a polynomial map associated Rudin-Shapiro substitution, of which the induced substitution is semi-primitive. But this is not yet important, since for any primitive substitution ξ over any finite alphabet, we will prove in §2.3 that there exist $l > 0$ and suitable polynomial map associated ξ^l (note that $\Omega_{\xi^l} = \Omega_\xi$, since their substitution sequences are equal) such that the corresponding induced substitutions are semi-primitive.

In [3], the existence of square word is used to estimate the Lyapunov exponent. But it is not a trivial condition, since for substitution sequences on three or more letter alphabet, the hypothesis may fail. The following is an example in [11] which contains no square word.

Let $\mathcal{A} = \{a, b\}$ be a two-letter alphabet, and $\xi(a) = ab$, $\xi(b) = ba$ the Thue-Morse substitution. The Thue-Morse sequence is the fixed point $u = \xi^\infty(a)$ of ξ, and it is well-known that u is $2 + \varepsilon$ free, that is to say, it has no subword of the form UUV with V a prefix of U.

Let $\mathcal{B} = \{A, B, C\}$, and define the substitution $\tau : \mathcal{B}^* \to \mathcal{B}^*$ as $\tau(A) = ABC$, $\tau(B) = AC$, $\tau(C) = B$. Let ζ be the fixed point of τ,i.e., $\zeta = \tau^\infty(A)$. Define a morphism $\delta : \mathcal{B}^* \to \mathcal{A}^*$ by $\delta(A) = abb$, $\delta(B) = ab$, $\delta(C) = a$. Extend δ in a natural way to sequences, and it is easy to see that $\delta(\zeta) = u$. So it is readily to prove that ζ is square-free, i.e. no word of form UU is contained in ζ, by $2 + \varepsilon$ free of u.

We will remove the hypothesis of existence of square word through further analysis.

In this paper, we will not prove the theorems which we needed and had already been proved in [3]. So combining with these theorems, we prove in §5 that

Theorem 1. *Let ξ be a primitive substitution with non-periodic substitution sequence on a finite alphabet $\mathcal{A}$. Let v be an injective map from $\mathcal{A}$ to $\mathbb{R}$, Ω a set of all two-side infinite sequence generated by ξ, for any $\omega \in \Omega$, let $H_{v,\omega}$ be the Schrödinger operator defined in (1), and let $\Sigma_\xi = \sigma(H_{v,\omega})$ be the spectrum of $H_{v,\omega}$. Then*

$$\Sigma_\xi = \{E \in \mathbb{R} | \gamma_{v,\omega}(E) = 0\}, \quad \forall \omega \in \Omega,$$

and is a Cantor set of zero Lebesgue measure.

Remark 2. We need the substitution sequence to be non-periodic since there are primitive substitutions, say for example $\xi(a) = a(ba)^n$, $\xi(b) = b(ab)^m$, with periodic substitution sequence and the spectrum for such potential is purely absolutely continuous.

2. SEMI-PRIMITIVITY

2.1. **Trace Polynomial.** A word $u \in \mathcal{A}^*$ is said to be a cyclic permutation of $v \in \mathcal{A}^*$, if there exist $w_1, w_2 \in \mathcal{A}^*$ such that $u = w_1w_2$ and $v = w_2w_1$. The cyclic permutation induces a equivalence relation, which is denoted by $\sim$.

Let $\mathcal{B}_1 = \{u \in \mathcal{A}^* : u \text{ contains no letter twice }\}$ and $\mathcal{B} = \mathcal{B}_1/\sim$. It is easy to see that $\mathcal{B}$ is a finite set, in fact $|\mathcal{B}| = K = \sum_{i=1}^{m} \frac{m!}{i\cdot(m-i)!}$. It is convenient to regard $\mathcal{B}$ as a subset of $\mathcal{A}^*$, and we write that

$$\mathcal{B} = \{\beta_1, \beta_2, \cdots, \beta_K\},$$

where $\beta_1, \cdots, \beta_K \in \mathcal{A}^*$.

Let $\mathrm{SL}(2,\mathbb{R})$ be the set of all real 2×2 matrices with determinant 1, $\mathrm{Hom}(\mathcal{A}^*, \mathrm{SL}(2,\mathbb{R}))$ be the set of all the homomorphism maps from $\mathcal{A}^*$ into $\mathrm{SL}(2,\mathbb{R})$. $\varphi \in \mathrm{Hom}(\mathcal{A}^*, \mathrm{SL}(2,\mathbb{R}))$ is determined by matrices $\{\varphi(\alpha_i)\}_{i=1}^m$, and for $u = u_1u_2\cdots u_l (u_i \in \mathcal{A})$,

$$\varphi(u) = \varphi(u_1)\varphi(u_2)\cdots\varphi(u_l).$$

Thus $\varphi(u)$ can be seen as a matrix gotten by replacing every letter in u to the responding matrix. We call $\varphi \in \mathrm{Hom}(\mathcal{A}^*, \mathrm{SL}(2,\mathbb{R}))$ a matrix representation.

Let u be a word, a polynomial P over $\mathbb{R}^K$ is said to be a trace polynomial of u, if for any matrix representation $\varphi \in \mathrm{Hom}(\mathcal{A}^*, \mathrm{SL}(2,\mathbb{R}))$, we have

$$\mathrm{tr}\,\varphi(u) = P(\mathrm{tr}\,\varphi(\beta_1), \mathrm{tr}\,\varphi(\beta_2), \cdots, \mathrm{tr}\,\varphi(\beta_K)),$$

where $\mathrm{tr}(M)$ denotes the trace of the matrix M.

For any $u \in \mathcal{A}^*$, such polynomial always exists, but may be not unique. In fact, only when $|\mathcal{A}| = 2$, such polynomial is unique; if $|\mathcal{A}| > 2$, there are infinitely many trace polynomials for each $u \in \mathcal{A}^*$, since there exist polynomials (say Λ) such that for any matrix representation φ,

$$\Lambda(\mathrm{tr}\,\varphi(\beta_1), \mathrm{tr}\,\varphi(\beta_2), \cdots, \mathrm{tr}\,\varphi(\beta_K)) \equiv 0.$$

We refer to [14, 20] for some details of these properties.

