Exponentially small splitting of separatrices for one and a half degrees of freedom Hamiltonian systems

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February 8, 2011

Abstract

In this paper we study the problem of exponentially small splitting of separatrices of one degree of freedom classical Hamiltonian systems with a non-autonomous perturbation which is fast and periodic in time. We provide the asymptotic formula for the measure of the splitting for both the so-called regular and singular cases. In the latter we show that Melnikov theory fails to predict correctly the first order of the splitting of separatrices.

1 Introduction

In this paper we consider the familiy of Hamiltonian systems of the form

$$H\left(x,y,\frac{t}{\varepsilon};\varepsilon\right) = H_0(x,y) + \mu\varepsilon^\eta H_1\left(x,y,\frac{t}{\varepsilon};\varepsilon\right) \tag{1}$$

where $H_0(x,y)$ is given by a classical Hamiltonian

$$H_0(x,y) = \frac{y^2}{2} + V(x)$$

and $H_1(x,y,\tau;\varepsilon)$ is a 2π-periodic time dependent Hamiltonian with zero average. We assume $H(x,y,\tau;\varepsilon)$ is analytic and we study the problem of the splitting of separatrices. The parameter ε is a small parameter but this is not the case for μ, which is considered a parameter of order one. The results in this paper are valid not only for μ small, but also for finite values of μ, generically for $\mu = 1$. We will see that the results are different depending on the other parameter $\eta \geq 0$, which appears in (1), and on the analytic properties of H.

When $\mu\varepsilon^\eta$ is small, the Hamiltonian system associated to H is a small perturbation of the Hamiltonian system associated to H_0:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -V'(x). \end{aligned} \tag{2}$$

We assume that this system has a hyperbolic or parabolic critical point at the origin with stable and unstable manifolds which coincide along a separatrix $(q_0(u), p_0(u))$. The coincidence of the stable and unstable invariant manifolds is not a generic phenomenon even for Hamiltonian systems of one and half degrees of freedom as (1). Therefore, one can expect that the homoclinic connection of (2) breaks down when we add the non-autonomous part to the system. Nevertheless, the symplectic structure ensures the existence of intersections between the perturbed invariant manifolds. The main goal of this paper is to provide an asymptotic formula for this splitting.

Our first observation is that, being the Hamiltonian H fast in time, averaging theory [AKN88, LM88] tells us that, even for $\mu\varepsilon^\eta = \mathcal{O}(1)$, the solutions of the Hamiltonian system associated to (1) are close to the solutions of (2). Therefore, the problem of measuring the splitting of separatrices can be considered, in general, for $\eta \geq 0$.

Due to the $2\pi\varepsilon$-periodicity of the Hamiltonian H it is convenient to consider the Poincaré map P_{t_0} defined in a Poincaré section $\Sigma_{t_0} = \{(x, y, t_0);\ (x, y) \in \mathbb{R}^2\}$. If $\mu = 0$, the phase portrait of P_{t_0} is given by the level curves of the Hamiltonian $H_0(x, y) = \frac{y^2}{2} + V(x)$. The origin is a hyperbolic or parabolic fixed point of P_{t_0} with stable and unstable curves given by the homoclinic connection $(q_0(u), p_0(u))$.

In the hyperbolic case, a classical result of averaging theory [AKN88, LM88] is that, for ε small enough, there exists a hyperbolic fixed point of P_{t_0} corresponding to a hyperbolic periodic orbit of H, and their stable and unstable invariant curves $C^s(t_0)$ and $C^u(t_0)$. These curves lie near the unperturbed separatrix. In the parabolic case our (standard) hypotheses will ensure that the origin will still be a fixed point with similar properties.

Generically the curves $C^s(t_0)$ and $C^u(t_0)$ intersect giving rise to some homoclinic points z_h. If this intersection is transversal at z_h, the curves enclose lobes whose area $\mathcal{A}$ does not depend on the homoclinic point we have chosen (see Figure 1). The measure of this area in terms of ε is the main purpose of this paper. Another quantity that can be used at homoclinic points to measure the transversality of the intersection is the angle between the curves $C^s(t_0)$ and $C^u(t_0)$, but this quantity depends on the chosen homoclinic point. An invariant quantity related to the angle is the so-called *Lazutkin invariant* (see, for instance [GLT91]).

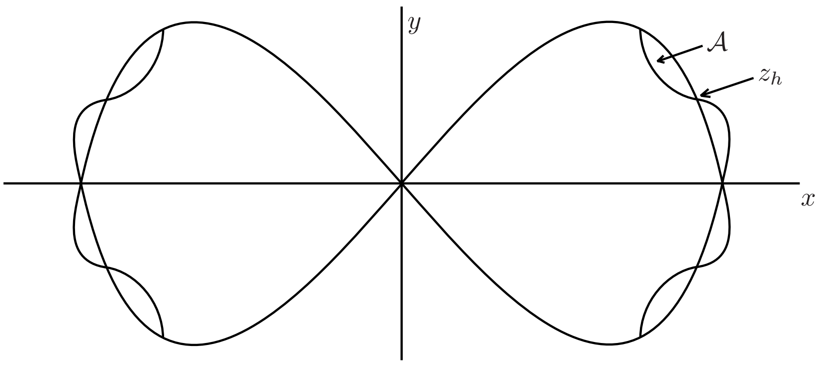

Figure 1: Splitting of separatrices.

Classical perturbation theory applied to our problem provides the so-called Melnikov potential (called also sometimes Poincaré Function, see for instance [DG00]), which is given by

$$L(t_0) = \int_{-\infty}^{+\infty} H_1\left(q_0(u), p_0(u), \varepsilon^{-1}(t_0 + u); 0\right) du.$$

Using this function, Poincaré, and later Melnikov [Mel63], proved that, if $\mu\varepsilon^\eta$ is small enough, non-degenerate critical points of L give rise to transversal intersections between the invariant curves $C^s(t_0)$ and $C^u(t_0)$, and the area between the lobes is given asymptotically by $L(t_0^1) - L(t_0^2)$, being t_0^1 and t_0^2 two consecutive critical points of L. In our setting, where $H_0(x, y)$ and $H_1(x, y, \tau; 0)$ will be either algebraic or trigonometric polynomials in (x, y), the Poincaré function L is asymptotically given by:

$$L(t_0) \simeq \varepsilon^\beta e^{-a/\varepsilon} g\left(\frac{t_0}{\varepsilon}\right), \qquad \varepsilon \to 0 \tag{3}$$

being $g(s)$ a 2π-periodic function, and $a > 0$, and β some computable constants. The constant a is independent of the perturbation: it turns out that the time parameterization of the unperturbed separatrix has always singularities in the complex plane (see [Fon95, BF04]) and the constant a is nothing but the imaginary part of the singularity closest to the real axis.

The straighforward application of Melnikov method to our Hamiltonian provides a formula for the area which reads:

$$\mathcal{A} \simeq \mu\varepsilon^\eta \left(K\varepsilon^\beta e^{-a/\varepsilon} + \mathcal{O}(\mu\varepsilon^\eta)\right), \quad \varepsilon \to 0 \tag{4}$$

with $K > 0$, a computable constant. Therefore, Melnikov potential is exponentially small in ε and a direct application of classical perturbation theory only ensures the validity of such an approximation if $\mu \ll e^{-a/\varepsilon}$.

We want to emphasize that the asymptotic size with respect to ε of the Melnikov potential is given by (3) provided $H_0(x, y)$ and $H_1(x, y, \tau; 0)$ are either algebraic or algebraic in y and trigonometric polynomials in x. The study of the Melnikov potential for general analytic Hamiltonian systems with fast periodic perturbations has not been previously done. This study strongly depends on the analyticity properties of the Hamiltonian H and therefore, even if the Poincaré function can be estimated for some concrete systems [LS80, MP94, SMH91], a general study of this function seems to require more powerful analytic tools. Even if to compute the first asymptotic order for general Hamiltonian systems seems nowadays a problem out of reach, to obtain (non-sharp) exponentially small upper bounds was already achieved by Neishtadt in [Neĭ84] using averaging techniques and by [FS90] using complex extensions of the invariant manifolds.

Once we know that the splitting is exponentially small, a natural question which arises is whether the Melnikov potential gives the correct asymptotic first order of the splitting. In comparison with the problem of giving exponentially small upper bounds for the splitting, this problem is much more intricate. The results in this direction strongly depend on the behavior of the homoclinic orbit $(q_0(u), p_0(u))$ around its complex singularities and on the analytical properties of the perturbation.

As we have already explained, inspecting formula (3), one sees that Melnikov theory works provided $\mu\varepsilon^\eta = o(\varepsilon^\beta e^{-a/\varepsilon})$. Namely, one needs the size of the perturbation to be exponentially small with respect to ε. This is not the natural setting and therefore the first works dealing with this problem tried to enlarge the size of the perturbation $\mu\varepsilon^\eta H_1$ for which Melnikov theory actually measures the splitting. In fact, under certain non-degeneracy conditions, it suffices to take η big enough and μ of order 1.

In other words, all the results validating the prediction of the Melnikov approach require some artificial condition about the smallness of the perturbation. The reason, roughly speaking, is the following. To prove that Melnikov theory gives asymptotically the first order of the splitting one needs to perform "complex perturbation theory". Namely, one looks for complex parameterizations $Z_\mu^{u,s}(u, t_0)$ of the perturbed invariant curves $C^{u,s}(t_0)$ of the Poincaré map P_{t_0} as a perturbation of the time-parameterization of the unperturbed separatrix $Z_0(u) = (q_0(u), p_0(u))$. One needs to prove that both perturbed parameterizations are close enough to the unperturbed one not only for real values of time u, as classical perturbation theory requires, but also in some complex domains. The main novelty in the proofs of exponentially small splitting, that was discovered by Lazutkin in his pioneer paper [Laz84], is that the perturbed and unperturbed manifolds, as well as the solutions of the variational equations along them, need to be close enough when one considers complex times in a domain which contains a real interval and which reaches a neighborhood of order ε of the singularities of the unperturbed homoclinic orbit. Clearly, when time is real, the homoclinic orbit is a bounded solution and it is easy to see that the perturbed invariant manifolds are close to it. However, when we reach a neighborhood of its singularities, the homoclinic itself blows up, and it is not clear that the perturbed invariant manifolds are close to it anymore. Of course assuming artificially that the perturbation is small enough (adding extra powers of η in the perturbative term) one can see that the manifolds are close to the unperturbed homoclinic in a complex domain which reaches a neighborhood of size ε of the singularities of the unperturbed homoclinic trajectory. Consequently the Melnikov approach, that is based on the fact that the perturbed manifolds are well approximated by the unperturbed homoclinic, still works. This was the approach used in [DS97, Gel97a, BF04] for $\eta > \ell$. The constant ℓ was called the *order* of the perturbation H_1. Roughly speaking, it is the order of the singularities of the unperturbed homoclinic trajectory $(q_0(u), p_0(u))$ closest to the real axis of the function $f(u) = H_1(q_0(u), p_0(u), t/\varepsilon; 0)$, for any $t \in \mathbb{R}$.

In the aforementioned works, the condition $\eta > \ell$ ensures that the perturbed parameterizations $Z_\mu^{u.s}$ are close to the parameterization of the unperturbed separatrix Z_0 even at a distance of order ε of the singularities of Z_0 closest to the real axis. Nevertheless, as we will see in this paper, the condition $\eta > \ell$ is sufficient but not necessary to ensure that Melnikov approach still predicts correctly the size of the splitting. What is important is the relative size between the homoclinic orbit Z_0 and the difference between the homoclinic and the perturbed manifolds, and analogously between the solutions of the corresponding variational equations. In other words, as the parameterizations of the invariant manifolds can be written as $Z_\mu^{u,s} = Z_0 + (Z_\mu^{u,s} - Z_0)$, the Melnikov method gives the correct asymptotic term for the size of the splitting provided the homoclinic Z_0 is bigger than the difference $Z_\mu^{u,s} - Z_0$. Call r to the order of the singularities of $p_0(u)$ closest to the real axis. Then, the size of $p_0(u)$ at points u which are ε-close to the singularities is $\mathcal{O}(\varepsilon^{-r})$. Looking at the relative size of $\mathrm{grad} H_0(q_0(u), p_0(u))$ and $\mu\varepsilon^\eta \mathrm{grad} H_1(q_0(u), p_0(u), \tau; \varepsilon)$, one can guess that the first one is strictly bigger than the second if $\eta - (\ell - r) > -r$. This new approach allows us to prove in this paper that $Z_0(u)$ is strictly bigger than $Z_\mu^{u,s}(u, t_0) - Z_0(u)$ provided $\eta > \ell - 2r$, even if u is at a distance ε of the singularity. For $\ell \geq 2r$, the condition for the parameterizations and for the solutions of the variational equations to be close coincide and is given $\eta > \eta^* = \ell - 2r$.

For $\ell < 2r$ we will not reach values of η such that $\ell - 2r < \eta < 0$. In fact, η will reach first the "natural" limit $\eta = 0$, where the partial derivatives of $H_0(x, y)$ and $H_1(x, y, \tau; \varepsilon)$ evaluated on the unperturbed homoclinic are not close even for real values. Even if for concrete examples [GOS10, Gel00] one can prove the existence of invariant manifolds and compute the size of their splitting for negative values of η, in this paper we deal with general Hamiltonians and $\eta \geq 0$. This implies that the unperturbed system and the perturbation can have the same size (see Section 1.1 for some problems which correspond to this natural limit $\eta = 0$). When $\eta = 0$, one can apply classical averaging theory to see that we are still in a perturbative setting and the perturbed real invariant manifolds are $\mu\varepsilon$-close to the real unperturbed separatrix. Nevertheless, as we will see in this paper, the solutions of the variational equations are not close enough in this case. This implies that, as is stated in Theorem 2.5, Melnikov formula does not give the correct first asymptotic term of the splitting generically.

In conclusion, (under certain non-degeneracy conditions) Melnikov theory gives the correct prediction provided

$$\eta > \eta^* = \max\{\ell - 2r, 0\}.$$

The so called "singular" case occurs when the difference $Z_\mu^{u,s}(u, t_0) - Z_0(u)$ has the same size as the unperturbed homoclinic $Z_0(u)$ when u reaches a neighborhood at a distance ε of the singularities of Z_0. Consequently, the invariant manifolds are not well approximated by the unperturbed homoclinic in this complex region. Let us note that this singular case can only happen if $\ell \geq 2r$ and $\eta = \eta^*$. In this case, we need to obtain a different approximation of the manifolds in this region of the complex plane. Close to the singularities of the homoclinic orbit, an equation for the leading term is obtained and it is called the *inner equation*. This is a non-integrable equation without parameters, which needs a deep study.

Summarizing, on the one hand, the invariant manifolds are well approximated by the unperturbed homoclinic in a complex region containg the real line. On the other hand, the inner equation provides a good approximation near the singularities $\pm ia$. Finally, matching techniques are required to match the different approximations obtained for the invariant manifolds. Roughly speaking, the difference between the solutions of the inner equation replaces the Melnikov potential in the asymptotic formula for the splitting.

The first author who dealt with this singular case was Lazutkin in [Laz84, Laz03]. He studied the splitting of separatrices of the Chirikov standard map, and gave the main idea that inspired all the works in the subject: as we explained above, one needs to deal with suitable complex parameterizations of the invariant manifolds. These parameterizations are analytic in a complex strip, whose size is limited by the singularities of the unperturbed homoclinic orbit. A complete proof was published years later by Gelfreich in [Gel99]. A fundamental tool in Lazutkin's work is the use of "flow box coordinates", called "straightening the flow" in [Gel00], around one of the manifolds. In this way, one obtains a periodic function whose values are related with the distance between the manifolds and whose zeros correspond to the intersections between them. Consequently, the result about exponentially small splitting is derived from some properties of analytic periodic functions bounded in complex strips (see, for instance, Proposition 2.7 in [DS97]).

In the papers [Sau01, LMS03] the authors introduced a different approach that avoided the "flow-box coordinates" of Lazutkin's method. The authors worked with the original variables of the problem and were able to measure the distance between the manifolds without using "flow box coordinates". The idea was the following: being both manifolds given by the graphs of suitable functions that are solutions of the same equation, their difference satisfies a linear equation and is bounded in some complex strip.

Studying the properties of bounded solutions of this linear equation, where periodicity also plays a role, one obtains exponentially small results.

The method in [Sau01, LMS03] uses the fact that, in the considered systems, the manifolds can be written as graphs of the gradient of generating functions in suitable domains. These generating functions are solutions of the Hamilton-Jacobi equation associated to system (1). Solving these partial differential equations one can obtain parameterizations of the global manifolds. Even if in this paper we deal with general Hamiltonian systems and then the invariant manifolds may not be graphs globally, we have adapted the method in [Sau01, LMS03]. We work with the Hamilton-Jacobi equation in suitable domains where the manifolds are given by graphs and then we measure the splitting there.

We want to emphasize that, as far as the authors know, there are no general results dealing with the singular case. The previous results in the singular case (see [Gel00, Tre97, Oli06, GOS10]) only dealt with particular examples.

In this paper we give results that contain the so-called regular case $\eta > \eta^*$ (see Section 2.1), in which the Melnikov formula predicts correctly the difference between the manifolds, but we also consider the so-called singular case $\eta = \eta^*$, in which the Melnikov formula does not predict correctly anymore the difference between the perturbed manifolds and one has to consider an alternative formula.

Studying the phenomenon of splitting in general Hamiltonian systems we have found examples where the Melnikov theory does not predict correctly the formula for the area of the lobes (4) in several aspects. In section 2.2.4 we will see models where the constant K is not correctly given by the Melnikov formula, as it happens in some models studied before [Gel00, Tre97, Oli06, GOS10]. But we will also encounter examples where the Melnikov formula (4) neither predicts this constant nor the correct power β in (4).

We have seen that the behavior of the splitting is extremely sensitive on the sign of $\ell - 2r$ and the value of η. In consequence, one has to obtain the different first asymptotic orders separately, taking into account the properties of each case. We summarize the main particularities of each case:

- $\eta > \eta^* = \max(\ell - 2r, 0)$: under certain non-degeneracy conditions, the Melnikov formula (4) gives the correct first order, that is, the correct constants K, β and a.

- $\ell - 2r < 0$ and $\eta = 0$: there appears a correcting term which replaces K in the Melnikov formula. This term can be obtained through classical perturbation theory techniques. Since it never vanishes, the first asymptotic order is non-degenerate if and only if neither is the Melnikov potential. Note that in this case, for real values of the variables H is not a perturbation of H_0.

- $\ell - 2r > 0$ and $\eta = \eta^* = \ell - 2r$: as in the previous case, the Melnikov formula (4) fails to predict correctly the constant K. Nevertheless, the correcting term has a significantly different origin, since it comes from the study of the aforementioned *inner equation*. Let us note that the range $\eta \in [0, \ell - 2r)$ remains open.

- $\ell - 2r = 0$ and $\eta = 0$: as in the previous case, we need to consider an *inner equation* to obtain a candidate for the first asymptotic order. This candidate differs from the Melnikov formula by both the constant K and the exponent β. Nevertheless, to obtain the true first order, one has to make still one more modification to the formula given by the inner equation, which implies a change in the constant provided by the inner equation. Note, that the change in the exponent β is a substantial qualitative change in the behavior of the splitting. Even if this fact was already pointed out in [Bal06], the present paper, as far as the authors know, is the first work that rigorously proves that this phenomenon actually happens.

The structure of this paper goes as follows. First in Section 2 we introduce some notation, the hypotheses and we state the main results. In Section 3 we give some heuristic ideas of the proof and we compare our methods to those of some of the aforementioned previous results. Section 4 is devoted to describe the proof of the main theorem. To make this section more readable, the proof of the partial results obtained in this section are deferred to the following sections, that is, Sections 5-9.

1.1 Motivation of the problem

Throughout the paper we will use the classical notation $H_0 + \mu\varepsilon^\eta H_1$ which encounters the singular case $\eta = \eta^*$, where the value of η^* depends on the properties of H_0 and H_1. For general perturbations H_1 of general classical Hamiltonian systems $H_0(x, y) = \frac{y^2}{2} + V(x)$ this work finishes the general problem, initiated and partially solved in [DS97, Gel97a, BF04, BF05] for $\eta > \ell$, of the splitting of separatrices in the singular and regular cases $\eta \geq \eta^*$.

Nevertheless we want to stress that it is of particular interest the non-perturbative case $\mu = 1$ and $\eta = 0$. For instance, one can consider models of the form

$$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \widetilde{V}\left(x, \frac{t}{\varepsilon}\right) \tag{5}$$

taking $V(x) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{V}(x, \tau)d\tau$ and $H_1(x, y, \tau) = \widetilde{V}(x, \tau) - V(x)$. In this case, under certain generic hypotheses about V, which are specified in Section 2.1, our result in Theorem 2.6 provides a formula for the splitting even for $\eta^* = 0$.

Hamiltonian (5) is a particular case of periodic time dependent Hamiltonians $H(x, y, t/\varepsilon)$. After the change of time $t = \varepsilon\tau$, these systems become time dependent Hamiltonian systems with slow dynamics and classical averaging theory can be applied to them. So it is natural to split

$$H\left(x, y, \frac{t}{\varepsilon}\right) = H_0(x, y) + H_1\left(x, y, \frac{t}{\varepsilon}\right)$$

with $H_0(x, y) = \frac{1}{2\pi}\int_0^{2\pi} H(x, y, \tau)\, d\tau$ and $H_1(x, y, \tau) = H(x, y, \tau) - H_0(x, y)$.

Averaging theory tells us that, even if in this model $\eta = 0$, the solutions of the whole Hamiltonian system associated to H can be approximated, up to long times, by the solutions of H_0. Moreover, in the case that $H_0(x, y)$ has a hyperbolic critical point the stable and unstable invariant manifolds of H are ε-close to the stable and unstable manifolds of H_0. It is then natural to study if the homoclinic orbit of the Hamiltonian system (2) splits and to give an asymptotic formula, for ε small enough, of this splitting. Our result in Theorem 2.6 gives the splitting of these manifolds in this case for the value $\eta^* = 0$ for a wide class of Hamiltonians.

One also encounters the singular case for $\eta^* = 0$, when one studies the splitting of separatrices phenomenon in a resonance of one and a half degrees of freedom Hamiltonian systems which are close to completely integrable ones (in the sense of Liouville-Arnold). This setting does not fit exactly in our hypotheses but, as we will see in a forthcoming paper, the methods used in this paper can be easily adapted to that case (see section 2.3).

1.2 Historical remarks and related problems

Historically, the results about exponentially small splitting of separatrices can be classified into three groups: upper bounds, validation of the Melnikov approach and asymptotics for the singular case.

Some results, dealing with quite general systems, which obtain exponentially small upper bounds for the splitting for Hamiltonian systems. Neishtadt in [Neĭ84] gave exponentially small upper bounds for the splitting for two degrees of freedom Hamiltonian systems. For second order equations with a rapidly forced periodic term, several authors gave sharp exponentially small upper bounds in [Fon93, Fon95, FS96] and, for the higher dimensional case, the papers [Sau01, Sim94] gave (non-sharp) exponentially small upper bounds.

The Poincaré map of a non-autonomous Hamiltonian in the plane is a particular case of a planar area preserving map. For the Hamiltonian (1) the Poincaré map P is a near the identity area preserving map. Rigorous upper bounds for the splitting of area preserving maps close to the identity were given in [FS90].

The second group of results is concerned with the question about the validity of the asymptotics provided by the Melnikov theory. Several authors in the last 15 years have tried to ensure the validity of the formula provided by the Melnikov potential (3) to compute the asymptotic formula for the area $\mathcal{A}$.

The results in this direction strongly depend on the behavior of the homoclinic orbit around its complex singularities and on the analytical properties of the perturbation. For this reason, the existing results in this direction mostly deal with specific examples.

The most studied example in the literature has been the rapidly perturbed pendulum with a perturbation only depending on time,

$$\ddot{x} = \sin x + \mu\varepsilon^\eta \sin\frac{t}{\varepsilon},$$

which in our notation corresponds to $H_0(x, y) = y^2/2 + \cos x - 1$ and $H_1(x, t/\varepsilon) = -x\sin(t/\varepsilon)$. The first result concerning this system was obtained by Holmes, Marsden and Scheurle in [HMS88] (followed by [Sch89, Ang93]), where they confirmed the prediction of the Melnikov potential establishing exponentially small upper and lower bounds for the area $\mathcal{A}$ provided $\eta \geq 8$, which coincide with the Melnikov prediction. Later the work [EKS93] validated the same result for $\eta \geq 3$. Delshams and Seara established rigourosly the result in [DS92] for $\eta > 0$ and an analogous result for $\eta > 5$ was obtained by Gelfreich in [Gel94]. The latter two papers used a different approach inspired by [GLT91]. For a simplified perturbation in [Sau95] an alternative proof, using Parametric Resurgence, was done.

The only works which provide (partial) results for some general Hamiltonian as (1) taking η big enough, are [DS97, Gel97a, BF04, BF05]. In [DS97, Gel97a], a proof for the validity of the Melnikov method for general rapidly periodic Hamiltonian perturbations of a class of second order equations was given. The case of a perturbed second order equation with a parabolic point was studied in [BF04, BF05].

A Melnikov theory for twist maps can be found in [DR97] and some results about the validity of the prediction given by the Poincaré function for area preserving maps were given in [DR98].

The generalization of the splitting problem to higher dimensional systems has been achieved by several authors, mainly in the Hamiltonian case. See, for instance, [Eli94, Tre94, LMS03, DG00] and references therein. Some results about the validity of the Melnikov method for higher dimensional Hamiltonian systems can be found in [Gal94, CG94, DGJS97, GGM99, Sau01, DGS04]. Finally, in a non Hamiltonian setting, in [BS06] the splitting of a heteroclinic orbit for some degenerate unfoldings of the Hopf-zero singularity of vector fields in $\mathbb{R}^3$ was found.

As we already explained, all the results validating the prediction of the Melnikov approach require some artificial condition about the smallness of the perturbation. The third group of results deals with the so called "singular case" $\eta = \eta^*$ for which one needs to study the *inner equation* and use matching techniques to relate different approximations for the invariant manifolds.

The first result about splitting of separatrices in the singular case was done by V. Lazutkin in [Laz84] (see also [Laz03]) who gave a formula for the splitting for the Chirikov standard map (which can be reduced to a near the identity map through a simple scaling). In our notation, this case corresponds to $\eta^* = 0$,

After Lazutkin's work, some authors used his method and obtained results for the inner equation of several specific equations. In [GS01] there is a rigorous study of the inner equation of the Hénon map using Resurgence Theory [Éca81a, Éca81b], and in [BS08] the authors studied the inner system associated to the Hopf-zero singularity using functional analysis techniques. The corresponding inner equation for several periodically perturbed second order equations was given by Gelfreich in [Gel97b] and he called them Reference Systems. In [OSS03] there is a rigorous analysis of the inner equation for the Hamilton-Jacobi equation associated to a pendulum equation with perturbation term $H_1(x, t/\varepsilon) = (\cos x - 1)\sin(t/\varepsilon)$ by using Resurgence Theory. The only result which deals with the inner equation associated to general polynomial Hamiltonian like (1) is [Bal06], where this analysis is done using functional analysis techniques. Finally, in [MSS10b], the authors study the inner equation of the McMillan Map.

Besides the work of Lazutkin, there are very few works with rigorous proofs in the singular case. In [Gel00] there is a detailed sketch of the proof for the splitting of separatrices for the equation of a pendulum with perturbation $H_1(x, t/\varepsilon) = x\sin(t/\varepsilon)$ and $\eta^* = -2$. A complete rigorous proof which also cover some "under the limit" cases, that is $\eta < \eta^* = -2$ is done in [GOS10]. Numerical results about the splitting for this problem can be found in [BO93, Gel97b]. In [Oli06] it was obtained a rigorous proof for the pendulum with perturbation $H_1(x, t/\varepsilon) = (\cos x - 1)\sin(t/\varepsilon)$, for which $\eta^* = 0$. Treschev gave in [Tre97] an asymptotic formula for the splitting in the case of a pendulum with certain perturbations, for which $\eta^* = 0$, using a different method called Continuous Averaging. Concerning 2-

dimensional symplectic maps, the study of the splitting for the Hénon and McMillan maps have recently been completed in [BG10] and [MSS10a] respectively. In both cases, $\eta^* = 0$. Finally, in [GG10], combining numerical and analytical techniques, the authors study the Hamiltonian-Hopf bifurcation.

Another work dealing with a singular case is [Lom00], where the author proved the splitting of separatrices for a certain class of reversible systems in $\mathbb{R}^4$. A related problem about adiabatic invariants in the harmonic oscillator using matching techniques and Resurgence Theory was done in [BSSV98].

2 Notation and main results

In this section we present the main problem we consider, the hypotheses we assume and the rigorous statement of the main results.

2.1 Notation and hypotheses

We consider Hamiltonian systems with Hamiltonian function of the form

$$H\left(x, y, \frac{t}{\varepsilon}\right) = H_0(x, y) + \mu\varepsilon^\eta H_1\left(x, y, \frac{t}{\varepsilon}\right), \tag{6}$$

where

$$H_0(x, y) = \frac{y^2}{2} + V(x) \tag{7}$$

and V is either a polynomial or a trigonometric polynomial. In the first case we assume that

$$H_1(x, y, \tau) = \sum_{k+l=n}^{N} a_{kl}(\tau) x^k y^l \tag{8}$$

and in the second one

$$H_1(x, y, \tau) = a(\tau)x + \sum_{\substack{k=-N,\dots,N\\ l=0,\dots,N}} a_{kl}(\tau) e^{kix} y^l = \sum_{i+j\geq n} \widehat{a}_{ij}(\tau) x^i y^j, \tag{9}$$

where the second equality defines n and $\widehat{a}_{ij}$. Even if in the second case H_1 can have terms of the form $a(\tau)x$, we will refer to H_1 as a trigonometric polynomial. In both cases we will refer to n as the order of H_1.

The equations associated to the Hamiltonian (6) are

$$\begin{cases} \dot{x} = y + \mu\varepsilon^\eta \partial_y H_1\left(x, y, \frac{t}{\varepsilon}\right) \\ \dot{y} = -V'(x) - \mu\varepsilon^\eta \partial_x H_1\left(x, y, \frac{t}{\varepsilon}\right). \end{cases} \tag{10}$$

From now on, we call unperturbed system to the system defined by the Hamiltonian H_0 and we refer to H_1 as the perturbation. Let us observe that the term $a(\tau)x$ in (9) corresponds to a term in (10) which only depends on time.

We devote the rest of the section to state the hypotheses we assume on H.

2.1.1 Hypotheses on the unperturbed system

We assume the following hypotheses corresponding to the unperturbed system

HP1 $H_0(x, y) = y^2/2 + V(x)$, where V is either a polynomial or a trigonometric polynomial and satisfies one of the following conditions

HP1.1 H_0 has a hyperbolic critical point at $(0,0)$ with eigenvalues $\{\lambda, -\lambda\}$ with $\lambda > 0$, and then

$$V(x) = -\frac{\lambda^2}{2}x^2 + \mathcal{O}\left(x^3\right) \qquad \text{as } x \to 0.$$

HP1.2 H_0 has a parabolic critical point at $(0,0)$ and then

$$V(x) = v_m x^m + \mathcal{O}\left(x^{m+1}\right) \qquad \text{as } x \to 0, \tag{11}$$

for certain $m \in \mathbb{N}$, $m \geq 3$, which is called the order of V and $v_m \in \mathbb{R}$.

HP2 The critical point $(0,0)$ has stable and unstable invariant manifolds which coincide along a separatrix.

We denote by $(q_0(u), p_0(u))$ a real-analytic time parametrization of the separatrix with some chosen (fixed) initial condition. It is well known (see [Fon95] for the hyperbolic case and [BF04] for the parabolic one) that there exists $\rho > 0$ such that the parametrization $(q_0(u), p_0(u))$ is analytic in the complex strip $\{|\mathrm{Im}\, u| < \rho\}$.

We assume that there exists a real-analytic time parametrization of the separatrix $(q_0(u), p_0(u))$ analytic on $\{|\mathrm{Im}\, u| < a\}$ such that the only singularities of $(q_0(u), p_0(u))$ in the lines $\{\mathrm{Im}\, u = \pm a\}$ are $\pm ia$.

More precisely, Hypothesis **HP2** implies that one of the two following situations is satisfied (see the remarks in Section 2.1.3):

HP2.1 In the polynomial case, the singularities $\pm ia$ of the homoclinic orbit are branching points (or poles) of the same order, i.e. there exists an irreducible rational number $r = \alpha/\beta > 1$ (independent of the singularity) and $\nu > 0$ such that $(q_0(u), p_0(u))$ can be expressed as

$$\begin{aligned} q_0(u) &= -\frac{C_\pm}{(r-1)(u \mp ia)^{r-1}}\left(1 + \mathcal{O}\left((u \mp ia)^{1/\beta}\right)\right) \\ p_0(u) &= \frac{C_\pm}{(u \mp ia)^r}\left(1 + \mathcal{O}\left((u \mp ia)^{1/\beta}\right)\right) \end{aligned} \tag{12}$$

for $u \in \mathbb{C}$ and either $|u - ia| < \nu$ and $\arg(u - ia) \in (-3\pi/2, \pi/2)$ or $|u + ia| < \nu$ and $\arg(u + ia) \in (-\pi/2, 3\pi/2)$ respectively. Let us point out that the real-analytic character of $(q_0(u), p_0(u))$ implies that $C_- = \overline{C_+}$.

HP2.2 In the trigonometric case, $q_0(u)$ has logarithmic singularities at $\pm ia$ of the form $q_0(u) \sim \ln(u \mp ia)$ (where we take different branches of the logarithm whether we are close to $+ia$ or $-ia$: we take $\arg(u - ia) \in (-3\pi/2, \pi/2)$ and $\arg(u + ia) \in (-\pi/2, 3\pi/2)$ respectively). In this case, one can see that there exists $M \in \mathbb{N}$ such that, if $u \in \mathbb{C}$, $|u \mp ia| < \nu$,

$$\begin{aligned} \cos(q_0(u)) &= \frac{\widehat{C}^1_\pm}{(u \mp ia)^{2/M}}\left(1 + \mathcal{O}\left((u \mp ia)^{2/M}\right)\right) \\ \sin(q_0(u)) &= \frac{\widehat{C}^2_\pm}{(u \mp ia)^{2/M}}\left(1 + \mathcal{O}\left((u \mp ia)^{2/M}\right)\right) \\ p_0(u) &= \frac{C_\pm}{(u \mp ia)}\left(1 + \mathcal{O}\left((u \mp ia)^{2/M}\right)\right) \end{aligned} \tag{13}$$

with $\arg(u - ia) \in (-3\pi/2, \pi/2)$ and $\arg(u + ia) \in (-\pi/2, 3\pi/2)$ if we are dealing with the singularity $+ia$ or $-ia$ respectively. We also have that $C_+ = \overline{C_-} = \pm i2/M$.

For convenience, in the trigonometric case, we take the convention $r = 1$ and $\beta = M$.

2.1.2 Hypotheses on the perturbation

HP3 The function $H_1(x,y,\tau)$ is real-analytic in $(x,y,\tau)\in\mathbb{C}^2\times\mathbb{T}$ and 2π-periodic in τ. Furthermore, either it is a polynomial of the form (8) if $V(x)$ is a polynomial or it is a trigonometric polynomial of the form (9) if $V(x)$ is a trigonometric polynomial. Moreover, it has zero mean

$$\int_0^{2\pi} H_1(x,y,\tau)\,d\tau = 0.$$

HP4 Let us consider the order of H_1, n given in (8) or (9). We ask H_1 to satisfy:

HP4.1 In the hyperbolic case (H_0 satisfies **HP1.1**), $n\geq 1$.

HP4.2 In the parabolic case (H_0 satisfies **HP1.2**), $2n-2\geq m$.

Remark 2.1. *Let us point out that, in fact,* ***HP4.1*** *does not add any extra hypothesis on the Hamiltonian, since it can always be taken with $n\geq 1$ (the constant terms in (x,y) do not play any role).*

Let us consider the function $H_1(q_0(u),p_0(u),\tau)$ that is: H_1 evaluated on the separatrix. Then, we define ℓ to be the order of the branching points $\pm ia$, namely, the maximum of the orders of the branching points of the monomials of H_1. This parameter was already defined in [DS97, BF04]. Let us point out that ℓ can be simply defined as

$$\begin{aligned} \ell &= \max_{n\leq k+l\leq N}\{k(r-1)+lr; a_{kl}(\tau)\not\equiv 0\} \qquad \text{(polynomial case)}\\ \ell &= \max_{|k|\leq N,\ 0\leq l\leq N}\{2|k|/M+l; a_{kl}(\tau)\not\equiv 0\} \qquad \text{(trigonometric case).}\end{aligned} \tag{14}$$

Note that in the trigonometric case, if $H_1(x,y,\tau)=a(\tau)x$, then $H_1(q_0(u),p_0(u),\tau)$ has a logarithmic singularity (see Hypothesis **HP2.2**). In this case we make the convention $\ell=0$.

HP5 We assume $\eta\geq\eta^*=\max\{0,\ell-2r\}$.

In the case $\ell<2r$, we still have to assume a generic extra hypothesis. By the definition of ℓ in (14), we already know that $H_1(q_0(u),p_0(u),\tau)$ has branching points of degree ℓ at $u=\pm ia$. Therefore, if we consider the Fourier expansion

$$H_1(q_0(u),p_0(u),\tau)=\sum_{k\in\mathbb{Z}\setminus\{0\}} H_1^{[k]}(q_0(u),p_0(u))e^{ik\tau},$$

all the Fourier coefficients $H_1^{[k]}(q_0(u),p_0(u))$ have at $u=\pm ia$ a branching point of order less than or equal to ℓ. Next hypothesis, which will be only used in the case $\ell-2r<0$, requires a slightly stronger condition on the Fourier coefficients $H_1^{[\pm1]}(q_0(u),p_0(u))$, namely, that these coefficients also have these branchings points $u=\pm ia$ of order ℓ and not less.

HP6 If $\ell<2r$, the Fourier coefficients $H_1^{[\pm1]}(q_0(u),p_0(u))$ have branching points of order *exactly* ℓ at $u=\pm ia$. That is we are assuming that

$$A=\lim_{u\to ia}(u-ia)^\ell H_1^{[1]}(q_0(u),p_0(u))\neq 0.$$

Remark 2.2. *Hypothesis* ***HP6*** *is generic because it is equivalent to assume that some coefficient in the Laurent expansions of $H_1^{[\pm1]}(q_0(u),p_0(u))$ at the points $u=\pm ia$ is non-zero.*

2.1.3 Some remarks about the hypotheses

- Let us point out that the time parametrization of the separatrix has always singularities for complex time (see [Fon95] for the hyperbolic case and [BF04] for the parabolic one). The real restriction in **HP2** is that there exists only one singularity in the lines $\{\operatorname{Im} u = \pm a\}$.