Now we pay our attention to a special kind of trace polynomials (they have ever been got in [14, 3], we introduce them here in detail to get leading term). Their constructions are based on the facts that, for any matrices $A, B \in \mathrm{SL}(2,\mathbb{R})$,

$$\mathrm{tr}\,AB = \mathrm{tr}\,BA \tag{2}$$

$$ABA = (\mathrm{tr}\,AB)A + B - (\mathrm{tr}\,B)I \tag{3}$$

where I is unit.

Let u be a word over $\mathcal{A}$. There are two possibilities:

If u contains no letter in $\mathcal{A}$ twice, then a cyclic permutation of u (say β) belongs to $\mathcal{B}$, so by (2), for any matrix representation $\varphi \in \mathrm{Hom}(\mathcal{A}^*, \mathrm{SL}(2,\mathbb{R}))$,

$$\begin{aligned} \operatorname{tr}\varphi(u) &= \operatorname{tr}\varphi(\beta) \\ &= x_\beta(\operatorname{tr}\varphi(\beta_1), \operatorname{tr}\varphi(\beta_2), \cdots, \operatorname{tr}\varphi(\beta_K)) \end{aligned}$$

where x_β stands for the coordinate function on $\mathbb{R}^K$ corresponding to the index $\beta \in \mathcal{B}$. Thus x_β is a trace polynomial of u.

If u contains a word in $\mathcal{A}^*$ (say s) twice, then we can choose a cyclic permutation of u of the form sw_1sw_2, so by (2) and (3), for any matrix representation φ,

$$\begin{aligned} \operatorname{tr}\varphi(u) &= \operatorname{tr}\varphi(sw_1sw_2) \\ &= \operatorname{tr}\varphi(sw_1)\operatorname{tr}\varphi(sw_2) + \operatorname{tr}\varphi(w_1w_2) - \operatorname{tr}\varphi(w_1)\operatorname{tr}\varphi(w_2). \end{aligned} \tag{4}$$

We now proceed with each of the traces appearing in (4) just as before, that is, if a corresponding word (say sw_1) contains no letter twice, then one of its cyclic permutations belongs to $\mathcal{B}$. Whereas for words still contain a letter twice, (2) and (3) is again applied. Since that with each application of (3), the words that may appear become strictly shorter, so that this process necessarily terminates after a finite number of steps, then for any matrix representation φ,

$$\begin{aligned} \operatorname{tr}\varphi(u) &= \sum_{j=1}^n \prod_{i=1}^K [\operatorname{tr}\varphi(\beta_i)]^{n_{i,j}} \\ &= [\sum_{j=1}^n \prod_{i=1}^K x_{\beta_i}^{n_{i,j}}](\operatorname{tr}\varphi(\beta_1), \operatorname{tr}\varphi(\beta_2), \cdots, \operatorname{tr}\varphi(\beta_K)), \end{aligned}$$

and the polynomial $P = \sum_{j=1}^n \prod_{i=1}^K x_{\beta_i}^{n_{i,j}}$ is a trace polynomial of u.

In this paper, we call the trace polynomial which can be gotten in this way the *natural trace polynomial*. It is possible that there are more than one natural trace polynomials for one word, since different partitions of a word may give different polynomials.

Example 1: Let $u = abcacb$, the partitions $a(bc)a(cb)$ and $c(a)c(bab)$ result different natural trace polynomials $P = x_{abc}x_{acb} + x_bx_cx_{bc} - x_b^2 - x_c^2 - x_{bc}^2 + 2$ and $Q = x_{ac}x_{bc}x_{ab} + x_a^2 + x_{ac}^2 + x_{ab}^2 - x_ax_cx_{ac} - x_ax_bx_{ab} - 2$.

2.2. **$\mathcal{A}$-degree and Leading Terms.** We know that the trace polynomial of any word in two letter alphabet alway has leading term (see [21]). But for words in three or more letter alphabet, this is not true even for natural trace polynomial (see Example 1). Bovier and Ghez [3] introduced $\mathcal{A}$-degree of polynomials, which is different from the degree we usually mean. The benefit of the $\mathcal{A}$-degree is that it ensures the *natural trace polynomials* have leading term in sense of $\mathcal{A}$-degree.

For any monomial on $\mathbb{R}^K$ of the form $P = x_{\beta_{i_1}}x_{\beta_{i_2}}\cdots x_{\beta_{i_l}}$, where $\beta_{i_j} \in \mathcal{B}, j = 1, 2, \cdots, l$, let $W(P)$ denote the word

$$W(P) = \beta_{i_1}\beta_{i_2}\cdots\beta_{i_l} \in \mathcal{B}^*. \tag{5}$$

Also, $W(P)$ can be seen as a word in $\mathcal{A}^*$, the length of $W(P)$ as a word in $\mathcal{A}^*$ (i.e., number of letters in $\mathcal{A}$) is called the $\mathcal{A}$-degree of P and noted by

$$\mathrm{d}_{\mathcal{A}}(P) = |W(P)|_{\mathcal{A}}.$$

Although different arrays of factors in P may result different $W(P)$, it changes merely the order of letters in $W(P)$, and the $\mathcal{A}$-degree of a monomial is well-defined.

For any polynomial Q on $\mathbb{R}^K$, Q can be written as

$$Q = \sum_j c_jQ_j, \quad c_j \neq 0,$$

where Q_j is monomial with coefficient 1, and the $\mathcal{A}$-degree of Q is defined by

$$\mathrm{d}_{\mathcal{A}}\, Q = \max_j \mathrm{d}_{\mathcal{A}}\, Q_j.$$

Formula (3) and the construction of natural trace polynomial imply the following property. Before stating the proposition, we give some notations:

For any $v \in \mathcal{A}^*$,

$$L[v] = (|v|_{\alpha_1}, |v|_{\alpha_2}, \cdots, |v|_{\alpha_m}), \alpha_i \in \mathcal{A},$$

which is an $|\mathcal{A}|$-dimensional vector, $|v|_{\alpha_i}$ is the number of letter α_i contained in v. The "$>$" between two real vectors is defined as follows, if $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_m)$, then $x > y$ means that $x_i \geq y_i$ and there exists $1 \leq j \leq m$ such that $x_j > y_j$.