- The conditions satisfied in **HP2.1** and **HP2.2** are consequence of **HP2**. Indeed, let u^* be a singularity of $(q_0(u), p_0(u))$. We have that:

  – If V is a polynomial, let M be its degree. Then u^* is a branching points (or pole) of order $2/(M-2)$. That is, if u belongs to a neighborhood of u^*, then $(q_0(u), p_0(u))$ can be expressed as

  $$\begin{aligned} q_0(u) &= -\frac{C(M-2)}{2(u-u^*)^{2/(M-2)}}\left(1+\mathcal{O}\left((u-u^*)^{2/(M-2)}\right)\right) \\ p_0(u) &= \frac{C}{(u-u^*)^{M/(M-2)}}\left(1+\mathcal{O}\left((u-u^*)^{2/(M-2)}\right)\right) \end{aligned}$$

  for some constant $C \neq 0$. This fact is proved in [BF04].

  – If V is a trigonometric polynomial, let us call M to its degree. Then, for u belonging to a neighborhood of u^*, $(q_0(u), p_0(u))$ are of the form

  $$\begin{aligned} q_0(u) &= C\log\big(-i(u-u^*)\big) + \mathcal{O}\big((u-u^*)^{2/M}\big) \\ p_0(u) &= \frac{C}{(u-u^*)} + \mathcal{O}\big((u-u^*)^{2/M}\big) \end{aligned}$$

  with the constant $C = \pm i2/M$ depending on $\operatorname{Im} q_0(u) \to \mp\infty$ respectively. Indeed, first we note that, due to the fact that $\operatorname{Re} q_0(u) \in [0, 2\pi]$, the condition $|q_0(u)| \to +\infty$ as $u \to u^*$ forces to $|\operatorname{Im} q_0(u)| \to +\infty$ as u goes to u^*. Assume that $\operatorname{Im} q_0(u) \to -\infty$ as $u \to u^*$. We note that in this case, since $q_0(u)$ is a real analytic function, then $\overline{u^*}$ is also a singularity of q_0 and it satisfies $\operatorname{Im} q_0(u) \to +\infty$ as $u \to \overline{u^*}$. We perform the change of variables $x = i\log w$ and we emphasize that, if $\operatorname{Im} x \to -\infty$, then $w \to 0$. From the fact that

  $$\frac{dx}{du} = \sqrt{-2V(x)},$$

  we obtain that

  $$\frac{du}{dw} = iw^{M/2-1}(c_0 + \mathcal{O}(w))$$

  for some constant c_0. Henceforth, integrating both sides of the previous differential equation, we obtain $u - u^* = iw^{M/2}(c_1 + \mathcal{O}(w))$, for some constant c_1, which implies that $w = \big(-i(u-u^*)\big)^{2/M}\big(c_2 + \mathcal{O}\big((u-u^*)^{2/M}\big)\big)$ for a suitable constant c_2. and the results follows going back to the original variables.

- In fact, let us observe that the hypotheses considered about the expansions of $(q_0(u), p_0(u))$ given in (12) and (13) (**HP2.1** and **HP2.2**) are weaker than what usually happens when the potential V is a polynomial or a trigonometric polynomial as we have seen previously. This weakness comes from the fact that the second terms in the expansions are, in fact, of greater order. We assume this weaker hypothesis to show that our results could be applied to more general potentials as long as Hypothesis **HP2** is satisfied.

- From **HP2.1**, taking into account that the homoclinic connection is a solution of the unperturbed Hamiltonian system and identifying terms of the same order in $(u - ia)$, one can deduce that in the polynomial case, the degree of V is $2r/(r-1)$. In fact, there exists a constant $v_\infty \in \mathbb{R}$ such that

$$V(x) = v_\infty x^{\frac{2r}{r-1}}(1 + o(1)) \qquad \text{as } \; x \to \infty. \tag{15}$$

- Hipothesis **HP4.2** is to ensure that the parabolic critical point $(0, 0)$ of the unperturbed system persists when we add the perturbation and that it keeps its parabolic character. Therefore it is the natural hypothesis to deal with and it is the same one that was considered in [BF04]. Namely, if the perturbation has order n with $2n - 2 < m$, when the perturbation is added the system might undergo bifurcations and the invariant manifolds might even disappear. The only study done in one of these bifurcation cases can be found in [BF05].

- The form of the perturbed Hamiltonian H_1 considered is more restrictive than necessary. In fact, our result can be applied to any Hamiltonian of the form

$$H_1(x, y, \tau; \varepsilon) = \sum_{n=0}^{N} \varepsilon^n H_1^n(x, y, \tau)$$

  if the functions $H_1^n(q_0(u), p_0(u), \tau)$ have a singularity of order less or equal than $\ell + n$. In this case, the result would be the same but one has to slightly adapt the definition of the constant b in Theorem 2.6.

- Recall the Hamiltonian

$$H\left(x, y, \frac{t}{\varepsilon}\right) = H_0(x, y) + \mu\varepsilon^\eta H_1\left(x, y, \frac{t}{\varepsilon}\right).$$

  Let us point out that in the case $\ell - 2r \leq 0$, Hypothesis **HP5** corresponds to $\eta \geq 0$, which is optimal in the sense that it includes the case such that the perturbation is of the same order as the unperturbed system.

  The case $\ell = 2r$ is what typically happens in near integrable Hamiltonian systems close to a resonance and in general periodic systems with slow dynamics, therefore, in this sense Hypothesis **HP5** is optimal in the generic case.

  In the case $\ell - 2r > 0$ one may think to also ask $\eta \geq 0$. Nevertheless, our techniques only provide optimal exponentially upper bounds if $\eta - \ell + 2r \geq 0$.

  For lower values of η, that is $0 \leq \eta < \ell - 2r$, using similar tools as the ones presented in this paper, one could easily prove the existence of the perturbed invariant manifolds and obtain (non-optimal) exponentially small upper bounds for the difference between them. This case can be called *below the singular case* (see [GOS10]). To obtain an asymptotic formula for the difference between the invariant manifolds in the *below the singular case* is a problem which remains open. Some ideas to deal with this case by using averaging theory can be found in [GOS10].

- Hypothesis **HP6** for the case $\ell < 2r$ has an analogous condition for the general case $\ell \geq 2r$ which is stated in Theorem 2.6. Therefore we are not adding an extra hypothesis for the "easier" case $\ell < 2r$.

2.2 Main results

By Hypothesis **HP1**, system (6) with $\mu = 0$ has either a hyperbolic or parabolic point at the origin. In the second case, Hypothesis **HP4.2** ensures that the origin is also a critical point of the perturbed system ($\mu \neq 0$) which is also parabolic. In the hyperbolic case, the next theorem ensures that the unperturbed hyperbolic critical point of the unperturbed system becomes a hyperbolic periodic orbit which is close to the origin.

Theorem 2.3. *Let us assume Hypotheses **HP1.1**, **HP3**, **HP4.1**. Take $\eta \geq 0$ and fix any value $\mu_0 > 0$. Then, there exists $\varepsilon_0 > 0$ such that for any $|\mu| < \mu_0$ and $\varepsilon \in (0, \varepsilon_0)$, system (6) has a hyperbolic periodic orbit $(x_p(t/\varepsilon), y_p(t/\varepsilon))$ which satisfies that, for $t \in \mathbb{R}$,*

$$\left|x_p\left(\frac{t}{\varepsilon}\right)\right| + \left|y_p\left(\frac{t}{\varepsilon}\right)\right| \leq K|\mu|\varepsilon^{\eta+1}$$

for a constant $K > 0$ independent of ε and μ.

The proof of this theorem, which was done in [DS97] for $\eta > \ell$, is given in Section 5. An alternative proof for values of $\eta > -1/2$ without explicit bounds for the periodic orbit can be found in [Fon95]. For the case when perturbation only depends on time in [Fon93] the existence of the periodic orbit with explicit bounds was given for $\eta > -2$.

To use the same notation in both the hyperbolic and parabolic cases, in the latter one we define $(x_p, y_p) = (0, 0)$.

The next step is to study the stable and unstable invariant manifolds of the periodic orbit (x_p, y_p). In the unperturbed case (that is $\mu = 0$) we know that they coincide along the separatrix (q_0, p_0) given in **HP2**. When $\mu \neq 0$ they generically split.

To measure the splitting of the invariant manifolds let us consider the $2\pi\varepsilon$-Poincaré map P_{t_0} in a transversal section $\Sigma_{t_0} = \left\{(x, y, t_0); (x, y) \in \mathbb{R}^2\right\}$. This Poincaré map has a (hyperbolic or parabolic) fixed point $(x_p(t_0/\varepsilon), y_p(t_0/\varepsilon))$. We will see that this fixed point has stable and unstable invariant curves.

As P_{t_0} is an area preserving map, we measure the splitting giving an asymptotic formula for the area of the lobes generated by these curves between two transversal homoclinic points. Moreover, by the area preserving character of P_{t_0}, the area $\mathcal{A}$ of these lobes does not depend on the choice of the homoclinic points. Other quantities measuring the splitting, as the distance along a transversal section to the unperturbed separatrix, or the angle between these curves at an homoclinic point, can be easily derived from our work.

The quantitative measure of this splitting differs substantially depending whether $\ell - 2r < 0$ or $\ell - 2r \geq 0$ (see Hypothesis **HP2** for the definition of r and (14) for the definition of ℓ). Therefore, we split these results into two different theorems. First, Theorem 2.5 deals with the case $\ell - 2r < 0$ and then Theorem 2.6 deals with the case $\ell - 2r \geq 0$. We will give a complete description of the proof of these two theorems in Section 4. We also refer to Section 3 for an heuristic idea of the main features of the proof of our main results.

2.2.1 Main result for the case $\ell < 2r$

As we will see, in this case, the asymptotic coefficient for the area of the lobe between two consecutive homoclinic points is very related with the so-called Melnikov Potential. For this reason, first of all we are going to obtain an asymptotic formula for it.

The Melnikov Potential (called also sometimes Poincaré Function, see for instance [DG00]), is given by

$$L\left(u, \frac{t}{\varepsilon}\right) = \int_{-\infty}^{+\infty} H_1\left(q_0(u+s), p_0(u+s), \varepsilon^{-1}(t+s)\right) ds. \tag{16}$$

Let us point out that, by Hypothesis **HP4**, this integral is uniformly convergent. Moreover

$$L(u, \tau) = M(\tau - \varepsilon^{-1}u), \tag{17}$$

where M is the 2π-periodic function

$$M(s) = \int_{-\infty}^{+\infty} H_1\left(q_0(r), p_0(r), \varepsilon^{-1}r + s\right) dr = \sum_{k \neq 0} M^{[k]} e^{iks}$$

which, by **HP3**, has zero mean. Here $M^{[k]}$ denotes the k-Fourier coefficient of M.

In [DS97] (polar case) and [BF04] (branching point case), it was seen that Hypothesis **HP6** (together with Hypotheses **HP3** and **HP4**), allows us to give an asymptotic formula for the Fourier coefficients of M and henceforth we will obtain an asymptotic formula for the functions M and L.

Lemma 2.4 ([DS97, BF04]). *Let us assume Hypotheses **HP2**, **HP3**, **HP4** and **HP6**. Let*

$$f_0 = \frac{Ai^{-\ell-1}}{\Gamma(\ell)},$$

*where A is the constant defined by Hypothesis **HP6**. Then:*

1. *The first Fourier coefficients of M are given by:*

$$\overline{M^{[1]}} = M^{[-1]} = -\frac{1}{\varepsilon^{\ell-1}} f_0 e^{-\frac{a}{\varepsilon}} \left(1 + \mathcal{O}\left(\varepsilon^{\frac{1}{\beta}}\right)\right).$$

2. *If $|k| \neq 1$,*

$$M^{[k]} = \mathcal{O}\left(\frac{1}{\varepsilon^{\ell-1}} e^{-|k|\frac{a}{\varepsilon}}\right).$$

3. *For $u \in \mathbb{R}$ and $t \in \mathbb{R}$,*

$$L\left(u, \frac{t}{\varepsilon}\right) = -\frac{2}{\varepsilon^{\ell-1}} \mathrm{Re}\left(f_0 e^{-i\left(\frac{u-t}{\varepsilon}\right)}\right) e^{-\frac{a}{\varepsilon}} \left(1 + \mathcal{O}\left(\varepsilon^{\frac{1}{\beta}}\right)\right),$$

*where a and β are the constants defined in Hypothesis **HP2**.*

Theorem 2.5 (Main Theorem: case $\ell - 2r < 0$)**.** *Let us assume Hypotheses **HP1**-**HP6** and $\ell - 2r < 0$. Then, given $\mu_0 > 0$, there exists $\varepsilon_0 > 0$ such that for any $\mu \in \{|\mu| \leq \mu_0\}$ and $\varepsilon \in (0, \varepsilon_0)$ the invariant manifolds of the periodic orbit given in Theorem 2.3 split and the area of the corresponding lobes for the Poincaré map is given by*

$$\mathcal{A} = 4|\mu|\varepsilon^{\eta+1-\ell} e^{-\frac{a}{\varepsilon}} \left(\left|f_0 e^{iC(\mu,\varepsilon)}\right| + \mathcal{O}\left(\frac{1}{|\ln \varepsilon|}\right)\right),$$

where $C(\mu, \varepsilon)$ is a function defined in $\{|\mu| \leq \mu_0\} \times (0, \varepsilon_0)$, analytic in μ and such that $C(\mu, \varepsilon) = \mathcal{O}(\mu\varepsilon^\eta)$ uniformly in ε, and f_0 is given in Lemma 2.4.

2.2.2 Main result for the case $\ell \geq 2r$

The case $\ell \geq 2r$ is essentially different from the previous case $\ell < 2r$ in the sense that we are not able to have “a priori” estimates for the asymptotic coefficient of the area of the lobes between two consecutive homoclinic points. Such asymptotic coefficient depends on an unknown function ($f(\hat{\mu})$ in Theorem 2.6) which comes from the study of the difference between adequate approximations of the invariant manifolds near the singularities $\pm ia$. For that reason we note that, in this case, the analogous condition to Hypothesis **HP6** is $f(\hat{\mu}) \neq 0$, which is assumed in Theorem 2.6.

Theorem 2.6 (Main Theorem: case $\ell - 2r \geq 0$)**.** *Let us assume Hypotheses **HP1**-**HP5** and $\ell - 2r \geq 0$. Then, there exists an entire analytic function $f(\hat{\mu})$ such that, if $f(\hat{\mu}) \neq 0$ for $\hat{\mu} \in \{|\hat{\mu}| < \hat{\mu}_1\} \cap \mathbb{R}$ for some $\hat{\mu}_1 \in \mathbb{R}$, there exists $\varepsilon_0 > 0$ such that for any $(\varepsilon, \mu) \in \{(\varepsilon, \mu) \in \mathbb{R}^2;\ 0 < \varepsilon < \varepsilon_0,\ |\mu\varepsilon^{\eta-\ell+2r}| < \hat{\mu}_1\}$, the invariant manifolds of the periodic orbit given in Theorem 2.3 split and the area of the corresponding lobes for the Poincaré map is given by*

- *If $\ell - 2r > 0$,*

$$\mathcal{A} = 4|\mu|\varepsilon^{\eta+1-\ell} e^{-\frac{a}{\varepsilon}} \left(\left|f\left(\mu\varepsilon^{\eta+2r-\ell}\right)\right| + \mathcal{O}\left(\frac{1}{|\ln \varepsilon|^{\ell-2r}}\right)\right)$$

- *If $\ell - 2r = 0$,*

$$\mathcal{A} = 4|\mu|\varepsilon^{\eta+1-\ell} e^{-\frac{a}{\varepsilon} + \mu^2\varepsilon^{4\eta} \operatorname{Im} b \ln\frac{1}{\varepsilon}} \left(\left|f\left(\mu\varepsilon^{\eta}\right) e^{iC(\mu\varepsilon^\eta,\varepsilon)}\right| + \mathcal{O}\left(\frac{1}{|\ln \varepsilon|}\right)\right),$$

where $b \in \mathbb{C}$ is a constant completely determined by the Hamiltonian H and $C(\hat{\mu}, \varepsilon)$ is a function defined in $\{|\hat{\mu}| \leq \mu_1\} \times (0, \varepsilon_0)$, analytic in $\hat{\mu}$ and such that $C(\hat{\mu}, \varepsilon) = \mathcal{O}(\hat{\mu})$ uniformly in ε.

2.2.3 Some comments about the results

- The constant b appearing in Theorem 2.6 can be computed explicitly as it is showed in formula (75) in Proposition 4.15. In particular, $b = 0$ when the Hamiltonian H_1 in (8) and (9) does not depend on y. For this reason, in the previous results obtained in the singular case corresponding to $\eta = \ell - 2r = 0$, see [Tre97, Gel00, Oli06, GOS10], this term does not appear. The appearance of this logarithmic term in the asymptotic formula had already been detected in [Bal06]. Let us also point out that an analogous phenomenon happens in the analytic unfoldings of the Hopf-zero singularity (see [BS06, BS08]) and in weak resonances of area preserving maps [SV09].

- The constant $C(\hat{\mu}, \varepsilon)$ appearing in Theorems 2.5 and 2.6 also satisfies $C(\hat{\mu}, \varepsilon) = 0$ if the Hamiltonian H_1 in (8) and (9) does not depend on y. Moreover, it also satisfies $\lim_{\varepsilon \to 0} C(\hat{\mu}, \varepsilon) = C_0(\hat{\mu})$ for a certain function $C_0(\hat{\mu})$ which is entire analytic in $\hat{\mu}$. We prove this fact in Section 9.4 and we give an explicit expression of $C_0(\hat{\mu})$ in terms of several explicitly computable auxiliary functions.

- If one weakens Hypothesis **HP3** to admit Hamiltonian systems with $\mathcal{C}^1$ dependence on τ, one can get analogous results to the ones obtained in Theorems 2.5 and 2.6.

- **Comparison with Melnikov**. In the case $\ell - 2r < 0$, taking into account that by Theorem 2.5, $C(\mu, \varepsilon) = \mathcal{O}(\mu\varepsilon^\eta)$, one has that Melnikov theory predicts the area of the lobes correctly provided $\eta > 0$ even if in this case this theory cannot be directly applied due to the exponential smallness. If $\eta = 0$, it does not predict the area correctly in general. Nevertheless, since $C \equiv 0$ when the perturbation does not depend on y, in this case Melnikov theory predicts the asymptotic size of the area of the lobes even if $\eta = 0$, that is, when the perturbation has the same size as the integrable system.

  In the cases $\ell - 2r > 0$ and $\ell - 2r = 0$, we know that the function $f(\hat{\mu})$ appearing in Theorem 2.6, satisfies that for $\hat{\mu}$ small
  $$f(\hat{\mu}) = f_0 + \mathcal{O}(\hat{\mu}),$$
  where $f_0 \in \mathbb{C}$ is a constant independent of μ. In [Bal06], it is seen that the constant f_0 coincides with the constant that Melnikov theory gives in front of the exponential term (see Lemma 2.4).

  In other words, this means that for the case $\ell - 2r > 0$, if one assumes that $\eta > \ell - 2r$, Melnikov theory also predicts the asymptotic behavior of the area of the lobes correctly.

  In the case $\ell - 2r = 0$, f_0 also corresponds to the Melnikov theory prediction. Nevertheless, since a logarithmic term appears in the exponential, in the case $\eta = 0$ the Melnikov prediction is valid provided
  $$|\mu| \ll \frac{1}{\sqrt{|\ln \varepsilon|}}.$$
  Of course, if $b = 0$, as happens when the perturbation does not depend on y, the Melnikov prediction is valid for any μ small and independent of ε.

2.2.4 Examples

In this section we apply Theorems 2.5 and 2.6 to some examples. We consider the Duffing equation

$$H_0(x, y) = \frac{y^2}{2} - \frac{x^2}{2} - \frac{x^4}{4}$$

with different perturbations. The Duffing equation has two separatrices forming a figure eight, which are parameterized by

$$\Gamma^\pm(u) = (\pm q_0(u), p_0(u)) = \left(\pm \frac{\sqrt{2}}{\cosh u}, \mp \frac{\sqrt{2}\sinh u}{\cosh^2 u} \right).$$

The singularities of these separatrices which are closer to the real axis are $u = \pm i\pi/2$ and $r = 2$ (see the definition of r in Hypothesis **HP2**).

We consider two different types of perturbations and we study how the separatrix Γ^+ splits. The first perturbation that we consider is

$$H(x,y)=\frac{y^2}{2}-\frac{x^2}{2}-\frac{x^4}{4}+\mu\varepsilon^\eta x^n\sin\frac{t}{\varepsilon}$$

for $n\in\mathbb{N}$ and $\eta\geq 0$. Then the order of the perturbation is $\ell=n$ (see the definition of ℓ in (14)).

If one applies Melnikov theory to these Hamiltonian systems, obtains the following prediction for the area of the lobes

$$\mathcal{A}=|\mu|\varepsilon^\eta\frac{2^{\frac{n}{2}+2}\pi}{(n-1)!\,\varepsilon^{n-1}}e^{-\frac{\pi}{2\varepsilon}}+\mathcal{O}\left(\mu^2\varepsilon^{2\eta}\right). \tag{18}$$

For $n<4$, which corresponds to $\ell-2r<0$, one can apply Theorem 2.5 to see that Melnikov theory predicts correctly the area of the lobes for any $\eta\geq 0$. Namely,

$$\mathcal{A}=|\mu|\varepsilon^\eta\frac{2^{\frac{n}{2}+2}\pi}{(n-1)!\,\varepsilon^{n-1}}e^{-\frac{\pi}{2\varepsilon}}\left(1+\mathcal{O}\left(\frac{1}{|\ln\varepsilon|}\right)\right). \tag{19}$$

The case $n=4$ corresponds to $\ell=2r$. In this case, since the perturbation does not depend on y, we have that $b=0$ and $C=0$. Then, for any $\eta\geq 0$, the formula of the area is given by

$$\mathcal{A}=|\mu|\varepsilon^\eta\frac{4\left|f\left(\mu\varepsilon^\eta\right)\right|}{\varepsilon^{n-1}}e^{-\frac{\pi}{2\varepsilon}}\left(1+\mathcal{O}\left(\frac{1}{|\ln\varepsilon|}\right)\right), \tag{20}$$

where $f(\hat\mu)$ satisfies

$$f(\hat\mu)=\frac{2^{\frac{n}{2}}\pi i}{(n-1)!}+\mathcal{O}\left(\hat\mu\right). \tag{21}$$

Then, as we have explained in Section 2.2.3, Melnikov theory predicts correctly the area of the lobes provided $\eta>0$. For $\eta=0$ and fixed μ independent of ε, the first order depends on the full jet of $f(\hat\mu)$ and then the Melnikov function does not predict it correctly. Nevertheless both formulas (19) and (20) give a first order term of the same type.

The case $n>4$ corresponds to $\ell>2r$. Theorem 2.6 gives results for $\eta+4-n\geq 0$. Applying this theorem, one has that

$$\mathcal{A}=|\mu|\varepsilon^\eta\frac{4\left|f\left(\mu\varepsilon^{\eta+4-n}\right)\right|}{\varepsilon^{n-1}}e^{-\frac{\pi}{2\varepsilon}}\left(1+\mathcal{O}\left(\frac{1}{|\ln\varepsilon|^{n-4}}\right)\right)$$

where $f(\hat\mu)$ also satisfies (21). Then, for $\eta>n-4$ Melnikov theory predicts the area correctly and for $\eta=n-4$ and μ fixed and independent of ε it does not.

To see how the first asymptotic order of the area of the lobes changes when the perturbation depends on y and $\ell-2r=0$, we consider the following perturbation of the Duffing equation,

$$H(x,y)=\frac{y^2}{2}-\frac{x^2}{2}-\frac{x^4}{4}+\mu\left(x^4\sin\frac{t}{\varepsilon}+\lambda x^2y\cos\frac{t}{\varepsilon}\right)$$

with $\lambda\in\mathbb{R}$. For this example, Melnikov theory predicts that the area of the lobes is

$$\mathcal{A}=|\mu|\frac{4\pi}{3\varepsilon^3}|2+\sqrt{2}\lambda|e^{-\frac{\pi}{2\varepsilon}}+\mathcal{O}\left(\mu^2\right).$$

Note that if one takes $\lambda=0$, $\mathcal{A}$ coincides with (18) with $n=4$ and $\eta=0$. On the other hand, if one takes $\lambda=-\sqrt{2}$ the Melnikov function is degenerate since the first order vanishes.

For this perturbation $\ell=2r$ and therefore we can apply Theorem 2.6. Using formula (75), one can easily see that $b=-4\sqrt{2}\lambda i$. Therefore, the true first asymptotic order of the area of the lobes is given by

$$\mathcal{A}=|\mu|\frac{4}{\varepsilon^3}e^{-\frac{\pi}{2\varepsilon}-4\sqrt{2}\lambda\mu^2\ln\frac{1}{\varepsilon}}\left(\left|f(\mu)e^{iC(\mu,\varepsilon)}\right|+\mathcal{O}\left(\frac{1}{|\ln\varepsilon|}\right)\right), \tag{22}$$

where $C(\mu,\varepsilon)=\mathcal{O}(\mu)$ and $f(\mu)$ satisfies

$$f(\mu)=\frac{\pi i}{3}\left(2+\sqrt{2}\lambda\right)+\mathcal{O}\left(\mu\right).$$

One can take $\mu=1$ and write formula (22) as

$$\mathcal{A}=\frac{4}{\varepsilon^{3-4\sqrt{2}\lambda}}e^{-\frac{\pi}{2\varepsilon}}\left(\left|f(1)e^{iC(1,\varepsilon)}\right|+\mathcal{O}\left(\frac{1}{|\ln\varepsilon|}\right)\right).$$

Therefore, the correcting logarithmic term implies a drastic change in the degree of the polynomial term of the asymptotics. Note that one can take any $\lambda\in\mathbb{R}$ and then the polynomial term can change arbitrarily, both increasing or decreasing the degree. Finally, if one takes $\lambda=0$, one recovers formula (20).

2.3 Near integrable Hamiltonian systems of $1\frac{1}{2}$ degrees of freedom close to a resonance

The results obtained in this work can be easily adapted to study near integrable Hamiltonian systems of $1\frac{1}{2}$ degrees of freedom close to a resonance. Let us consider an analytic Hamiltonian system with Hamiltonian

$$h(x,I,\tau)=h_0(I)+\delta h_1(x,I,\tau),\tag{23}$$

where $\delta\ll 1$ is a small parameter, $(x,\tau)\in\mathbb{T}^2$, $I\in\mathbb{R}$ and h_1 is a trigonometric polynomial as a function of x. When $\delta=0$, the Hamiltonian system is completely integrable (in the sense of Liouville-Arnold) and the phase space is foliated by invariant tori with frequency $\omega(I)=(\partial_I h_0(I),1)$.

In particular, if for certain I, there exists $k\in\mathbb{Z}^2$ such that $\omega(I)\cdot k=0$, the corresponding torus is foliated by periodic orbits. When $\delta>0$ (but small enough), it is a well known fact that typically this torus, a resonant torus, breaks down.

Let us consider the simplest setting and let us assume that

$$h_0(I)=\frac{I^2}{2}+G(I)\qquad\text{with } G(I)=\mathcal{O}\left(I^3\right).$$

Then $I=0$ corresponds to the resonant vector $\omega(0)=(0,1)$. To study the dynamics of the perturbed system around this resonance, one usually performs the rescaling

$$I=\sqrt{\delta}y\quad\text{and}\quad\tau=\frac{t}{\sqrt{\delta}}$$

and takes $\varepsilon=\sqrt{\delta}$ as a new parameter. Then, one obtains the Hamiltonian

$$H(x,y,t)=\frac{y^2}{2}+\frac{1}{\varepsilon^2}G(\varepsilon y)+V(x)+F\left(x,\frac{t}{\varepsilon}\right)+R\left(x,\varepsilon y,\frac{t}{\varepsilon}\right),$$

where

$$\begin{aligned}V(x)&=\langle h_1(x,0,\tau)\rangle=\frac{1}{2\pi}\int_0^{2\pi}h_1(x,0,\tau)\,d\tau\\ F(x,\tau)&=h_1(x,0,\tau)-\langle h_1(x,0,\tau)\rangle\\ R(x,I,\tau)&=h_1(x,I,\tau)-h_1(x,0,\tau),\end{aligned}$$

which can be written as

$$H\left(x,y,\frac{t}{\varepsilon}\right)=H_0(x,y)+\mu H_1\left(x,y,\frac{t}{\varepsilon},\varepsilon\right)$$

with

$$\begin{aligned}H_0(x,y)&=\frac{y^2}{2}+V(x)\\ H_1(x,y,\tau,\varepsilon)&=F(x,\tau)+\frac{1}{\varepsilon^2}G(\varepsilon y)+R(x,\varepsilon y,\tau).\end{aligned}$$

Here μ is in fact a fake parameter, since we are interested in the case $\mu = 1$. This system is similar to the ones considered in this paper. Let us point out also that, by definition, $\varepsilon^{-2}G(\varepsilon y)$ and $R(x, \varepsilon y, \tau)$ are of order ε.

Let us assume that the Hamiltonian H satisfies Hypotheses **HP1-HP4** and instead of **HP5** satisfies the alternative hypothesis that V, which is a trigonometric polynomial, has the same degree as h_1 in (23) as a function of x. Then, using the tools considered in this paper, one can give an asymptotic formula analogous to the one given in Theorem 2.6. Let us point out that in this setting, even if the terms $\varepsilon^{-2}G(\varepsilon y)$ and $R(x, \varepsilon y, \tau)$ are of order ε and therefore smaller than $F(x, \tau)$, the function $f(\mu)$ appearing in Theorem 2.6 depends not only on F but also on the full jet in y of G and R. The reason is that these terms become of the same order as $V(x)$ and $F(x, \tau)$ close to the singularities of the unperturbed separatrix. Moreover, for these systems, the first asymptotic order also has the logarithmic term in the exponential as it happens in Theorem 2.6 for $\ell - 2r = 0$. We plan to study rigorously these kind of systems in future work.

3 Heuristic ideas of the proof

The rigorous proofs of asymptotic formulas for measuring the splitting of separatrices require a significant amount of technicalities. For the convenience of the reader, even though in Section 4 we give a precise description of the entire proof of Theorems 2.5 and 2.6, we first devote this section to give an heuristic description of our strategy explaining the main differences respect to the ones already used in the literature. We also explain the main novelties we have introduced to overcome the difficulties that our general setting involves.

3.1 Measuring the splitting by using generating functions

The main idea of the method that we are going to explain, was introduced in [LMS03, Sau01], based on ideas by Poincaré [Poi99]. Roughly speaking, if the invariant manifolds can be expressed in a suitable way, then the area of the lobes generated by the perturbed manifolds between two consecutive homoclinic points and also the distance between the manifolds can be simply computed by the difference between two functions.

Let us explain this approach in more detail. Assume that we are in the Hamiltonian setting given in Section 2. As the main goal is to measure the distance of the stable and unstable manifolds of the periodic orbit $(x_p(t/\varepsilon), y_p(t/\varepsilon))$ in a Poincaré section Σ_{t_0}, it is useful to obtain these manifolds as graphs. The stable and unstable manifolds of the unperturbed system can be expressed as graphs as

$$y = \varphi(x, t/\varepsilon) = y_p(t/\varepsilon) + \partial_x S^{s,u}(x - x_p(t/\varepsilon), t/\varepsilon)$$

in some complex domains, where the functions $S^{s,u}$ are called generating functions. As it is pointed out in [Sau01] the generating functions $S^{s,u}(q, \tau)$ are solutions of the Hamilton-Jacobi equation associated to our Hamiltonian system after the change of variables

$$q = x - x_p(t/\varepsilon), \quad p = y - y_p(t/\varepsilon)$$

and the change of time $\tau = t/\varepsilon$.

Then, to measure the distance between the stable and the unstable manifolds in a Poincaré section we just need to compute:

$$d(q, t_0) = \partial_q S^u(q, t_0/\varepsilon) - \partial_q S^s(q, t_0/\varepsilon) \tag{24}$$

and it is standard that the area of the lobes is given by

$$\mathcal{A} = S^u(q_2, t_0/\varepsilon) - S^s(q_2, t_0/\varepsilon) - (S^u(q_1, t_0/\varepsilon) - S^s(q_1, t_0/\varepsilon)), \tag{25}$$

where q_1, q_2 are the coordinates of two consecutive homoclinic points in the section Σ_{t_0}. Note that, thanks to the symplectic structure, $\mathcal{A}$ does not depend on t_0.

We perform the change of variables $q = q_0(u)$, where $q_0(u)$ is the first component of the unperturbed homoclinic orbit. In this way, we work with the function

$$T^{u,s}(u,\tau) = S^{u,s}(q_0(u),\tau)$$

that is, we write the perturbed manifolds as functions of time τ and the "time over the homoclinic orbit" u, which parameterizes the unperturbed homoclinic orbit. These functions satisfy a new Hamilton-Jacobi equation, which is easier to deal with.

Let us consider their difference

$$\Delta(u,\tau) = T^u(u,\tau) - T^s(u,\tau).$$

The first observation is that, when $\mu = 0$, $\partial_u T^{u,s}(u,\tau) = \partial_u T_0(u) = p_0(u)\partial_q S^{u,s}(q_0(u),\tau) = (p_0(u))^2$ which corresponds to the parameterization of the unperturbed separatrix. Then, by analyticity with respect to the regular parameter μ, we have that $\Delta(u,\tau) = \mathcal{O}(\mu)$.

The second observation is that, as the experts in this area know, $\Delta(u,\tau)$ is exponentially small in the singular parameter ε. To obtain sharp estimates of $\Delta(u,\tau)$, we need to bound it, and consequently $T^u(u,\tau)$ and $T^s(u,\tau)$, in a region of the complex plane that, on one hand, contains a segment of the real line having two values of u giving rise to two consecutive homoclinic points and, on the other hand, intersects a neighborhood sufficiently close to the singularities $\pm ia$ of $T_0(u)$.

Assume that we can construct parameterizations $T^{u,s}(u,\tau)$ of the perturbed invariant manifolds satisfying both that they are 2π-periodic with respect to τ and that they are real-analytic and bounded in some complex domain which contains two real values of u which give rise to two consecutive homoclinic points. Now we are going to explain how an exponentially small upper bound of the difference Δ can be derived. The first point is that, being T^u and T^s solutions of the same partial differential equation (with different boundary conditions), $\Delta(u,\tau)$ satisfies a homogeneous linear partial differential equation. One can see that this equation is conjugated to $(\varepsilon\partial_u + \partial_\tau)Y(u,\tau) = 0$. Let us assume for a moment that Δ is a solution of this equation. In fact, in Theorems 4.21 and 4.17, we will see that this is true after a suitable change of variables. Then, being Δ 2π-periodic in τ we obtain that $\Delta(u,\tau) = \Lambda(u/\varepsilon - \tau)$, where $\Lambda(s)$ is a 2π-periodic function. This fact implies that

$$\Delta(u,\tau) = \sum_{k\in\mathbb{Z}} \Lambda_k e^{-ik\frac{u}{\varepsilon}} e^{ik\tau}.$$

Now, a bound $|\Delta(u,\tau)| \leq M$ for $|\operatorname{Im} u| \leq a'$, automatically gives

$$|\Lambda_k| \leq M e^{-|k|\frac{a'}{\varepsilon}}$$

which implies that $|\Delta(u,\tau) - \Lambda_0| \leq 2Me^{-\frac{a'}{\varepsilon}}$ for real values of u. The bigger the size of the strip where we can bound $|\Delta(u,\tau)|$ the smaller the exponential that gives the bound for real values of u. Note that the constant Λ_0 does not appear neither in the formula of the area (25), nor in the formula of the distance (24) If we use Melnikov theory the expected exponential exponent is a, where $\pm ai$ are the singularities of T_0. Then, to obtain sharp bounds, it would be enough to take $a' = a - \varepsilon$.

In some cases, the change of variables which conjugates the original partial differential equation for $\Delta(u,\tau)$ with $(\varepsilon\partial_u + \partial_\tau)Y(u,\tau) = 0$ is not close enough to the identity. This fact implies the appearance of the constant $C(\mu,\varepsilon)$ and the logarithmic term in the asymptotic formulas obtained in Theorems 2.5 and 2.6. This change of variables is obtained, essentially, studying the variational equation along the perturbed invariant manifolds. Therefore, the existence of these terms, which were not present in the Melnikov prediction, shows that, to study the exponentially small splitting of separatrices, it is not enough to look for the first order approximations of the invariant manifolds close to the singularities. One has to look also for the first order of certain solutions of the variational equation of the perturbed invariant manifolds close to the singularities. In fact, these terms appear when these certain solutions of the variational equation of the perturbed invariant manifolds close to the singularities are not well approximated by the solutions of the variational equation of the unperturbed separatrix.

Then, roughly speaking one can conclude that Melnikov theory gives the correct answer if:

- The perturbed invariant manifolds are well approximated by the unperturbed separatrix close to the singularity.
- The solutions of the variational equation along the perturbed invariant manifold are well approximated by certain solutions of the variational equation along the unperturbed separatrix.

In all the other cases, the splitting is given by an alternative formula. This fact, is explained in more detail Section 3.4.

3.2 The *boomerang domains*

In this paper we deal with general Hamiltonian whose invariant manifolds, in general, are not global graphs over q. Therefore, the approach explained in the previous section cannot be used straightforwardly. Nevertheless, we will see that there are always regions in the phase space where both manifolds are graphs and we will use one of these regions to measure the splitting. Consequently, being the area of the lobes an invariant quantity, this will give the wanted result.

As we have explained, we are forced to find parameterizations $T^{u,s}$ of the invariant manifolds which have to be analytic in a common complex domain which reaches points at a distance ε of the singularities. Moreover we also need to guarantee that our domain contains an open set of real values of u (this will be enough to ensure that the domain contains u_1 and u_2 that give rise to homoclinic points since they are ε close).

To this end let us observe that we have no hope to construct parameterizations $T^{u,s}(u,\tau)$ for values of u such that $p_0(u) = 0$, at least in a general case. In fact, the unperturbed homoclinic connection can be expressed as $\mathrm{graph}\{p = \sqrt{-2V(q)}\} \cup \mathrm{graph}\{p = -\sqrt{-2V(q)}\}$. Then if $p_0(u_0) = 0$, for some value u_0, the unperturbed homoclinic connection cannot be expressed as a graph over the base in the original variables (q,p) in a neighborhood of $(q_0(u_0), 0)$. This fact implies that the Hamilton-Jacobi equation that $T^{u,s}$ has to satisfy is not defined for $u = u_0$.

We will always keep in mind that we need to check this condition ($p_0(u) \neq 0$) if we want to use the parameterizations $T^{u,s}$.

For this reason we define the following *boomerang domains* (see Figure 2), in which $p_0(u) \neq 0$, and hence the functions $T^{s,u}$ will be well defined in them.

$$
\begin{aligned}
D^s_{\kappa,d} =& \{u \in \mathbb{C};\ |\mathrm{Im}\, u| < \tan\beta_1 \mathrm{Re}\, u + a - \kappa\varepsilon, |\mathrm{Im}\, u| < \tan\beta_2 \mathrm{Re}\, u + a - \kappa\varepsilon,\\
& |\mathrm{Im}\, u| > \tan\beta_2 \mathrm{Re}\, u + a - d\}\\
D^u_{\kappa,d} =& \{u \in \mathbb{C};\ |\mathrm{Im}\, u| < -\tan\beta_1 \mathrm{Re}\, u + a - \kappa\varepsilon, |\mathrm{Im}\, u| < \tan\beta_2 \mathrm{Re}\, u + a - \kappa\varepsilon,\\
& |\mathrm{Im}\, u| > \tan\beta_2 \mathrm{Re}\, u + a - d\}\\
& \cup \{u \in \mathbb{C};\ |\mathrm{Im}\, u| < -\tan\beta_1 \mathrm{Re}\, u + a - \kappa\varepsilon, |\mathrm{Im}\, u| > -\tan\beta_2 \mathrm{Re}\, u + a - d,\\
& \mathrm{Re}\, u < 0\},
\end{aligned}
\tag{26}
$$

where $\beta_1 \in (0, \pi/2)$ is any fixed angle.