Proposition 3. *Let P be a natural trace polynomial of $u \in \mathcal{A}^*$, and*

$$\begin{array}{l} P = \sum_{i=1}^{l} c_i P_i, \quad c_i \neq 0 \\ \mathrm{d}_{\mathcal{A}}\, P_i \geq \mathrm{d}_{\mathcal{A}}\, P_{i+1}, \quad i = 1, 2, \cdots, l-1, \end{array}$$

where P_i is monomial with coefficient 1, then we have

(i) $c_1 = 1$,
(ii) $L[W(P_1)] > L[W(P_i)], \quad i = 2, 3, \cdots, l$,
(iii) $L[W(P_1)] = L[u]$,
(iv) $\mathrm{d}_{\mathcal{A}}\, P = |u|$.

Remark 4. By the property (ii), we know that the $\mathcal{A}$-degree of P_1 is greater than any other P_i. We call P_1 the leading term of P, denoted as $\widetilde{P}$. Moreover, property (ii) is stronger than what leading term means. In fact, if the trace polynomial hold only leading term, but without property (ii), we can hardly proceed the estimating in (10), so do analogous estimatings in [3].

Proof of the proposition We prove by induction on length of u.

If $|u| = 1$, then P must be a coordinate function, so it is a monomial with coefficient 1, the four properties listed above are satisfied.

Assume for $1 \leq |u| < n$, these properties are satisfied. We want to check these properties for $|u| = n$. In fact, we only need to prove the case when u contains at least one letter in $\mathcal{A}$ twice.

Suppose P is a natural trace polynomial of u, and sw_1sw_2 is the partition in the first step of the construction of P, where s may be a word in $\mathcal{A}^*$. Then by (4) for any matrix representation φ, there exist natural trace polynomials $Q_1, \cdots, Q_5$ corresponding to the words sw_1, sw_2, w_1w_2, w_1, w_2 respectively such that

$$P = Q_1Q_2 + Q_3 - Q_4Q_5.$$

By hypothesis of the induction, Q_i satisfies the four properties. Let $\widetilde{Q_i}$ be the leading term of Q_i. So only $L[W(\widetilde{Q_1}\widetilde{Q_2})]$, $L[W(\widetilde{Q_3})]$ and $L[W(\widetilde{Q_4}\widetilde{Q_5})]$ can be the maximal vector. Since

$$\begin{array}{rcl} L[W(\widetilde{Q_1}\widetilde{Q_2})] & = & L[W(\widetilde{Q_1})] + L[W(\widetilde{Q_2})] = L[sw_1] + L[sw_2] = L[u], \\ L[W(\widetilde{Q_3})] & = & L[w_1w_2], \\ L[W(\widetilde{Q_4}\widetilde{Q_5})] & = & L[w_1w_2], \end{array}$$

we have

$$L[W(\widetilde{Q_1}\widetilde{Q_2})] > L[W(\widetilde{Q_3})] = L[W(\widetilde{Q_4}\widetilde{Q_5})]$$

So P satisfies the above four properties, and $\widetilde{P} = \widetilde{Q_1}\widetilde{Q_2}$ be the leading term of P. $\square$

Remark 5. Let $u \in \mathcal{A}^*$, for two different natural trace polynomial P and Q of u, it may happen that $\widetilde{P} \neq \widetilde{Q}$. In **Example 1**, we have $\widetilde{P} = x_{abc}x_{acb}$ and $\widetilde{Q} = x_{ab}x_{ac}x_{bc}$.

2.3. **Polynomial Map and the Induced Substitution.** Let ξ be a substitution over $\mathcal{A}$, and assume that $\alpha_1 = a$, a is the first letter of the word $\xi(a)$ and $|\xi(a)| > 2$.

A polynomial map associated to the substitution ξ is defined as follows,

$$\Phi(x) = (\Phi_{\beta_1}(x), \Phi_{\beta_2}(x), \cdots, \Phi_{\beta_K}(x)), \quad \forall x \in \mathbb{R}^K$$

where Φ_{β_i} is a natural trace polynomial of $\xi(\beta_i)$. Such polynomial map may be not unique, depending on the natural trace polynomials we choose.

Since any Φ_{β_i} holds leading term $\widetilde{\Phi}_{\beta_i}$ of coefficient 1, we can define the reduced map of Φ on $\mathbb{R}^K$ by,

$$\widetilde{\Phi}(x) = (\widetilde{\Phi}_{\beta_1}(x), \widetilde{\Phi}_{\beta_2}(x), \cdots, \widetilde{\Phi}_{\beta_K}(x)), \quad x \in \mathbb{R}^K,$$

and a induced substitution ϕ of $\widetilde{\Phi}$ over $\mathcal{B}^*$:

$$\phi(\beta) = W(\widetilde{\Phi}_\beta), \quad \beta \in \mathcal{B},$$

where $W(\widetilde{\Phi}_\beta)$ is the word associated the monomial $\widetilde{\Phi}_\beta$, as defined in (5). As discussed before, $\phi(\beta) = W(\widetilde{\Phi}_\beta)$ can only be determined up to the order of the letters in $\phi(\beta)$. But what we consider is the semi-primitivity of ϕ, which has no relation with the order of letters, and if one induced substitution of $\widetilde{\Phi}$ is semi-primitive, so is the others.

Definition 1. *A substitution ϕ over $\mathcal{B}$ is said to be semi-primitive if*

(i) there exists a subset $\mathcal{C} \subset \mathcal{B}$, such that any $\beta \in \mathcal{C}$, $\phi(\beta)$ contains only letters in $\mathcal{C}$, and $\phi|_{\mathcal{C}}$ is primitive (i.e. there exists $k \in \mathbb{N}$ such that for any $\beta \in \mathcal{C}$, $\phi^k(\beta)$ contains all letters in $\mathcal{C}$);

(ii) there exists $l \in \mathbb{N}$, such that for all $\beta \in \mathcal{B}$, $\phi^l(\beta)$ contain letters in $\mathcal{C}$.

We give a sufficient and necessary condition for a substitution to be semi-primitive.

Proposition 6. *A substitution ϕ over an alphabet $\mathcal{B}$ is semi-primitive if and only if there exists a letter $\beta_0 \in \mathcal{B}$ and an integer k such that $\phi^k(\beta)$ contains β_0 for any letter $\beta \in \mathcal{B}$, and the length of $\phi^n(\beta_0)$ tends to infinity with n.*

Proof. It is readily to prove the necessary part, so we prove the sufficient part in the following.

If $k = 1$. For any integer n and $\beta \in \mathcal{B}$, let $N(\beta, n)$ be the number of β contained in the word $\phi^n(\beta_0)$, i.e.

$$N(\beta, n) := |\phi^n(\beta_0)|_\beta.$$

The number $N(\beta, n)$ is increasing with n, since β_0 is contained in $\phi(\beta_0)$ and hence $\phi^n(\beta_0)$ is a subword of $\phi^{n+1}(\beta_0)$.

Let

$$\mathcal{C} := \{\beta \in \mathcal{B} : \lim_{n\to\infty} N(\beta, n) = \infty\}.$$

$\mathcal{C}$ is not empty, since the length of $\phi^n(\beta_0)$ tends to infinity with n.

For any $\gamma \in \mathcal{C}$, $\phi(\gamma)$ contains only letters in $\mathcal{C}$: suppose that a letter $\beta \in \mathcal{B}$ is contained in $\phi(\gamma)$, then $N(\beta, n+1) \geq N(\gamma, n)$, so $\beta \in \mathcal{C}$.

Since $\mathcal{C}$ is a finite set, there exists an integer l, such that $\phi^l(\beta_0)$ contains all letters in $\mathcal{C}$. For any $\beta \in \mathcal{B}$, since β_0 is contained in $\phi(\beta)$, $\phi^l(\beta_0)$ is a subword of $\phi^{l+1}(\beta)$. Thus $\phi^{l+1}(\beta)$ contains all letters in $\mathcal{C}$. This means ϕ is semi-primitive.

If $k \neq 1$, the above argument means ϕ^k is semi-primitive. Suppose that ϕ^k is primitive on non-empty set $\mathcal{C}$. To prove ϕ is semi-primitive, we only need to prove that for any γ

in $\mathcal{C}$, $\phi(\gamma)$ contains only letters in $\mathcal{C}$. Suppose there exists a letter γ_0 in $\mathcal{C}$ such that $\phi(\gamma_0)$ contains a $\beta_1 \in \mathcal{B}$ with $\beta_1 \notin \mathcal{C}$. By semi-primitivity of ϕ^k, there exists an integer l such that, for any $\gamma \in \mathcal{C}$, $\phi^{kl}(\gamma)$ contains all letters in $\mathcal{C}$. Then for any $\gamma \in \mathcal{C}$, $\phi^{kl+1}(\gamma)$ contains β_1 and all letters in $\mathcal{C}$. Then for any integer $n \geq l$ and any $\gamma \in \mathcal{C}$, $\phi^{(kl+1)nk}(\gamma)$ contains β_1. This lead to a contradiction. □

It is a direct corollary of the proposition that the induced substitution corresponding to a primitive substitution over two-letter alphabet (say $\{a, b\}$) is semi-primitive, since we only need to let $\beta_0 = ab$. But for three or more letter alphabet, it is not so obvious. We prove the following corollary by choosing suitable polynomial map associated with primitive substitution.

Proposition 7. *If ξ is primitive, then there exists $l \in \mathbb{N}$ and a polynomial map associated with ξ^l, such that the corresponding induced substitution is semi-primitive.*

Proof. Since ξ is primitive, there exists a integer t such that for any $\alpha \in \mathcal{A}$, $\xi^t(\alpha)$ contains the letter a at least twice.

We choose a suitable polynomial map associated with ξ^{2t}.

(i) Since $\xi^t(a)$ contains at least two a, we can write $\xi^t(a) = aw_1aw_2$, where $w_1, w_2 \in \mathcal{A}^*$. We fix a natural trace polynomial P_{aw_1} of aw_1, and fix a letter $\beta_0 \in \mathcal{B}$ which is contained in the word $W(\widetilde{P}_{aw_1}) \in \mathcal{B}^*$.

(ii) For any $\beta \in \mathcal{B}$, since $\xi^t(\beta)$ contains at least one a, $\xi^{2t}(\beta)$ contains aw_1a, then we can choose a cyclic permutation of $\xi^{2t}(\beta)$ of form aw_1aw_β. Since for any matrix representation φ,

$$\begin{aligned} \operatorname{tr}\varphi(\xi^{2t}(\beta)) &= \operatorname{tr}\varphi(aw_1aw_\beta) \\ &= \operatorname{tr}\varphi(aw_1)\operatorname{tr}\varphi(aw_\beta) + \operatorname{tr}\varphi(w_1w_\beta) - \operatorname{tr}\varphi(w_1)\operatorname{tr}\varphi(w_\beta), \end{aligned}$$

we choose natural trace polynomials $P_{aw_\beta}, P_{w_1w_\beta}, P_{w_1}, P_{w_\beta}$ of the words $aw_\beta, w_1w_\beta, w_1, w_\beta$ respectively, then

$$\Phi_\beta := P_{aw_1}P_{aw_\beta} + P_{w_1w_\beta} - P_{w_1}P_{w_\beta}$$

is a natural trace polynomial of $\xi^{2t}(\beta)$. Thus we get a polynomial map Φ associated with the substitution ξ^{2t}, and the reduced map $\widetilde{\Phi}$ of Φ with

$$\widetilde{\Phi}_\beta = \widetilde{P}_{aw_1}\widetilde{P}_{aw_\beta}.$$

Set a induced substitution of $\widetilde{\Phi}$ by

$$\phi(\beta) = W(\widetilde{P}_{aw_1})W(\widetilde{P}_{aw_\beta}) = W_{aw_1}W(\widetilde{P}_{aw_\beta}). \tag{6}$$

For any $\beta \in \mathcal{B}$, $W(\widetilde{P}_{aw_1})$ is a subword of $\phi(\beta)$, and so β_0 is contained in $\phi(\beta)$.