To choose β_2 we use the following. First let us point out that the zeros of $p_0(u)$ are isolated in $\mathbb{C}$. Moreover, close to the singularities $u = \pm ia$, $p_0(u)$ can not vanish. Then, in order to assure that $p_0(u)$ does not vanish in the whole domains $D^s_{\kappa,d}$ and $D^u_{\kappa,d}$, one has to choose an angle β_2 such that $\beta_2 > \beta_1$ has a positive lower bound independent of ε and μ and such that the lines $|\mathrm{Im}\, u| = \tan\beta_2 \mathrm{Re}\, u + a$ do not contain any zero of $p_0(u)$. Then, taking $\varepsilon > 0$ and $d > 0$ independent of ε, both small enough, one can guarantee that $p_0(u)$ does not vanish neither in $D^s_{\kappa,d}$ nor in $D^u_{\kappa,d}$.

We will use these *boomerang domains* as fundamental domains to measure the splitting. It is important to emphasize that both $D^s_{\kappa,d}$ and $D^u_{\kappa,d}$ reach a neighborhood of the singularities $\pm ia$ of size ε.

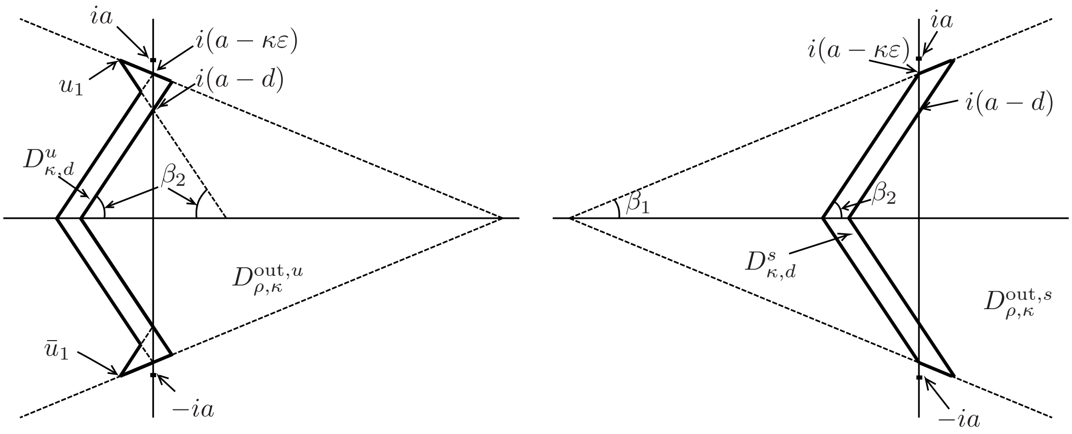

Figure 2: The *boomerang domains* $D^u_{\kappa,d}$ and $D^s_{\kappa,d}$ defined in (26).

Remark 3.1. *Let us observe that the domains* $D^u_{\kappa,d}$ *and* $D^s_{\kappa,d}$ *have different shape. We will give all the proofs in the unstable case. All of them are analogous, and even simpler, in the stable one.*

To study the difference between the manifolds, we consider $\Delta(u,\tau) = T^u(u,\tau) - T^s(u,\tau)$ in the domain $R_{\kappa,d} = D^s_{\kappa,d} \cap D^u_{\kappa,d}$ which is defined as

$$R_{\kappa,d} = \{u \in \mathbb{C};\ |\mathrm{Im}\, u| < \tan\beta_2 \mathrm{Re}\, u + a - \kappa\varepsilon, |\mathrm{Im}\, u| > \tan\beta_2 \mathrm{Re}\, u + a - d, \\ |\mathrm{Im}\, u| < -\tan\beta_1 \mathrm{Re}\, u + a - \kappa\varepsilon\}\,. \tag{27}$$

We recall that $p_0(u) \neq 0$ if $u \in R_{\kappa,d}$ and hence we can use the functions $T^{s,u}$ in this domain.

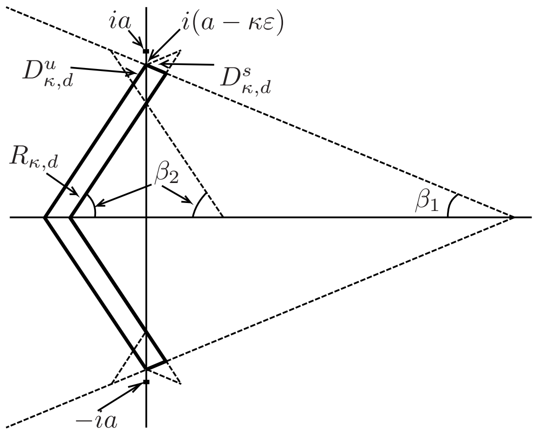

Figure 3: The domain $R_{\kappa,d}$ defined in (27).

The domain $R_{\kappa,d}$, where we measure the difference between the invariant manifolds, is considerably different from the ones used in previous works (see for instance [Sau01]), where the analogous domains look like a diamonds. In [Sau01], the author considers systems for which the unperturbed separatrix is a graph globally and then they can work in such wide domains. Nevertheless, we have to restrict ourselves to this narrower *fundamental domain* $R_{\kappa,d}$ to ensure that $p_0(u) \neq 0$ on it.

Once we have the difference Δ in $R_{\kappa,d}$, using the arguments exposed in the previous subsection one can obtain exponentially small upper bounds for Δ.

Recall that our goal is to give an asymptotic formula for the area of the lobe between two consecutive homoclinic points. Henceforth, once we found the first asymptotic term of Δ, which we call Δ_0, we will use the arguments exposed in the previous section to bound the difference $\Delta(u,\tau) - \Delta_0(u,\tau)$. We will come back to the problem of finding Δ_0 in Section 3.4.

3.3 Parameterizations of the invariant manifolds of the perturbed system

In this section we are going to explain the strategy we use for proving the existence of $T^{u,s}$ in the corresponding boomerang domains $D^{u,s}_{\kappa,d}$. In fact we will always deal with $\partial_u T^{u,s}$.

We begin our construction near the origin $(q,p) = (0,0)$. In terms of the new variable u this corresponds to take $\mathrm{Re}\, u$ near $-\infty$ for the unstable invariant manifold and near $+\infty$ for the stable one.

Given $\rho_1 \geq 0$, we consider the following domains:

$$\begin{aligned} D^u_{\infty,\rho_1} &= \{u \in \mathbb{C}; \mathrm{Re}\, u < -\rho_1\} \\ D^s_{\infty,\rho_1} &= \{u \in \mathbb{C}; \mathrm{Re}\, u > \rho_1\}. \end{aligned} \tag{28}$$

It is not difficult to prove that the constant ρ_1 can be taken big enough so that $p_0(u)$ does not vanish in these domains. Henceforth the Hamilton-Jacobi formulation is allowed in these domains (see (47) and (48)). The first result is Theorem 4.3, where we prove the existence of $\partial_u T^{s,u}$ and we see that both are well approximated by $\partial_u T_0$ in $D^{u,s}_{\infty,\rho_1}$. This result gives the existence of local invariant manifolds and, moreover, provides suitable properties of them.

In the case that $p_0(u) \neq 0$ the next step is to extend $\partial_u T^{u,s}$ to the so-called *outer domains* (see Figure 4) defined by

$$\begin{aligned} D^{\mathrm{out},u}_{\rho,\kappa} &= \{u \in \mathbb{C}; |\mathrm{Im}\, u| < -\tan\beta_1 \mathrm{Re}\, u + a - \kappa\varepsilon, \mathrm{Re}\, u > -\rho\} \\ D^{\mathrm{out},s}_{\rho,\kappa} &= \left\{u \in \mathbb{C}; -u \in D^{\mathrm{out},u}_{\rho,\kappa}\right\}, \end{aligned} \tag{29}$$

where $\kappa > 0$, which might depend on ε, is such that $a - \kappa\varepsilon > 0$. The constant ρ will be taken $\rho > \rho_1$, in order to ensure that $D^*_{\infty,\rho_1} \cap D^{\mathrm{out},*}_{\rho,\kappa} \neq \emptyset$ for $* = u, s$. Since we have already proved the existence of local invariant manifolds defined in $D^{u,s}_{\infty,\rho_1}$, therefore $\partial_u T^{u,s}$ are defined in $D^*_{\infty,\rho_1} \cap D^{\mathrm{out},*}_{\rho,\kappa}$ for $* = u, s$.

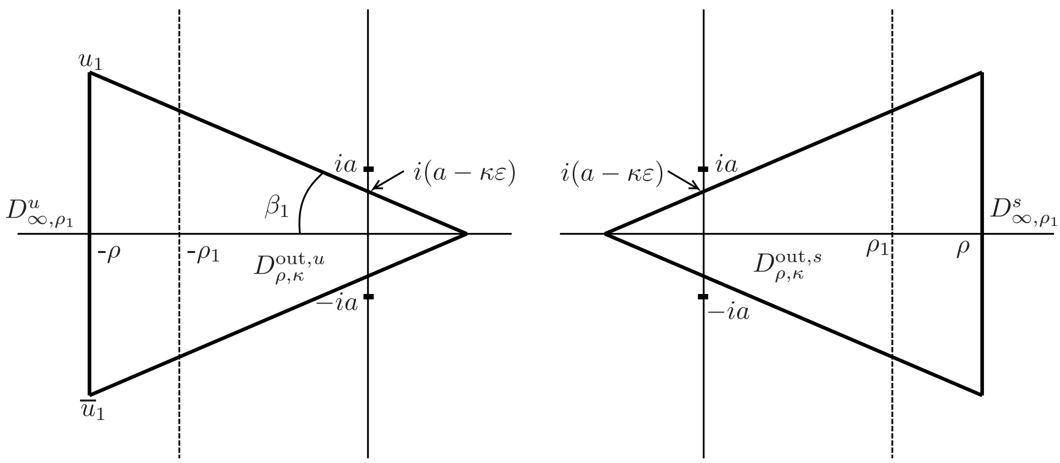

Figure 4: The outer domains $D^{\mathrm{out},u}_{\rho,\kappa}$ and $D^{\mathrm{out},s}_{\rho,\kappa}$ defined in (29).

In Theorem 4.4 it is proved that $\partial_u T^{u,s}(u,\tau)$ can be extended to the *outer domain* $D^{\mathrm{out},*}_{\rho,\kappa}$, $* = u, s$, and that is well approximated (in some norm) by $\partial_u T_0(u)$ there.

In the case that $p_0(u)$ can vanish in the outer domains the procedure becomes a little technical. The main idea is to use parameterizations of the invariant manifolds of the form $(Q(u,\tau), P(u,\tau))$ to extend them to a new domain where $p_0(u) \neq 0$ does not vanish anymore and that overlaps with the *boomerang domain* $D^{u,s}_{\kappa,d}$ (see Theorem 4.6). It is important to emphasize here that these new domains are still far away from the singularities $\pm ia$ of $T_0(u)$, henceforth these parameterizations are straightforward (see Theorem 4.7). Once we have proved the existence of the parameterizations of the invariant manifolds for values of u far from the singularities but inside the *boomerang domains* $D^{u,s}_{\kappa,d}$, we can recover the generating functions $\partial_u T^{u,s}(u,\tau)$ and extend them to the whole *boomerang domains* $D^{u,s}_{\kappa,d}$ in Theorem 4.8.

We want to emphasize here that

- We are able to extend the manifolds up to a distance of order ε of the singularities in all the cases without using any inner equation even in the singular case $\ell - 2r \geq 0$ and $\eta = \ell - 2r$.
- The outer domain $D^{\mathrm{out},*}_{\rho,\kappa}$ contains the *boomerang domain* $D^{*}_{\kappa,d}$ for $* = u, s$.

3.4 The asymptotic first order of Δ

Even though we have proved the existence of the invariant manifolds in the *boomerang domains*, we need some extra information to detect the asymptotic first order of this difference. The main idea is that functions which are of algebraic order with respect to ε near the singularities $\pm ia$ are exponentially small for real values of u. Thus, the main point to compute the difference and capture the asymptotic first order is to be able to give the main terms of this difference close to the singularities, concretely, up to distance of order ε of the singularities. For that we need to give better approximations of the generating functions $T^{u,s}(u,\tau)$ near of the singularities $\pm ia$ of the homoclinic connection.

To this end, we define the so-called *inner domains* (see Figure 5), which are defined as

$$\begin{aligned}
D^{\mathrm{in},+,u}_{\kappa,c} &= \{u \in \mathbb{C}; \operatorname{Im} u > -\tan\beta_1(\operatorname{Re} u + c\varepsilon^{\gamma}) + a, \operatorname{Im} u < -\tan\beta_2 \operatorname{Re} u + a - \kappa\varepsilon, \\
&\qquad \operatorname{Im} u < -\tan\beta_0 \operatorname{Re} u + a - \kappa\varepsilon\} \\
D^{\mathrm{in},-,u}_{\kappa,c} &= \left\{u \in \mathbb{C}; \bar{u} \in D^{\mathrm{in},+,u}_{\kappa,c}\right\} \\
D^{\mathrm{in},+,s}_{\kappa,c} &= \left\{u \in \mathbb{C}; -\bar{u} \in D^{\mathrm{in},+,u}_{\kappa,c}\right\} \\
D^{\mathrm{in},-,s}_{\kappa,c} &= \left\{u \in \mathbb{C}; -u \in D^{\mathrm{in},+,u}_{\kappa,c}\right\}
\end{aligned} \tag{30}$$

for $\kappa > 0$, $c > 0$ and $\gamma \in (0,1)$. On the other hand, β_1 and β_2 are the angles considered in the definition of the boomerang domains in (26) and β_0 is any angle satisfying that $\beta_1 - \beta_0$ has a positive lower bound independent of ε and μ. Let us observe that, if $u \in D^{\mathrm{in},\pm,*}_{\kappa,c}$, $* = u, s$, then $\mathcal{O}(\kappa\varepsilon) \leq |u \mp ia| \leq \mathcal{O}(\varepsilon^{\gamma})$.

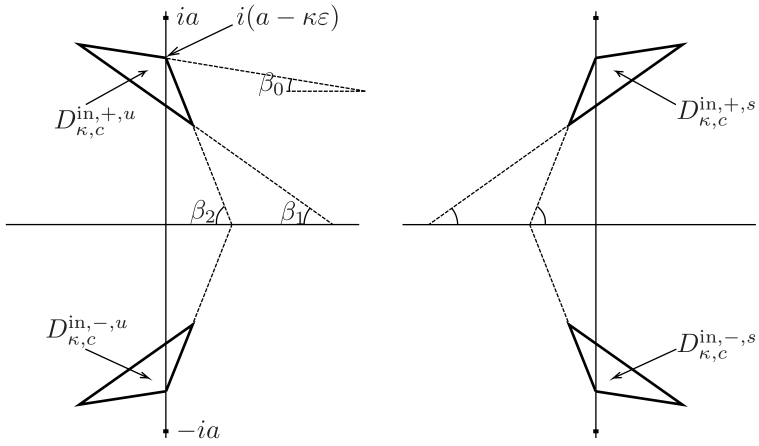

Figure 5: The *inner domains* defined in (30).

Let us observe that simply rewriting $\mu := \mu\varepsilon^{\eta-\eta^*}$, one can include the regular case ($\eta > \eta^*$) in the singular one. This is very convenient since one can prove the results for both cases at the same time. Therefore, from now on in this section, we will focus on the singular case.

When studying the functions $\partial_u T^{u,s}$ evaluated in the inner domains, one can distinguish the cases $\ell-2r<0$ or $\ell-2r\geq 0$. The difference between these two cases, roughly speaking, is that, when $\ell-2r<0$, the approximation of the manifolds in the inner domain is still given by the first order perturbation theory as is stated in Proposition 4.18. In the case $\ell-2r\geq 0$ this fact is not true anymore.

Analyzing $\partial_u T^{u,s}$ close to the singularity ia, one can see that, if $u-ia=\mathcal{O}(\varepsilon)$, then $\partial_u T^{u,s}$ is of order $\mathcal{O}(1/\varepsilon^{2r})$. For this reason we perform the change of variables $u=ia+\varepsilon z$ and we study the functions $\psi^{u,s}(z,\tau)=\varepsilon^{2r-1}T^{u,s}(ia+\varepsilon z,\tau)$. The first order in ε of these functions verifies the so called *inner equation* which does not depend on ε. Their solutions $\psi_0^{u,s}(z,\tau)$ were studied in [Bal06]. Then, in Theorem 4.16 we provide a bound for $|\psi^{u,s}(z,\tau)-\psi_0^{u,s}(z,\tau)|$. This is known as *complex matching*.

It is important to emphasize here that we have not needed to use the inner solutions $\psi_0^{u,s}(z,\tau)$ to extend our functions $T^{u,s}$ to the inner domains since we already knew their existence. Henceforth to bound $|\psi^{u,s}(z,\tau)-\psi_0^{u,s}(z,\tau)|$ we have exploited the same idea as the one for studying the difference $\Delta=T^u-T^s$. Let us explain it with more detail. As we have explained in Section 3.3, we have already proved the existence of generating functions $T^{u,s}$ in the whole *boomerang domains*, therefore, we can translate their properties to the new functions $\psi^{u,s}(z,\tau)=\varepsilon^{2r-1}T^{u,s}(ia+\varepsilon z,\tau)$. Now we consider the difference $\Delta\psi^{u,s}=\partial_z\psi^{u,s}-\partial_z\psi_0$. Such a function (which is known) satisfies a non-homogeneous linear equation which can be "easily" studied. Summarizing, we have just obtained an "a posteriori" bound of $\Delta\psi^{u,s}$. This makes our *complex matching* considerably simpler because we just need to use Gronwall-like techniques.

In both cases $\ell-2r<0$ and $\ell-2r\geq 0$, we have now accurate approximations for $T^{u,s}$ near the singularities. Let us call them $T_0^{u,s}$. The first order asymptotics for the difference $\Delta=T^u-T^s$ comes from $T_0^u-T_0^s$ after a change of variables. Recall that, as we have explained in Section 3.1, in some cases, this change of variables implies an additional correcting term in $T_0^u-T_0^s$. Finally, we bound the remainder by using the techniques explained in Section 3.1.

4 Description of the proofs of Theorems 2.5 and 2.6

We devote this section to prove Theorems 2.5 and 2.6.

4.1 Basic notations

First, we introduce some basic notations which will be used through the paper.

We denote by $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$ the real 1-dimensional torus and by

$$\mathbb{T}_\sigma=\{\tau\in\mathbb{C}/(2\pi\mathbb{Z});|\mathrm{Im}\,\tau|<\sigma\},$$

with $\sigma>0$, the torus with a complex strip.

Given a function $h:D\times\mathbb{T}_\sigma\to\mathbb{C}$, where $D\subset\mathbb{C}$ is an open set, we denote its Fourier series by

$$h(u,\tau)=\sum_{k\in\mathbb{Z}}h^{[k]}(u)e^{ik\tau}$$

and its average by

$$\langle h\rangle(u)=h^{[0]}(u)=\frac{1}{2\pi}\int_0^{2\pi}h(u,\tau)\,d\tau.$$

In any Banach space $(\mathcal{X},\|\cdot\|)$, we define the following balls

$$\begin{aligned}B(R)&=\{x\in\mathcal{X};\|x\|<R\}\\ \overline{B}(R)&=\{x\in\mathcal{X};\|x\|\leq R\}.\end{aligned}$$

By Hypothesis **HP3**, the Hamiltonian H in (6) is analytic in $\tau=t/\varepsilon$. By the compactness of $\mathbb{T}$, actually there exists a constant σ_0 such that H is continuous in $\overline{\mathbb{T}}_{\sigma_0}$ and analytic in $\mathbb{T}_{\sigma_0}$. From now on, we fix $0<\sigma<\sigma_0$.

Throughout the proof of Theorems 2.5 and 2.6 we will use the analyticity in μ. We fix an arbitrary value $\mu_0 > 0$. Even if we do not write it explicitly, all functions which appear in this paper will be analytic in $\mu \in B(\mu_0)$.

From now on, we work with the fast time $\tau = t/\varepsilon$. Then, denoting $' = d/d\tau$, we have the system

$$\left\{ \begin{array}{ll} x' & = \varepsilon \left(y + \mu\varepsilon^\eta \partial_y H_1 \left(x, y, \tau \right) \right) \\ y' & = -\varepsilon \left(V'(x) + \mu\varepsilon^\eta \partial_x H_1 \left(x, y, \tau \right) \right), \end{array} \right. \tag{31}$$

In order to make Sections 4-9 more readable, we will denote by K any constant independent of μ and ε to state all the bounds.

4.2 The periodic orbit

In the parabolic case, Hypothesis **HP4.2** on H_1 implies that the origin is still a critical point of the perturbed system (31). In the hyperbolic case, the next theorem states the existence and useful properties of a hyperbolic periodic orbit close to the origin of the perturbed system.

Theorem 4.1. *Let us assume Hypotheses **HP1.1**, **HP3**, **HP4.1** and $\eta \geq 0$. Then, there exists $\varepsilon_0 > 0$ such that for any $|\mu| < \mu_0$ and $\varepsilon \in (0, \varepsilon_0)$, system (31) has a 2π-periodic orbit $(x_p(\tau), y_p(\tau)) : \mathbb{T}_\sigma \to \mathbb{C}^2$ which is real-analytic and satisfies*

$$\sup_{\tau \in \mathbb{T}_\sigma} \left(|x_p(\tau)| + |y_p(\tau)| \right) \leq b_0 |\mu| \varepsilon^{\eta+1},$$

where $b_0 > 0$ is a constant independent of ε and μ.

This theorem is proved in Section 5.

Once we know the existence of the periodic orbit, we perform the time dependent change of variables

$$\left\{ \begin{array}{l} q = x - x_p(\tau) \\ p = y - y_p(\tau) \end{array} \right. \tag{32}$$

which transforms system (31) into a Hamiltonian system with Hamiltonian function $\varepsilon \widehat{H}(q, p, \tau)$:

$$\begin{aligned} \widehat{H}(q, p, \tau) = & \frac{p^2}{2} + V\left(q + x_p(\tau)\right) - V\left(x_p(\tau)\right) - V'\left(x_p(\tau)\right) q \\ & + \mu\varepsilon^\eta \widehat{H}_1(q, p, \tau) \end{aligned} \tag{33}$$

with

$$\begin{aligned} \widehat{H}_1(q, p, \tau) = & H_1(x_p(\tau) + q, y_p(\tau) + p, \tau) - H_1(x_p(\tau), y_p(\tau), \tau) \\ & - DH_1(x_p(\tau), y_p(\tau), \tau) \begin{pmatrix} q \\ p \end{pmatrix}, \end{aligned} \tag{34}$$

where we have denoted $DH_1 = (\partial_x H_1, \partial_y H_1)$. We have added the terms $V\left(x_p(\tau)\right)$ and $H_1(x_p(\tau), y_p(\tau), \tau)$ for convenience. Note that they don't generate any term in the differential equations associated to $\widehat{H}$.

Since $|(x_p(\tau), y_p(\tau))| = \mathcal{O}\left(\mu\varepsilon^{\eta+1}\right)$, $\widehat{H}_1$ can be split as

$$\widehat{H}_1(q, p, \tau) = \widehat{H}_1^1(q, p, \tau) + \varepsilon \widehat{H}_1^2(q, p, \tau),$$

where

$$\widehat{H}_1^1(q, p, \tau) = H_1(q, p, \tau) - H_1(0, 0, \tau) - DH_1(0, 0, \tau) \begin{pmatrix} q \\ p \end{pmatrix}$$

and $\widehat{H}_1^2(q,p,\tau)$ is the remaining part. In fact, we can give a more precise formula for $\widehat{H}_1^1$ and $\widehat{H}_1^2$ in both the polynomial and the trigonometric cases:

$$\begin{aligned}
\widehat{H}_1^1(q,p,\tau) &= \sum_{2\leq k+l\leq N} a_{kl}(\tau)q^k p^l && \text{(polynomial case)}\\
\widehat{H}_1^1(q,p,\tau) &= \sum_{k=-N,\dots,N} a_{k0}(\tau)\left(e^{ikq}-1-ikq\right) \\
&\quad+\sum_{k=-N,\dots,N} a_{k1}(\tau)\left(e^{ikq}-1\right)p+\sum_{\substack{k=-N,\dots,N\\ l=2,\dots,N}} a_{kl}(\tau)e^{ikq}p^l, && \text{(trigonometric case)}
\end{aligned} \tag{35}$$

where a_{kl} are the functions defined in (8) and (9) and have zero average, that is

$$\langle \widehat{H}_1^1\rangle = 0. \tag{36}$$

Let us point out that $\widehat{H}_1^1$ is H_1 subtracting its linear terms in (x,y), and hence it is of order $n=2$.

The Hamiltonian $\widehat{H}_1^2$ is given by:

$$\begin{aligned}
\widehat{H}_1^2(q,p,\tau) &= \sum_{2\leq k+l\leq N-1} c_{kl}(\tau)q^k p^l && \text{(polynomial case)}\\
\widehat{H}_1^2(q,p,\tau) &= \sum_{k=-N,\dots,N} c_{k0}(\tau)\left(e^{ikq}-1-ikq\right) \\
&\quad+\sum_{k=-N,\dots,N} c_{k1}(\tau)\left(e^{ikq}-1\right)p+\sum_{\substack{k=-N,\dots,N\\ l=2,\dots,N-1}} c_{kl}(\tau)e^{ikq}p^l, && \text{(trigonometric case)}
\end{aligned} \tag{37}$$

where c_{kl} are 2π-periodic functions which, in general, do not have zero average. As we will see in Corollary 5.6 the functions c_{kl} are 2π-periodic and satisfy

$$|c_{kl}(\tau)|\leq K|\mu|\varepsilon^{\eta}. \tag{38}$$

In the case that the unperturbed Hamiltonian has a parabolic point at the origin, since $(x_p,y_p)=(0,0)$, we have that $c_{kl}=0$.

Remark 4.2. *The periodic orbit $(x_p(\tau),y_p(\tau))$, and consequently the Hamiltonians $\widehat{H}$, $\widehat{H}_1$, $\widehat{H}_1^1$, $\widehat{H}_1^2$ depend on the parameters μ, ε. We will not write this dependence explicitly but we will emphasize it when necessary. In particular, the Hamiltonian $\widehat{H}_1^1$ is simply the limit, when $\varepsilon\to 0$, of $\widehat{H}^1$.*

4.3 Different parameterizations of the invariant manifolds

The next step is to prove the existence of the unstable and stable invariant manifolds of the periodic orbit given in Theorem 4.1.

We will consider two different strategies to find suitable parameterizations of these invariant manifolds depending on the domain we are. On the one hand, when it is possible, we will follow [LMS03, Sau01] (see also [GOS10]), and we will write the invariant manifolds as graphs of suitable generating functions which are solutions of a Hamilton-Jacobi equation in appropriate variables. On the other hand, when this is not possible, we will obtain suitable parameterizations of invariant manifolds.

To introduce the first method, let us consider the symplectic change of variables (see [Bal06])

$$\left\{\begin{array}{l} q=q_0(u)\\ p=\dfrac{w}{p_0(u)}, \end{array}\right. \tag{39}$$

where $(q_0(u),p_0(u))$ is the parameterization of the homoclinic orbit given in Hypothesis **HP2**. This is a well defined change for any $u\in\mathbb{C}$ such that $p_0(u)\neq 0$ and leads to a new Hamiltonian given by

$$\varepsilon\overline{H}(u,w,\tau)=\varepsilon\widehat{H}\left(q_0(u),\frac{w}{p_0(u)},\tau\right), \tag{40}$$

where $\widehat{H}$ is the Hamiltonian defined in (33).

Let us recall that when $\mu = 0$, $\widehat{H}$ becomes H_0 defined in (7). Then, the separatrix of the unperturbed system ($\mu = 0$) for $\overline{H}$ can be parameterized as a graph as $w = p_0(u)^2$.

If we want to obtain parameterizations of the perturbed invariant manifolds, we can take into account the well known fact that, locally, they are Lagrangian and can be obtained as graphs of some functions which are solutions of the Hamilton-Jacobi equation.

Therefore, we look for the invariant manifolds as graphs of generating functions which are solutions of the Hamilton-Jacobi equation associated to the Hamiltonian $\varepsilon\overline{H}$. That is, we write them as $w = \partial_u T^{u,s}(u,\tau)$, where the functions $T^{u,s}$ satisfy

$$\partial_\tau T(u,\tau) + \varepsilon\overline{H}(u, \partial_u T(u,\tau), \tau) = 0 \tag{41}$$

and certain limiting properties.

The solutions of this equation give parameterizations of the invariant manifolds, which, in the original variables, read

$$(q,p) = \left(q_0(u), \frac{\partial_u T^{u,s}(u,\tau)}{p_0(u)}\right). \tag{42}$$

Notice that in variables (q,p) the condition $p_0(u) = \dot{q}_0(u) \neq 0$ ensures that the manifolds can be written as graphs over the variable q through the functions $S^{u,s}(q,\tau) = T^{u,s}(q_0^{-1}(q),\tau)$ wich verify the classical Hamilton-Jacobi equation associated to the Hamiltonian $\widehat{H}(q,p,\tau)$.

When this method cannot be used, that is when $p_0(u)$ can vanish, we look for the invariant manifolds as parameterizations:

$$(q,p) = (Q(v,\tau), P(v,\tau)) \tag{43}$$

in such a way that $(Q(v+t, \tau+\frac{t}{\varepsilon}), P(v+t,\tau+\frac{t}{\varepsilon}))$ are solutions of the differential equation associated to the Hamiltonian (33). These kind of parameterizations were used in [DS92, DS97, Gel97a, Gel00, BF04, BF05].

Then, it is straightforward to see ([Gel97a]) that (Q,P) has to satisfy

$$\mathcal{L}_\varepsilon \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} P + \mu\varepsilon^\eta \partial_p \widehat{H}_1(Q,P,\tau) \\ -\left(V'(Q + x_p(\tau)) - V'(x_p(\tau))\right) - \mu\varepsilon^\eta \partial_q \widehat{H}_1(Q,P,\tau) \end{pmatrix}, \tag{44}$$

where $\mathcal{L}_\varepsilon$ is the operator

$$\mathcal{L}_\varepsilon = \varepsilon^{-1}\partial_\tau + \partial_v \tag{45}$$

and $\widehat{H}_1$ is the Hamiltonian defined in (34).

Both parameterizations (42), (44) satisfy that, fixing $\tau = \tau_*$, they give parameterizations of the invariant curves of the fixed point of the 2π-Poincaré map from the section $\tau = \tau^*$ to the section $\tau = \tau^* + 2\pi$.

4.4 Existence of the local invariant manifolds

In this section we will find the local invariant manifolds of the origin of the Hamiltonian system (33).

First, we recall the behavior of the separatrix $(q_0(u), p_0(u))$ as $\operatorname{Re} u \to \pm\infty$, which is substantially different depending whether $(0,0)$ is a hyperbolic or a parabolic point of the unperturbed system.

In the hyperbolic case, by Hypothesis **HP1.1**, close to $x = 0$ the potential behaves as

$$V(x) = -\frac{\lambda^2}{2}x^2 + \mathcal{O}(x^3). \tag{46}$$

Therefore, $\{\lambda, -\lambda\}$ are the eigenvalues of the critical point. Moreover, there exist constants $c_\pm \neq 0$ such that as $\operatorname{Re} u \to \mp\infty$ the separatrix behaves as

$$\begin{aligned} q_0(u) &= c_\pm e^{\pm\lambda u} + \mathcal{O}\left(e^{\pm 2\lambda u}\right), \\ p_0(u) &= \pm\lambda c_\pm e^{\pm\lambda u} + \mathcal{O}\left(e^{\pm 2\lambda u}\right). \end{aligned} \tag{47}$$

In the parabolic case, using Hypothesis **HP1.2**, in [BF04] it is seen that there exists a constant c_0 such that as $\operatorname{Re} u \to \mp\infty$ the separatrix behaves as

$$
\begin{aligned}
q_0(u) &= \frac{c_0}{u^{\frac{2}{m-2}}} + \mathcal{O}\left(\frac{1}{u^\nu}\right), \\
p_0(u) &= -\frac{2c_0}{(m-2)u^{\frac{m}{m-2}}} + \mathcal{O}\left(\frac{1}{u^{\nu+1}}\right),
\end{aligned}
\tag{48}
$$

where m is the order of the potential (11) and $\nu > 2/(m-2)$.

We look for the parameterizations of the local invariant manifolds in the domains $D^{u,s}_{\infty,\rho}$ defined in (28).

By (47) and (48), the constant ρ can be taken big enough so that $p_0(u)$ does not vanish in these domains. Then, as we explained in Section 4.3, we can look for the invariant manifolds by means of generating functions $T^{u,s}$ defined in $D^*_{\infty,\rho}$ with $* = u, s$ respectively, which are solutions of the Hamilton-Jacobi equation (41). Moreover, we impose the asymptotic conditions

$$\lim_{\operatorname{Re} u \to -\infty} p_0^{-1}(u) \cdot \partial_u T^u(u,\tau) = 0 \quad \text{(for the unstable manifold)} \tag{49}$$

$$\lim_{\operatorname{Re} u \to +\infty} p_0^{-1}(u) \cdot \partial_u T^s(u,\tau) = 0 \quad \text{(for the stable manifold).} \tag{50}$$

The functions $T^{u,s}$ will give us parameterizations of the invariant manifolds of $(0,0)$ of the Hamiltonian system with Hamiltonian (33) of the form (42) and thus, these asymptotic conditions ensure that the invariant manifolds tend to the critical point $(0,0)$ as $\operatorname{Re} u \to \pm\infty$.

We note that when $\mu = 0$ a solution of (41) satisfying both asymptotic conditions (49) and (50) is

$$T_0(u) = \int_{-\infty}^{u} p_0^2(v)\, dv, \tag{51}$$

which corresponds to the the unperturbed separatrix.

The next theorem gives the existence of the invariant manifolds in the domains $D^*_{\infty,\rho}$ with $* = u, s$ defined in (28). We state the results for the unstable invariant manifold. The stable one has analogous properties.

Theorem 4.3. *Let us assume Hypotheses **HP1.1**, **HP3**, **HP4** and take $\eta \geq 0$. Let $\rho_1 > 0$ be a real number big enough such that $p_0(u) \neq 0$ for $u \in D^u_{\infty,\rho_1}$. Then, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$ and $\mu \in B(\mu_0)$, the Hamilton-Jacobi equation (41) has a unique (modulo an additive constant) real-analytic solution in $D^u_{\infty,\rho_1} \times \mathbb{T}_\sigma$ satisfying the asymptotic condition (49).*

Moreover, there exists a real constant $b_1 > 0$ independent of ε and μ, such that for $(u,\tau) \in D^u_{\infty,\rho_1} \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u,\tau) - \partial_u T_0(u)| \leq b_1 |\mu| \varepsilon^{\eta+1}.$$

The asymptotic behavior of the invariant manifolds when $\operatorname{Re} u \to +\infty$ is qualitatively different for the hyperbolic case and the parabolic case. For this reason we prove separately Theorem 4.3 for these two cases. We deal with the hyperbolic case in Section 6.1 and with the parabolic case in Section 6.2.

In the rest of the paper we will assume the whole set of Hypotheses **HP1**, **HP2**, **HP3**, **HP4**, **HP5** and **HP6**.

4.5 The global invariant manifolds

The next step is to extend the invariant manifolds to a wider domain which contains a region close to the singularities $\pm ia$ of the separatrix (see Hypothesis **HP2**). In the general case the function $p_0(u)$ might vanish and therefore, the symplectic change (39) is not well defined. For this reason one cannot use the Hamilton-Jacobi equation (41) anymore. Instead we look for parameterizations

$$(q,p) = (Q^{u,s}(v,\tau), P^{u,s}(v,\tau))$$

which are solutions of the partial differential equation (44).

Nevertheless, there are some cases, as happens for the classical pendulum, in which $p_0(u)$ does not vanish for $u \in \mathbb{C}$, and then one can use the Hamilton-Jacobi equation in the whole domain, which makes the proof of Theorems 2.5 and 2.6 remarkably simpler. Section 4.5.1 is devoted to this simpler case and Section 4.5.2 to the general one.

4.5.1 The global invariant manifolds in the case $p_0(u) \neq 0$

In this section we extend the parameterizations (42) of the invariant manifolds to the outer domains $D^{\mathrm{out},*}_{\rho,\kappa}$, $* = u, s$, (see Figure 4) defined by (29), in the case that $p_0(u) \neq 0$. We emphasize that these domains reach a region which is a distance of $\mathcal{O}(\varepsilon)$ to the singularities $u = \pm ia$ of the unperturbed separatrix.

The constant ρ will be taken $\rho > \rho_1$, where ρ_1 is the constant given by Theorem 4.3, in order to ensure that $D^u_{\infty,\rho_1} \cap D^{\mathrm{out},u}_{\rho,\kappa} \neq \emptyset$.

Since in this section we are assuming that $p_0(u) \neq 0$ in the whole *outer domain*, the symplectic change of variables (39) is still well defined there. Then, it is enough to look for the analytic continuation of the generating functions $T^{u,s}$ obtained in Theorem 4.3.

Theorem 4.4. *Let ρ_1 be the constant considered in Theorem 4.3 and let us consider ρ_2 such that $\rho_2 > \rho_1$, $\kappa_1 > 0$ big enough and $\varepsilon_0 > 0$ small enough. Then, for $\mu \in B(\mu_0)$, $\varepsilon \in (0, \varepsilon_0)$, the function $T^u(u,\tau)$ obtained in Theorem 4.3 can be analytically extended to the domain $D^{\mathrm{out},u}_{\rho_2,\kappa_1} \times \mathbb{T}_\sigma$.*

Moreover, there exists a real constant $b_2 > 0$ independent of ε and μ, such that for $(u,\tau) \in D^{\mathrm{out},u}_{\rho_2,\kappa_1} \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u,\tau) - \partial_u T_0(u)| \leq \frac{b_2 |\mu| \varepsilon^{\eta+1}}{|u^2 + a^2|^{\ell+1}},$$

where T_0 is the unperturbed separatrix, given in (51).

The proof of this theorem is given in Section 7.1. The results for the stable manifold are analogous.