Since $L[\phi^n(\beta_0)] = L[\xi^{2tn}(\beta_0)]$, the length of $\phi^n(\beta_0)$ tends to infinity with n as a word in $\mathcal{B}^*$. So according to Proposition 6, ϕ is semi-primitive. □

3. Periodic approach

By Proposition 7, for any primitive substitution ξ, there exists an integer l, and a polynomial map associated with ξ^l, such that the corresponding induced substitution is semi-primitive. Let $\tau = \xi^l$, $\Phi = (\Phi_{\beta_1}, \Phi_{\beta_2}, \cdots, \Phi_{\beta_K})$ be the corresponding polynomial map, $\widetilde{\Phi}$ be the reduced map of Φ, and ϕ be its induced substitution. Since ϕ is semi-primitive, we can suppose that ϕ is primitive on nonempty set $\mathcal{C} \subset \mathcal{B}$. Since ξ^l and ξ have same fixed points, so $\Omega_\tau = \Omega_{\xi^l} = \Omega_\xi$, we discuss τ directly instead of ξ in the following.

Fix a letter $\gamma_0 \in \mathcal{C}$. Let

$$\mathcal{U}_n := \{x \in \mathbb{R}^K : \forall m \geq n\ |\Phi^n_{\gamma_0}(x)| > 2\},$$
$$\mathcal{U} := \bigcup_{n=0}^{\infty} \operatorname{int}(\mathcal{U}_n),$$

where $\operatorname{int}(\mathcal{U}_n)$ is the interior of $\mathcal{U}_n$.

For any $E \in \mathbb{R}$, let $\varphi_{v,E}$ be a matrix representation associated E defined by,

$$\varphi_{v,E}(\alpha) := \begin{bmatrix} E - v(\alpha) & -1 \\ 1 & 0 \end{bmatrix}, \tag{7}$$

which is often called transfer matrix. Define the point $x_{v,E} \in \mathbb{R}^K$ by,

$$x_{v,E} = (\operatorname{tr}\varphi_{v,E}(\beta_1), \operatorname{tr}\varphi_{v,E}(\beta_2), \cdots, \operatorname{tr}\varphi_{v,E}(\beta_K)) \tag{8}$$

For any set $X \subset \mathbb{R}^K$, let

$$\mathcal{E}_v(X) := \{E \in \mathbb{R} :\ x_{v,E} \in X\} \tag{9}$$

Then we have

Lemma 8. *$\mathcal{E}_v(\mathcal{U}) \subset \rho(H_{v,\omega})$.*

Where $\rho(H_{v,\omega})$ is resolvent set of $H_{v,\omega}$, which is the complement of the spectrum. The proof of this lemma was given by Sütö [18] and then Bellissard *et al.* [2]. Essentially it is contained in Theorem VIII.24a of Reed and Simon, Vol.1 [16].

Theorem A2. ([16]) *Let $\{A_n\}_{n=1}^{\infty}$ and A be self-adjoint operators and suppose that $A_n \to A$ in the strong resolvent sense. Then if $a < b$ and $(a,b) \cap \sigma(A_n) = \emptyset$ for all n, then $(a,b) \cap \sigma(A) = \emptyset$.*

Proof of lemma 8. For any two-side infinite sequence ω generated by τ, there exist two increasing sequence $\{t_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ of positive integers such that t_n and $|\tau^{p_n}(\gamma_0)| - t_n$ tends to ∞ with n, and ω is the limit of $\{\omega^{(n)}\}$, where $\omega^{(n)}$ is periodic sequence in $\mathcal{A}^{\mathbb{Z}}$ with period $|\tau^{p_n}\gamma_0|$ such that

$$\omega_i^{(n)} = (\tau^{p_n}\gamma_0)_{i+t_n}, \quad 1 - t_n \leq i \leq |\tau^{p_n}\gamma_0| - t_n.$$

So let $A_n = H_{v,\omega^{(n)}}$, $A_n \to H_{v,\omega}$ in the strong resolvent sense, by Theorem **A**2

$$\operatorname{int}\left(\bigcap_{n=k}^{\infty} \rho(A_n)\right) \subset \rho(H_{v,\omega}), \quad \forall k > 0.$$

Notice that

$$\rho(A_n) = \{E : |\operatorname{tr}\varphi_{v,E}(\tau^{p_n}\gamma_0)| > 2\} = \{E : |\Phi^{p_n}_{\gamma_0}(x_{v,E})| > 2\}$$

This proves the lemma. □

In principle we would like to prove also the converse of Lemma 8, which would allow us to compute the spectrum of H. If $\mathcal{U}$ is such that Lyapunov exponent vanishes for $E \in V(\mathcal{U}^c)$, then not only the converse of Lemma 8 hold , but also, applying some general results of Kotani [10], the spectrum has zero Lebesgue measure. However, while the definition of $\mathcal{U}$ is convenient to prove lemma 8, it is inconvenient to describe $\mathcal{U}$ in more detail. Bovier and Ghez [3] define a set $\widetilde{\mathcal{U}}$ that a priori is contained in $\mathcal{U}$ but that is big enough such that for energies $E \in \mathcal{E}_v(\widetilde{\mathcal{U}})^c$, the Lyapunov exponent vanishes.

4. Lyapunov exponent

Notice that for any $\beta \in \mathcal{B}$, $\mathrm{d}_{\mathcal{A}}\,\Phi_\beta^n = |\tau^n(\beta)|$. For notational convenience, we denote

$$\rho^{(n)}(\beta, x) := |\Phi_\beta^n(x)|^{\frac{1}{|\tau^n(\beta)|}}, \quad \beta \in \mathcal{B}$$

and

$$\rho_{\max}^{(n)}(x) := \max_{\beta\in\mathcal{B}} \rho^{(n)}(\beta, x).$$

We will introduce some theorems in [3].

For $0 < \varepsilon < 1$, $c > 0$ and $n \in \mathbb{N}$, let

$$\widetilde{\mathcal{U}}_{\varepsilon,c,n} = \{x \in \mathbb{R}^K \,|\, \min_{i\in\mathcal{C}} \rho^{(n)}(i, x) > [\rho_{\max}^{(n)}(x)]^{1-\varepsilon} \text{ and } [\min_{i\in\mathcal{C}} \rho^{(n)}(i, x)]^{\min_{\alpha\in\mathcal{A}} |\tau^n(\alpha)|} > c\}.$$

Lemma A3. ([3]) *There exists $n_0 < \infty$ such that for all $\varepsilon > 0$ small enough and $\varepsilon' > \varepsilon$, there exists $1 < c < \infty$ such that for all $n \geq n_0$ and for all $n' \geq n$, $\widetilde{\mathcal{U}}_{\varepsilon,c,n} \subset \widetilde{\mathcal{U}}_{\varepsilon',c,n'}$.*

Corollary A4. ([3]) *Let $0 < \varepsilon < 1$ and $c > 2$ and n_0 be chosen that the conclusion of Lemma A3 holds. Then, for all $n \geq 0$,*

$$\widetilde{\mathcal{U}}_{\varepsilon,c,n} \subset \mathrm{int}(\mathcal{U}_n).$$

Moreover, defining (for a given choice of ε, c and n_0)

$$\widetilde{\mathcal{U}} := \bigcup_{n=n_0}^{\infty} \widetilde{\mathcal{U}}_{\varepsilon,c,n},$$

we have that

$$\widetilde{\mathcal{U}} \subset \mathcal{U}.$$

Proposition A5. ([3]) *If $x \in \widetilde{\mathcal{U}}^c$, then for all $\beta \in \mathcal{B}$,*

$$\limsup_{n\to\infty} \frac{1}{|\tau^n(\beta)|} \ln |\Phi_\beta^n(x)| \leq 0.$$

In the following we manage to remove hypothesis of the existence of the square word from Theorem A1.

Proposition 9. *For any matrix representation $\varphi \in \mathrm{Hom}(\mathcal{A}^*, \mathrm{SL}(2,\mathbb{R}))$, define the point $x \in \mathbb{R}^K$ by,*

$$x = (\mathrm{tr}\,\varphi(\beta_1), \mathrm{tr}\,\varphi(\beta_2), \cdots, \mathrm{tr}\,\varphi(\beta_K)).$$

If $x \notin \widetilde{\mathcal{U}}$, then for any word $u \in \mathcal{A}^*$,

$$\limsup_{n\to\infty} \frac{1}{|\tau^n(u)|} \ln |\,\mathrm{tr}\,\varphi(\tau^n u)| \leq 0$$

Proof. Choose a natural trace polynomial P of the word u. Then by definition of trace polynomial,

$$\mathrm{tr}\,\varphi(\tau^n u) = P \circ \Phi^n(x),$$

P can be written as

$$P = \sum_{i=1}^{l} c_i P_i, \quad c_i \neq 0,$$
$$\mathrm{d}_{\mathcal{A}}\, P_i \geq \mathrm{d}_{\mathcal{A}}\, P_{i+1}, \quad i = 1, 2, \cdots, l-1,$$

where P_i is monomial with coefficient 1. Suppose $P_i = \prod_{j=1}^{K} x_{\beta_k}^{n_{i,j}}$, $n_{i,j} \geq 0$. By Propostion 3(iv), $|u| = \sum_{j=1}^{K} n_{1,j}|\beta_j|$, so $|\tau^n u| = \sum_{j=1}^{K} n_{1,j}|\tau^n \beta_j|$. By Proposition 3(ii), for any $1 < i < l$, $\sum_{j=1}^{K} n_{i,j}|\beta_j| \leq |u| - 1$. Moreover, there exists α_i such that for any $n > 0$,

$$\sum_{j=1}^{K} n_{i,j}|\tau^n \beta_j| \leq |\tau^n u| - |\tau^n \alpha_i| \leq |\tau^n u| - \min_{\alpha \in \mathcal{A}} |\tau^n \alpha|, \tag{10}$$

this implies that

$$\begin{aligned} |P \circ \Phi^n(x)| \leq \; & |P_1 \circ \Phi^n(x)| + \textstyle\sum_{j=2}^{l} |c_j| \cdot |P_j \circ \Phi^n(x)| \\ = \; & \textstyle\prod_{j=1}^{K} [\Phi_{\beta_j}^n(x)]^{n_{1,j}} + \sum_{i=2}^{l} |c_i| \prod_{j=1}^{K} [\Phi_{\beta_j}^n(x)]^{n_{i,j}} \\ = \; & \textstyle\prod_{j=1}^{K} [\rho^{(n)}(\beta_j, x)]^{n_{1,j}|\tau^n \beta_j|} + \sum_{i=2}^{l} |c_i| \prod_{j=1}^{K} [\rho^{(n)}(\beta_j, x)]^{n_{i,j}|\tau^n \beta_j|} \\ \leq \; & [\rho_{\max}^{(n)}(x)]^{|\tau^n u|} + \textstyle\sum_{j=2}^{l} |c_j| [\rho_{\max}^{(n)}(x)]^{|\tau^n u| - \min_{\alpha \in \mathcal{A}} |\tau^n \alpha|} \\ \leq \; & M \max\{[\rho_{\max}^{(n)}(x)]^{|\tau^n u|}, 1\}, \end{aligned}$$

where $M = 1 + \sum_{i=2}^{l} |c_i|$, so

$$|\operatorname{tr} \varphi(\tau^n u)|^{\frac{1}{|\tau^n(u)|}} \leq M^{\frac{1}{|\tau^n(u)|}} \max\{\rho_{\max}^{(n)}(x), 1\}.$$