4.5.2 The global invariant manifolds for the general case

We devote this section to obtain parameterizations of the global invariant manifolds for the general case, that is, considering Hamiltonian systems for which $p_0(u)$ might vanish in the outer domains defined in (29). In this case, we cannot use the Hamilton-Jacobi equation (41) in the whole outer domains. For this reason, we look first for parameterizations

$$(q,p) = (Q^{u,s}(v,\tau), P^{u,s}(v,\tau))$$

which are solutions of the partial differential equation (44). Our strategy will be:

- To obtain the parameterizations $(Q^{u,s}(v,\tau), P^{u,s}(v,\tau))$ in a transition domain (Theorem 4.5).
- To extend them up to a region where we can ensure that $p_0(u)$ does not vanish (Theorem 4.6).
- To recover in this new region the representations (42) through the generating function $T^{u,s}$ of the manifolds, which are solution of the Hamilton-Jacobi equation (41) (Theorem 4.7).
- To extend the generating function $\partial_u T^{u,s}(u,\tau)$ up to a distance of order ε of the singularity, as it was done in the easier case $p_0(u) \neq 0$ in Theorem 4.4 (Theorem 4.8).

First we are going to construct the two dimensional parameterizations of the invariant manifolds from the parameterizations of the local invariant manifolds given in Theorem 4.3, which were obtained by using the Hamilton-Jacobi equation. We look for them in the transition domains

$$\begin{aligned} I^u_{\rho,\bar\rho} &= D^{\mathrm{out},u}_{\kappa,\bar\rho} \cap D^u_{\infty,\rho} \\ I^s_{\rho,\bar\rho} &= D^{\mathrm{out},s}_{\kappa,\bar\rho} \cap D^s_{\infty,\rho} \end{aligned} \tag{52}$$

with $\bar{\rho} > \rho$ (see Figure 6). Taking into account the change of variables (39), it is natural to look for the parameterizations of the invariant manifolds $(Q^{u,s}, P^{u,s})$ of the form

$$\begin{aligned} Q^{u,s}(v,\tau) &= q_0\left(v + \mathcal{U}^{u,s}(v,\tau)\right) \\ P^{u,s}(v,\tau) &= \frac{\partial_u T^{u,s}\left(v + \mathcal{U}^{u,s}(v,\tau)\right)}{p_0(v + \mathcal{U}^{u,s}(v,\tau))}, \end{aligned} \tag{53}$$

where $\mathcal{U}^{u,s}$ define a change of variables $u = v + \mathcal{U}^{u,s}(v,\tau)$ in such a way that $(Q^{u,s}, P^{u,s})$ satisfy the system of equations (44).

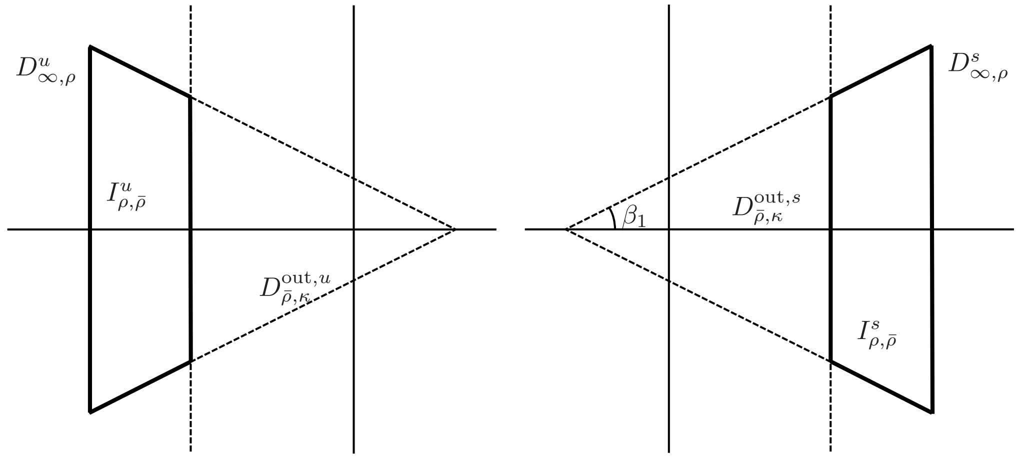

Figure 6: The transition domains $I^u_{\rho,\bar{\rho}}$ and $I^s_{\rho,\bar{\rho}}$ defined in (52).

The next theorem ensures that this change of variables exists and it is well defined in the transition domains $I^*_{\rho,\bar{\rho}}$, $* = u, s$. We state it for $\mathcal{U}^u$. The results for $\mathcal{U}^s$ are analogous.

Theorem 4.5. *Let ρ_1 be the constant considered in Theorem 4.3 and let us consider ρ_3 and ρ_4 such that $\rho_4 > \rho_3 > \rho_1$ and ε_0 small enough (which might depend on ρ_i, $i = 1, 2, 3$). Then, for $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in B(\mu_0)$, there exists a real-analytic function $\mathcal{U}^u : I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma \to \mathbb{C}$ such that*

- *There exists a constant $b_3 > 0$ independent of ε and μ such that for $(v,\tau) \in I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$,*

$$|\mathcal{U}^u(v,\tau)| \leq b_3|\mu|\varepsilon^{\eta+1}.$$

- *If $(v,\tau) \in I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$, then $v + \mathcal{U}^u(v,\tau) \in D^u_{\infty,\rho_1}$.*
- *The parameterizations of the invariant manifolds $(Q^u(v,\tau), P^u(v,\tau))$ in (53) satisfy the system of equations (44) and there exists a constant $b_4 > 0$ such that for $(v,\tau) \in I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$,*

$$\begin{aligned} |Q^u(v,\tau) - q_0(v)| &\leq b_4|\mu|\varepsilon^{\eta+1} \\ |P^u(v,\tau) - p_0(v)| &\leq b_4|\mu|\varepsilon^{\eta+1}, \end{aligned}$$

*where (q_0, p_0) is the parametrization of the unperturbed separatrix given in Hypothesis **HP2**.*

The proof of this theorem is deferred to section 7.2.2.

Having the parameterizations $(Q^{u,s}(v,\tau), P^{u,s}(v,\tau))$ in the transition domains $I^*_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$ for $* = u, s$, we extend them till we arrive to a region where we can ensure that $p_0(u)$ does not vanish anymore. This region consists of a piece of the *boomerang domains* defined in (26) (see Figure 2), in which $p_0(u) \neq 0$, and hence the parameterizations (42) will be well defined in them.

The next step is to extend the parameterizations $(Q^{u,s}(v,\tau), P^{u,s}(v,\tau))$ provided in Theorem 4.5 up to domains which intersect the *boomerang domains* $D^u_{\kappa,d}$ and $D^s_{\kappa,d}$ respectively. To this end, we define the following domains

$$\begin{aligned}\widetilde{D}^{\mathrm{out},u}_{\rho,d,\kappa} &= D^{\mathrm{out},u}_{\rho,\kappa} \cap \left\{ u \in \mathbb{C}; |\mathrm{Im}\, u| < -\tan\beta_2 \mathrm{Re}\, u + a - \frac{d}{2} \right\} \\ \widetilde{D}^{\mathrm{out},s}_{\rho,d,\kappa} &= D^{\mathrm{out},s}_{\rho,\kappa} \cap \left\{ u \in \mathbb{C}; |\mathrm{Im}\, u| > \tan\beta_2 \mathrm{Re}\, u + a - \frac{d}{2} \right\},\end{aligned} \tag{54}$$

which are depicted in Figure 7.

We want to emphasize that to extend the parameterizations $(Q^{u,s}(v,\tau), P^{u,s}(v,\tau))$ to these new domains, has no technical difficulties since they are far from the singularities $u = \pm ia$. Consequently the next theorem is a classical perturbative result.

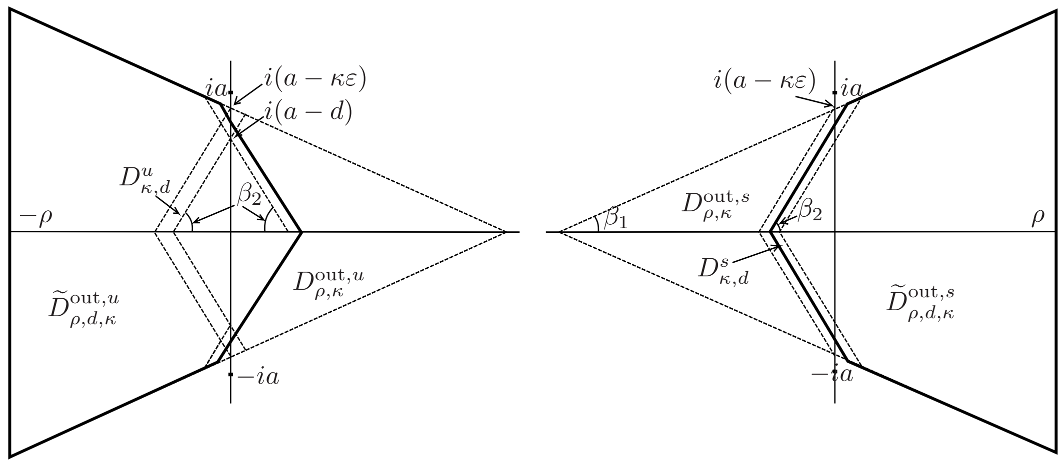

Figure 7: The domains $\widetilde{D}^{\mathrm{out},u}_{\rho,d,\kappa}$ and $\widetilde{D}^{\mathrm{out},s}_{\rho,d,\kappa}$ defined in (54).

The next theorem gives the results for the unstable manifold. For the stable one are analogous.

Theorem 4.6. *Let ρ_4 and κ_1 be the constants considered in Theorems 4.5 and 4.4, $d_0 > 0$ and $\varepsilon_0 > 0$ small enough. Then, for $\mu \in B(\mu_0)$ and $\varepsilon \in (0, \varepsilon_0)$, there exist functions $(Q^u(v,\tau), P^u(v,\tau))$ defined in $\widetilde{D}^{\mathrm{out},u}_{\rho_4,d_0,\kappa_1} \times \mathbb{T}_\sigma$ satisfying equation (44) and such that they are the analytic continuation of the parameterizations of the invariant manifolds obtained in Theorem 4.5.*

Moreover, there exists a constant $b_5 > 0$ independent of ε and μ such that for $(v,\tau) \in \widetilde{D}^{\mathrm{out},u}_{\rho_4,d_0,\kappa_1} \times \mathbb{T}_\sigma$,

$$\begin{aligned}|Q^u(v,\tau) - q_0(v)| &\leq b_5 |\mu| \varepsilon^{\eta+1} \\ |P^u(v,\tau) - p_0(v)| &\leq b_5 |\mu| \varepsilon^{\eta+1}.\end{aligned}$$

The proof of this theorem is given in Section 7.2.3.

Theorem 4.6 provides parameterizations of the invariant manifolds of the form (43) in the domains $\widetilde{D}^{\mathrm{out},u}_{\rho,d,\kappa}$ and $\widetilde{D}^{\mathrm{out},s}_{\rho,d,\kappa}$. In particular, they are defined in the following transition domains, which are depicted in Figure 8.

$$\begin{aligned}I^{\mathrm{out},u}_{\kappa,d} &= \widetilde{D}^{\mathrm{out},u}_{\rho,d,\kappa} \cap D^u_{\kappa,d} \\ I^{\mathrm{out},s}_{\kappa,d} &= \widetilde{D}^{\mathrm{out},s}_{\rho,d,\kappa} \cap D^s_{\kappa,d},\end{aligned} \tag{55}$$

where, by construction, $p_0(u)$ does not vanish. Then, we can use these domains as transition domains where we can go back to the parameterizations (42) and where the Hamilton-Jacobi equation (41) can be used. To obtain them, we look for changes of variables $v = u + \mathcal{V}^{u,s}(u,\tau)$ which satisfy

$$Q^{u,s}(u + \mathcal{V}^{u,s}(u,\tau), \tau) = q_0(u), \tag{56}$$

where $Q^{u,s}$ are the first components of the parameterizations obtained in Theorem 4.6. Once we have them, we will define the generating functions $T^{u,s}$ which give the parameterizations (42). Let us observe that if $p_0(u)$ does not vanish in the outer domains, the changes of variables $v = u + \mathcal{V}^{u,s}(u,\tau)$ are defined in the whole domain and they are the inverse of the changes $u = v + \mathcal{U}^{u,s}(v,\tau)$ obtained in Theorem 4.5.

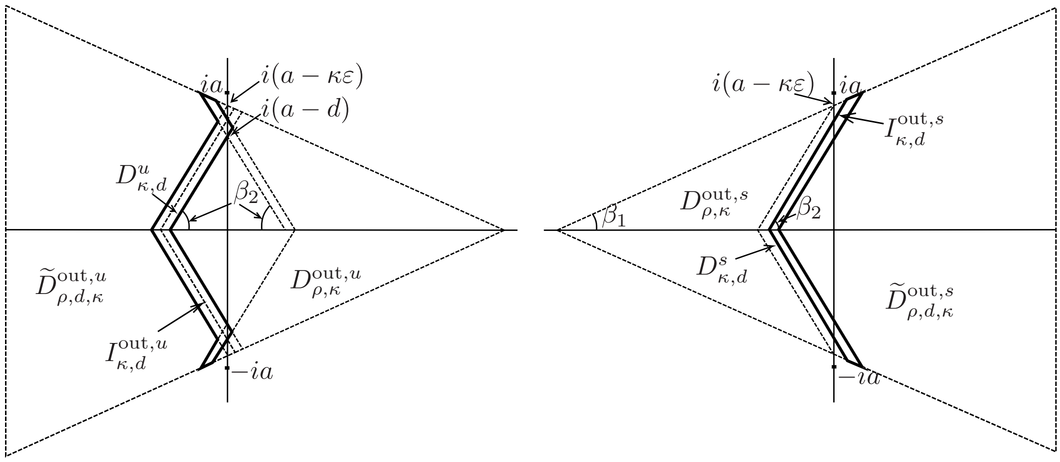

Figure 8: The domains $I^{\text{out},u}_{\kappa,d}$ and $I^{\text{out},s}_{\kappa,d}$ defined in (55).

We state the next theorem for the unstable manifold in the domain $I^{\text{out},u}_{\kappa,d}$. The stable case is analogous.

Theorem 4.7. *Let d_0, κ_1, ρ_4 be the constants given in Theorem 4.6, $\kappa_2 > \kappa_1$, $d_1 < d_0$ and $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in B(\mu_0)$, and increasing κ_1 if necessary,*

- *There exists a real-analytic function $\mathcal{V}^u : I^{\text{out},u}_{\kappa_2,d_1} \times \mathbb{T}_\sigma \to \mathbb{C}$ which satisfies (56). Moreover, if $(u,\tau) \in I^{\text{out},u}_{\kappa_2,d_1} \times \mathbb{T}_\sigma$, then $u + \mathcal{V}^u(u,\tau) \in I^{\text{out},u}_{\kappa_1,d_0}$ and*

$$|\mathcal{V}^u(u,\tau)| \leq b_6|\mu|\varepsilon^{\eta+1}.$$

  with b_6 a constant independent of μ and ε.

- *There exists a generating function $T^u : I^{\text{out},u}_{\kappa_2,d_1} \times \mathbb{T}_\sigma \to \mathbb{C}$ such that*

$$\partial_u T^u(u,\tau) = p_0(u) P^u(u + \mathcal{V}^u(u,\tau), \tau),$$

  where P^u is the function obtained in Theorem 4.6, and satisfies equation (41). Then, we have that $(q,p) = (q_0, p_0(u)^{-1}\partial_u T^u(u,\tau))$ is a parametrization of the unstable invariant manifold of the form (42). Moreover, there exists a constant $b_7 > 0$ such that, for $(u,\tau) \in I^{\text{out},u}_{\kappa_2,d_1} \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u,\tau) - \partial_u T_0(u)| \leq b_7|\mu|\varepsilon^{\eta+1}.$$

This theorem is proved in Section 7.2.4.

The final step is to extend the just obtained parameterizations of the form (42) to the whole *boomerang domains* $D^u_{\kappa,d}$ and $D^s_{\kappa,d}$ defined in (26) (see also Figure 2). In particular the whole *boomerang domains* contain points up to a distance $\kappa\varepsilon$ of the singularities $\pm ia$. The next theorem gives the results for the unstable manifold. The stable one satisfies analogous ones.

Theorem 4.8. *Let κ_2 and d_1 be the constants given in Theorem 4.7, $d_2 < d_1$, $\kappa_3 > \kappa_2$ big enough and $\varepsilon_0 > 0$ small enough. Then, for $\mu \in B(\mu_0)$ and $\varepsilon \in (0, \varepsilon_0)$, the function $T^u(u,\tau)$ obtained in Theorem 4.7 can be analytically extended to the domain $D^u_{\kappa_3,d_2} \times \mathbb{T}_\sigma$.*

Moreover, there exists a real constant $b_8 > 0$ independent of ε and μ, such that for $(u,\tau) \in D^u_{\kappa_3,d_2} \times \mathbb{T}_\sigma$,

$$|\partial_u T^u(u,\tau) - \partial_u T_0(u)| \le \frac{b_8|\mu|\varepsilon^{\eta+1}}{|u^2+a^2|^{\ell+1}},$$

where T_0 is the unperturbed separatrix given in (51).

The proof of this theorem is given in Section 7.2.5.

Remark 4.9. *Let us point out that these domains satisfy $D^u_{\kappa,d} \subset D^{\mathrm{out},u}_{\rho,\kappa}$ and $D^s_{\kappa,d} \subset D^{\mathrm{out},s}_{\rho,\kappa}$ if ρ is big enough. Therefore, in the case that $p_0(u)$ does not vanish, Theorem 4.4 ensures that the functions $T^{u,s}$ are already defined in $D^u_{\kappa,d}$ and $D^s_{\kappa,d}$ respectively.*

Let us observe that, if ε is small enough, $D^{\mathrm{in},\pm,s}_{\kappa,c} \subset D^s_{\kappa,d}$ and $D^{\mathrm{in},\pm,u}_{\kappa,c} \subset D^u_{\kappa,d}$.

After Theorem 4.4 and 4.8 there is no difference between the case $p_0(u) \neq 0$, when the invariant manifolds can be written as graphs globally, and the general case when p_0 can vanish: we have found *boomerang domains* which intersect the real line and which reach neighborhoods of size $\kappa\varepsilon$ of the singularities where both manifolds can be written as graphs. This will be the starting point in our strategy to measure the distance between the invariant manifolds.

4.6 The asymptotic first order of $\partial_u T^{u,s}$ close to the singularities $\pm ia$

Theorems 4.4 and 4.8 are valid for $\eta \ge \max\{0, \ell - 2r\}$. We notice that when $\ell \le 2r$ the result is true for $\eta \ge 0$, but only if $\ell < 2r$ Theorems 4.4 and 4.8 give a *classical* perturbative result with respect to the singular parameter ε, in the sense that the main term of $\partial_u T^{u,s}$ is given by the unperturbed separatrix $\partial_u T_0$ in the whole *outer* domains. This fact is not true anymore in the case $\ell - 2r \ge 0$ and then we will have to look for different approximations of the invariant manifolds close to the singularities $u = \pm ia$, by using suitable solutions of the so-called *inner* equations. Consequently, the case $\ell < 2r$ is easier to deal with, because it is a regular case and there is no need of using *inner* equations to obtain a better approximation of $\partial_u T^{u,s}$ near the singularities $\pm ia$ of T_0.

We separate both cases $\ell < 2r$ and $\ell \ge 2r$ in the corresponding sections below.

4.6.1 The asymptotic first order of $\partial_u T^{u,s}$ for the case $\ell < 2r$

In this section we will assume that $\ell < 2r$ and henceforth we are dealing with values of $\eta \ge 0$.

To obtain the main term of $\partial_u T^{u,s} - \partial_u T_0$ we just need to use classical perturbation theory even in the *inner domains* $D^{\mathrm{in},\pm,*}_{\kappa,c}$, $* = u, s$, defined in (30) (see Figure 5). Let us observe that, if $u \in D^{\mathrm{in},\pm,*}_{\kappa,c}$, $* = u, s$, then $\mathcal{O}(\kappa\varepsilon) \le |u \mp ia| \le \mathcal{O}(\varepsilon^\gamma)$.

The next proposition gives the first order asymptotic terms of $\partial_u T^{u,s} - \partial_u T_0$ close to $u = ia$, that is in $D^{\mathrm{in},+,*}_{\kappa,c}$, $* = u, s$. The study close to $u = -ia$ can be done analogously.

Proposition 4.10. *Let us assume $\ell - 2r < 0$ and $0 < \gamma < \min\{1, \frac{\ell+1}{r+1}\}$ where γ is the constant involved in the definition of the inner domains in* (30). *Let us consider the constant κ_3 given by Theorem 4.8 and $c_1 > 0$ and let us define the constant*

$$\nu^* = \min\{\nu_1^*, \nu_2^*, 1 - \max\{0, \ell - 2r + 1\}, r, \ell, \ell + 1 - (r+1)\gamma\} > 0,$$

where

$$\nu_1^* = \min\{(2r-\ell)\gamma, 1\}$$
$$\nu_2^* = \begin{cases} \ell(1-\gamma) & \text{if } \ell > 0 \\ 1-\gamma & \text{if } \ell = 0 \end{cases}.$$

Let us also define the functions

$$
\begin{aligned}
\mathcal{T}_0^u(u,\tau) =& -\mu\varepsilon^\eta \int_{-\infty}^0 H_1(q_0(u+t), p_0(u+t), \tau+\varepsilon^{-1}t)\, dt \\
\mathcal{T}_0^s(u,\tau) =& -\mu\varepsilon^\eta \int_{+\infty}^0 H_1(q_0(u+t), p_0(u+t), \tau+\varepsilon^{-1}t)\, dt,
\end{aligned}
\tag{57}
$$

where H_1 is the function defined in (8) *and* (9) *and $(q_0(u), p_0(u))$ is the parameterization of the unperturbed separatrix given in Hypothesis **HP2**. Then,there exists $\varepsilon_0 > 0$ and a constant $b_9 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in B(\mu_0)$ the following bounds are satisfied.*

- *If $(u,\tau) \in D^{\mathrm{in},+,u}_{\kappa_3,c_1} \times \mathbb{T}_\sigma$,*

$$|\partial_u T^u(u,\tau) - \partial_u T_0(u) - \partial_u \mathcal{T}_0^u(u,\tau)| \le b_9|\mu|\varepsilon^{\eta-\ell+\nu^*}.$$

- *If $(u,\tau) \in D^{\mathrm{in},+,s}_{\kappa_3,c_1} \times \mathbb{T}_\sigma$,*

$$|\partial_u T^s(u,\tau) - \partial_s T_0(u) - \partial_u \mathcal{T}_0^s(u,\tau)| \le b_9|\mu|\varepsilon^{\eta-\ell+\nu^*}.$$

This proposition is proved in Section 7.1.

4.6.2 The first asymptotic order of $\partial_u T^{u,s}$ for the case $\ell \ge 2r$

Theorems 4.4 and 4.8 give the existence of parameterizations of the invariant manifolds of the form (42) in D^s_{κ,d_2} and D^u_{κ,d_2} for ε small enough and κ big enough. Nevertheless, when $\eta = \ell - 2r$ the parameterizations of the perturbed invariant manifolds are not well approximated by the unperturbed separatrix when u is at a distance of order $\mathcal{O}(\varepsilon)$ of the singularities $u = \pm ia$. For this reason, to obtain the first asymptotic order of the difference between the manifolds, we need to look for better approximations $T^{u,s}$ in the inner domains defined in (30). We obtain them through a singular limit. Since we are dealing with the case $\eta \ge \ell - 2r$, the first step is to define a new parameter

$$\hat\mu = \mu\varepsilon^{\eta-(\ell-2r)}. \tag{58}$$

Then, the Hamiltonian $\widehat{H}$ reads

$$
\begin{aligned}
\widehat{H}(q,p,\tau) =& \frac{p^2}{2} + V(q + x_p(\tau)) - V(x_p(\tau)) - V'(x_p(\tau))\, q \\
&+ \hat\mu\varepsilon^{\ell-2r}\widehat{H}_1(q,p,\tau)
\end{aligned}
\tag{59}
$$

and, from $\widehat{H}$, one can define the Hamiltonian $\overline{H}$ in (40) using again the change (39). On the other hand, from Theorems 4.4 and 4.8, one can obtain bounds for the parameterizations of the invariant manifolds in terms of $\hat\mu$ and ε. We state them for the unstable manifold. The stable manifold satisfies analogous bounds.

Corollary 4.11. *Let us consider the constants κ_3 and d_2 defined in Theorem 4.8. Then the function T^u obtained in Theorems 4.4 and 4.8, which is defined for $(u,\tau) \in D^u_{\kappa_3,d_2} \times \mathbb{T}_\sigma$, satisfies*

$$|\partial_u T^u(u,\tau) - \partial_u T_0(u)| \le \frac{b_8|\hat\mu|\varepsilon^{\ell-2r+1}}{|u^2+a^2|^{\ell+1}},$$

where T_0 is the unperturbed separatrix given in (51).

We want to study the invariant manifolds close to the singularities $u = \pm ia$, that is, in the inner domains defined in (30). Since the study of both invariant manifolds close either to $u = ia$ or $u = -ia$ is analogous, we only study them in the domain $D^{\mathrm{in},+,u}_{\kappa,c}$. Then, we consider the change of variables

$$z = \varepsilon^{-1}(u - ia). \tag{60}$$

The variable z is called the *inner variable*, in contraposition to the *outer variable* u. We note that, by definition of T_0 in (51) and using the expansion around the singularities of $p_0(u)$ in (12) and (13), we have that

$$\partial_u T_0(\varepsilon z + ia) = \frac{C_+^2}{\varepsilon^{2r} z^{2r}} \left(1 + \mathcal{O}\left((\varepsilon z)^{1/\beta}\right)\right)$$

and, using the results of Corollary 4.11, we have that

$$|\partial_u T^{u,s}(\varepsilon z + ia, \tau) - \partial_u T_0(\varepsilon z + ia)| \le K \frac{|\hat{\mu}|}{\varepsilon^{2r}|z|^{\ell+1}}.$$

Hence, in order to catch the terms of the same order in ε, we scale the generating function as

$$\psi^{u,s}(z,\tau) = \varepsilon^{2r-1} C_+^{-2} T^{u,s}(ia + \varepsilon z, \tau). \tag{61}$$

Then, the Hamilton-Jacobi equation (41) reads

$$\partial_\tau \psi + \varepsilon^{2r} C_+^{-2} \overline{H}\left(ia + \varepsilon z, \varepsilon^{-2r} C_+^2 \partial_z \psi, \tau\right) = 0, \tag{62}$$

where $\overline{H}$ is the Hamiltonian function defined in (40). The corresponding Hamiltonian is

$$\mathcal{H}(z,w,\tau) = \varepsilon^{2r} C_+^{-2} \overline{H}\left(ia + \varepsilon z, \varepsilon^{-2r} C_+^2 w, \tau\right). \tag{63}$$

We study equation (62) in the domain $\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c} \times \mathbb{T}_\sigma$, where

$$\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c} = \left\{z \in \mathbb{C}; ia + \varepsilon z \in D^{\mathrm{in},+,u}_{\kappa,c}\right\}. \tag{64}$$

To study equation (62), as a first step it is natural to study it in the limit case $\varepsilon = 0$. In the polynomial case it reads

$$\partial_\tau \psi_0 + \frac{1}{2} z^{2r} \left(\partial_z \psi_0\right)^2 - \frac{1}{2z^{2r}} + \frac{\hat{\mu}}{z^\ell} \sum_{(r-1)k + rl = \ell} \frac{C_+^{k+l-2}}{(1-r)^k} a_{kl}(\tau) \left(z^{2r} \partial_z \psi_0\right)^l = 0. \tag{65}$$

The solutions of this equation were studied in detail in [Bal06], where equation (65) was rewritten as

$$\partial_\tau \psi_0 + \frac{1}{2} z^{2r} \left(\partial_z \psi_0\right)^2 - \frac{1}{2z^{2r}} + \frac{\hat{\mu}}{z^\ell} \sum_{l=0}^{N} A_l(\tau) \left(z^{2r} \partial_z \psi_0\right)^l = 0, \tag{66}$$

where

$$A_l(\tau) = \sum_{(r-1)k + rl = \ell} \frac{C_+^{k+l-2}}{(1-r)^k} a_{kl}(\tau), \tag{67}$$

and a_{kl} are the coefficients of H_1 in (8) and C_+ is given in **HP2**. This equation is in fact the Hamilton-Jacobi equation associated to the non-autonomous Hamiltonian

$$\mathcal{H}_0(z,w,\tau) = \frac{1}{2} z^{2r} w^2 - \frac{1}{2z^{2r}} + \frac{\hat{\mu}}{z^\ell} \sum_{l=0}^{N} A_l(\tau) \left(z^{2r} w\right)^l, \tag{68}$$

which satisfies that $\mathcal{H} \to \mathcal{H}_0$ as $\varepsilon \to 0$, where $\mathcal{H}$ is the Hamiltonian function defined in (63).

In the trigonometric case, an analogous equation to (65) is obtained. There are only two differences. First, one has to consider the definition of ℓ given in (14) associated to this type of systems. Secondly, in the trigonometric case, the coefficients in front of $a_{kl}(\tau)$ are expressed in terms of the coefficients $\widehat{C}^1_\pm$, $\widehat{C}^2_\pm$ and $C_\pm$ in (13). Taking into account these facts, one can also define the analogous functions A_l.

The solutions of the Hamilton-Jacobi equation (66) were studied in [Bal06] in the complex domains

$$\begin{aligned} \mathcal{D}^{+,u}_{\kappa,\theta} &= \{z \in \mathbb{C}; |\mathrm{Im}\, z| > \theta \,\mathrm{Re}\, z + \kappa\} \\ \mathcal{D}^{+,s}_{\kappa,\theta} &= \left\{z \in \mathbb{C}; -z \in \mathcal{D}^{+,u}_{\kappa,\theta}\right\} \end{aligned} \tag{69}$$

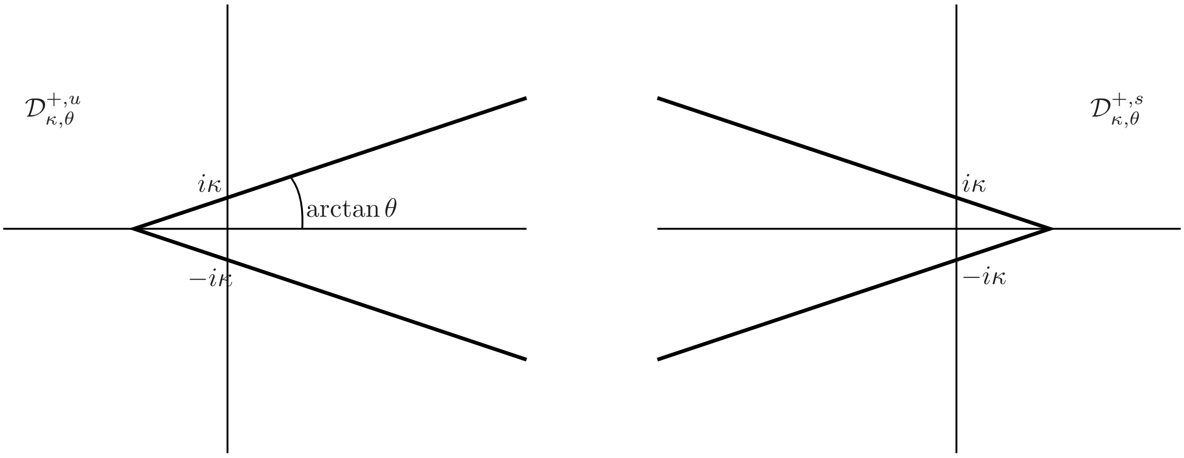

Figure 9: The domains $\mathcal{D}^{+,u}_{\kappa,\theta}$ and $\mathcal{D}^{+,s}_{\kappa,\theta}$ defined in (69).

for $\kappa > 0$ and $\theta > 0$. Let us observe that, for any $c > 0$, $\mathcal{D}^{\mathrm{in},+,*}_{\kappa,c} \subset \mathcal{D}^{+,*}_{\kappa,\tan\beta_2}$ for $* = u, s$. Nevertheless, since through the proof we will have to change the slope of the domains $\mathcal{D}^{+,*}_{\kappa,\theta}$, we start with a certain fixed slope $\theta_0 < \tan\beta_2$ which will be determined *a posteriori*.

The difference between the stable and unstable manifolds of the inner equation was studied in the intersection domain

$$\mathcal{R}^{+}_{\kappa,\theta} = \mathcal{D}^{+,u}_{\kappa,\theta} \cap \mathcal{D}^{+,s}_{\kappa,\theta} \cap \{z \in \mathbb{C}; \operatorname{Im} z < 0\}. \tag{70}$$

The next theorem gives the main results obtained in [Bal06] about the solutions of equation (66) and

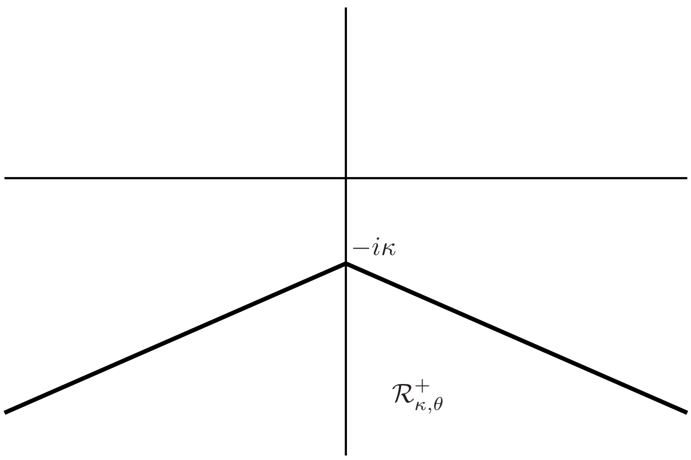

Figure 10: The domain $\mathcal{R}^{+}_{\kappa,\theta}$ defined in (70).

their difference.

Theorem 4.12. *Let us consider any fixed $\theta_0 > 0$. Then, for $\hat\mu \in B(\hat\mu_0)$ the following statements are satisfied:*

1. *There exists $\kappa_4 > 0$ such that, equation (66) has solutions $\psi_0^* : \mathcal{D}^{+,*}_{\kappa_4,\theta_0} \times \mathbb{T}_\sigma \to \mathbb{C}$, $* = u, s$, of the form*

$$\psi_0^{u,s}(z,\tau) = -\frac{1}{(2r-1)z^{2r-1}} + \hat\mu\overline{\psi}_0^{u,s}(z,\tau) + K^{u,s}, \quad K^{u,s} \in \mathbb{C} \tag{71}$$

where $\overline{\psi}_0^{u,s}$ *are analytic functions in all their variables. Moreover, the derivatives of* $\overline{\psi}_0^{u,s}$ *are uniquely determined by the condition*

$$\sup_{(z,\tau)\in\mathcal{D}^{+,*}_{\kappa_4,\theta_0}\times\mathbb{T}_\sigma}\left|z^{\ell+1}\partial_z\overline{\psi}_0^{*}(z,\tau)\right|<\infty$$

for $*=u,s$*. In fact, one can choose* $\overline{\psi}_0^{u,s}$ *such that*

$$\sup_{(z,\tau)\in\mathcal{D}^{+,*}_{\kappa_4,\theta_0}\times\mathbb{T}_\sigma}\left|z^{\ell}\overline{\psi}_0^{*}(z,\tau)\right|<\infty$$

for $*=u,s$.

2. *There exists* $\kappa_5>\kappa_4$*, analytic functions* $\left\{\chi^{[k]}(\hat{\mu})\right\}_{k\in\mathbb{Z}^-}$ *defined on* $B(\hat{\mu}_0)$ *and* $g:\mathcal{R}^+_{\kappa_5,2\theta_0}\times\mathbb{T}_\sigma\to\mathbb{C}$ *such that two solutions* $\psi_0^{u,s}$ *of equation* (66) *of the form given in* (71) *with* $K^u=K^s$*, satisfy*

$$\left(\psi_0^u-\psi_0^s\right)(z,\tau)=\hat{\mu}\sum_{k<0}\chi^{[k]}(\hat{\mu})e^{ik(z-\tau+\hat{\mu}g(z,\tau))}. \tag{72}$$

Moreover, the function g *satisfies that*

$$\sup_{(z,\tau)\in\mathcal{R}^+_{\kappa_5,2\theta_0}\times\mathbb{T}_\sigma}\left|z^{\ell-2r}g(z,\tau)\right|<\infty \qquad \text{if } \ell>2r$$

$$\sup_{(z,\tau)\in\mathcal{R}^+_{\kappa_5,2\theta_0}\times\mathbb{T}_\sigma}\left|(\ln|z|)^{-1}g(z,\tau)\right|<\infty \qquad \text{if } \ell=2r.$$

The proof of Theorem 4.12 is given in [Bal06].

Remark 4.13. *Following the proofs of [Bal06], it can be easily seen that the analytic functions* $\left\{\chi^{[k]}(\hat{\mu})\right\}_{k\in\mathbb{Z}^-}$ *are entire.*

For the case $\ell-2r=0$ we will need better knowledge of the function g given by Theorem 4.12. The next proposition gives its first asymptotic terms. First, we define certain functions which will be used in the statement of the next proposition. Let us consider the functions A_j defined in (67), then we define

$$Q_j(\tau)=\sum_{k=j}^{N}\begin{pmatrix}k\\ j\end{pmatrix}A_k(\tau), \tag{73}$$

and functions F_j such that

$$\partial_\tau F_j=Q_j \quad\text{and}\quad \langle F_j\rangle=0, \tag{74}$$

which are periodic since $\langle Q_j\rangle=0$.

Remark 4.14. *The functions* $Q_j(\tau)$ *can be also defined intrinsically either* $\widehat{H}_1^1$ *is a polynomial or a trigonometric polynomial, as*

$$Q_j(\tau)=\frac{1}{j!}C_+^{j-2}\lim_{u\to ia}(u-ia)^{\ell-rj}\partial_p^j\widehat{H}_1^1(q_0(u),p_0(u),\tau),$$

where $\widehat{H}_1^1$ *is the Hamiltonian defined in* (35) *and* C_+ *is given in* (12) *and* (13).

Proposition 4.15. *Let us consider the constant*

$$b=2r\left\langle Q_0F_1+2F_0Q_2\right\rangle, \tag{75}$$

where Q_j and F_j are the functions defined in (73) *and* (74) *respectively. Then, when $\ell - 2r = 0$, the function g obtained in Theorem 4.12, is of the form*

$$g(z,\tau) = -F_1(\tau) - \hat{\mu} b \ln z + \overline{g}(z,\tau)$$

and $\overline{g}$ satisfies

$$\sup_{(z,\tau)\in\mathcal{R}^+_{\kappa_5,2\theta_0}\times\mathbb{T}_\sigma} |z\overline{g}(z,\tau)| < \infty.$$

To have a better knowledge of the parameterizations of the invariant manifolds in the inner domains $\mathcal{D}^{\mathrm{in},+,*}_{\kappa,c}$, $* = u,s$ in (64), we need to compare the parameterizations $\psi^{u,s}$, which are solutions of (62) with $\psi_0^{u,s}$ which are solutions of (65) and have been given in Theorem 4.12.

Since we have to use the functions and results obtained in Theorem 4.12, we need that $\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c} \subset \mathcal{D}^{+,u}_{\kappa,2\theta_0}$. To this end, we impose

$$\theta_0 = \frac{\tan\beta_2}{2}.$$

We state the next theorem for the unstable invariant manifold. The stable manifold satisfies analogous properties.