Since by Proposition A5, $\limsup_{n \to \infty} \rho_{\max}^{(n)}(x) \leq 1$, this proves the proposition. $\square$

Proposition 10. *Let $\omega = \{a_i\}_{i=-\infty}^{\infty}$ be a sequence generated by the primitive substitution τ. For any $E \in \mathbb{R}$, let $\varphi_{v,E}$ be the matrix representation associated E defined in* (7). *Let $x_{v,E}$ be the point defined in* (8). *If $x_{v,E} \notin \widetilde{\mathcal{U}}$, then $\gamma_{v,\omega}(E) = \gamma_{v,\omega}^+(E) = \gamma_{v,\omega}^-(E) = 0$, where*

$$\begin{aligned} \gamma_{v,\omega}^+(E) &= \lim_{n \to \infty} \tfrac{1}{n} \ln \|\varphi_{v,E}(a_1 \cdots a_n)\|, \\ \gamma_{v,\omega}^-(E) &= \lim_{n \to \infty} \tfrac{1}{n} \ln \|\varphi_{v,E}(a_{-n+1} \cdots a_0)\|. \end{aligned}$$

Proof. Firstly, we discuss a little more than a limit of subsequence, i.e., we prove that

$$\limsup_{n \to \infty} \max_{\alpha \in \mathcal{A}} \|\varphi_{v,E}(\tau^n \alpha)\|^{\frac{1}{|\tau^n(\alpha)|}} = 1,$$

for convenient we write

$$R_{\max}^{(n)} := \max_{\alpha \in \mathcal{A}} \|\varphi_{v,E}(\tau^n \alpha)\|^{\frac{1}{|\tau^n(\alpha)|}}.$$

Note that for any $n \in \mathbb{N}$, $R_{\max}^{(n)} \geq 1$. So we only need to prove that for any $\delta > 0$, there exists $N > 0$, such that for all $n > N, R_{\max}^{(n)} < 1 + \delta$. Now fix a $\delta > 0$.

Using the Schwarz inequality, we have $\|\varphi_{v,E}(\tau^{n+1}\alpha)\| \leq [R_{\max}^{(n)}]^{|\tau^{n+1}(\alpha)|}$.

By definition of τ, for any $\alpha \in \mathcal{A}$, $\tau(\alpha)$ contains word of the form aw_1a, where w_1 can be empty word. So $\tau^{n+l}(\alpha)$ contains $\tau^n(aw_1a)$ for any $l \geq 1$. By (3) we have

$$\begin{aligned} \varphi_{v,E}[\tau^n(aw_1a)] = \; & \varphi_{v,E}[\tau^n(a)]\varphi_{v,E}[\tau^n(w_1)]\varphi_{v,E}[\tau^n(a)] \\ = \; & \{\operatorname{tr} \varphi_{v,E}[\tau^n(aw_1)]\}\varphi_{v,E}[\tau^n(a)] + \varphi_{v,E}[\tau^n(w_1)] - \{\operatorname{tr} \varphi_{v,E}[\tau^n(w_1)]\}I. \end{aligned}$$

Since $x_{v,E} \notin \widetilde{\mathcal{U}}$, by Proposition 9, for both $u = aw_1$ and $u = w_1$, there exists N_0 such that for all $n > N_0$,

$$|\operatorname{tr} \varphi(\tau^n u)| \leq (1+\delta)^{\frac{1}{2}|\tau^n(u)|} \quad \text{and} \quad (1+\delta)^{\frac{1}{4}|\tau^n(a)|} > 3.$$

So for a sufficiently large N_1, and for all $n > N_1$

$$\begin{aligned}
\left\|\varphi_{v,E}(\tau^{n+l}(\alpha))\right\| \leq\ & \{|\operatorname{tr}\varphi_{v,E}[\tau^n(aw_1)]|\}[R^{(n)}_{\max}]^{|\tau^{n+l}\alpha|-|\tau^n(aw_1)|} + [R^{(n)}_{\max}]^{|\tau^{n+l}\alpha|-2|\tau^n(a)|} \\
& +\{\operatorname{tr}\varphi_{v,E}[\tau^n(w_1)]\}[R^{(n)}_{\max}]^{|\tau^{n+l}\alpha|-|\tau^n(aw_1a)|} \\
\leq\ & (1+\delta)^{\frac{1}{2}|\tau^n(aw_1)|}[R^{(n)}_{\max}]^{|\tau^{n+l}\alpha|-|\tau^n(aw_1)|} + [R^{(n)}_{\max}]^{|\tau^{n+l}\alpha|-2|\tau^n(a)|} \\
& +(1+\delta)^{\frac{1}{2}|\tau^n(w_1)|}[R^{(n)}_{\max}]^{|\tau^{n+l}\alpha|-|\tau^n(aw_1a)|} \\
\leq\ & 3[\max\{1+\delta, R^{(n)}_{\max}\}]^{|\tau^{n+l}\alpha|-\frac{1}{2}|\tau^n(a)|} \\
\leq\ & [\max\{1+\delta, R^{(n)}_{\max}\}]^{|\tau^{n+l}\alpha|-\frac{1}{4}|\tau^n(a)|} \\
\leq\ & [\max\{1+\delta, R^{(n)}_{\max}\}]^{|\tau^{n+l}\alpha|(1-\frac{1}{8}b\theta^{-l})},
\end{aligned}$$

where $b \geq 1$, $\theta > 1$, is determined by τ, since by the primitivity of τ, we have (refer to [15]),

$$\lim_{n\to\infty}\frac{\max_{\alpha\in\mathcal{A}}|\tau^n(\alpha)|}{\min_{\alpha\in\mathcal{A}}|\tau^n(\alpha)|} = b \geq 1, \quad \lim_{n\to\infty}\frac{|\tau^{n+1}(\alpha)|}{|\tau^n(\alpha)|} = \theta > 1, \quad \alpha\in\mathcal{A}.$$

We can choose $l > 1$ such that $0 < \frac{1}{8}b\theta^{-l} < 1$. So for any $n > N_1$, there exists $0 < c < 1$ such that

$$R^{(n+l)}_{\max} \leq [\max\{1+\delta, R^{(n)}_{\max}\}]^{1-c},$$

this implies that there exists $N > 0$, such that for all $n > N$,

$$R^{(n)}_{\max} \leq 1+\delta.$$

Now we prove $\gamma^+_{v,\omega}(E) = 0$.

For any $\varepsilon > 0$, choose n_0 such that

$$R^{(n_0)}_{\max} < (1+\varepsilon)^{\frac{1}{2}}$$

and let $M = \max\limits_{\alpha\in\mathcal{A}}|\tau^{n_0}(\alpha)|$ there exists large enough $N > 0$, such that

$$[R^{(0)}_{\max}]^{\frac{M}{N}} < (1+\varepsilon)^{\frac{1}{4}}$$

For any $n > N$. Since $\omega = \{a_i\}_{i=-\infty}^{\infty}$ is generated by τ, as in the proof of Lemma 8, there exist two integers $t \geq 0$ and $p \geq n_0$, such that $a_i = (\tau^p a)_{i+t}$, for any $1 \leq i \leq n$. So there exists integers r, s, l and a word $b_1b_2\cdots b_l$ such that

$$a_1a_2\cdots a_n = a_1a_2\cdots a_r\tau^{n_0}(b_1b_2\cdots b_l)a_{s+1}a_{s+2}\cdots a_n$$

where $r < M, n-s < M$ and $s = r + |\tau^{n_0}(b_1b_2\cdots b_l)|$. Then

$$\begin{aligned}
\|\varphi_{v,E}(v_1\cdots v_n)\|^{\frac{1}{n}} =\ & \|\varphi_{v,E}[a_1a_2\cdots a_r]\varphi_{v,E}[\tau^{n_0}(b_1b_2\cdots b_l)]\varphi_{v,E}[a_{s+1}a_{s+2}\cdots a_n]\| \\
\leq\ & [R^{(n_0)}_{\max}]^{\frac{s-r}{n}}[R^{(0)}_{\max}]^{\frac{2M}{N}} \\
<\ & 1+\varepsilon,
\end{aligned}$$

this implies $\gamma^+_{v,\omega}(E) = 0$.

In the same way we can prove that $\gamma^-_{v,\omega}(E) = 0$, so $\gamma_{v,\omega}(E) = 0$. □

Proposition 11. $\sigma(H_{v,\omega}) \subset \{E : \gamma_{v,\omega}(E) = 0\}$.

Proof By Proposition 10, we have $\mathcal{E}_v(\widetilde{\mathcal{U}})^c \subset \{E : \gamma_{v,\omega}(E) = 0\}$. On the other hand, by Lemma 8 and Corollary A4,

$$\sigma(H_{v,\omega}) \subset \mathcal{E}_v(\mathcal{U})^c \subset \mathcal{E}_v(\widetilde{\mathcal{U}})^c,$$

so that finally we have a chain of inclusions

$$\sigma(H_{v,\omega}) \subset \mathcal{E}_v(\mathcal{U})^c \subset \mathcal{E}_v(\widetilde{\mathcal{U}})^c \subset \{E : \gamma_{v,\omega}(E) = 0\}$$

□

5. Proof of Theorem 1

Before proving Theorem 1, we introduce two theorems due to Kotani [10] and Pastur [13].

Let μ be any probability measure on Ω such that T is ergodic. Thus (Ω, T, μ) is an ergodic dynamical system. Denote by $\gamma_{v,\omega}(E)$ the Lyapunov exponent corresponding to $H_{v,\omega}$. The ergodicity of the translation T implies that $\gamma_{v,\omega}(E)$ exists μ-almost surely and equals a constant, and $\gamma_{v,\mu}(E) := \int d\mu(\omega)\gamma_{v,\omega}(E)$. Note that

$$N_\mu := \{E|\gamma_{v,\mu}(E) = 0\}.$$

Kotani [10] showed the following :

Lemma A6. ([10]) *Let ω be a sequence taking a finite number of values, and let μ be an ergodic measure on its hull. Then the set N_μ has Lebesgue measure zero, unless the support of μ is a finite set.*

Theorem A7. ([13]) *Let μ be an ergodic measure on (Ω, T), then for any $\lambda \in \mathbb{R}$,*

$$\mu(\{\omega \in \Omega|\lambda \textit{ is an eigenvalue of } H_{v,\omega}\}) = 0.$$

Proof of Theorem 1. By Proposition 11, $\sigma(H_{v,\omega}) \subset \{E : \gamma_{v,\omega}(E) = 0\}$, and so

$$\sigma(H_{v,\omega}) \subset N_\mu.$$

Since (Ω, T) is strictly ergodic, Ω is not a finite set, and v is injective, $\sigma(H_{v,\omega})$ is of Lebesgue measure zero by Lemma A6. So the spectrum contains no interval.

Now to prove that $\sigma(H_{v,\omega})$ is an Cantor set, it suffices to show that it contains no isolated point. A proof of this result has been given in [19], and we will give another proof here.

In fact suppose λ is an isolated point of $\sigma(H_{v,\omega})$, then it is an eigenvalue of $H_{v,\omega'}$ for any $\omega' \in \Omega$. Then

$$\mu(\{\omega \in \Omega|\lambda \text{ is an eigenvalue of } H_{v,\omega}\}) = 0,$$

this contradicts Pastur's Theorem A7. □

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