Theorem 4.16. *Let $\gamma \in (0,1)$, the constants κ_3 and κ_5 defined in Theorems 4.8 and 4.12, $c_1 > 0$ and $\varepsilon_0 > 0$ small enough and $\kappa_6 > \max\{\kappa_3,\kappa_5\}$ big enough, which might depend on the previous constants. Then, for $\varepsilon \in (0,\varepsilon_0)$ and $\hat{\mu} \in B(\hat{\mu}_0)$, there exists a constant $b_{10} > 0$ such that for $(z,\tau) \in \mathcal{D}^{\mathrm{in},+,u}_{\kappa_6,c_1} \times \mathbb{T}_\sigma$*

$$|\partial_z\psi^u(z,\tau) - \partial_z\psi^u_0(z,\tau)| \leq \frac{b_{10}\varepsilon^{\frac{1}{\beta}}}{|z|^{2r-\frac{1}{\beta}}},$$

*where γ enters in the definition of $\mathcal{D}^{\mathrm{in},+,u}_{\kappa_6,c_1}$, $r = \alpha/\beta$ has been defined in Hypothesis **HP2**, ψ^u_0 is given in Theorem 4.12 and ψ^u is the scaling of the generating function T^u given in* (61).

The proof of this theorem is given in Section 8.

4.7 Study of the difference between the invariant manifolds

Once we have obtained parameterizations of the invariant manifolds of the form (42) in the domains $D^s_{\kappa_3,d_2}$ and $D^u_{\kappa_3,d_2}$ and studied their first order approximations close to the singularities, the next step is to study their difference.

We devote Section 4.7.1 to study the (easier) case $\ell - 2r < 0$ and then in Section 4.7.2 we consider the case $\ell - 2r \geq 0$.

4.7.1 Study of the difference between the invariant manifolds for the case $\ell - 2r < 0$

We are going to proceed to study the difference $\partial_u T^u(u,\tau) - \partial_u T^s(u,\tau)$. Recall that in the case $\ell - 2r < 0$, Hypothesis **HP5** becomes $\eta \geq 0$. Therefore our study includes the non perturbative case $\eta = 0$.

To study the difference between the manifolds, we define

$$\Delta(u,\tau) = T^u(u,\tau) - T^s(u,\tau) \tag{76}$$

in the domain $R_{\kappa,d} = D^s_{\kappa,d} \cap D^u_{\kappa,d}$ which is defined in (27).

We recall that $p_0(u) \neq 0$ if $u \in R_{\kappa,d}$ and hence we can use the Hamilton-Jacobi equation in this domain.

Subtracting equation (41) for both T^u and T^s, one can see that Δ satisfies the partial differential equation

$$\widetilde{\mathcal{L}}_\varepsilon \xi = 0, \tag{77}$$

where

$$\widetilde{\mathcal{L}}_\varepsilon = \varepsilon^{-1}\partial_\tau + (1 + G(u,\tau))\partial_u \tag{78}$$

with

$$G(u,\tau) = \frac{1}{2p_0(u)}\left(\partial_u T_1^u(u,\tau) + \partial_u T_1^s(u,\tau)\right)$$
$$+\frac{\mu\varepsilon^\eta}{p_0(u)}\int_0^1 \partial_p \widehat{H}_1\left(q_0(u), p_0(u) + \frac{s\partial_u T_1^u(u,\tau) + (1-s)\partial_u T_1^s(u,\tau)}{p_0(u)}, \tau\right) ds, \tag{79}$$

where $\widehat{H}_1$ is the function defined in (34) and $T_1^{u,s}(u,\tau) = T^{u,s}(u,\tau) - T_0(u)$ with $\partial_u T_0(u) = p_0^2(u)$ and $T^{u,s}$ are given in Theorems 4.4 and 4.8.

Following [Bal06], to obtain the asymptotic expression of the difference Δ, we take advantage from the fact that it is a solution of the homogeneous linear partial differential equation (77). In [Bal06] it is seen that if (77) has a solution ξ_0 such that $(\xi_0(u,\tau),\tau)$ is injective in $R_{\kappa,d}\times\mathbb{T}_\sigma$, then any solution of equation (77) defined in $R_{\kappa,d}\times\mathbb{T}_\sigma$ can be written as $\xi = \Upsilon\circ\xi_0$ for some function Υ.

Following this approach, we begin by looking for a solution of the form

$$\xi_0(u,\tau) = \varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau) \tag{80}$$

being $\mathcal{C}$ a function 2π-periodic in τ, such that $(\xi_0(u,\tau),\tau)$ is injective in $R_{\kappa,d}\times\mathbb{T}_\sigma$.

From now on the parameter κ will be play an important role in our computations. The next results will deal with big values of $\kappa=\kappa(\varepsilon)$ such that $\kappa\varepsilon < a$. In particular, in Theorem 4.19 we will use $\kappa = \mathcal{O}(\log(1/\varepsilon))$.

Theorem 4.17. *Let $d_2>0$ and $\kappa_3>0$ the constants defined in Theorem 4.8, $d_3<d_2$, $\varepsilon_0>0$ small enough and $\kappa_7>\kappa_3$ big enough, which might depend on the previous constants. Then, for $\varepsilon\in(0,\varepsilon_0)$, $\mu\in B(\mu_0)$ and any $\kappa\geq\kappa_7$ such that $\varepsilon\kappa<a$, there exists a real-analytic function $\mathcal{C}: R_{\kappa,d_3}\times\mathbb{T}_\sigma\to\mathbb{C}$ such that $\xi_0(u,\tau) = \varepsilon^{-1}u-\tau+\mathcal{C}(u,\tau)$ is a solution of (77) and*

$$(\xi_0(u,\tau),\tau) = \left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau),\tau\right)$$

is injective.

Moreover, there exists a constant $b_{11}>0$ independent of μ, ε and κ, such that for $(u,\tau)\in R_{\kappa,d_3}\times\mathbb{T}_\sigma$,

$$|\mathcal{C}(u,\tau)| \leq b_{11}|\mu|\varepsilon^\eta$$
$$|\partial_u\mathcal{C}(u,\tau)| \leq b_{11}\kappa^{-1}|\mu|\varepsilon^{\eta-1}.$$

To study the first order of the difference between the invariant manifolds, we need a better knowledge of the behavior of the function $\mathcal{C}$ in the inner domains defined in (30). The next proposition gives the first order asymptotic terms of $\mathcal{C}$ close to $u=ia$, that is in $D^{\mathrm{in},+,u}_{\kappa,c}\cap D^{\mathrm{in},+,s}_{\kappa,c}$. The study close to $u=-ia$ can be done analogously.

Proposition 4.18. *Let κ_7 be given by Theorem 4.17 and $c_1>0$. Then, for any $\varepsilon_0>0$ and $\kappa>\kappa_7$ such that $\kappa\varepsilon<a$, there exist a constant $C(\mu,\varepsilon)$ defined for $(\mu,\varepsilon)\in B(\mu_0)\times(0,\varepsilon_0)$ and depending real-anallytically in μ and a constant $b_{12}>0$ such that $|C(\mu,\varepsilon)|\leq b_{12}|\mu|\varepsilon^\eta$ and, if $(u,\tau)\in\left(D^{\mathrm{in},+,u}_{\kappa,c_1}\cap D^{\mathrm{in},+,s}_{\kappa,c_1}\right)\times\mathbb{T}_\sigma$,*

$$|\mathcal{C}(u,\tau) - C(\mu,\varepsilon)| \leq \frac{b_{12}|\mu|\varepsilon^\eta}{\kappa}.$$

The proofs of Theorem 4.17 and Proposition 4.18 are done in Section 9.3.

As we have explained, since $\Delta = T^u - T^s$ is a solution of the same homogeneous partial differential equation as ξ_0 given in Theorem 4.17, there exists a function Υ such that $\Delta = \Upsilon\circ\xi_0$, which gives

$$\Delta(u,\tau) = \Upsilon\left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)\right). \tag{81}$$

Since Δ is 2π-periodic in τ, we notice that the function Υ is 2π-periodic in its variable. Therefore, considering the Fourier series of Υ we obtain

$$\Delta(u,\tau) = \sum_{k\in\mathbb{Z}} \Upsilon^{[k]} e^{ik\left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)\right)}. \tag{82}$$

Now we are going to find the first asymptotic term of Δ. Let us first observe that the Melnikov Potential defined in (16) can be defined through the functions $\mathcal{T}_0^{u,s}$, given in (57), as

$$\mathcal{T}_0^u(u,\tau) - \mathcal{T}_0^s(u,\tau) = -\mu\varepsilon^\eta L(u,\tau). \tag{83}$$

Moreover by (17),

$$L(u,\tau) = \sum_{k\in\mathbb{Z}} M^{[k]} e^{ik\left(\varepsilon^{-1}u-\tau\right)}. \tag{84}$$

In [DS97] (for the hyperbolic case) and [BF04] (for the parabolic case), it was seen that for $\eta > \ell$, the function L gives the leading term of the difference between manifolds. Nevertheless, for the general case $\eta \geq 0$, one has to modify slightly this function to obtain the correct first order. Let us define

$$\Delta_0(u,\tau) = \sum_{k\in\mathbb{Z}} \Upsilon_0^{[k]} e^{ik\left(\varepsilon^{-1}u-\tau+\mathcal{C}(u,\tau)\right)}, \tag{85}$$

where

$$\begin{aligned} \Upsilon_0^{[k]} &= -\mu\varepsilon^\eta M^{[k]} e^{-ikC(\mu,\varepsilon)} \quad \text{if} \quad k<0 \\ \Upsilon_0^{[0]} &= 0 \\ \Upsilon_0^{[k]} &= -\mu\varepsilon^\eta M^{[k]} e^{-ik\overline{C}(\mu,\varepsilon)} \quad \text{if} \quad k>0, \end{aligned} \tag{86}$$

where $C(\mu,\varepsilon)$ is the constant obtained in Proposition 4.18 and $\overline{C}(\mu,\varepsilon)$ is its complex conjugate. Let us point out that, by Proposition 4.18, these coefficients satisfy

$$\Upsilon_0^{[k]} = -\mu\varepsilon^\eta M^{[k]}\left(1+\mathcal{O}\left(\mu\varepsilon^\eta\right)\right).$$

Next theorem shows that this function Δ_0 gives the first asymptotic order of (76). From now on, in this subsection, we consider real values of $\tau \in \mathbb{T} = \mathbb{T}_\sigma \cap \mathbb{R}$. In this setting it can be easily seen that the function Δ_0 is real-analytic in u.

Theorem 4.19. *Let us consider the mean value of Υ, $\Upsilon^{[0]}$, defined in* (82)*, $s < \nu^*$ where ν^* is the constant defined in Proposition 4.10 and $\varepsilon_0 > 0$ small enough. Then, there exists a constant $b_{13} > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$ and $\mu \in B(\mu_0)\cap\mathbb{R}$ and $(u,\tau) \in \left(R_{s\ln(1/\varepsilon),d_3}\cap\mathbb{R}\right)\times\mathbb{T}$, the following statements are satisfied:*

$$\begin{aligned} \left|\Delta(u,\tau) - \Upsilon^{[0]} - \Delta_0(u,\tau)\right| &\leq \frac{b_{13}|\mu|\varepsilon^{\eta+1-\ell}}{|\ln\varepsilon|} e^{-\frac{a}{\varepsilon}} \\ \left|\partial_u\Delta(u,\tau) - \partial_u\Delta_0(u,\tau)\right| &\leq \frac{b_{13}|\mu|\varepsilon^{\eta-\ell}}{|\ln\varepsilon|} e^{-\frac{a}{\varepsilon}} \\ \left|\partial_u^2\Delta(u,\tau) - \partial_u^2\Delta_0(u,\tau)\right| &\leq \frac{b_{13}|\mu|\varepsilon^{\eta-1-\ell}}{|\ln\varepsilon|} e^{-\frac{a}{\varepsilon}}. \end{aligned}$$

We observe that, assuming Hypothesis **HP6**, Lemma 2.4 gives the first order of the function L. Then, using this lemma, the definition of the coefficients $\Upsilon_0^{[k]}$ in (86) and Proposition 4.18, one can deduce a simpler leading term of Δ in (76). For this purpose let us define the function

$$\Delta_{00}(u,\tau) = \frac{2\mu\varepsilon^\eta}{\varepsilon^{\ell-1}} e^{-\frac{a}{\varepsilon}} \mathrm{Re}\left(f_0 e^{iC(\mu,\varepsilon)} e^{-i\left(\frac{u}{\varepsilon}-\tau+\mathcal{C}(u,\tau)\right)}\right), \tag{87}$$

where $C(\mu,\varepsilon)$ is the constant given in Proposition 4.18 and $\mathcal{C}$ is the function given by Theorem 4.17.

Corollary 4.20. *There exists a constant $b_{14} > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $\mu \in B(\mu_0) \cap \mathbb{R}$ and $(u, \tau) \in \left(R_{s\ln(1/\varepsilon), d_3} \cap \mathbb{R}\right) \times \mathbb{T}$, the following statements are satisfied.*

$$\left|\Delta(u,\tau) - \Upsilon^{[0]} - \Delta_{00}(u,\tau)\right| \leq \frac{b_{14}\,|\mu|\,\varepsilon^{\eta+1-\ell}}{|\ln\varepsilon|} e^{-\frac{a}{\varepsilon}}$$
$$|\partial_u\Delta(u,\tau) - \partial_u\Delta_{00}(u,\tau)| \leq \frac{b_{14}\,|\mu|\,\varepsilon^{\eta-\ell}}{|\ln\varepsilon|} e^{-\frac{a}{\varepsilon}}$$
$$\left|\partial_u^2\Delta(u,\tau) - \partial_u^2\Delta_{00}(u,\tau)\right| \leq \frac{b_{14}\,|\mu|\,\varepsilon^{\eta-1-\ell}}{|\ln\varepsilon|} e^{-\frac{a}{\varepsilon}}.$$

We devote the rest of this section to prove Theorem 4.19, from which, using also Lemma 2.4, Corollary 4.20 is a direct consequence.

Proof of Theorem 4.19. For the first part of the proof we consider complex values of $\mu \in B(\mu_0)$ and later we will restrict to $\mu \in B(\mu_0) \cap \mathbb{R}$. We define

$$\widetilde{\Upsilon}(\zeta) = \sum_{k\in\mathbb{Z}} \widetilde{\Upsilon}^{[k]} e^{ik\zeta},$$

where $\widetilde{\Upsilon}^{[k]} = \Upsilon^{[k]} - \Upsilon_0^{[k]}$. By (82) and (85), the function $\widetilde{\Delta}(u,\tau) = \Delta(u,\tau) - \Delta_0(u,\tau)$ can be written as

$$\widetilde{\Delta}(u,\tau) = \widetilde{\Upsilon}\left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)\right) = \sum_{k\in\mathbb{Z}} \widetilde{\Upsilon}^{[k]} e^{ik\left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)\right)}. \tag{88}$$

Therefore, to obtain the bounds of Theorem 4.19, it is crucial to bound $\left|\widetilde{\Upsilon}^{[k]}\right|$.

The first step is to obtain a bound of $\widetilde{\Delta}(u,\tau)$ for $(u,\tau) \in R_{s\ln\frac{1}{\varepsilon}, d_3} \times \mathbb{T}$. First we bound this term for $(u,\tau) \in \left(R_{s\ln\frac{1}{\varepsilon}, d_3} \cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon}, c_1} \cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon}, c_1}\right) \times \mathbb{T}$. Recalling the definitions in (76), (57), (83), (84), (85) and (86), we split $\widetilde{\Delta}$ as

$$\widetilde{\Delta}(u,\tau) = \widetilde{\Delta}_1^u(u,\tau) - \widetilde{\Delta}_1^s(u,\tau) + \widetilde{\Delta}_2(u,\tau) + \widetilde{\Delta}_3(u,\tau)$$

with

$$\widetilde{\Delta}_1^{u,s}(u,\tau) = T^{u,s}(u,\tau) - T_0(u) - \mathcal{T}_0^{u,s}(u,\tau) \tag{89}$$
$$\widetilde{\Delta}_2(u,\tau) = -\mu\varepsilon^\eta \sum_{k<0} M^{[k]} e^{ik\left(\varepsilon^{-1}u-\tau\right)} \left(1 - e^{ik(\mathcal{C}(u,\tau) - C(\mu,\varepsilon))}\right) \tag{90}$$
$$\widetilde{\Delta}_3(u,\tau) = -\mu\varepsilon^\eta \sum_{k>0} M^{[k]} e^{ik\left(\varepsilon^{-1}u-\tau\right)} \left(1 - e^{ik\left(\mathcal{C}(u,\tau) - \overline{C}(\mu,\varepsilon)\right)}\right). \tag{91}$$

Applying Proposition 4.10, one can see that for $(u,\tau) \in \left(R_{s\ln\frac{1}{\varepsilon}, d_3} \cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon}, c_1} \cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon}, c_1}\right) \times \mathbb{T}$,

$$\left|\partial_u\widetilde{\Delta}_1^{u,s}(u,\tau)\right| \leq K|\mu|\varepsilon^{\eta-\ell+\nu^*},$$

where $\nu^* > 0$ is a constant defined in that proposition.

To bound $\widetilde{\Delta}_2$, it is enough to apply Lemma 2.4, Theorem 4.17 and Proposition 4.18 to obtain that for $(u,\tau) \in \left(R_{s\ln\frac{1}{\varepsilon}, d_3} \cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon}, c_1} \cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon}, c_1}\right) \times \mathbb{T}$,

$$\left|\partial_u\widetilde{\Delta}_2(u,\tau)\right| \leq \frac{K|\mu|^2\varepsilon^{2\eta-\ell+s}}{|\ln\varepsilon|}.$$

Finally, to bound $\partial_u\widetilde{\Delta}_3$, it is enough to take into account again Lemma 2.4, Theorem 4.17 and Proposition 4.18. Then, one can see that for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon},c_1}\cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon},c_1}\right)\times\mathbb{T}$,

$$\left|\partial_u\widetilde{\Delta}_3(u,\tau)\right|\leq K|\mu|^2\varepsilon^{2\eta-\ell-s}e^{-\frac{2a}{\varepsilon}}.$$

Therefore, from the bounds of $\widetilde{\Delta}_1^{u,s}$, $\widetilde{\Delta}_2$ and $\widetilde{\Delta}_3$ and recalling that by hypothesis $s<\nu^*$, we have that for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon},c_1}\cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon},c_1}\right)\times\mathbb{T}$,

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right|\leq\frac{K|\mu|\varepsilon^{\eta-\ell+s}}{|\ln\varepsilon|}. \tag{92}$$

Reasoning analogously, one can see that for

$$(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},-,s}_{s\ln\frac{1}{\varepsilon},c_1}\cap D^{\mathrm{in},-,u}_{s\ln\frac{1}{\varepsilon},c_1}\right)\times\mathbb{T},$$

the function $\partial_u\widetilde{\Delta}$ satisfies

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right|\leq\frac{K|\mu|\varepsilon^{\eta-\ell+s}}{|\ln\varepsilon|}. \tag{93}$$

Finally, for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{out},s}_{c_1\varepsilon^\gamma,\rho_4}\cap D^{\mathrm{out},u}_{c_1\varepsilon^\gamma,\rho_4}\right)\times\mathbb{T}$, we decompose $\widetilde{\Delta}(u,\tau)=(T^u(u,\tau)-T_0(u))-(T^s(u,\tau)-T_0(u))-\Delta_0(u,\tau)$. Using Theorems 4.4, 4.8, and 4.17 and also Lemma 2.4, one can easily see that

$$|\partial_u\Delta(u,\tau)|\leq K|\mu|\varepsilon^{\eta+1-\gamma(\ell+1)}$$

provided $|u-ia|\geq\mathcal{O}(\varepsilon^\gamma)$. This bound is smaller than (92) and (93) due to the fact that $(\ell+1)(1-\gamma)>\nu^*>s$ (see Proposition 4.10 for the definition of μ^*).

Taking into account (92) and (93), one can conclude that for $\mu\in B(\mu_0)\cap\mathbb{R}$,

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right|\leq\frac{K|\mu|\varepsilon^{\eta-\ell+s}}{|\ln\varepsilon|}. \tag{94}$$

The second step of the proof is to consider the change of variables $(w,\tau)=(u+\varepsilon\mathcal{C}(u,\tau),\tau)$. By Theorem 4.17, one can easily see that it is a diffeomorphism from $R_{s\ln(1/\varepsilon),d_3}\times\mathbb{T}$ onto its image $\widetilde{R}\times\mathbb{T}$. Denoting by $\widetilde{\Upsilon}'$ the derivative of the function $\widetilde{\Upsilon}$ (see (88)), we define the function

$$\Theta(w,\tau)=\widetilde{\Upsilon}'\left(\varepsilon^{-1}w-\tau\right),$$

on $\widetilde{R}\times\mathbb{T}$ which, by construction, satisfies

$$\Theta(u+\varepsilon\mathcal{C}(u,\tau),\tau)=\left(\frac{1}{\varepsilon}+\partial_u\mathcal{C}(u,\tau)\right)^{-1}\partial_u\widetilde{\Delta}(u,\tau). \tag{95}$$

Moreover, as $\Theta(w,\tau)$ is periodic in τ, it can be also written as

$$\Theta(w,\tau)=\sum_{k\in\mathbb{Z}}\Theta^{[k]}(w)e^{ik\tau}.$$

Then, for any $w\in\widetilde{R}$, the Fourier coefficients satisfy

$$ik\widetilde{\Upsilon}^{[k]}=\Theta^{[-k]}(w)e^{-ik\frac{w}{\varepsilon}}.$$

Now, taking advantage of the fact that the coefficients $\widetilde{\Upsilon}^{[k]}$ do not depend on w, we will obtain sharp bounds for the coefficients $\widetilde{\Upsilon}^{[k]}$ with $k<0$. Since we are dealing with real analytic functions, the coefficients $\widetilde{\Upsilon}^{[k]}$ with $k > 0$ will satisfy the same bounds. Let us consider $w = w^* = u^* + \varepsilon\mathcal{C}(u^*, 0)$ with $u^* = i(a - s\varepsilon \ln(1/\varepsilon))$. Then,

$$\begin{aligned}\left|\widetilde{\Upsilon}^{[k]}\right| &\leq |k|^{-1} \sup_{w\in\widetilde{R}} \left|\Theta^{[-k]}(w)\right| e^{-\frac{|k|}{\varepsilon}\left(a-s\varepsilon\ln\frac{1}{\varepsilon}\right)-|k|\mathrm{Im}\,(\mathcal{C}(u^*,0))} \\ &\leq |k|^{-1} \sup_{(w,\tau)\in\widetilde{R}\times\mathbb{T}} |\Theta(w,\tau)|\, e^{-\frac{|k|}{\varepsilon}\left(a-s\varepsilon\ln\frac{1}{\varepsilon}\right)-|k|\mathrm{Im}\,(\mathcal{C}(u^*,0))}.\end{aligned}$$

Then, taking into account (95) and Theorem 4.17, we have that for $k < 0$,

$$\left|\widetilde{\Upsilon}^{[k]}\right| \leq K\varepsilon \sup_{(u,\tau)\in R_{s\ln(1/\varepsilon),d_3}\times\mathbb{T}} \left|\partial_u\widetilde{\Delta}(u,\tau)\right| e^{-\frac{|k|}{\varepsilon}\left(a-s\varepsilon\ln\frac{1}{\varepsilon}\right)-|k|\mathrm{Im}\,(\mathcal{C}(u^*,0))}.$$

Therefore, to obtain the bounds for $\widetilde{\Upsilon}^{[k]}$ with $k < 0$, it only remains to use bounds (94) and the properties of $\mathcal{C}$ given in Theorem 4.17 and Proposition 4.18. Then, we obtain that for $k < 0$

$$\left|\widetilde{\Upsilon}^{[k]}\right| \leq \frac{K|\mu|\varepsilon^{\eta}e^{-\frac{a}{\varepsilon}}}{|\ln\varepsilon|\varepsilon^{\ell-1}} e^{-\frac{|k|-1}{\varepsilon}\left(a+\varepsilon s\log\varepsilon+b_{11}|\mu|\varepsilon^{\eta+1}\right)}.$$

Finally, the bounds of $\widetilde{\Upsilon}^{[k]}$ lead easily to the desired bounds of $\widetilde{\Delta}(u,\tau)$ for $(u,\tau) \in \left(R_{s\ln(1/\varepsilon),d_3} \cap \mathbb{R}\right) \times \mathbb{T}$. □

4.7.2 Study of the difference between the invariant manifolds for the case $\ell - 2r \geq 0$

Recall that when $\ell - 2r \geq 0$, Hypothesis **HP5** becomes $\eta \geq \ell - 2r$. For this reason, as we did in Section 4.6.2, we will denote $\hat{\mu} = \mu\varepsilon^{\eta-\ell+2r}$.

As we have done for the case $\ell - 2r < 0$ in Section 4.7.1, we consider the function $\Delta(u,\tau) = T^u(u,\tau) - T^s(u,\tau)$ defined in (76) in the domain $R_{\kappa,d} = D^s_{\kappa,d} \cap D^u_{\kappa,d}$ defined in (27) (see also Figure 3).

Now Δ satisfies the partial differential equation

$$\widetilde{\mathcal{L}}_\varepsilon\xi = 0, \tag{96}$$

where $\widetilde{\mathcal{L}}_\varepsilon$ is the operator defined in (78) and G now is

$$\begin{aligned}G(u,\tau) =& \frac{1}{2p_0(u)}\left(\partial_u T_1^u(u,\tau) + \partial_u T_1^s(u,\tau)\right) \\ &+ \frac{\hat{\mu}\varepsilon^{\ell-2r}}{p_0(u)} \int_0^1 \partial_p\widehat{H}_1\left(q_0(u), p_0(u) + \frac{s\partial_u T_1^u(u,\tau) + (1-s)\partial_u T_1^s(u,\tau)}{p_0(u)}, \tau\right) ds,\end{aligned} \tag{97}$$

where $\widehat{H}_1$ is the function defined in (34) and $T^{u,s}(u,\tau) = T_0(u) + T_1^{u,s}(u,\tau)$ with $\partial_u T_0(u) = p_0^2(u)$ and $T_1^{u,s}$ are given in Theorems 4.4 and 4.8. Let us point out that the only difference between the function G defined in (97) from the one defined in (79) is the dependence on the parameters. The first one depends on μ and ε whereas the second one depends on $\hat{\mu}$, which has been defined in terms of μ and ε in (58).

As we have done in Section 4.7.1, to obtain the asymptotic expression of the difference Δ, we look for a solution ξ_0 of (77) of the form

$$\xi_0(u,\tau) = \varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)$$

with $\mathcal{C}$ a function 2π-periodic in τ, such that $(\xi_0(u,\tau),\tau)$ is injective in $R_{\kappa,d} \times \mathbb{T}_\sigma$. Then, we will write Δ as $\xi = \Upsilon \circ \xi_0$ for some function Υ.

Theorem 4.21. *Let us consider the constants $d_2 > 0$ defined in Theorem 4.8 and $\kappa_6 > 0$ in Theorem 4.16, $d_3 < d_2$ and $\varepsilon_0 > 0$ small enough and $\kappa_8 > \kappa_6$ big enough, which might depend on the previous constants. Then, for $\varepsilon \in (0,\varepsilon_0)$, $\mu \in B(\mu_0)$ and any $\kappa \geq \kappa_8$ such that $\varepsilon\kappa < a$, there exists a real-analytic function $\mathcal{C}(u,\tau) : R_{\kappa,d_3} \times \mathbb{T}_\sigma \to \mathbb{C}$ such that $\xi_0(u,\tau) = \varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)$ is solution of (96) and*

$$(\xi_0(u,\tau),\tau) = \left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau),\tau\right)$$

is injective.

Moreover, there exists a constant $b_{15} > 0$ independent of μ, ε and κ, such that for $(u,\tau) \in R_{\kappa,d_3} \times \mathbb{T}_\sigma$,

- *If $\ell - 2r > 0$,*

$$|\mathcal{C}(u,\tau)| \le \frac{b_{15}\,|\hat\mu|\,\varepsilon^{\ell-2r}}{|u^2+a^2|^{\ell-2r}}$$
$$|\partial_u\mathcal{C}(u,\tau)| \le \frac{b_{15}\,|\hat\mu|\,\varepsilon^{\ell-2r-1}}{\kappa\,|u^2+a^2|^{\ell-2r}}.$$

- *If $\ell - 2r = 0$,*

$$|\mathcal{C}(u,\tau)| \le b_{15}\,|\hat\mu|\ln\left|u^2+a^2\right|$$
$$|\partial_u\mathcal{C}(u,\tau)| \le \frac{b_{15}\,|\hat\mu|}{|u^2+a^2|}.$$

To study the first order of the difference between the invariant manifolds when $\ell - 2r = 0$, we need a better knowledge of the behavior of the function $\mathcal{C}$ in the inner domains (30). The next proposition gives the first order asymptotic terms of $\mathcal{C}$ close to $u = ia$. The study close to $u = -ia$ can be done analogously.

Proposition 4.22. *Assume $\ell = 2r$. Let c_1 be a constant as in Theorem 4.16. We consider $c_2 > c_1$ and*

$$\frac{\beta}{\beta+1} < \gamma < 1, \tag{98}$$

*where $r = \alpha/\beta$ has been defined in Hypothesis **HP2**.*

Then, for any $\varepsilon_0 > 0$, there exist a constant $C(\hat\mu,\varepsilon)$ defined for $(\hat\mu,\varepsilon) \in B(\hat\mu_0)\times(0,\varepsilon_0)$ depending real-analytically in $\hat\mu$ and a constant $b_{16} > 0$ such that $|C(\hat\mu,\varepsilon)| \le b_{16}|\hat\mu|$ and, if $(u,\tau) \in \left(D^{\mathrm{in},+,u}_{\kappa_8,c_2} \cap D^{\mathrm{in},+,s}_{\kappa_8,c_2}\right)\times \mathbb{T}_\sigma$,

$$\left|\mathcal{C}(u,\tau) - C(\hat\mu,\varepsilon) + \mu F_1(\tau) + \hat\mu^2 b\ln(u-ia)\right| \le \frac{b_{16}|\hat\mu|\varepsilon}{|u-ia|}.$$

We recall that γ enters in the definitions of $D^{\mathrm{in},+,u}_{\kappa_8,c_2}$ and $D^{\mathrm{in},+,s}_{\kappa_8,c_2}$, $\mathcal{C}$ is the function given in Theorem 4.21 and the function F_1 and the constant b have been defined in (74) and (75) respectively.

Therefore, if we consider the function g given in Theorem 4.12, by Proposition 4.15, there exists a constant $b_{17} > 0$ such that, if $(u,\tau) \in \left(D^{\mathrm{in},+,u}_{\kappa_8,c_2} \cap D^{\mathrm{in},+,s}_{\kappa_8,c_2}\right)\times \mathbb{T}_\sigma$,

$$\left|\mathcal{C}(u,\tau) - C(\hat\mu,\varepsilon) + \hat\mu^2 b\ln\varepsilon - \hat\mu g\left(\varepsilon^{-1}(u-ia),\tau\right)\right| \le \frac{b_{17}|\hat\mu|\varepsilon}{|u-ia|}.$$

The proofs of Theorem 4.21 and Proposition 4.22 are done in Section 9.3.

As we have explained in Section 4.7.1, since Δ is a solution of the same homogeneous linear partial differential equation as ξ_0 given by Theorem 4.21, there exists a 2π-periodic function Υ such that $\Delta = \Upsilon\circ\xi_0$, which gives

$$\Delta(u,\tau) = \Upsilon\left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)\right). \tag{99}$$

and considering its Fourier series we have

$$\Delta(u,\tau) = \sum_{k\in\mathbb{Z}} \Upsilon^{[k]} e^{ik\left(\varepsilon^{-1}u-\tau+\mathcal{C}(u,\tau)\right)}. \tag{100}$$

Now we are going to find the first asymptotic term of Δ which will be strongly related with $(\psi_0^u - \psi_0^s)(\varepsilon^{-1}(u-ia),\tau)$, being $\psi_0^{u,s}$ the solutions of the inner equation given in Theorem 4.12. We introduce the auxiliary function

$$\Delta_0^+(u,\tau) = \sum_{k<0} \Upsilon_0^{[k]} e^{ik\left(\varepsilon^{-1}u-\tau+\mathcal{C}(u,\tau)\right)} \tag{101}$$

with

$$\Upsilon_0^{[k]} = \frac{C_+^2 \hat\mu}{\varepsilon^{2r-1}} \chi^{[k]}(\hat\mu) e^{-\frac{|k|a}{\varepsilon}} \qquad \text{if } \ell - 2r > 0 \tag{102}$$

$$\Upsilon_0^{[k]} = \frac{C_+^2 \hat\mu}{\varepsilon^{2r-1}} \chi^{[k]}(\hat\mu) e^{-\frac{|k|a}{\varepsilon} - i|k|(-C(\hat\mu,\varepsilon) + \hat\mu^2 b \ln \varepsilon)} \qquad \text{if } \ell - 2r = 0, \tag{103}$$

where $\left\{\chi^k(\hat\mu)\right\}_{k<0}$ are the coefficients given in Theorem 4.12 and $C(\hat\mu, \varepsilon)$ and b are the constants obtained in Propositions 4.22 and 4.15 respectively. The scaling C_+^2/ε^{2r-1} comes from the inner change in (61).

We also introduce

$$\Delta_0^-(u,\tau) = \sum_{k>0} \Upsilon_0^{[k]} e^{ik\left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)\right)}$$

with

$$\Upsilon_0^{[k]} = \frac{\overline{C}_+^2 \hat\mu}{\varepsilon^{2r-1}} \overline{\chi}^{[-k]}(\hat\mu) e^{-\frac{|k|a}{\varepsilon}} \qquad \text{if } \ell - 2r > 0 \tag{104}$$

$$\Upsilon_0^{[k]} = \frac{\overline{C}_+^2 \hat\mu}{\varepsilon^{2r-1}} \overline{\chi}^{[-k]}(\hat\mu) e^{-\frac{|k|a}{\varepsilon} + i|k|(-\overline{C}(\hat\mu,\varepsilon) + \hat\mu^2 \overline{b} \ln \varepsilon)} \qquad \text{if } \ell - 2r = 0. \tag{105}$$

The function $\Delta_0^-(u,\tau)$ corresponds to the difference of the solutions of the inner equation close to $u = -ia$ if $\hat\mu, \tau \in \mathbb{R}$. We note that, taking $\tau, \hat\mu \in \mathbb{R}$, Δ_0^- is nothing but the complex conjugate of Δ_0^+. In fact, as we know that Δ is a real analytic function in the u variable for real values of $\hat\mu, \tau$, we can define Δ_0^- as the function that satisfies that $\Delta_0 = \Delta_0^+ + \Delta_0^-$ is also a real analytic function in the same sense as explained before for Δ.

We will see that the first order of Δ is given by

$$\Delta_0(u,\tau) = \Delta_0^+(u,\tau) + \Delta_0^-(u,\tau). \tag{106}$$

Let us point out that it can be written as

$$\Delta_0(u,\tau) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \Upsilon_0^{[k]} e^{ik\left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)\right)}, \tag{107}$$

where $\Upsilon_0^{[k]}$ are defined either by (102) and (104) in the case $\ell - 2r > 0$ or by (103) and (105) in the case $\ell - 2r = 0$. For convenience we introduce $\Upsilon_0^{[0]} = 0$. From now on, in this subsection, we consider real values of $\tau \in \mathbb{T}_\sigma \cap \mathbb{R}$.

Theorem 4.23. *Let us consider the mean value of Υ, $\Upsilon^{[0]}$, defined in* (100)*, $s < 1/\beta$, where $r = \alpha/\beta$ is defined in Hypothesis **HP2**, and $\varepsilon_0 > 0$ small enough. Then, there exists a constant $b_{18} > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\hat\mu \in B(\hat\mu_0) \cap \mathbb{R}$ and $(u,\tau) \in \left(R_{s\ln(1/\varepsilon), d_3} \cap \mathbb{R}\right) \times \mathbb{T}$, the following statements are satisfied.*

- *If $\ell - 2r > 0$,*

$$\left|\Delta(u,\tau) - \Upsilon^{[0]} - \Delta_0(u,\tau)\right| \le \frac{b_{18}|\hat\mu|}{\varepsilon^{2r-1}|\ln\varepsilon|^{\ell-2r}} e^{-\frac{a}{\varepsilon}}$$

$$\left|\partial_u\Delta(u,\tau) - \partial_u\Delta_0(u,\tau)\right| \le \frac{b_{18}|\hat\mu|}{\varepsilon^{2r}|\ln\varepsilon|^{\ell-2r}} e^{-\frac{a}{\varepsilon}}$$

$$\left|\partial_u^2\Delta(u,\tau) - \partial_u^2\Delta_0(u,\tau)\right| \le \frac{b_{18}|\hat\mu|}{\varepsilon^{2r+1}|\ln\varepsilon|^{\ell-2r}} e^{-\frac{a}{\varepsilon}}.$$

- *If $\ell - 2r = 0$,*

$$\left|\Delta(u,\tau) - \Upsilon^{[0]} - \Delta_0(u,\tau)\right| \le \frac{b_{18}|\hat\mu|}{\varepsilon^{2r-1}|\ln\varepsilon|} e^{-\frac{a}{\varepsilon} + \hat\mu^2 \operatorname{Im} b \ln\varepsilon}$$

$$\left|\partial_u\Delta(u,\tau) - \partial_u\Delta_0(u,\tau)\right| \le \frac{b_{18}|\hat\mu|}{\varepsilon^{2r}|\ln\varepsilon|} e^{-\frac{a}{\varepsilon} + \hat\mu^2 \operatorname{Im} b \ln\varepsilon}$$

$$\left|\partial_u^2\Delta(u,\tau) - \partial_u^2\Delta_0(u,\tau)\right| \le \frac{b_{18}|\hat\mu|}{\varepsilon^{2r+1}|\ln\varepsilon|} e^{-\frac{a}{\varepsilon} + \hat\mu^2 \operatorname{Im} b \ln\varepsilon}.$$

We observe that $\partial_u\Delta_0$ gives the correct asymptotic prediction of $\partial_u\Delta$ if $\Upsilon_0^{[-1]}\neq 0$. In fact, we only need this coefficient to give a simpler leading term of the asymptotic formula. For this purpose let us define the function

$$f(\hat\mu)=C_+^2\chi^{[-1]}(\hat\mu)\,, \tag{108}$$

where C_+ is the constant defined in (12) or (13) and $\chi^{[-1]}(\hat\mu)$ is the constant given in Theorem 4.12. Let us point out that the zeros of $f(\hat\mu)$ correspond to the zeros of $\chi^{[-1]}(\hat\mu)$. We define

$$\Delta_{00}(u,\tau)=\frac{2\hat\mu}{\varepsilon^{2r-1}}e^{-\frac{a}{\varepsilon}}\mathrm{Re}\left(f(\hat\mu)e^{-i\left(\frac{u}{\varepsilon}-\tau+\mathcal{C}(u,\tau)\right)}\right)\qquad \text{if } \ell-2r>0 \tag{109}$$

$$\Delta_{00}(u,\tau)=\frac{2\hat\mu}{\varepsilon^{2r-1}}e^{-\frac{a}{\varepsilon}}\mathrm{Re}\left(f(\hat\mu)e^{-i\left(\hat\mu^2 b\ln\varepsilon-C(\hat\mu,\varepsilon)\right)}e^{-i\left(\frac{u}{\varepsilon}-\tau+\mathcal{C}(u,\tau)\right)}\right)\qquad \text{if } \ell-2r=0, \tag{110}$$

where b is the constant defined in (75), $C(\mu,\varepsilon)$ the constant given in Proposition 4.22 and $\mathcal{C}$ the function given by Theorem 4.21.

Corollary 4.24. *There exists a constant $b_{19}>0$ such that for $\varepsilon\in(0,\varepsilon_0)$, $\hat\mu\in B(\hat\mu_0)\cap\mathbb{R}$ and $(u,\tau)\in\left(R_{s\ln(1/\varepsilon),d_3}\cap\mathbb{R}\right)\times\mathbb{T}$, the following statements are satisfied.*

- *If $\ell-2r>0$,*

$$\begin{aligned}\left|\Delta(u,\tau)-\Upsilon^{[0]}-\Delta_{00}(u,\tau)\right|&\leq\frac{b_{19}|\hat\mu|}{\varepsilon^{2r-1}|\ln\varepsilon|^{\ell-2r}}e^{-\frac{a}{\varepsilon}}\\ \left|\partial_u\Delta(u,\tau)-\partial_u\Delta_{00}(u,\tau)\right|&\leq\frac{b_{19}|\hat\mu|}{\varepsilon^{2r}|\ln\varepsilon|^{\ell-2r}}e^{-\frac{a}{\varepsilon}}\\ \left|\partial_u^2\Delta(u,\tau)-\partial_u^2\Delta_{00}(u,\tau)\right|&\leq\frac{b_{19}|\hat\mu|}{\varepsilon^{2r+1}|\ln\varepsilon|^{\ell-2r}}e^{-\frac{a}{\varepsilon}}.\end{aligned}$$

- *If $\ell-2r=0$,*

$$\begin{aligned}\left|\Delta(u,\tau)-\Upsilon^{[0]}-\Delta_{00}(u,\tau)\right|&\leq\frac{b_{19}\,|\hat\mu|}{\varepsilon^{2r-1}|\ln\varepsilon|}e^{-\frac{a}{\varepsilon}+\hat\mu^2\mathrm{Im}\,b\ln\varepsilon}\\ \left|\partial_u\Delta(u,\tau)-\partial_u\Delta_{00}(u,\tau)\right|&\leq\frac{b_{19}\,|\hat\mu|}{\varepsilon^{2r}|\ln\varepsilon|}e^{-\frac{a}{\varepsilon}+\hat\mu^2\mathrm{Im}\,b\ln\varepsilon}\\ \left|\partial_u^2\Delta(u,\tau)-\partial_u^2\Delta_{00}(u,\tau)\right|&\leq\frac{b_{19}\,|\hat\mu|}{\varepsilon^{2r+1}|\ln\varepsilon|}e^{-\frac{a}{\varepsilon}+\hat\mu^2\mathrm{Im}\,b\ln\varepsilon}.\end{aligned}$$

We devote the rest of this section to prove Theorem 4.23, from which Corollary 4.24 is a direct consequence.

Proof of Theorem 4.23. For the first part of the proof we consider complex values of $\hat\mu\in B(\hat\mu_0)$ and later we will restrict to $\hat\mu\in B(\hat\mu_0)\cap\mathbb{R}$. By (100) and (107), the function $\widetilde\Delta(u,\tau)=\Delta(u,\tau)-\Delta_0(u,\tau)$ can be written as

$$\widetilde\Delta(u,\tau)=\widetilde\Upsilon\left(\varepsilon^{-1}u-\tau+\mathcal{C}(u,\tau)\right)=\sum_{k\in\mathbb{Z}}\widetilde\Upsilon^{[k]}e^{ik\left(\varepsilon^{-1}u-\tau+\mathcal{C}(u,\tau)\right)}, \tag{111}$$

where $\widetilde\Upsilon^{[k]}=\Upsilon^{[k]}-\Upsilon_0^{[k]}$. Therefore, to obtain the bounds of Theorem 4.23, it is crucial to bound $\left|\widetilde\Upsilon^{[k]}\right|$.

The first step is to obtain a bound of $\widetilde\Delta(u,\tau)$ for $(u,\tau)\in R_{s\ln\frac{1}{\varepsilon},d_3}\times\mathbb{T}$. First we bound this term for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon},c_2}\cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon},c_2}\right)\times\mathbb{T}$. Recalling the definitions of (76), (106), (101) and (72), we split $\widetilde\Delta$ as

$$\widetilde\Delta(u,\tau)=\widetilde\Delta_1^u(u,\tau)-\widetilde\Delta_1^s(u,\tau)+\widetilde\Delta_2(u,\tau)+\widetilde\Delta_3(u,\tau)$$

with

$$\begin{aligned}
\widetilde{\Delta}_1^{u,s}(u,\tau) &= T^{u,s}(u,\tau) - \frac{C_+^2}{\varepsilon^{2r-1}}\psi_0^{u,s}\left(\frac{u-ia}{\varepsilon},\tau\right)\\
&= \frac{C_+^2}{\varepsilon^{2r-1}}\left(\psi^{u,s}\left(\frac{u-ia}{\varepsilon},\tau\right)-\psi_0^{u,s}\left(\frac{u-ia}{\varepsilon},\tau\right)\right) \qquad (112)\\
\widetilde{\Delta}_2(u,\tau) &= \frac{C_+^2}{\varepsilon^{2r-1}}\left(\psi_0^{u}\left(\frac{u-ia}{\varepsilon},\tau\right)-\psi_0^{s}\left(\frac{u-ia}{\varepsilon},\tau\right)\right)-\Delta_0^+(u,\tau) \qquad (113)\\
\widetilde{\Delta}_3(u,\tau) &= -\Delta_0^-(u,\tau). \qquad (114)
\end{aligned}$$

Applying Theorem 4.16, one can see that for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon},c_2}\cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon},c_2}\right)\times\mathbb{T}$,

$$\left|\partial_u\widetilde{\Delta}_1^{u,s}(u,\tau)\right|\leq\frac{K\varepsilon^{\frac{1}{\beta}-2r}}{|\ln\varepsilon|^{2r-\frac{1}{\beta}}}.$$

To bound $\widetilde{\Delta}_2$, one has to proceed in different ways, depending on whether $\ell-2r>0$ or $\ell-2r=0$. For the first case, let us point out that,

$$\widetilde{\Delta}_2(u,\tau)=\sum_{k<0}\Upsilon_0^{[k]}\left(e^{ik\left(\varepsilon^{-1}u-\tau+\hat{\mu}g\left(\varepsilon^{-1}(u-ia),\tau\right)\right)}-e^{ik\left(\varepsilon^{-1}u-\tau+\mathcal{C}(u,\tau)\right)}\right).$$

Then, applying Theorems 4.12 and 4.21 and the mean value theorem one obtains that for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon},c_2}\cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon},c_2}\right)\times\mathbb{T}$,

$$\left|\partial_u\widetilde{\Delta}_2(u,\tau)\right|\leq\frac{K|\hat{\mu}|^2\varepsilon^{s-2r}}{|\ln\varepsilon|^{\ell-2r}}.$$

For the case $\ell-2r=0$, taking into account the definition of $\Upsilon_0^{[k]}$ in (103),

$$\widetilde{\Delta}_2(u,\tau)=\frac{C_+^2\hat{\mu}}{\varepsilon^{2r-1}}\sum_{k<0}\chi^{[k]}(\hat{\mu})\left(e^{ik\left(\varepsilon^{-1}(u-ia)-\tau+\hat{\mu}g\left(\varepsilon^{-1}(u-ia),\tau\right)\right)}-e^{ik\left(\varepsilon^{-1}(u-ia)-\tau+\mathcal{C}(u,\tau)-C(\hat{\mu},\varepsilon)+\hat{\mu}^2b\ln\varepsilon\right)}\right).$$

By Theorems 4.12 and 4.21 and Proposition 4.22 for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon},c_2}\cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon},c_2}\right)\times\mathbb{T}$, we have that

$$\left|\partial_u\widetilde{\Delta}_2(u,\tau)\right|\leq\frac{K|\hat{\mu}|^2\varepsilon^{s-2r}}{|\ln\varepsilon|^{1+\mathrm{Im}\,(\hat{\mu}^2b)}}.$$

Finally, to bound $\partial_u\widetilde{\Delta}_3$, it is enough to take into account (72). Then, one can see that for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon},c_2}\cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon},c_2}\right)\times\mathbb{T}$,

$$\begin{aligned}
\left|\partial_u\widetilde{\Delta}_3(u,\tau)\right| &\leq K|\hat{\mu}|\varepsilon^{-s-2r}e^{-\frac{2a}{\varepsilon}} && \text{provided } \ell-2r>0\\
\left|\partial_u\widetilde{\Delta}_3(u,\tau)\right| &\leq K|\hat{\mu}|\varepsilon^{-s-2r}e^{-\frac{2a}{\varepsilon}+2\mathrm{Im}\left(\hat{\mu}^2b\right)\ln\varepsilon+\mathrm{Im}\left(\hat{\mu}^2b\right)\ln\ln\frac{1}{\varepsilon}} && \text{provided } \ell-2r=0.
\end{aligned}$$

Therefore, from the bounds of $\widetilde{\Delta}_1^{u,s}$, $\widetilde{\Delta}_2$ and $\widetilde{\Delta}_3$ and recalling that by hypothesis $s<1/\beta$, we have that for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},+,s}_{s\ln\frac{1}{\varepsilon},c_2}\cap D^{\mathrm{in},+,u}_{s\ln\frac{1}{\varepsilon},c_2}\right)\times\mathbb{T}$,

$$\begin{aligned}
\left|\partial_u\widetilde{\Delta}(u,\tau)\right| &\leq \frac{K\varepsilon^{s-2r}}{|\ln\varepsilon|^{\ell-2r}} && \text{provided } \ell-2r>0\\
\left|\partial_u\widetilde{\Delta}(u,\tau)\right| &\leq \frac{K\varepsilon^{s-2r}}{|\ln\varepsilon|^{1+\mathrm{Im}\,(\hat{\mu}^2b)}} && \text{provided } \ell-2r=0.
\end{aligned}$$

Moreover, taking into account that $\partial_u\widetilde{\Delta}(u,\tau)$ depends analytically on $\hat{\mu}$ and moreover satisfies $\partial_u\widetilde{\Delta}(u,\tau)\Big|_{\hat{\mu}=0} = 0$, one can apply Schwartz Lemma to obtain

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right| \leq \frac{K|\hat{\mu}|\varepsilon^{s-2r}}{|\ln\varepsilon|^{\ell-2r}} \qquad \text{provided } \ell-2r>0 \tag{115}$$

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right| \leq \frac{K|\hat{\mu}|\varepsilon^{s-2r}}{|\ln\varepsilon|^{1+\mathrm{Im}\,(\hat{\mu}^2 b)}} \qquad \text{provided } \ell-2r=0. \tag{116}$$

Reasoning analogously, one can see that for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{in},-,s}_{s\ln\frac{1}{\varepsilon},c_2}\cap D^{\mathrm{in},-,u}_{s\ln\frac{1}{\varepsilon},c_2}\right)\times\mathbb{T}$, the function $\partial_u\widetilde{\Delta}$ satisfies

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right| \leq \frac{K|\hat{\mu}|\varepsilon^{s-2r}}{|\ln\varepsilon|^{\ell-2r}} \qquad \text{provided } \ell-2r>0 \tag{117}$$

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right| \leq \frac{K|\hat{\mu}|\varepsilon^{s-2r}}{|\ln\varepsilon|^{1-\mathrm{Im}\,\left(\hat{\mu}^2 \overline{b}\right)}} \qquad \text{provided } \ell-2r=0. \tag{118}$$

Finally, by Theorems 4.4, 4.8, 4.12 and 4.21, one can easily see that the bound of $\partial_u\widetilde{\Delta}(u,\tau)$ for $(u,\tau)\in\left(R_{s\ln\frac{1}{\varepsilon},d_3}\cap D^{\mathrm{out},s}_{c_2\varepsilon^\gamma,\rho_4}\cap D^{\mathrm{out},u}_{c_2\varepsilon^\gamma,\rho_4}\right)\times\mathbb{T}$ is smaller than (115) and (117) (case $\ell-2r>0$) and (116) and (118) (case $\ell-2r=0$), provided $|u-ia|\geq\mathcal{O}(\varepsilon^\gamma)$.

Taking into account (115) and (117) (case $\ell-2r>0$) and (116) and (118) (case $\ell-2r=0$), one can conclude that for $\hat{\mu}\in B(\hat{\mu}_0)\cap\mathbb{R}$,

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right| \leq \frac{K|\hat{\mu}|\varepsilon^{s-2r}}{|\ln\varepsilon|^{\ell-2r}} \qquad \text{provided } \ell-2r>0 \tag{119}$$

$$\left|\partial_u\widetilde{\Delta}(u,\tau)\right| \leq \frac{K|\hat{\mu}|\varepsilon^{s-2r}}{|\ln\varepsilon|^{1+\hat{\mu}^2\,\mathrm{Im}\,b}} \qquad \text{provided } \ell-2r=0. \tag{120}$$

Analogously to the proof of Theorem 4.19, the second step is to consider the change of variables $(w,\tau)=(u+\varepsilon\mathcal{C}(u,\tau),\tau)$ and the auxiliary function

$$\Theta(w,\tau)=\widetilde{\Upsilon}'\left(\varepsilon^{-1}w-\tau\right),$$

to obtain a bound for the Fourier coefficients of $\widetilde{\Upsilon}$:

$$\left|\widetilde{\Upsilon}^{[k]}\right| \leq K\varepsilon \sup_{(u,\tau)\in R_{s\ln(1/\varepsilon),d_3}\times\mathbb{T}}\left|\partial_u\widetilde{\Delta}(u,\tau)\right| e^{-\frac{|k|}{\varepsilon}\left(a-s\varepsilon\ln\frac{1}{\varepsilon}\right)-|k|\mathrm{Im}\,(\mathcal{C}(u^*,0))}.$$

Therefore, to obtain the bounds for $\widetilde{\Upsilon}^{[k]}$ with $k<0$, it only remains to use bounds (119) and (120) and the properties of $\mathcal{C}$ given in Theorem 4.21 and Proposition 4.22. Then, we obtain that for $k<0$

$$\left|\widetilde{\Upsilon}^{[k]}\right| \leq \frac{K|\hat{\mu}|}{\varepsilon^{2r-1}\,|\ln\varepsilon|^{\ell-2r}} e^{-|k|\frac{a}{\varepsilon}+(|k|-1)s\ln\frac{1}{\varepsilon}} \qquad \text{provided } \ell-2r>0$$

$$\left|\widetilde{\Upsilon}^{[k]}\right| \leq \frac{K|\hat{\mu}|}{\varepsilon^{2r-1}\,|\ln\varepsilon|} e^{-|k|\left(\frac{a}{\varepsilon}-\mathrm{Im}\,\left(\hat{\mu}^2 b\right)\ln\varepsilon\right)+(|k|-1)\left(s\ln\frac{1}{\varepsilon}+\mathrm{Im}\,\left(\hat{\mu}^2 b\right)\ln\ln\frac{1}{\varepsilon}\right)} \qquad \text{provided } \ell-2r=0.$$

Since $\partial_u\widetilde{\Delta}(u,\tau)$ and $\mathcal{C}(u,\tau)$ are real-analytic for $(\mu,\tau)\in\mathbb{R}$, the coefficients $\widetilde{\Upsilon}^{[k]}$ for $k>0$ satisfy the same bounds. Finally, the bounds of $\widetilde{\Upsilon}^{[k]}$ lead easily to the desired bounds of $\widetilde{\Delta}(u,\tau)$ for $(u,\tau)\in\left(R_{s\ln(1/\varepsilon),d_3}\cap\mathbb{R}\right)\times\mathbb{T}$. $\square$

4.8 Computation of the area of the lobes: proof of Theorems 2.5 and 2.6

Once we have obtained the first asymptotic term of the difference between the manifolds in Corollaries 4.20 and 4.24, we use it to prove the existence of transversal homoclinic points and to compute an asymptotic formula of the area of the lobes between two consecutive ones.

Let us fix $\mu \in \mathbb{R}$ such that $f(\mu) \neq 0$. Let us fix also a transversal section corresponding to $\tau = \tau_0 \in \mathbb{R}$. It can be easily seen that the consecutive zeros of $\partial_u \Delta_{00}(u, \tau_0)$ (see (87), (109) and (110)) are $\mathcal{O}(\varepsilon)$ close and therefore taking ε small enough, in $\left(R_{s\ln(1/\varepsilon),d_3} \cap \mathbb{R}\right)$ there exist at least two consecutive zeros u_- and u_+ in $\left(R_{s\ln(1/\varepsilon),d_3} \cap \mathbb{R}\right)$, which depend on τ_0. It can be easily checked that the function Δ_{00} evaluated at these points satisfies

$$\Delta_{00}(u_+, \tau_0) = -\Delta_{00}(u_-, \tau_0) \tag{121}$$

and

$$|\Delta_{00}(u_\pm, \tau_0)| = 2\mu\varepsilon^{\eta+1-\ell} \left|f(\mu\varepsilon^{\eta-\ell+2r})\right| e^{-\frac{a}{\varepsilon}} \qquad \text{if } \ell - 2r > 0 \tag{122}$$

$$|\Delta_{00}(u_\pm, \tau_0)| = 2\mu\varepsilon^{\eta+1-\ell} \left|f(\mu)e^{iC(\mu,\varepsilon)}\right| e^{-\frac{a}{\varepsilon}+\mu^2 \mathrm{Im}\, b \ln\varepsilon} \qquad \text{if } \ell - 2r = 0 \tag{123}$$

$$|\Delta_{00}(u_\pm, \tau_0)| = 2\mu\varepsilon^{\eta+1-\ell} \left|f_0 e^{iC(\mu,\varepsilon)}\right| e^{-\frac{a}{\varepsilon}} \qquad \text{if } \ell - 2r < 0. \tag{124}$$

In the case $\ell - 2r > 0$ we have undone the change of parameter (58).

By Corollary 4.24, since by hypothesis $f(\mu) \neq 0$, we can apply the implicit function theorem to see that close to u_- and u_+, there exist zeros u_-^* and u_+^* of the function $\partial_u \Delta(u, \tau_0)$ defined in (76). These zeros correspond to transversal homoclinic points and they satisfy

$$u_\pm^* = u_\pm + \mathcal{O}\left(\frac{\varepsilon}{|\ln\varepsilon|^{\nu_\ell}}\right), \tag{125}$$

where $\nu_\ell = \ell - 2r$ for $\ell > 2r$ and $\nu_\ell = 1$ for $\ell \leq 2r$.

To compute the area $\mathcal{A}$ of the lobes between u_- and u_+ it is enough to point out that, since the change (39) is symplectic, it preserves area. Then recalling the definition of Δ in (76), the area is just given by

$$\mathcal{A} = \left|\int_{u_-^*}^{u_+^*} \partial_u \Delta(u, \tau_0)\, du\right| = \left|\Delta(u_+^*, \tau_0) - \Delta(u_-^*, \tau_0)\right|.$$

Finally, if $\ell < 2r$ we use Corollary 4.20 and Theorem 4.19 and also formulas (121), (124) and (125) to obtain the asymptotic formula for $\mathcal{A}$ stated in Theorem 2.5. In the same way, if $\ell \geq 2r$ using Corollary 4.24, Theorem 4.21 and formulas (121), (122) (if $\ell - 2r > 0$) or (123) (if $\ell - 2r = 0$) and (125), we derive the results in Theorem 2.6.

5 Existence of the periodic orbit in the hyperbolic case: proof of Theorem 4.1

In this section we prove Theorem 4.1. We look for a periodic orbit $(x, y) = (x_p(\tau), y_p(\tau))$ which is close to the hyperbolic critical point of the unperturbed system $(0, 0)$.

By **HP1.1**, the differential of the unperturbed hyperbolic critical point is

$$\varepsilon A_0 = \varepsilon \begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}. \tag{126}$$

Then, defining $z = (x, y)$ and considering the differential operator

$$\mathcal{D}_0 z(\tau) = \frac{d}{d\tau} z(\tau), \tag{127}$$

we look for the periodic orbit as a 2π-periodic solution of the following equation,

$$(\mathcal{D}_0 - \varepsilon A_0)\, z = \varepsilon F(z,\tau), \tag{128}$$

where

$$F(z,\tau) = \begin{pmatrix} \mu\varepsilon^\eta \partial_y H_1(x,y,\tau) \\ -\mu\varepsilon^\eta \partial_x H_1(x,y,\tau) - \big(V'(x) + \lambda^2 x\big) \end{pmatrix}.$$

We split F in constant, linear and higher order terms with respect to z

$$F(z,\tau) = F_0(\tau) + F_1(\tau)z + F_2(z,\tau) \tag{129}$$

with

$$F_0(\tau) = \begin{pmatrix} \mu\varepsilon^\eta \partial_y H_1(0,0,\tau) \\ -\mu\varepsilon^\eta \partial_x H_1(0,0,\tau) \end{pmatrix} \tag{130}$$

$$F_1(\tau) = \begin{pmatrix} \mu\varepsilon^\eta \partial_{yx} H_1(0,0,\tau) & \mu\varepsilon^\eta \partial_{yy} H_1(0,0,\tau) \\ -\mu\varepsilon^\eta \partial_{xx} H_1(0,0,\tau) & -\mu\varepsilon^\eta \partial_{xy} H_1(0,0,\tau) \end{pmatrix} \tag{131}$$

$$F_2(z,\tau) = F(z,\tau) - F_0(\tau) - F_1(\tau)z. \tag{132}$$

We devote the rest of the section to obtain a solution of equation (128). First in Section 5.1 we define a Banach space we will use and we state some technical properties. Then, in Section 5.2 we prove Theorem 4.1.

5.1 Banach spaces and technical lemmas

For analytic functions $z : \mathbb{T}_\sigma \to \mathbb{C}$, $z(\tau) = \sum_{k\in\mathbb{Z}} z^{[k]} e^{ik\tau}$, we define the Fourier norm

$$\|z\|_\sigma = \sum_{k\in\mathbb{Z}} \left| z^{[k]} \right| e^{|k|\sigma}.$$

Then, we define the function space endowed with the previous norm

$$\mathcal{S}_\sigma = \{z : \mathbb{T}_\sigma \to \mathbb{C};\ \text{real-analytic}\ , \|z\|_\sigma < \infty\} \tag{133}$$

which is a Banach algebra. We also consider the product space $\mathcal{S}_\sigma \times \mathcal{S}_\sigma$ with the induced norm

$$\|(z_1,z_2)\|_{1,\sigma} = \|z_1\|_\sigma + \|z_2\|_\sigma\,.$$

Remark 5.1. *Let us consider the classical supremmum norm*

$$\|z\|_{\infty,\sigma} = \sup_{\tau\in\overline{\mathbb{T}}_\sigma} |z(\tau)|\,.$$

Then, it is a well known fact (see for instance [Sau01]) that for any $\sigma_1 < \sigma_2$, the supremmum and the Fourier norm satisfy the following relation

$$\|z\|_{\sigma_1} < K\left(1 + \frac{1}{\sigma_2 - \sigma_1}\right)\|z\|_{\infty,\sigma_2}$$

Therefore, since we are assuming that there exists $\sigma_0 > 0$ such that the functions a_{kl} defined in (8) *and* (9) *are $\mathcal{C}^0$ in $\overline{\mathbb{T}}_{\sigma_0}$ and analytic in $\mathbb{T}_{\sigma_0}$, we can deduce that for any $\sigma < \sigma_0$ such that $\sigma_0 - \sigma$ has a positive lower bound independent of ε, they satisfy*

$$\|a_{kl}\|_\sigma < K.$$

We will use this fact without mentioning it, in the rest of the section and also in Sections 6.1 to 9.

Since we deal with vector functions, we also consider the norm for 2×2 matrices induced by $\|\cdot\|_{1,\sigma}$. Let us consider $B=\left(b^{ij}\right)$ a 2×2 matrix such that $b^{ij}\in\mathcal{S}_\sigma$. Then, the induced matrix norm is given by

$$\|B\|_{1,\sigma}=\max_{j=1,2}\left\{\left\|b^{1j}\right\|_\sigma+\left\|b^{2j}\right\|_\sigma\right\}.$$

The next lemma gives some properties of this norm.

Lemma 5.2. *The following statements are satisfied.*

1. *If $h\in\mathcal{S}_\sigma\times\mathcal{S}_\sigma$ and $B=\left(b^{ij}\right)$ is a 2×2 matrix with $b^{ij}\in\mathcal{S}_\sigma$, then $Bh\in\mathcal{S}_\sigma\times\mathcal{S}_\sigma$ and*
$$\|Bh\|_{1,\sigma}\le\|B\|_{1,\sigma}\|h\|_{1,\sigma}.$$
2. *If $B_1=\left(b_1^{ij}\right)$ and $B_2=\left(b_2^{ij}\right)$ are 2×2 matrices which satisfy $b_1^{ij},b_2^{ij}\in\mathcal{S}_\sigma$, then*
$$\|B_1B_2\|_{1,\sigma}\le\|B_1\|_{1,\sigma}\|B_2\|_{1,\sigma}.$$

Throughout this section, we will need to solve equations of the form $(\mathcal{D}_0-\varepsilon A_0)z=w$. For that, we will invert the operator $\mathcal{D}_0-\varepsilon A_0$ acting on $\mathcal{S}_\sigma\times\mathcal{S}_\sigma$. Considering the Fourier series of $z(\tau)=(z_1(\tau),z_2(\tau))$, one has that

$$\mathcal{D}_0(z)(\tau)=\sum_{\kappa\in\mathbb{Z}}ikz^{[k]}e^{ik\tau}.$$

Then, one can invert $\mathcal{D}_0-\varepsilon A_0$ as

$$\mathcal{G}_0(w)(\tau)=-\sum_{k\in\mathbb{Z}}\frac{1}{k^2+\lambda^2\varepsilon^2}\begin{pmatrix} ikw_1^{[k]}+\varepsilon w_2^{[k]}\\ \varepsilon\lambda^2w_1^{[k]}+ikw_2^{[k]}\end{pmatrix}e^{ik\tau}. \tag{134}$$

Lemma 5.3. *The operator $\mathcal{G}_0:\mathcal{S}_\sigma\times\mathcal{S}_\sigma\to\mathcal{S}_\sigma\times\mathcal{S}_\sigma$ in* (134) *is well defined, and for $w\in\mathcal{S}_\sigma\times\mathcal{S}_\sigma$,*

$$\|\mathcal{G}_0(w)\|_{1,\sigma}\le\frac{K}{\varepsilon}\|w\|_{1,\sigma}.$$

Moreover, if $\langle w\rangle=0$,

$$\|\mathcal{G}_0(w)\|_{1,\sigma}\le K\|w\|_{1,\sigma}.$$

We finally state a technical lemma which will be used in Section 5.2. Its proof is straightforward.

Lemma 5.4. *The functions F_0, F_1 and F_2 defined in* (130), (131) *and* (132) *respectively satisfy the following properties.*

1. *$F_0\in\mathcal{S}_\sigma\times\mathcal{S}_\sigma$, $\langle F_0\rangle=0$ and*
$$\|F_0\|_{1,\sigma}\le K|\mu|\varepsilon^\eta.$$
2. *$F_1=\left(F_1^{ij}\right)$ satisfies $F_1^{ij}\in\mathcal{S}_\sigma$, $\langle F_1^{ij}\rangle=0$ and*
$$\|F_1\|_{1,\sigma}\le K|\mu|\varepsilon^\eta.$$
3. *If $z,z'\in B(\nu)\subset\mathcal{S}_\sigma$ with $\nu\ll 1$, then*
$$\|F_2(z',\tau)-F_2(z,\tau)\|_\sigma\le K\nu\|z'-z\|_\sigma.$$

5.2 Proof of Theorem 4.1

We rewrite Theorem 4.1 in terms of the Banach space (133).

Proposition 5.5. *Let $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0, \varepsilon_0)$, equation* (128) *has a solution $(x_p, y_p) \in \mathcal{S}_\sigma$. Moreover, there exists a constant $b_0 > 0$ such that*

$$\|(x_p, y_p)\|_{1,\sigma} \leq b_0 |\mu| \varepsilon^{\eta+1}.$$

Corollary 5.6. *The change of variables* (32) *transforms the Hamiltonian system with Hamiltonian* (6) *to a new Hamiltonian system with Hamiltonian* (33).

Moreover, the functions c_{ij} in the definition of (33) *(see also* (37)*) satisfy*

$$\|c_{ij}\|_\sigma \leq K |\mu| \varepsilon^\eta.$$

We devote the rest of the section to prove Proposition 5.5. We obtain the solution of equation (128) through a fixed point argument. To obtain a contractive operator, first we have to perform a change of variables, which actually it is only needed in the case $\ell - 2r = 0$.

Let us consider a function $\overline{F}_1$ which satisfies $\langle \overline{F}_1 \rangle = 0$ and $\partial_\tau \overline{F}_1 = F_1$, where F_1 is the function in (131). The function $\overline{F}_1$ can be defined as

$$\overline{F}_1(\tau) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{ik} F_1^{[k]} e^{ik\tau}$$

and satisfies

$$\left\|\overline{F}_1\right\|_{1,\sigma} \leq \|F_1\|_{1,\sigma}. \tag{135}$$

We perform the change of variables

$$z = \left(\mathrm{Id} + \varepsilon \overline{F}_1(\tau)\right) \overline{z} \tag{136}$$

and then equation (128) becomes

$$\left(\mathcal{D}_0 - \varepsilon A_0\right) \overline{z} = \overline{F}(\overline{z}, \tau), \tag{137}$$

where

$$\begin{aligned}
\overline{F}(\overline{z}, \tau) =& \varepsilon \left(\mathrm{Id} + \varepsilon \overline{F}_1(\tau)\right)^{-1} F_0(\tau) \\
&+ \varepsilon^2 \left(\mathrm{Id} + \varepsilon \overline{F}_1(\tau)\right)^{-1} \left(A_0 \overline{F}_1(\tau) - \overline{F}_1(\tau) A_0 + \overline{F}_1(\tau) F_1(\tau)\right) \overline{z} \\
&+ \varepsilon \left(\mathrm{Id} + \varepsilon \overline{F}_1(\tau)\right)^{-1} F_2\left(\left(\mathrm{Id} + \varepsilon \overline{F}_1(\tau)\right) \overline{z}, \tau\right).
\end{aligned} \tag{138}$$

Since the operator $\mathcal{G}_0$ defined in (134) is a left inverse of $\mathcal{D}_0 - \varepsilon A_0$, we look for a solution of equation (137) as a fixed point of the operator

$$\mathcal{F}_0 = \mathcal{G}_0 \circ \overline{F}. \tag{139}$$

Then Proposition 5.5 follows from the following lemma.

Lemma 5.7. *Let $\varepsilon_0 > 0$ small enough. Then, there exists a constant $b_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the operator $\mathcal{F}_0$ in* (139) *is contractive from $\overline{B}\left(b_0 |\mu| \varepsilon^{\eta+1}\right) \subset \mathcal{S}_\sigma \times \mathcal{S}_\sigma$ to itself.*

Then, $\mathcal{F}_0$ has a unique fixed point $\overline{z}^* \in \overline{B}\left(b_0 |\mu| \varepsilon^{\eta+1}\right) \subset \mathcal{S}_\sigma \times \mathcal{S}_\sigma$.

Proof. It is easily checked that $\mathcal{F}_0$ sends $\mathcal{S}_\sigma \times \mathcal{S}_\sigma$ into itself. To see that it is contractive we first consider $\mathcal{F}_0(0)$, which can be split as

$$\mathcal{F}_0(0) = \varepsilon \mathcal{G}_0\left(F_0\right) - \varepsilon^2 \mathcal{G}_0\left(\left(\mathrm{Id} + \varepsilon \overline{F}_1\right)^{-1} \overline{F}_1 F_0\right).$$

By Lemma 5.4, $\langle F_0 \rangle = 0$ and $\|F_0\|_{1,\sigma} \leq K |\mu| \varepsilon^\eta$. Then, applying Lemma 5.3, one has that

$$\left\|\mathcal{G}_0\left(F_0\right)\right\|_{1,\sigma} \leq K |\mu| \varepsilon^\eta.$$

For the second term, considering also (135) and Lemmas 5.2, 5.3 and 5.4, one can proceed analogously to obtain

$$\left\|\mathcal{G}_0\left(\left(\mathrm{Id}+\varepsilon\overline{F}_1\right)^{-1}\overline{F}_1F_0\right)\right\|_{1,\sigma}\leq K|\mu|\varepsilon^{2\eta-1}.$$

Therefore, there exists a constant $b_0>0$ such that

$$\|\mathcal{F}_0(0)\|_{1,\sigma}\leq\frac{b_0}{2}|\mu|\varepsilon^{\eta+1}.$$

Let us consider now $z^1,z^2\in\overline{B}\left(b_0|\mu|\varepsilon^{\eta+1}\right)\subset\mathcal{S}_\sigma\times\mathcal{S}_\sigma$. Then, by Lemmas 5.3, 5.2 and 5.4, and reducing ε if necessary, one can see that,

$$\begin{aligned}\left\|\mathcal{F}_0\left(z^2\right)-\mathcal{F}_0\left(z^1\right)\right\|_{1,\sigma}&\leq K|\mu|\varepsilon^{\eta+1}\left\|z^2-z^1\right\|_{1,\sigma}\\&\leq\frac{1}{2}\left\|z^2-z^1\right\|_{1,\sigma}.\end{aligned}$$

Then, $\mathcal{F}_0:\overline{B}\left(b_0|\mu|\varepsilon^{\eta+1}\right)\to\overline{B}\left(b_0|\mu|\varepsilon^{\eta+1}\right)\subset\mathcal{S}_\sigma\times\mathcal{S}_\sigma$ and is contractive. Therefore, it has a unique fixed point $\overline{z}^*$. □

Proof of Proposition 5.5. It is enough to take

$$z^*(\tau)=\left(\mathrm{Id}+\varepsilon\overline{F}_1(\tau)\right)\overline{z}^*(\tau),$$

which satisfies equation (128) and satisfies the desired bound (increasing b_0 slightly if necessary). □

6 Local invariant manifolds: proof of Theorem 4.3

Since the proof for both invariant manifolds is analogous, we only deal with the unstable case. We look for a solution of equation (41) satisfying the asymptotic condition (49). We look for it as a perturbation of the unperturbed separatrix

$$T_0(u)=\int_{-\infty}^u p_0^2(v)\,dv \tag{140}$$

and therefore we work with $T_1(u,\tau)=T(u,\tau)-T_0(u)$.

Replacing T in equation (41) and taking into account that $V(q_0(u))=-p_0^2(u)/2$, it is straightforward to see that the equation for T_1 reads

$$\mathcal{L}_\varepsilon T_1=\mathcal{F}\left(\partial_uT_1,u,\tau\right), \tag{141}$$

where $\mathcal{L}_\varepsilon$ is the operator defined in (45) and

$$\begin{aligned}\mathcal{F}(w,u,\tau)=\;&-\frac{w^2}{2p_0^2(u)}-\Big(V(q_0(u)+x_p(\tau))-V(x_p(\tau))-V(q_0(u))-V'(x_p(\tau))q_0(u)\Big)\\&-\mu\varepsilon^\eta\widehat{H}_1\left(q_0(u),p_0(u)+\frac{w}{p_0(u)},\tau\right),\end{aligned}$$

where $\widehat{H}_1$ is the function defined in (34).

We split $\mathcal{F}$ into constant, linear and higher order terms in w as

$$\mathcal{F}(w,u,\tau)=A(u,\tau)+\left(B_1(u,\tau)+B_2(u,\tau)\right)w+C(w,u,\tau), \tag{142}$$

with

$$\begin{aligned}
A(u,\tau) =& -\left(V(q_0(u)+x_p(\tau)) - V(x_p(\tau)) - V(q_0(u)) - V'(x_p(\tau))q_0(u)\right) \\
& - \mu\varepsilon^\eta \widehat{H}_1\left(q_0(u), p_0(u), \tau\right), && (143)\\
B_1(u,\tau) =& -\mu\varepsilon^\eta p_0^{-1}(u)\partial_p \widehat{H}_1^1(q_0(u), p_0(u), \tau), && (144)\\
B_2(u,\tau) =& -\mu\varepsilon^{\eta+1} p_0^{-1}(u)\partial_p \widehat{H}_1^2(q_0(u), p_0(u), \tau), && (145)\\
C(w,u,\tau) =& -\frac{w^2}{2p_0^2(u)} - \mu\varepsilon^\eta \widehat{H}_1\left(q_0(u), p_0(u) + \frac{w}{p_0(u)}, \tau\right) \\
& + \mu\varepsilon^\eta \frac{w}{p_0(u)}\partial_p \widehat{H}_1(q_0(u), p_0(u), \tau) + \mu\varepsilon^\eta \widehat{H}_1\left(q_0(u), p_0(u), \tau\right), && (146)
\end{aligned}$$

where $\widehat{H}_1^1$ and $\widehat{H}_1^2$ are the functions defined in (35) and (37) respectively.

6.1 Local invariant manifolds in the hyperbolic case

In this section we prove the existence of suitable representations of the unstable and stable invariant manifolds in the domains $D^u_{\infty,\rho}\times\mathbb{T}_\sigma$ and $D^s_{\infty,\rho}\times\mathbb{T}_\sigma$ respectively under the hypothesis that the unperturbed Hamiltonian system has a hyperbolic critical point at the origin.

6.1.1 Banach spaces and technical lemmas

This subsection is devoted to define the Banach spaces which will be used in Section 6.1.2. We also state some of their useful properties.

We define some norms for functions defined in a domain $D^u_{\infty,\rho}$ with $\rho \geq 0$. Given $\alpha \geq 0$, $\rho \geq 0$ and an analytic function $h : D^u_{\infty,\rho} \to \mathbb{C}$, we consider

$$\|h\|_{\alpha,\rho} = \sup_{u\in D^u_{\infty,\rho}} \left|e^{-\alpha u} h(u)\right|.$$

Moreover for analytic functions $h : D^u_{\infty,\rho}\times\mathbb{T}_\sigma \to \mathbb{C}$ which are 2π-periodic in τ, we consider the corresponding Fourier norm

$$\|h\|_{\alpha,\rho,\sigma} = \sum_{k\in\mathbb{Z}} \left\|h^{[k]}\right\|_{\alpha,\rho} e^{|k|\sigma}.$$

We consider, thus, the following function space

$$\mathcal{H}_{\alpha,\rho,\sigma} = \{h : D^u_{\infty,\rho}\times\mathbb{T}_\sigma \to \mathbb{C};\ \text{real-analytic}, \|h\|_{\alpha,\rho,\sigma} < \infty\}, \tag{147}$$

which can be checked that is a Banach space for any fixed $\alpha > 0$ and $\sigma > 0$.

In the next lemma, we state some properties of these Banach spaces.

Lemma 6.1. *The following statements hold:*

1. *If $\alpha_1 \geq \alpha_2 \geq 0$, then $\mathcal{H}_{\alpha_1,\rho,\sigma} \subset \mathcal{H}_{\alpha_2,\rho,\sigma}$ and*

$$\|h\|_{\alpha_2,\rho,\sigma} \leq \|h\|_{\alpha_1,\rho,\sigma}.$$

2. *If $\alpha_1, \alpha_2 \geq 0$, then, for $h \in \mathcal{H}_{\alpha_1,\rho,\sigma}$ and $g \in \mathcal{H}_{\alpha_2,\rho,\sigma}$, we have that $hg \in \mathcal{H}_{\alpha_1+\alpha_2,\rho,\sigma}$ and*

$$\|hg\|_{\alpha_1+\alpha_2,\rho,\sigma} \leq \|h\|_{\alpha_1,\rho,\sigma}\|g\|_{\alpha_2,\rho,\sigma}.$$

3. *Let us consider $\alpha \geq 0$ and $\rho' > \rho > 0$ such that $\rho' - \rho$ has a positive lower bound independent of ε. Then for $h \in \mathcal{H}_{\alpha,\rho,\sigma}$ we have that $\partial_u h \in \mathcal{H}_{\alpha,\rho',\sigma}$ and*

$$\|\partial_u h\|_{\alpha,\rho',\sigma} \leq K\|h\|_{\alpha,\rho,\sigma}.$$

Throughout this section we are going to solve equations of the form $\mathcal{L}_\varepsilon h = g$, where $\mathcal{L}_\varepsilon$ is the differential operator defined in (45). Note that if $\alpha > 0$, $\mathrm{Ker}\mathcal{L}_\varepsilon = \{0\}$ and hence $\mathcal{L}_\varepsilon$ is invertible. It turns out that its inverse is $\mathcal{G}_\varepsilon$ defined by

$$\mathcal{G}_\varepsilon(h)(u,\tau) = \int_{-\infty}^{0} h(u+t, \tau + \varepsilon^{-1}t)\, dt. \tag{148}$$

We also introduce

$$\overline{\mathcal{G}}_\varepsilon(h)(u,\tau) = \partial_u \left[\mathcal{G}_\varepsilon(h)(u,\tau)\right]. \tag{149}$$

We will consider $\mathcal{G}_\varepsilon$ defined in $\mathcal{H}_{\alpha,\rho,\sigma}$ with $\alpha > 0$ in order the integral in (148) to be convergent.

Lemma 6.2. *Let $\alpha > 0$. Then, the operators $\mathcal{G}_\varepsilon$ and $\overline{\mathcal{G}}_\varepsilon$ in* (148) *and* (149) *respectively satisfy the following properties.*

1. *$\mathcal{G}_\varepsilon$ is linear from $\mathcal{H}_{\alpha,\rho,\sigma}$ to itself, commutes with ∂_u and $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon = \mathrm{Id}$.*

2. *If $h \in \mathcal{H}_{\alpha,\rho,\sigma}$, then*
$$\|\mathcal{G}_\varepsilon(h)\|_{\alpha,\rho,\sigma} \leq K\|h\|_{\alpha,\rho,\sigma}.$$
Furthermore, if $\langle h\rangle = 0$, then
$$\|\mathcal{G}_\varepsilon(h)\|_{\alpha,\rho,\sigma} \leq K\varepsilon\|h\|_{\alpha,\rho,\sigma}.$$

3. *If $h \in \mathcal{H}_{\alpha,\rho,\sigma}$, then $\overline{\mathcal{G}}_\varepsilon(h) \in \mathcal{H}_{\alpha,\rho,\sigma}$ and*
$$\left\|\overline{\mathcal{G}}_\varepsilon(h)\right\|_{\alpha,\rho,\sigma} \leq K\|h\|_{\alpha,\rho,\sigma}.$$

Proof. It follows the same lines as the proof of Lemma 5.5 in [GOS10]. □

Finally, we state a technical lemma about estimates of the functions A, B_1, B_2 and C defined in (143), (144), (145) and (146) respectively. First we fix ρ_0 big enough such that $p_0(u) \neq 0$ in D^u_{∞,ρ_0}, where D^u_{∞,ρ_0} is the domain defined in (28).

Lemma 6.3. *Let $\{\lambda, -\lambda\}$ be the eigenvalues of the hyperbolic critical point of the unperturbed Hamiltonian system and $\overline{\mathcal{G}}_\varepsilon$ the operator defined in Lemma 6.2. Let us fix ρ_0 big enough such that $p_0(u) \neq 0$ in D^u_{∞,ρ_0}. Then, for any $\rho > \rho_0$, the functions A, B_1, B_2 and C defined in* (143), (144), (145) *and* (146) *satisfy the following properties,*

1. *$A, \partial_u A \in \mathcal{H}_{2\lambda,\rho,\sigma}$ and satisfy*
$$\left\|\overline{\mathcal{G}}_\varepsilon(A)\right\|_{2\lambda,\rho,\sigma} \leq K|\mu|\varepsilon^{\eta+1}, \quad \|\partial_u A\|_{2\lambda,\rho,\sigma} \leq K|\mu|\varepsilon^{\eta}. \tag{150}$$

2. *$B_1, \partial_u B_1, B_2 \in \mathcal{H}_{0,\rho,\sigma}$ and satisfy*
$$\|B_1\|_{0,\rho,\sigma} \leq K|\mu|\varepsilon^{\eta}, \quad \|\partial_u B_1\|_{0,\rho,\sigma} \leq K|\mu|\varepsilon^{\eta}, \quad \|B_2\|_{0,\rho,\sigma} \leq K|\mu|\varepsilon^{\eta+1}. \tag{151}$$

3. *Let us consider $h_1, h_2 \in B(\nu) \subset \mathcal{H}_{2\lambda,\rho,\sigma}$. Then,*
$$\|C(h_2,u,\tau) - C(h_1,u,\tau)\|_{2\lambda,\rho,\sigma} \leq K\nu\|h_2 - h_1\|_{2\lambda,\rho,\sigma}.$$

Proof. For the first bounds, we split $A = A_1 + A_2 + A_3$ as

$$A_1(u,\tau) = -\left(V(q_0(u) + x_p(\tau)) - V(x_p(\tau)) - V(q_0(u)) - V'(x_p(\tau))q_0(u)\right) \tag{152}$$
$$A_2(u,\tau) = -\mu\varepsilon^{\eta}\widehat{H}_1^1(q_0(u), p_0(u), \tau) \tag{153}$$
$$A_3(u,\tau) = -\mu\varepsilon^{\eta+1}\widehat{H}_1^2(q_0(u), p_0(u), \tau), \tag{154}$$

where $\widehat{H}_1^1$ and $\widehat{H}_1^2$ are the functions defined in (35) and (37).

For A_1, using the mean value theorem and Hypothesis **HP1.1**, one can see that

$$\begin{aligned} A_1(u,\tau) = &- q_0^2(u)\int_0^1 \left(V''\left(x_p(\tau)+s_1q_0(u)\right) - V''\left(s_1q_0(u)\right)\right)(1-s_1)\,ds_1 \\ = &- q_0^2(u)x_p(\tau)\int_0^1\int_0^1 V'''\left(s_2x_p(\tau)+s_1q_0(u)\right)(1-s_1)\,ds_1ds_2. \end{aligned} \tag{155}$$

Therefore, $A_1 \in \mathcal{H}_{2\lambda,\rho,\sigma}$ and $\|A_1\|_{2\lambda,\rho,\sigma} \le K|\mu|\varepsilon^{\eta+1}$. Applying Lemma 6.2, we obtain $\|\overline{\mathcal{G}}_\varepsilon(A_1)\|_{2\lambda,\rho,\sigma} \le K|\mu|\varepsilon^{\eta+1}$.

For the other terms, let us point out that, by construction, $\widehat{H}_1^1$ and $\widehat{H}_1^2$ are quadratic in (q,p) and therefore $A_2, A_3 \in \mathcal{H}_{2\lambda,\rho,\sigma}$. To bound $\overline{\mathcal{G}}_\varepsilon(A_2)$, let us point out that $\langle A_2\rangle = 0$ and therefore, taking into account that A_2 is analytic in $D^u_{\infty,\rho_0}\times\mathbb{T}_\sigma$ and $\rho>\rho_0$, by Lemmas (6.1) and 6.2,

$$\|\overline{\mathcal{G}}_\varepsilon(A_2)\|_{2\lambda,\rho,\sigma} \le K\varepsilon\|A_2\|_{2\lambda,\rho,\sigma} \le K|\mu|\varepsilon^{\eta+1}.$$

On the other hand, since by Corollary 5.6, $\|A_3\|_{2\lambda,\rho,\sigma} \le K|\mu|^2\varepsilon^{2\eta+1}$, we have that $\|\overline{\mathcal{G}}_\varepsilon(A_3)\|_{2\lambda,\rho,\sigma} \le K|\mu|^2\varepsilon^{2\eta+1}$. Therefore

$$\left\|\overline{\mathcal{G}}_\varepsilon(A)\right\|_{2\lambda,\rho,\sigma} \le K|\mu|\varepsilon^{\eta+1}.$$

The bound for $\partial_u A$ can be obtained just differentiating A_i, $i=1,2,3$.

The other bounds are straightforward. $\square$

6.1.2 Proof of Theorem 4.3 in the hyperbolic case

We devote this section to prove Theorem 4.3 for the case in which the unperturbed Hamiltonian has a hyperbolic critical point. First we rewrite it in terms of the Banach spaces defined in (147).

Proposition 6.4. *Let $\{\lambda,-\lambda\}$ be the eigenvalues of the unperturbed hyperbolic critical point, $\rho_1>0$ big enough and $\varepsilon_0>0$ small enough. Then, for $\varepsilon\in(0,\varepsilon_0)$, there exists a function $T_1(u,\tau)$ defined in $D^u_{\infty,\rho_1}\times\mathbb{T}_\sigma$ which satisfies equation* (141) *and the asymptotic condition* (49)*. Moreover, there exists a constant $b_1>0$ such that*

$$\|\partial_u T_1\|_{2\lambda,\rho_1,\sigma} \le b_1|\mu|\varepsilon^{\eta+1}.$$

Theorem 4.3 is a straightforward consequence of this proposition.

Let us observe that the operator $\mathcal{F}$ defined in (142) has linear terms in w which are not small when $\eta=0$. Therefore, if one wants to prove the existence of T through a fixed point argument, first we must look for a change of variables. Let us point out that this change of variables is not necessary for the case $\eta>0$.

Lemma 6.5. *Let us consider $\rho_1>\rho_0'>\rho_0>0$, where ρ_0 is big enough such that $p_0(u)\neq 0$ for $u\in D^u_{\infty,\rho_0}$. Then, for $\varepsilon>0$ small enough, there exists a function $g\in\mathcal{H}_{0,\rho_0',\sigma}$ such that $\langle g\rangle=0$ and is solution of*

$$\mathcal{L}_\varepsilon g = -B_1(v,\tau), \tag{156}$$

where $\mathcal{L}_\varepsilon$ is the operator defined in (45) *and B_1 is the function defined in* (144)*. Moreover, it satisfies that*

$$\|g\|_{0,\rho_0',\sigma} \le K|\mu|\varepsilon^{\eta+1}, \quad \|\partial_v g\|_{0,\rho_0',\sigma} \le K|\mu|\varepsilon^{\eta+1}$$

and $v+g(v,\tau)\in D^u_{\infty,\rho_0}$ for $(v,\tau)\in D^u_{\infty,\rho_0'}\times\mathbb{T}_\sigma$.

Furthermore, $(u,\tau)=(v+g(v,\tau),\tau)$ is invertible and its inverse is of the form $(v,\tau)=(u+h(u,\tau),\tau)$, where h is a function defined for $(u,\tau)\in D^u_{\infty,\rho_1}\times\mathbb{T}_\sigma$ and satisfies that $h\in\mathcal{H}_{0,\rho_1,\sigma}$,

$$\|h\|_{0,\rho_1,\sigma} \le K|\mu|\varepsilon^{\eta+1}$$

and that $u+h(u,\tau)\in D^u_{\infty,\rho_0'}$ for $(u,\tau)\in D^u_{\infty,\rho_1}\times\mathbb{T}_\sigma$.

Proof. From the definition of B_1 in (144) we have that $\langle B_1\rangle = 0$. On the other hand, using the definition of $\widehat{H}_1^1$ and λ in (35) and (46) respectively, B_1 can be split as

$$B_1(v,\tau) = B_{10}(\tau) + B_{11}(v,\tau),$$

where, using (47),

$$B_{10}(\tau) = \lim_{\mathrm{Re}\, v\to-\infty} B_1(v,\tau) = -\mu\varepsilon^\eta\left(\frac{a_{11}(\tau)}{\lambda} + 2a_{02}(\tau)\right)$$

and $B_{11}(v,\tau) = B_1(v,\tau) - B_{10}(\tau)$. Both terms have zero mean. Moreover, $B_{10} \in \mathcal{H}_{0,\rho_0',\sigma}$ and satisfies $\|B_{10}\|_{0,\rho_0',\sigma} \le K|\mu|\varepsilon^\eta$ and $B_{11} \in \mathcal{H}_{\lambda,\rho_0',\sigma}$ and satisfies $\|B_{11}\|_{\lambda,\rho_0',\sigma} \le K|\mu|\varepsilon^\eta$.

Since $B_{10}(\tau) = \sum_{k\in\mathbb{Z}\setminus\{0\}} B_{10}^{[k]} e^{ik\tau}$ has zero average, we can define a 2π-periodic primitive with zero average as

$$\overline{B}_{10}(\tau) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{B_{10}^{[k]}}{ik} e^{ik\tau}$$

which satisfies $\|\overline{B}_{10}\|_{0,\rho_0',\sigma} \le K|\mu|\varepsilon^\eta$.

By the linearity of equation (156), we can take g as

$$g(v,\tau) = -\varepsilon \overline{B}_{10}(\tau) - \mathcal{G}_\varepsilon\left(B_{11}\right)(v,\tau),$$

where $\mathcal{G}_\varepsilon$ is the operator defined in (148). Moreover, using the first statement of Lemma 6.1 and Lemma 6.2,

$$\|g\|_{0,\rho_0',\sigma} \le \varepsilon\left\|\overline{B}_{10}\right\|_{0,\rho_0',\sigma} + \left\|\mathcal{G}_\varepsilon\left(B_{11}\right)\right\|_{\lambda,\rho_0',\sigma} \le K|\mu|\varepsilon^{\eta+1} + K\varepsilon\left\|B_{11}\right\|_{\lambda,\rho_0',\sigma} \le K|\mu|\varepsilon^{\eta+1}.$$

Moreover, by Lemma 6.2,

$$\partial_v g = -\partial_v \mathcal{G}_\varepsilon\left(B_{11}\right) = -\mathcal{G}_\varepsilon\left(\partial_v B_{11}\right)$$

and then,

$$\|\partial_v g\|_{0,\rho_0',\sigma} \le \|\partial_v g\|_{\lambda,\rho_0',\sigma} = \left\|\mathcal{G}_\varepsilon\left(\partial_v B_{11}\right)\right\|_{\lambda,\rho_0',\sigma} \le K\varepsilon\left\|\partial_v B_{11}\right\|_{\lambda,\rho_0',\sigma} \le K|\mu|\varepsilon^{\eta+1}.$$

Since $\|g\|_{0,\rho_0',\sigma} \le K|\mu|\varepsilon^{\eta+1}$, we have that $v + g(v,\tau) \in D^u_{\infty,\rho_0}$ for $(v,\tau) \in D^u_{\infty,\rho_0'} \times \mathbb{T}_\sigma$ provided ε is small enough and $\rho_0' > \rho_0$.

To obtain the inverse change and its properties it is straightforward. □

If we apply the change of variables $u = v + g(v,\tau)$ to equation (141), one can see that

$$\widehat{T}_1(v,\tau) = T_1\left(v + g(v,\tau),\tau\right)$$

is solution of

$$\mathcal{L}_\varepsilon \widehat{T}_1 = \widehat{\mathcal{F}}\left(\partial_v \widehat{T}_1\right), \tag{157}$$

where

$$\widehat{\mathcal{F}}(h)(v,\tau) = \widehat{A}(v,\tau) + \widehat{B}(v,\tau)h(v,\tau) + \widehat{C}(h(v,\tau),v,\tau), \tag{158}$$

with

$$\widehat{A}(v,\tau) = A\left(v + g(v,\tau),\tau\right) \tag{159}$$

$$\widehat{B}(v,\tau) = \frac{B_1\left(v+g(v,\tau),\tau\right) - B_1(v,\tau) + B_2\left(v+g(v,\tau),\tau\right)}{1+\partial_v g(v,\tau)} \tag{160}$$

$$\widehat{C}(w,v,\tau) = C\left(\frac{1}{1+\partial_v g(v,\tau)}w, v+g(v,\tau),\tau\right). \tag{161}$$

where the functions $A(u,\tau)$, $B_1(u,\tau)$ and $B_2(u,\tau)$ are defined in (143), (144) and (145).

We look for $\widehat{T}_1$ by using a fixed point argument for $\partial_v\widehat{T}_1$ instead of $\widehat{T}_1$ itself. Therefore, we look for a fixed point of the operator

$$\overline{\mathcal{F}} = \overline{\mathcal{G}}_\varepsilon \circ \widehat{\mathcal{F}}, \tag{162}$$

where $\overline{\mathcal{G}}_\varepsilon$ is the operator in (149), in the Banach space $\mathcal{H}_{2\lambda,\rho_0',\sigma}$ defined in (147).

Lemma 6.6. *Let ρ_0' be defined in Lemma 6.5 and $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0, \varepsilon_0)$ there exists a function $\widehat{T}_1(v, \tau)$ defined in $D^u_{\infty,\rho_0'} \times \mathbb{T}_\sigma$ such that $\partial_v \widehat{T}_1 \in \mathcal{H}_{2\lambda,\rho_0',\sigma}$ is a fixed point of the operator (162). Furthermore, there exists a constant $b_1 > 0$ such that,*

$$\left\| \partial_v \widehat{T}_1 \right\|_{2\lambda,\rho_0',\sigma} \leq b_1 |\mu| \varepsilon^{\eta+1}.$$

Proof. It is straightforward to see that $\overline{\mathcal{F}}$ is well defined from $\mathcal{H}_{2\lambda,\rho_0',\sigma}$ to itself. We are going to prove that there exists a constant $b_1 > 0$ such that $\overline{\mathcal{F}}$ sends $\overline{B}(b_1|\mu|\varepsilon^{\eta+1}) \subset \mathcal{H}_{2\lambda,\rho_0',\sigma}$ to itself and it is contractive there.

Let us first consider $\overline{\mathcal{F}}(0)$. From the definition of $\overline{\mathcal{F}}$ in (162) and the definition of $\widehat{\mathcal{F}}$ in (158), we have that

$$\overline{\mathcal{F}}(0)(v, \tau) = \overline{\mathcal{G}}_\varepsilon \left(\widehat{A} \right) (v, \tau) = \overline{\mathcal{G}}_\varepsilon (A)(v, \tau) + \overline{\mathcal{G}}_\varepsilon \left(\widehat{A} - A \right) (v, \tau).$$

The first term was already bounded in Lemma 6.3. For the second one, it is enough to use mean value theorem and Lemmas 6.3 and 6.5 to bound $\partial_u A$ and g respectively, to obtain

$$\| A(v + g(v, \tau), \tau) - A(v, \tau) \|_{2\lambda,\rho_0',\sigma} \leq K |\mu|^2 \varepsilon^{2\eta+1}.$$

Thus, applying Lemma 6.2, there exists constant $b_1 > 0$ such that

$$\left\| \overline{\mathcal{F}}(0) \right\|_{2\lambda,\rho_0',\sigma} \leq \frac{b_1}{2} |\mu| \varepsilon^{\eta+1}.$$

Let us consider now, $h_1, h_2 \in \overline{B}(b_1|\mu|\varepsilon^{\eta+1}) \in \mathcal{H}_{2\lambda,\rho_0',\sigma}$. Then, using the properties of $\overline{\mathcal{G}}_\varepsilon$ in Lemma 6.2 and the definition of $\widehat{\mathcal{F}}$ in (158)

$$\begin{aligned} \left\| \overline{\mathcal{F}}(h_2) - \overline{\mathcal{F}}(h_1) \right\|_{2\lambda,\rho_0',\sigma} &\leq K \left\| \widehat{\mathcal{F}}(h_2) - \widehat{\mathcal{F}}(h_1) \right\|_{2\lambda,\rho_0',\sigma} \\ &\leq K \left\| \widehat{B} \cdot (h_2 - h_1) + \widehat{C}(h_2, u, \tau) - \widehat{C}(h_1, u, \tau) \right\|_{2\lambda,\rho_0',\sigma}. \end{aligned}$$

Taking into account the definitions of $\widehat{B}$ and $\widehat{C}$ in (160) and (161) respectively and applying Lemmas 6.1, 6.3 and 6.5, we obtain

$$\left\| \overline{\mathcal{F}}(h_2) - \overline{\mathcal{F}}(h_1) \right\|_{2\lambda,\rho_0',\sigma} \leq K |\mu| \varepsilon^{\eta+1} \| h_2 - h_1 \|_{2\lambda,\rho_0',\sigma}.$$

Therefore, reducing ε if necessary, $\mathrm{Lip}\overline{\mathcal{F}} \leq 1/2$ and therefore $\overline{\mathcal{F}}$ is contractive from the ball $\overline{B}(b_1|\mu|\varepsilon^{\eta+1}) \subset \mathcal{H}_{2\lambda,\rho_0',\sigma}$ into itself, and it has a unique fixed point h^*. Since it satisfies

$$|h^*(v, \tau)| \leq b_1 |\mu| \varepsilon^{\eta+1} e^{2\lambda \mathrm{Re}\, v}$$

for $(v, \tau) \in D^u_{\infty,\rho_0'} \times \mathbb{T}_\sigma$, we can take $\widehat{T}_1$ as

$$\widehat{T}_1(v, \tau) = \int_{-\infty}^{v} h^*(w, \tau)\, dw.$$

□

Finally, to prove Proposition 6.4 from Lemma 6.6, it is enough to consider the change $v = u + h(u, \tau)$ obtained in Lemma 6.5, take $T_1(u, \tau) = \widehat{T}_1(u + h(u, \tau), \tau)$ and increase slightly b_1 if necessary.

6.2 Local invariant manifolds in the parabolic case

We devote this section to prove the existence of suitable representations of the unstable and stable invariant manifolds in the domains $D^u_{\infty,\rho} \times \mathbb{T}_\sigma$ and $D^s_{\infty,\rho} \times \mathbb{T}_\sigma$ respectively, under the hypotheses that the unperturbed Hamiltonian system has a parabolic critical point at the origin. We proceed as we have done in Section 6.1 for the hyperbolic case, that is, solving equation (141). Let us point out that in the parabolic case, by Hypothesis **HP4.2**, the perturbation is taken in such a way that the periodic orbit remains at the origin.

6.2.1 Banach spaces and technical lemmas

Given $\alpha \geq 0$, $\rho \geq 0$ and an analytic function $h : D^u_{\infty,\rho} \to \mathbb{C}$, we define

$$\|h\|_{\alpha,\rho} = \sup_{u \in D^u_{\infty,\rho}} |u^\alpha h(u)| .$$

Moreover for analytic functions $h : D^u_{\infty,\rho} \times \mathbb{T}_\sigma \to \mathbb{C}$ which are 2π-periodic in τ, we define the corresponding Fourier norm

$$\|h\|_{\alpha,\rho,\sigma} = \sum_{k\in\mathbb{Z}} \left\| h^{[k]} \right\|_{\alpha,\rho} e^{|k|\sigma}.$$

We introduce, thus, the following function space

$$\mathcal{P}_{\alpha,\rho,\sigma} = \{h : D^u_{\infty,\rho} \times \mathbb{T}_\sigma \to \mathbb{C};\ \text{real-analytic}, \|h\|_{\alpha,\rho,\sigma} < \infty\}, \tag{163}$$

which can be checked that is a Banach space for any fixed $\alpha \geq 0$.

In the next lemma, we state some properties of these Banach spaces.

Lemma 6.7. *The following statements hold:*

1. *If $\alpha_1 \geq \alpha_2 \geq 0$, then $\mathcal{P}_{\alpha_1,\rho,\sigma} \subset \mathcal{P}_{\alpha_2,\rho,\sigma}$ and*
$$\|h\|_{\alpha_2,\rho,\sigma} \leq \|h\|_{\alpha_1,\rho,\sigma}.$$
2. *If $\alpha_1, \alpha_2 \geq 0$, then, for $h \in \mathcal{P}_{\alpha_1,\rho,\sigma}$ and $g \in \mathcal{P}_{\alpha_2,\rho,\sigma}$, we have that $hg \in \mathcal{P}_{\alpha_1+\alpha_2,\rho,\sigma}$ and*
$$\|hg\|_{\alpha_1+\alpha_2,\rho,\sigma} \leq \|h\|_{\alpha_1,\rho,\sigma}\|g\|_{\alpha_2,\rho,\sigma}.$$

As in Section 6.1, we need to use the operators $\mathcal{G}_\varepsilon$ and $\bar{\mathcal{G}}_\varepsilon$ formally defined in (148) and (149) respectively.

Lemma 6.8. *The operators $\mathcal{G}_\varepsilon$ and $\overline{\mathcal{G}}_\varepsilon$ acting on the spaces $\mathcal{P}_{\alpha,\rho,\sigma}$ with $\alpha > 1$ satisfy the following properties.*

1. *For any $\alpha > 1$, $\mathcal{G}_\varepsilon : \mathcal{P}_{\alpha,\rho,\sigma} \to \mathcal{P}_{\alpha-1,\rho,\sigma}$ is well defined and linear continuous. Moreover, commutes with ∂_u and $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon = \mathrm{Id}$.*
2. *If $h \in \mathcal{P}_{\alpha,\rho,\sigma}$ for some $\alpha > 1$, then*
$$\|\mathcal{G}_\varepsilon(h)\|_{\alpha-1,\rho,\sigma} \leq K\|h\|_{\alpha,\rho,\sigma}.$$
Furthermore, if $h \in \mathcal{P}_{\alpha,\rho,\sigma}$ for some $\alpha > 0$ and $\langle h \rangle = 0$, then
$$\|\mathcal{G}_\varepsilon(h)\|_{\alpha,\rho,\sigma} \leq K\varepsilon\|h\|_{\alpha,\rho,\sigma}.$$
3. *If $h \in \mathcal{P}_{\alpha,\rho,\sigma}$ for some $\alpha \geq 1$, then $\overline{\mathcal{G}}_\varepsilon(h) \in \mathcal{P}_{\alpha,\rho,\sigma}$ and*
$$\left\|\overline{\mathcal{G}}_\varepsilon(h)\right\|_{\alpha,\rho,\sigma} \leq K\|h\|_{\alpha,\rho,\sigma}.$$

We also state a technical lemma about properties of the functions A, B_1 and C defined in (143), (144) and (146) respectively. Let us point out that now the function B_2 defined in (145) satisfies $B_2 = 0$ since, by hypothesis, the perturbation fixes the periodic orbit at the origin.

We first fix $\rho_0 > 0$ such that $p_0(u)$ does not vanish in D^u_{∞,ρ_0} and we define the constant

$$\alpha_0 = \frac{2n}{m-2} > 1, \tag{164}$$

where m is the order of the potential (11) and n is the order of the perturbation (8). We observe that $q_0(u) \in \mathcal{P}_{\frac{2}{m-2},\rho,\sigma}$ and $p_0(u) \in \mathcal{P}_{\frac{m}{m-2},\rho,\sigma}$ for any ρ big enough and any $\sigma > 0$.

Lemma 6.9. *Let us consider $\rho > \rho_0$. Then, the functions A, B_1 and C defined in* (143), (144) *and* (146) *satisfy the following properties,*

1. *$A \in \mathcal{P}_{\alpha_0,\rho,\sigma}$ and $\partial_u A \in \mathcal{P}_{\alpha_0+1,\rho,\sigma}$. Moreover, $\langle A\rangle = \langle \partial_u A\rangle = 0$ and*
$$\|\partial_u A\|_{\alpha_0+1,\rho,\sigma} \le K|\mu|\varepsilon^\eta, \quad \left\|\overline{\mathcal{G}}_\varepsilon(A)\right\|_{\alpha_0+1,\rho,\sigma} \le K|\mu|\varepsilon^{\eta+1}. \tag{165}$$

2. *$B_1 \in \mathcal{P}_{\frac{2n-m-2}{m-2},\rho,\sigma}$ and $\partial_u B_1 \in \mathcal{P}_{\frac{2n-m-2}{m-2}+1,\rho,\sigma}$. Moreover, they satisfy*
$$\|B_1\|_{\frac{2n-m-2}{m-2},\rho,\sigma} \le K|\mu|\varepsilon^\eta, \quad \|\partial_u B_1\|_{\frac{2n-m-2}{m-2}+1,\rho,\sigma} \le K|\mu|\varepsilon^\eta. \tag{166}$$

3. *Let us consider $h_1, h_2 \in B(\nu) \subset \mathcal{P}_{\alpha_0+1,\rho,\sigma}$ with $\nu \ll 1$. Then,*
$$\|C(h_2,u,\tau) - C(h_1,u,\tau)\|_{\alpha_0+1,\rho,\sigma} \le K\nu\|h_2 - h_1\|_{\alpha_0+1,\rho,\sigma}.$$

Proof. We prove the lemma in the polynomial case. The trigonometric one can be done analogously. For the first statement, let us point out that in the parabolic case the periodic orbit is located at the origin by Hypothesis **HP4.2**. Then
$$A(u,\tau) = -\mu\varepsilon^\eta H_1(q_0(u), p_0(u), \tau),$$
where H_1 is the function defined in (8) and has zero mean. On the other hand, it is clear that the monomial with lowest order as $\operatorname{Re} u \to +\infty$ corresponds to $a_{n0}q_0^n(u)$ which behaves as
$$a_{n0}(\tau)q_0^n(u) \sim \frac{1}{u^{\alpha_0}}.$$
Then $A \in \mathcal{P}_{\alpha_0,\rho,\sigma}$, that implies $\partial_u A \in \mathcal{P}_{\alpha_0+1,\rho,\sigma}$ and
$$\|\partial_u A\|_{\alpha_0+1,\rho,\sigma} \le K|\mu|\varepsilon^\eta.$$
Moreover, by Lemma 6.8,
$$\left\|\overline{\mathcal{G}}_\varepsilon(A)\right\|_{\alpha_0+1,\rho,\sigma} = \|\mathcal{G}_\varepsilon(\partial_u A)\|_{\alpha_0+1,\rho,\sigma} \le K|\mu|\varepsilon^{\eta+1}.$$
For the second statement, let us recall that
$$B_1(u,\tau) = -\mu\varepsilon^\eta \sum_{\substack{i+j=n\\ j\ge 1}}^{N} a_{ij}(\tau)q_0^i(u)p_0^{j-2}(u).$$
As $\operatorname{Re} u \to -\infty$, the monomials of B_1 behave as
$$a_{ij}(\tau)q_0^i(u)p_0^{j-2}(u) \sim u^{-\left(\frac{2}{m-2}i + \left(\frac{2}{m-2}+1\right)(j-2)\right)}.$$
Taking into account that $2n-2 \ge m$ by Hypothesis **HP5** and that $i+j \ge n$ and $j \ge 1$,
$$\begin{aligned}\frac{2}{m-2}i + \left(\frac{2}{m-2}+1\right)(j-2) &= \frac{2}{m-2}(i+j) + j - \frac{2m}{m-2}\\ &\ge \frac{2n}{m-2} + 1 - \frac{2m}{m-2}.\end{aligned}$$
Therefore $B_1 \in \mathcal{P}_{\frac{2n-m-2}{m-2},\rho,\sigma}$ and satisfies $\|B_1\|_{\frac{2n-m-2}{m-2},\rho,\sigma} \le K|\mu|\varepsilon^\eta$. For $\partial_u B_1$, it is enough to differentiate. For the case $2n-2 > m$ we have that $\partial_u B_1 \in \mathcal{P}_{\frac{2n-m-2}{m-2}+1,\rho,\sigma}$. In the case $2n-2 = m$ we have that
$$\partial_u B_1 \in \mathcal{P}_{\frac{1}{m-2}+1,\rho,\sigma} \subset \mathcal{P}_{\frac{2n-m-2}{m-2}+1,\rho,\sigma}.$$

In both cases, we have that $\|\partial_u B_1\|_{\frac{2n-m-2}{m-2}+1,\rho,\sigma} \leq K|\mu|\varepsilon^\eta$.

We bound the third term in the polynomial case. The trigonometric case could be done analogously. We split $C = C_1 + C_2$ as

$$C_1(w,u,\tau) = -\frac{w^2}{2p_0^2(u)}$$
$$C_2(w,u,\tau) = -\mu\varepsilon^\eta \sum_{\substack{i+j=n\\ j\geq 1}}^{N} a_{ij}(\tau)q_0^i(u)p_0^j(u)\left(\left(1+\frac{w}{p_0^2(u)}\right)^j - 1 - j\frac{w}{p_0^2(u)}\right).$$

Let $h_1, h_2 \in B(\nu) \subset \mathcal{P}_{\alpha_0+1,\rho,\sigma}$. Then, for the first term,

$$\begin{aligned}\|C_1(h_2,u,\tau) - C_1(h_1,u,\tau)\|_{\alpha_0+1,\rho,\sigma} &\leq K\left\|p_0(u)^{-2}(h_2+h_1)\right\|_{0,\rho,\sigma}\|h_2-h_1\|_{\alpha_0+1,\rho,\sigma}\\ &\leq K\|h_2+h_1\|_{2m/(m-2),\rho,\sigma}\|h_2-h_1\|_{\alpha_0+1,\rho,\sigma}.\end{aligned}$$

By Hypotheses **HP5**, we have $2n-2 \geq m$ which implies $2m/(m-2) \leq \alpha_0+1$ and therefore

$$\|h_2+h_1\|_{2m/(m-2),\rho,\sigma} \leq \|h_2+h_1\|_{\alpha_0+1,\rho,\sigma} \leq K\nu.$$

Reasoning analogously, one can see that

$$\|C_2(h_2,u,\tau) - C_2(h_1,u,\tau)\|_{\alpha_0+1,\rho,\sigma} \leq K|\mu|\varepsilon^\eta\nu\|h_2-h_1\|_{\alpha_0+1,\rho,\sigma}.$$

□

6.2.2 Proof of Theorem 4.3 in the parabolic case

We devote this section to prove Theorem 4.3 for the case in which the unperturbed Hamiltonian has a parabolic critical point. First we rewrite it in terms of the Banach spaces defined in (163).

Proposition 6.10. *Let us consider the constant α_0 defined in* (164)*, $\rho_1 > 0$ big enough and $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0,\varepsilon_0)$, there exists a function $T_1(u,\tau)$ defined in $D^u_{\infty,\rho_1} \times \mathbb{T}_\sigma$ which satisfies equation* (141) *and the asymptotic condition* (49)*. Moreover, $\partial_u T_1 \in \mathcal{P}_{\alpha_0+1,\rho_1,\sigma}$ and there exists a constant $b_1 > 0$ such that*

$$\|\partial_u T_1\|_{\alpha_0+1,\rho_1,\sigma} \leq b_1|\mu|\varepsilon^{\eta+1}.$$

Theorem 4.3 is a straightforward consequence of this proposition.

The proof of this proposition follows the same steps as the proof of Proposition 6.4. We choose constants $\rho_0 < \rho_0' < \rho_1$.

The first step is to perform a change of variables which reduces the size of the linear term of $\mathcal{F}$ in (142). This change is not necessary for the case $\eta > 0$.

Lemma 6.11. *Let us consider $\rho_0' > \rho_0$. Then, for $\varepsilon > 0$ small enough, there exists a function $g \in \mathcal{P}_{0,\rho_0',\sigma}$ such that $\langle g\rangle = 0$ and is a solution of* (156)*. Moreover, it satisfies that*

$$\|g\|_{0,\rho_0',\sigma} \leq K|\mu|\varepsilon^{\eta+1}, \quad \|\partial_v g\|_{0,\rho_0',\sigma} \leq K|\mu|\varepsilon^{\eta+1},$$

and $v + g(v,\tau) \in D^u_{\infty,\rho_0}$ for $(v,\tau) \in D^u_{\infty,\rho_0'} \times \mathbb{T}_\sigma$.

Furthermore, $(u,\tau) = (v+g(v,\tau),\tau)$ is invertible and its inverse is of the form $(v,\tau) = (u+h(u,\tau),\tau)$, where h is a function defined for $(u,\tau) \in D^u_{\infty,\rho_1} \times \mathbb{T}_\sigma$ and satisfies that $h \in \mathcal{P}_{0,\rho_1,\sigma}$,

$$\|h\|_{0,\rho_1,\sigma} \leq K|\mu|\varepsilon^{\eta+1}$$

and that $u + h(u,\tau) \in D^u_{\infty,\rho_0'}$ for $(u,\tau) \in D^u_{\infty,\rho_1} \times \mathbb{T}_\sigma$.

Proof. Since $B_1 \in \mathcal{P}_{\frac{2n-m-2}{m-2},\rho,\sigma}$ and it might happen that $\frac{2n-m-2}{m-2} < 1$, we cannot apply directly Lemma 6.8 to invert $\mathcal{L}_\varepsilon$. Let us observe that, by Lemma 6.9, $\langle B_1 \rangle = 0$ and then we can define a function $\overline{B}_1$ such that

$$\partial_\tau \overline{B}_1 = B_1 \quad \text{and} \quad \langle \overline{B}_1 \rangle = 0,$$

which satisfies $\|\overline{B}_1\|_{\frac{2n-m-2}{m-2},\rho,\sigma} \leq K|\mu|\varepsilon^\eta$.

We can define g as

$$g(v,\tau) = -\varepsilon \overline{B}_1(v,\tau) + \varepsilon \mathcal{G}_\varepsilon \left(\partial_v \overline{B}_1\right)(v,\tau).$$

Then, applying Lemmas 6.8 and 6.9 one obtains the bounds for g and $\partial_v g$.

The proof of the other statements is analogous to the proof of Lemma 6.5. □

As in Section 6.1.2, we define

$$\widehat{T}_1(v,\tau) = T_1(v + g(v,\tau),\tau),$$

which is a solution of (157). Then, we look for $\partial_v \widehat{T}_1$ as a fixed point of the operator (162) in the Banach space $\mathcal{P}_{\alpha_0+1,\rho_0',\sigma}$.

Lemma 6.12. *Let us consider the constant α_0 defined in* (164) *and $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0,\varepsilon_0)$ there exists a function $\widehat{T}_1(v,\tau)$ defined in $D^u_{\infty,\rho_0'} \times \mathbb{T}_\sigma$ such that $\partial_v \widehat{T}_1 \in \mathcal{P}_{\alpha_0+1,\rho_0',\sigma}$ is a fixed point of the operator* (162)*. Furthermore, there exists a constant $b_1 > 0$ such that*

$$\left\|\partial_v \widehat{T}_1\right\|_{\alpha_0+1,\rho_0',\sigma,0} \leq b_1|\mu|\varepsilon^{\eta+1}.$$

Proof. It is straightforward to see that $\overline{\mathcal{F}}$ is well defined from $\mathcal{P}_{\alpha_0+1,\rho_0',\sigma}$ to itself. We are going to prove that there exists a constant $b_1 > 0$ such that $\overline{\mathcal{F}}$ is contractive in $\overline{B}(b_1|\mu|\varepsilon^{\eta+1}) \subset \mathcal{P}_{\alpha_0+1,\rho_0',\sigma}$.

Let us consider first $\overline{\mathcal{F}}(0)$. From the definition of $\overline{\mathcal{F}}$ in (162) and the definition of $\widehat{\mathcal{F}}$ in (158), we have that

$$\overline{\mathcal{F}}(0)(v,\tau) = \overline{\mathcal{G}}_\varepsilon\left(\widehat{A}(v,\tau)\right) = \overline{\mathcal{G}}_\varepsilon\left(A(v,\tau)\right) + \overline{\mathcal{G}}_\varepsilon\left(A(v+g(v,\tau),\tau) - A(v,\tau)\right).$$

The first term has been bounded in Lemma 6.9. For the second one, we apply Lemmas 6.9 and 6.11 and the mean value theorem to obtain

$$\|A(v+g(v,\tau),\tau) - A(v,\tau)\|_{\alpha_0+1,\rho_0',\sigma} \leq \|\partial_u A\|_{\alpha_0+1,\rho_0,\sigma}\|g\|_{0,\rho_0',\sigma} \leq K|\mu|^2\varepsilon^{2\eta+1}.$$

Thus, applying Lemma 6.8, there exists a constant $b_1 > 0$ such that

$$\left\|\overline{\mathcal{F}}(0)\right\|_{\alpha_0+1,\sigma} \leq \frac{b_1}{2}|\mu|\varepsilon^{\eta+1}.$$

Let us consider now, $h_1, h_2 \in \overline{B}(b_1|\mu|\varepsilon^{\eta+1}) \subset \mathcal{P}_{\alpha_0+1,\rho_0',\sigma}$. Then, using the properties of $\overline{\mathcal{G}}_\varepsilon$ in Lemma 6.8 and the definition of $\widehat{\mathcal{F}}$ in (158),

$$\begin{aligned}\left\|\overline{\mathcal{F}}(h_2) - \overline{\mathcal{F}}(h_1)\right\|_{\alpha_0+1,\rho_0',\sigma} &\leq K\left\|\widehat{\mathcal{F}}(h_2) - \widehat{\mathcal{F}}(h_1)\right\|_{\alpha_0+1,\rho_0',\sigma} \\ &\leq K\left\|\widehat{B}\cdot(h_2-h_1) + \widehat{C}(h_2,v,\tau) - \widehat{C}(h_1,v,\tau)\right\|_{\alpha_0+1,\rho_0',\sigma}.\end{aligned}$$

Taking into account the definitions of $\widehat{B}$ and $\widehat{C}$ in (160) and (161), recalling that $B_2 = 0$ and applying Lemmas 6.7, 6.9 and 6.11,we obtain

$$\left\|\overline{\mathcal{F}}(h_2) - \overline{\mathcal{F}}(h_1)\right\|_{\alpha_0+1,\rho_0',\sigma} \leq K|\mu|\varepsilon^{\eta+1}\|h_2 - h_1\|_{\alpha_0+1,\rho_0',\sigma}.$$

Then, reducing ε if necessary, $\mathrm{Lip}\overline{\mathcal{F}} < 1/2$ and then $\overline{\mathcal{F}}$ is contractive from $\overline{B}\left(b_1|\mu|\varepsilon^{\eta+1}\right) \subset \mathcal{P}_{\alpha_0+1,\sigma}$ to itself and has a unique fixed point h^*. Moreover, since it satisfies

$$|h^*(v,\tau)| \leq b_1|\mu|\varepsilon^{\eta+1}\frac{1}{|v|^{\alpha_0+1}}$$

for $(v,\tau)\in D^u_{\infty,\rho_0'}\times\mathbb{T}_\sigma$, we can define $\widehat{T}_1$ as

$$\widehat{T}_1(v,\tau)=\int_{-\infty}^{v}h^*(w,\tau)\,dw.$$

□

To prove Proposition 6.10 from Lemma 6.12, as we have proceeded in Section 6.1.2, it is enough to consider the change of variables $v=u+h(u,\tau)$ obtained in Lemma 6.11, take $T_1(u,\tau)=\widehat{T}_1(u+h(u,\tau),\tau)$ and increase slightly b_1 if necessary.

7 Invariant manifolds in the outer domains: proof of Theorems 4.4 and 4.8

7.1 Invariant manifolds in the outer domains when $p_0(u)\neq 0$: proof of Theorem 4.4

In this section we prove the existence of the invariant manifolds in the domains $D^{\mathrm{out},*}_{\rho,\kappa}\times\mathbb{T}_\sigma$ for $*=u,s$ defined in (29) provided $p_0(u)\neq 0$ in these domains. Since the proof for both invariant manifolds is analogous, we deal only with the unstable case.

First in Section 7.1.1 we define some Banach spaces and we state some technical lemmas. Then, in Section 7.1.2 we prove Theorem 4.4. Finally, we devote Sections 8.1 and 7.3 to prove Propositions 4.10 and 7.23.

7.1.1 Banach spaces and technical lemmas

We start by defining some norms. Given $\nu\in\mathbb{R}$ and an analytic function $h:D^{\mathrm{out},u}_{\rho,\kappa}\to\mathbb{C}$, where $D^{\mathrm{out},u}_{\rho,\kappa}$ is the domain defined in (29), we consider

$$\|h\|_{\nu,\rho,\kappa}=\sup_{u\in D^{\mathrm{out},u}_{\rho,\kappa}}\left|\left(u^2+a^2\right)^{\nu}h(u)\right|.$$

Moreover for analytic functions $h:D^{\mathrm{out},u}_{\rho,\kappa}\times\mathbb{T}_\sigma\to\mathbb{C}$ which are 2π-periodic in τ, we consider the corresponding Fourier norm

$$\|h\|_{\nu,\rho,\kappa,\sigma}=\sum_{k\in\mathbb{Z}}\left\|h^{[k]}\right\|_{\nu,\rho,\kappa}e^{|k|\sigma}.$$

We consider, thus, the following function space

$$\mathcal{E}_{\nu,\rho,\kappa,\sigma}=\{h:D^{\mathrm{out},u}_{\rho,\kappa}\times\mathbb{T}_\sigma\to\mathbb{C};\ \text{real-analytic},\|h\|_{\nu,\rho,\kappa,\sigma}<\infty\},\tag{167}$$

which can be checked that is a Banach space for any $\nu\in\mathbb{R}$.

If there is no danger of confusion about the domain $D^{\mathrm{out},u}_{\rho,\kappa}$, we will denote

$$\|\cdot\|_{\nu,\sigma}=\|\cdot\|_{\nu,\rho,\kappa,\sigma}\quad\text{and}\quad\mathcal{E}_{\nu,\sigma}=\mathcal{E}_{\nu,\rho,\kappa,\sigma}.$$

In the next lemma, we state some properties of these Banach spaces. In the estimates we will make explicit the dependence of the constants with respect to κ.

Lemma 7.1. *The following statements hold:*

1. *If $\nu_1\geq\nu_2$, then $\mathcal{E}_{\nu_1,\sigma}\subset\mathcal{E}_{\nu_2,\sigma}$ and moreover if $h\in\mathcal{E}_{\nu_1,\sigma}$,*

$$\|h\|_{\nu_2,\sigma}\leq K(\kappa\varepsilon)^{\nu_2-\nu_1}\|h\|_{\nu_1,\sigma}.$$

2. *If $\nu_1 \leq \nu_2$, then $\mathcal{E}_{\nu_1,\sigma} \subset \mathcal{E}_{\nu_2,\sigma}$ and moreover if $h \in \mathcal{E}_{\nu_1,\sigma}$,*

$$\|h\|_{\nu_2,\sigma} \leq K\|h\|_{\nu_1,\sigma}.$$

3. *If $h \in \mathcal{E}_{\nu_1,\sigma}$ and $g \in \mathcal{E}_{\nu_2,\sigma}$, then $hg \in \mathcal{E}_{\nu_1+\nu_2,\sigma}$ and*

$$\|hg\|_{\nu_1+\nu_2,\sigma} \leq \|h\|_{\nu_1,\sigma}\|g\|_{\nu_2,\sigma}.$$

4. *Let $\rho' < \rho$ be such that $\rho - \rho'$ has a positive lower bound independent of ε, κ' and κ such that $\kappa < \kappa' < 0$ and $h \in \mathcal{E}_{\nu,\rho,\kappa,\sigma}$. Then $\partial_u h \in \mathcal{E}_{\nu,\rho',\kappa',\sigma}$ and satisfies*

$$\|\partial_u h\|_{\nu,\rho',\kappa',\sigma} \leq \frac{K}{\varepsilon\,|\kappa' - \kappa|}\|h\|_{\nu,\rho,\kappa,\sigma}.$$

Throughout this section we are going to solve equations of the form $\mathcal{L}_\varepsilon h = g$, where $\mathcal{L}_\varepsilon$ is the differential operator defined in (45). Note that $\mathcal{L}_\varepsilon$ acting on $\mathcal{E}_{\nu,\rho}$ is not invertible. Indeed for any smooth function f, $f(u/\varepsilon - \tau) \in \mathrm{Ker}\mathcal{L}_\varepsilon$. We consider a left-inverse of the operator $\mathcal{L}_\varepsilon$, which we call $\mathcal{G}_\varepsilon$, defined acting on the Fourier coefficients. Let us consider $u_1, \bar{u}_1 \in \mathbb{C}$ the vertices of the domain $D^{\mathrm{out},u}_{\rho,\kappa}$ (see Figure 4). Then, we define $\mathcal{G}_\varepsilon$ as

$$\mathcal{G}_\varepsilon(h)(u,\tau) = \sum_{k\in\mathbb{Z}} \mathcal{G}_\varepsilon(h)^{[k]}(u)e^{ik\tau}, \tag{168}$$

where its Fourier coefficients are given by

$$\begin{aligned}
\mathcal{G}_\varepsilon(h)^{[k]}(u) &= \int_{\bar{u}_1}^{u} e^{ik\varepsilon^{-1}(t-u)} h^{[k]}(t)\,dt && \text{for } k < 0\\
\mathcal{G}_\varepsilon(h)^{[0]}(u) &= \int_{-\rho}^{u} h^{[0]}(t)\,dt &&\\
\mathcal{G}_\varepsilon(h)^{[k]}(u) &= \int_{u_1}^{u} e^{ik\varepsilon^{-1}(t-u)} h^{[k]}(t)\,dt && \text{for } k > 0.
\end{aligned}$$

Remark 7.2. *Let us observe that the definition of the operator $\mathcal{G}_\varepsilon$ depends on the domain, since in its definition we use its vertices u_1, $\bar{u}_1$ and also ρ.*

Lemma 7.3. *The operator $\mathcal{G}_\varepsilon$ in (168) satisfies the following properties.*

1. *If $h \in \mathcal{E}_{\nu,\sigma}$ for some $\nu \geq 0$, then $\mathcal{G}_\varepsilon(h) \in \mathcal{E}_{\nu,\sigma}$ and*

$$\|\mathcal{G}_\varepsilon(h)\|_{\nu,\sigma} \leq K\|h\|_{\nu,\sigma}.$$

Furthermore, if $\langle h \rangle = 0$,

$$\|\mathcal{G}_\varepsilon(h)\|_{\nu,\sigma} \leq K\varepsilon\,\|h\|_{\nu,\sigma}.$$

2. *If $h \in \mathcal{E}_{\nu,\sigma}$ for some $\nu > 1$, then $\mathcal{G}_\varepsilon(h) \in \mathcal{E}_{\nu-1,\sigma}$ and*

$$\|\mathcal{G}_\varepsilon(h)\|_{\nu-1,\sigma} \leq K\|h\|_{\nu,\sigma}.$$

3. *If $h \in \mathcal{E}_{\nu,\sigma}$ for some $\nu \in (0,1)$, then $\mathcal{G}_\varepsilon(h) \in \mathcal{E}_{0,\sigma}$ and*

$$\|\mathcal{G}_\varepsilon(h)\|_{0,\sigma} \leq K\|h\|_{\nu,\sigma}.$$

4. *If $h \in \mathcal{E}_{\nu,\sigma}$ for some $\nu \geq 0$, then $\mathcal{G}_\varepsilon(\partial_u h) \in \mathcal{E}_{\nu,\sigma}$ and*

$$\|\mathcal{G}_\varepsilon(\partial_u h)\|_{\nu,\sigma} \leq K\|h\|_{\nu,\sigma}.$$

5. *If $h \in \mathcal{X}_{\nu,\sigma}$ for some $\nu \geq 0$, $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon(h) = h$ and*

$$\mathcal{G}_\varepsilon \circ \mathcal{L}_\varepsilon(h)(v,\tau) = h(v,\tau) - \sum_{k<0} e^{ik\varepsilon^{-1}(-u_1-u)} h^{[k]}(-u_1) - h^{[0]}(u_0) - \sum_{k>0} e^{ik\varepsilon^{-1}(u_1-u)} h^{[k]}(u_1)$$

6. *If $h \in \mathcal{X}_{\nu,\sigma}$ for some $\nu \geq 0$, $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon(h) = h$ and*

$$\mathcal{G}_\varepsilon \circ \mathcal{L}_\varepsilon(h)(v,\tau) = h(v,\tau) - \sum_{k<0} e^{ik\varepsilon^{-1}(-u_1-u)} h^{[k]}(-u_1) - h^{[0]}(u_0) - \sum_{k>0} e^{ik\varepsilon^{-1}(u_1-u)} h^{[k]}(u_1)$$

Proof. It is a consequence of Lemma 5.5 in [GOS10]. □

7.1.2 Proof of Theorem 4.4

We prove Theorem 4.4, looking for the analytic continuation of the function $T_1 = T - T_0$ obtained in Propositions 6.4 and 6.10 as a solution of equation (141). First we rewrite the result in terms of the Banach spaces defined in (167).

Proposition 7.4. *Let ρ_1 be the constant considered in Theorem 4.3 and let us consider $\rho_2 > \rho_1$, $\varepsilon_0 > 0$ small enough and $\kappa_1 > 0$ big enough. Then, for $\varepsilon \in (0, \varepsilon_0)$, there exists a function $T_1 \in \mathcal{E}_{\ell+1,\rho_2,\kappa_1,\sigma}$ which satisfies equation* (141) *and is the analytic continuation of the analytic function T_1 obtained in Propositions 6.4 and 6.10. Moreover, there exists a constant $b_2 > 0$ such that*

$$\|\partial_u T_1\|_{\ell+1,\rho_2,\kappa_1,\sigma} \leq b_2 |\mu| \varepsilon^{\eta+1}.$$

This proposition gives the existence of the invariant manifolds in $D^{\mathrm{out},*}_{\rho_2,\kappa_1} \times \mathbb{T}_\sigma$, $* = u, s$.

We devote the rest of the section to prove Proposition 7.4.

First, we state a technical lemma about properties of the functions A, B_1, B_2 and C defined in (143), (144), (145) and (146) respectively.

Lemma 7.5. *Let us consider any $\rho > 0$ and $\kappa > 0$. Then, the functions A, B_1, B_2 and C defined in* (143), (144), (145) *and* (146) *satisfy the following properties.*

1. *$A \in \mathcal{E}_{\ell,\rho,\kappa,\sigma}$ and $\partial_u A \in \mathcal{E}_{\ell+1,\rho,\kappa,\sigma}$. Moreover $\partial_u A$ satisfies*

$$\begin{aligned} \|\partial_u A\|_{\ell+1,\rho,\kappa,\sigma} &\leq K|\mu|\varepsilon^\eta \\ \|\mathcal{G}_\varepsilon(\partial_u A)\|_{\ell+1,\rho,\kappa,\sigma} &\leq K|\mu|\varepsilon^{\eta+1}. \end{aligned} \tag{169}$$

2. *If $\ell - 2r < 0$, $B_1, \partial_u B_1, B_2 \in \mathcal{E}_{0,\rho,\kappa,\sigma}$ and satisfy $\langle B_1 \rangle = 0$ and*

$$\begin{aligned} \|B_1\|_{0,\rho,\kappa,\sigma} &\leq K|\mu|\varepsilon^\eta \\ \|\partial_u B_1\|_{\max\{0,\ell-2r+1\},\rho,\kappa,\sigma} &\leq K|\mu|\varepsilon^\eta \\ \|B_2\|_{0,\rho,\kappa,\sigma} &\leq K|\mu|^2\varepsilon^{2\eta+1}. \end{aligned} \tag{170}$$

3. *If $\ell - 2r \geq 0$, $B_1, B_2 \in \mathcal{E}_{\ell-2r,\rho,\kappa,\sigma}$, $\partial_u B_1 \in \mathcal{E}_{\ell-2r+1,\rho,\kappa,\sigma}$ and satisfy $\langle B_1 \rangle = 0$ and*

$$\begin{aligned} \|B_1\|_{\ell-2r,\rho,\kappa,\sigma} &\leq K|\mu|\varepsilon^\eta \\ \|\partial_u B_1\|_{\ell-2r+1,\rho,\kappa,\sigma} &\leq K|\mu|\varepsilon^\eta \\ \|B_2\|_{\ell-2r,\rho,\kappa,\sigma} &\leq K|\mu|^2\varepsilon^{2\eta+1}. \end{aligned} \tag{171}$$

4. *Let us consider $h_1, h_2 \in B(\nu) \subset \mathcal{E}_{\ell+1,\rho,\kappa,\sigma}$ with $\nu \ll 1$. Then,*

   - *If $\ell - 2r < 0$,*

$$\|C(h_2,u,\tau) - C(h_1,u,\tau)\|_{\ell+1,\rho,\kappa,\sigma} \leq K \frac{\nu}{\varepsilon^{\max\{0,\ell-2r+1\}}} \|h_2 - h_1\|_{\ell+1,\rho,\kappa,\sigma}.$$

- *If $\ell - 2r \geq 0$,*

$$\|C(h_2, u, \tau) - C(h_1, u, \tau)\|_{2\ell-2r+2,\rho,\kappa,\sigma} \leq K\nu \|h_2 - h_1\|_{\ell+1,\rho,\kappa,\sigma}.$$

Proof. For the first bounds, we split $A = A_1 + A_2 + A_3$, where A_i, $i = 1, 2, 3$, are the functions defined in (152), (153) and (154) respectively.

Using (155) and (15), one can see that $A_1 \in \mathcal{E}_{r+1,\rho,\delta,\sigma} \subset \mathcal{E}_{\ell+1,\rho,\delta,\sigma}$ and

$$\|A_1\|_{\ell+1,\rho,\delta,\sigma} \leq \|A_1\|_{r+1,\rho,\delta,\sigma} \leq K|\mu|\varepsilon^{\eta+1}. \tag{172}$$

Applying Lemma 6.2, we obtain $\|\mathcal{G}_\varepsilon(\partial_u A_1)\|_{\ell+1,\rho,\delta,\sigma} \leq K|\mu|\varepsilon^{\eta+1}$.

For the other terms, let us point out that, by the definition of ℓ, $A_2, A_3 \in \mathcal{E}_{\ell,\rho,\delta,\sigma}$. Therefore $\partial_u A_2, \partial_u A_3 \in \mathcal{E}_{\ell+1,\rho,\delta,\sigma}$ and satisfy $\|\partial_u A_2\|_{\ell+1,\rho,\delta,\sigma} \leq K|\mu|\varepsilon^{\eta}$ and $\|\partial_u A_3\|_{\ell+1,\rho,\delta,\sigma} \leq K|\mu|^2\varepsilon^{2\eta+1}$.

To bound $\mathcal{G}_\varepsilon(A_2)$, let us point out that $\langle A_2 \rangle = 0$ and then, by Lemma 6.2,

$$\|\mathcal{G}_\varepsilon (\partial_u A_2)\|_{\ell+1,\rho,\delta,\sigma} \leq K\varepsilon \|\partial_u A_2\|_{\ell+1,\rho,\delta,\sigma} \leq K|\mu|\varepsilon^{\eta+1}.$$

Applying again Lemma 6.2 we have $\|\mathcal{G}_\varepsilon(\partial_u A_3)\|_{\ell+1,\rho,\delta,\sigma} \leq K|\mu|^2\varepsilon^{2\eta+1}$. Therefore

$$\|\mathcal{G}_\varepsilon(\partial_u A)\|_{\ell+1,\rho,\delta,\sigma} \leq K|\mu|^2\varepsilon^{2\eta+1}.$$

The other bounds are straightforward. □

To prove Proposition 7.4, we proceed as in the proofs of Propositions 6.4 and 6.10. That is, we first perform a change of variables which reduces the size of the linear terms of $\mathcal{F}$ in (142). Let us point out, that for the proof of Proposition 7.4 we could look for this change as the analytic continuation of the changes obtained in Lemmas 6.5 and 6.11. Nevertheless, since we want the proof of Theorem 4.4 be also valid for Theorem 4.8, we look for a change g which is not necessarily continuation of the one obtained in Lemmas 6.5 and 6.11.

Lemma 7.6. *Let us consider $\kappa_1 > \kappa_0' > \kappa_0 > 0$ and $\rho_1'' > \rho_1' > \rho_2 > \rho_0'$, where ρ_0' is the constant defined in Lemmas 6.5 and 6.11. Then, for $\varepsilon > 0$ small enough and κ_0' big enough, there exists a function g which is solution of* (156) *and satisfies:*

- *If $\ell - 2r < 0$, $g \in \mathcal{E}_{0,\rho_1',\kappa_0',\sigma}$ and*

$$\begin{aligned} \|g\|_{0,\rho_1',\kappa_0',\sigma} &\leq K|\mu|\varepsilon^{\eta+1} \\ \|\partial_v g\|_{0,\rho_1',\delta,\sigma} &\leq K|\mu|\varepsilon^{\eta+1} \end{aligned}$$

- *If $\ell - 2r \geq 0$, $g \in \mathcal{E}_{\ell-2r,\rho_1',\kappa_0',\sigma}$ and*

$$\begin{aligned} \|g\|_{\ell-2r,\rho_1',\delta,\sigma} &\leq K|\mu|\varepsilon^{\eta+1} \\ \|\partial_v g\|_{\ell-2r+1,\rho_1',\delta,\sigma} &\leq K|\mu|\varepsilon^{\eta+1} \end{aligned}$$

Moreover, $v + g(v,\tau) \in D^{\mathrm{out},u}_{\rho_1'',\kappa_0}$ for $(v,\tau) \in D^{\mathrm{out},u}_{\rho_1',\kappa_0'} \times \mathbb{T}_\sigma$.

Furthermore, the change of variables $(u,\tau) = (v+g(v,\tau),\tau)$ is invertible and its inverse is of the form $(v,\tau) = (u + h(u,\tau),\tau)$. The function h is defined in the domain $D^{\mathrm{out},u}_{\rho_2,\kappa_1} \times \mathbb{T}_\sigma$ and it satisfies

- *If $\ell - 2r < 0$*

$$\|h\|_{0,\rho_2,\kappa_1,\sigma} \leq K|\mu|\varepsilon^{\eta+1}.$$

- *If $\ell - 2r \geq 0$*

$$\|h\|_{\ell-2r,\rho_2,\kappa_1,\sigma} \leq K|\mu|\varepsilon^{\eta+1}.$$

Moreover, $u + h(u,\tau) \in D^{\mathrm{out},u}_{\rho_1',\kappa_0'}$ for $(u,\tau) \in D^{\mathrm{out},u}_{\rho_2,\kappa_1} \times \mathbb{T}_\sigma$.

In the case $\ell - 2r < 0$ we need more precise bounds of both functions g and h restricted to the inner domain $D^{\mathrm{in},+,u}_{\kappa_1,c}$ defined in (30). These bounds are given in next corollary.

Corollary 7.7. *Let us assume $\ell - 2r < 0$ and let us consider a constant $c_1 > 0$. Then, the functions g ad h obtained in Lemma 7.6 restricted to the inner domain $D^{\mathrm{in},+,u}_{\kappa_1,c_1}$ satisfy the following bounds*

$$\sup |g(u,\tau)|_{(u,\tau)\in D^{\mathrm{in},+,u}_{\kappa_1,c_1}\times\mathbb{T}_\sigma} \le K|\mu|\varepsilon^{\eta+1+\nu_1^*} \quad \text{and} \quad \sup |h|_{(u,\tau)\in D^{\mathrm{in},+,u}_{\kappa_1,c_1}\times\mathbb{T}_\sigma} \le K|\mu|\varepsilon^{\eta+1+\nu_1^*}$$

with $\nu_1^* = \min\{2r-\ell)\gamma, 1\}$.

Proof of Lemma 7.6 and corollary 7.7. To define g, let us recall first that, by Lemma 7.5, $\langle B_1\rangle = 0$. Then we can define a function $\overline{B}_1$ such that $\partial_\tau \overline{B}_1 = B_1$ and $\langle \overline{B}_1\rangle = 0$. Then, one can see that a solution of equation (156), can be given by

$$g(v,\tau) = -\varepsilon\overline{B}_1(v,\tau) + \varepsilon\mathcal{G}_\varepsilon(\partial_v\overline{B}_1)(v,\tau), \tag{173}$$

where $\mathcal{G}_\varepsilon$ is the integral operator defined in (168).

By Lemma 7.5, if $\ell - 2r \ge 0$ one has:

$$\begin{aligned} \left\|\overline{B}_1\right\|_{\ell-2r,\rho_2,\kappa_0',\sigma} &\le K|\mu|\varepsilon^\eta \\ \left\|\partial_v\overline{B}_1\right\|_{\ell-2r+1,\rho_2,\kappa_0',\sigma} &\le K|\mu|\varepsilon^\eta. \end{aligned} \tag{174}$$

If $-1 \le \ell - 2r < 0$

$$\begin{aligned} \left\|\overline{B}_1\right\|_{0,\rho_2,\kappa_0',\sigma} &\le K|\mu|\varepsilon^\eta \\ \left\|\partial_v\overline{B}_1\right\|_{\ell-2r+1,\rho_2,\kappa_0',\sigma} &\le K|\mu|\varepsilon^\eta. \end{aligned} \tag{175}$$

Finally, if $\ell - 2r < -1$

$$\begin{aligned} \left\|\overline{B}_1\right\|_{0,\rho_2,\kappa_0',\sigma} &\le K|\mu|\varepsilon^\eta \\ \left\|\partial_v\overline{B}_1\right\|_{0,\rho_2,\kappa_0',\sigma} &\le K|\mu|\varepsilon^\eta. \end{aligned} \tag{176}$$

From these inequalities, using Lemma 7.3 we conclude that:

$$\left\|g(v,\tau) + \varepsilon\overline{B}_1(v,\tau)\right\|_{\max\{\ell-2r+1,0\},\rho_2,\kappa_0',\sigma} \le K\mu\varepsilon^{\eta+2},$$

which, together with (174) when $\ell - 2r \ge 0$ and with (175) and (176) when $\ell - 2r < 0$, gives the desired bounds for g. For the proof of the bound of $\partial_v g$ it is enough to apply again Lemmas 7.3 and 7.5 and (174).

The rest of the statements are straightforward.

To proof corollary 7.7 we just need to use the definition of B_1 in (144), and observe that it has a singularity or order $\ell - 2r$ if $\ell - 2r \ge 0$ and a zero of order $2r - \ell$ if $\ell - 2r \le 0$. □

Once we have the change g, we proceed as in Section 6.1.2, defining

$$\widehat{T}_1(v,\tau) = T_1(v + g(v,\tau),\tau) \tag{177}$$

which is solution of (157), that is:

$$\mathcal{L}_\varepsilon\widehat{T}_1 = \widehat{\mathcal{F}}\left(\partial_v\widehat{T}_1\right).$$

We look for it using a fixed point argument on $\partial_v\widehat{T}_1$. Nevertheless, since we want $\partial_u T_1$ to be the analytic continuation of the function $\partial_u T_1$ obtained in Propositions 6.4 and 6.10, we have to impose *initial conditions*. Nevertheless, since we invert $\mathcal{L}_\varepsilon$ by using the operator $\mathcal{G}_\varepsilon$ defined in (168) adapted to the domain $D^{\mathrm{out},u}_{\rho_1',\delta}\times\mathbb{T}_\sigma$, we consider a different initial condition depending on the Fourier coefficient. Recall that we are looking for $\partial_v\widehat{T}_1$ defined in $D^{\mathrm{out},u}_{\rho_1',\delta}\times\mathbb{T}_\sigma$. Thus, we define

$$\begin{aligned} A_0(v,\tau) =& \sum_{k<0}\partial_v\widehat{T}_1^{[k]}(\overline{v}_1)\, e^{-ik\varepsilon^{-1}(v-\overline{v}_1)}e^{ik\tau}\\ &+\sum_{k>0}\partial_v\widehat{T}_1^{[k]}(v_1)\, e^{-ik\varepsilon^{-1}(v-v_1)}e^{ik\tau}\\ &+\partial_v\widehat{T}_1^{[0]}(-\rho_1'), \end{aligned}\tag{178}$$

where $v_1,\overline{v}_1$ are the vertices of the outer domain $D^{\mathrm{out},u}_{\rho_1',\delta}$ (see Figure 4) and $\partial_v\widehat{T}_1$ can be obtained differentiating (177), since T_1 is already known in a neighborhood of these points. Note that $v_1,\overline{v}_1,\rho_1'\in D^u_{\infty,\rho_1}$. Applying the bounds obtained in Propositions 6.4 and 6.10 and Lemma 7.6, one can see that

$$\|A_0\|_{0,\rho_1',\kappa_0',\sigma}\le K|\mu|\varepsilon^{\eta+1}.\tag{179}$$

Let us define $S(v,\tau)$ as the solution of

$$S(v,\tau)=A_0(v,\tau)+\mathcal{G}_\varepsilon\left(\partial_v\widehat{\mathcal{F}}(S)\right)(v,\tau),$$

where $\mathcal{G}_\varepsilon$ and $\widehat{\mathcal{F}}$ are the operators defined in (168) and (158) respectively. Let us point out that the definition of $\widehat{\mathcal{F}}$ involves the functions $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ defined in (159), (160) and (161). Even if we keep the same notation, now the definitions involve the function g obtained in Lemma 7.6 instead of the ones given in Lemmas 6.5 and Lemma 6.11.

We will see that S is the analytic continuation of the function $\partial_u T_1(v+g(v,\tau),\tau)(1+\partial_v g(v,\tau))^{-1}$, where T_1 is obtained from Propositions 6.4 and 6.10.

Thus, we look for a fixed point $S\in\mathcal{E}_{\ell+1,\rho_1',\kappa_0',\sigma}$ of the operator

$$\mathcal{J}(S)(v,\tau)=A_0(v,\tau)+\mathcal{G}_\varepsilon\left(\partial_v\widehat{\mathcal{F}}(S)\right)(v,\tau).\tag{180}$$

Lemma 7.8. *Let us consider $\varepsilon_0>0$ small enough and $\kappa_0'>\kappa_0$ big enough. Then, for $\varepsilon\in(0,\varepsilon_0)$, there exists a function $S\in\mathcal{E}_{\ell+1,\rho_1',\kappa_0',\sigma}$ defined in $D^{\mathrm{out},u}_{\rho_1',\kappa_0'}\times\mathbb{T}_\sigma$ such that it is a fixed point of the operator* (180) *and is the analytic continuation of the function $\partial_u T_1(v+g(v,\tau),\tau)(1+\partial_v g(v,\tau))^{-1}$ where T_1 is obtained from Propositions 6.4 and 6.10 and g is given in Lemma 7.6.*

Moreover, there exists a constant $b_2>0$ such that

$$\|S\|_{\ell+1,\rho_1',\kappa_0',\sigma}\le b_2|\mu|\varepsilon^{\eta+1}.$$

Proof. We recall that, during the proof, g is the function given in Lemma 7.6.

It is straightforward to see that $\mathcal{J}$ is well defined from $\mathcal{E}_{\ell+1,\rho_1',\delta,\sigma}$ to itself. We are going to prove that there exists a constant $b_2>0$ such that $\mathcal{J}$ is contractive in $\overline{B}(b_2|\mu|\varepsilon^{\eta+1})\subset\mathcal{E}_{\ell+1,\rho_1',\kappa_0',\sigma}$.

First, let us consider $\mathcal{J}(0)$. From the definition of $\mathcal{J}$ in (180) and the definition of $\widehat{\mathcal{F}}$ in (158), we have

$$\mathcal{J}(0)(v,\tau)=A_0(v,\tau)+\mathcal{G}_\varepsilon\left(\partial_v\widehat{A}(v,\tau)\right),$$

where $\widehat{A}$ is the function in (159).

Taking into account the definition of $\widehat{A}$, we split $\mathcal{J}(0)$ as

$$\mathcal{J}(0)(v,\tau)=A_0(v,\tau)+\mathcal{G}_\varepsilon\left(\partial_v A(v,\tau)\right)+\mathcal{G}_\varepsilon\left(\partial_v\left[A(v+g(v,\tau),\tau)-A(v,\tau)\right]\right),$$

where a is given in (143). The first term has already been bounded in (179) and the second one in Lemma 7.5. For the third one, using ρ_1'' defined in Lemma 7.6, and applying Lemmas 7.3, 7.5 and 7.6 and the mean value theorem,

$$\begin{aligned}\|\mathcal{G}_\varepsilon\left(\partial_v\left[A(v+g(v,\tau),\tau)-A(v,\tau)\right]\right)\|_{\ell+1,\rho_1',\kappa_0',\sigma} &\leq \|A(v+g(v,\tau),\tau)-A(v,\tau)\|_{\ell+1,\rho_1',\kappa_0',\sigma}\\ &\leq \|\partial_u A\|_{\ell+1,\rho_1'',\kappa_0\varepsilon,\sigma}\|g\|_{0,\rho_1',\kappa_0',\sigma}\\ &\leq K|\mu|^2\varepsilon^{2\eta+1}\end{aligned}$$

Thus, there exists a constant $b_2>0$ such that

$$\|\mathcal{J}(0)\|_{\ell+1,\rho_1',\kappa_0',\sigma}\leq \frac{b_2}{2}|\mu|\varepsilon^{\eta+1}.$$

Let us consider now, $h_1,h_2\in\overline{B}(b_2|\mu|\varepsilon^{\eta+1})\subset\mathcal{E}_{\ell+1,\rho_1',\kappa_0',\sigma}$. Then, using the definitions of $\mathcal{J}$ and $\widehat{\mathcal{F}}$ in (180) and (158) respectively, and applying Lemma 7.3,

$$\begin{aligned}\|\mathcal{J}(h_2)-\mathcal{J}(h_1)\|_{\ell+1,\rho_1',\kappa_0',\sigma} &\leq K\left\|\widehat{\mathcal{F}}(h_2)-\widehat{\mathcal{F}}(h_1)\right\|_{\ell+1,\rho_1',\kappa_0',\sigma}\\ &\leq K\left\|\widehat{B}\cdot(h_2-h_1)+\widehat{C}(h_2,v,\tau)-\widehat{C}(h_1,v,\tau)\right\|_{\ell+1,\rho_1',\kappa_0',\sigma}.\end{aligned}$$

To bound the Lipschitz constant of $\mathcal{J}$, one has to take into account the definitions of $\widehat{B}$ and $\widehat{C}$ in (160) and (161) respectively, and to apply Lemmas 7.5 and 7.6. We bound it in different ways depending whether $\ell-2r<0$ or $\ell-2r\geq 0$. In the first case we obtain

$$\|\mathcal{J}(h_2)-\mathcal{J}(h_1)\|_{\ell+1,\rho_1',\kappa_0',\sigma}\leq K|\mu|\varepsilon^{\eta+1-\max\{0,\ell-2r+1\}}\|h_2-h_1\|_{\ell+1,\rho_1',\kappa_0',\sigma},$$

and in the second,

$$\|\mathcal{J}(h_2)-\mathcal{J}(h_1)\|_{\ell+1,\rho_1',\kappa_0',\sigma}\leq K|\mu|\frac{\varepsilon^{\eta-(\ell-2r)}}{(\kappa_0')^{\ell-2r+1}}\|h_2-h_1\|_{\ell+1,\rho_1',\kappa_0',\sigma}.$$

Therefore, since $\eta\geq\max\{0,\ell-2r\}$, taking $\varepsilon<\varepsilon_0$ and κ_0' big enough, $\mathrm{Lip}\mathcal{J}<1/2$ and then $\mathcal{J}$ is contractive in $\overline{B}(b_2|\mu|\varepsilon^{\eta+1})\subset\mathcal{E}_{\ell+1,\rho_1',\kappa_0',\sigma}$ and it has a unique fixed point $S(v,\tau)$.

Now, we have to prove that $S(v,\tau)$ is the analytic continuation of the function $\widetilde{S}(v,\tau)=\partial_u T_1(v+g(v,\tau),\tau)(1+\partial_v g(v,\tau))^{-1}$ obtained from Propositions 6.4 and 6.10. First let us observe that operator (180) is well defined for functions in $\left(D^u_{\infty,\rho_1}\cap D^{\mathrm{out},u}_{\rho_1',\kappa_0'}\right)\times\mathbb{T}_\sigma$. Moreover, both functions $S(v,\tau)$ and $\widetilde{S}(v,\tau)$ are defined in $\left(D^u_{\infty,\rho_1}\cap D^{\mathrm{out},u}_{\rho_1',\kappa_0'}\right)\times\mathbb{T}_\sigma$ and for (v,τ) in this domain both are fixed points of operator (180) and

$$\left\|\widetilde{S}\right\|_{\ell+1,\sigma}\leq b_1\mu\varepsilon^{\eta+1}.$$

Then, using the norms defined in Section 7.1.1 but for functions defined in $\left(D^u_{\infty,\rho_1}\cap D^{\mathrm{out},u}_{\rho_1',\kappa_0'}\right)\times\mathbb{T}_\sigma$, one can see that

$$\begin{aligned}\left\|S(v,\tau)-\widetilde{S}(v,\tau)\right\|_{\ell+1,\sigma} &\leq \left\|\mathcal{J}\left(S(v,\tau)\right)-\mathcal{J}\left(\widetilde{S}(v,\tau)\right)\right\|_{\ell+1,\sigma}\\ &\leq\frac{1}{2}\left\|S(v,\tau)-\widetilde{S}(v,\tau)\right\|_{\ell+1,\sigma}.\end{aligned}$$

Then $S(v,\tau)=\widetilde{S}(v,\tau)$ for $(v,\tau)\in\left(D^u_{\infty,\rho_1}\cap D^{\mathrm{out},u}_{\rho_1',\kappa_0'}\right)\times\mathbb{T}_\sigma$ and $S(v,\tau)$ is the analytic continuation of the function $\partial_u T_1(v+g(v,\tau),\tau)(1+\partial_v g(v,\tau))^{-1}$ to $D^{\mathrm{out},u}_{\rho_1',\kappa_0'}\times\mathbb{T}_\sigma$. Finally, it is enough to point out that one can easily recover $\widehat{T}_1$ from S. $\square$

Proof of Proposition 7.4. To prove Proposition 7.4 from Lemma 7.8, it is enough to consider the change of variables $v=u+h(u,\tau)$ obtained in Lemma 7.6 and to take $T_1(u,\tau)=\widehat{T}_1(u+h(u,\tau),\tau)$ which by construction is the analytic continuation of the function T_1 obtained in Propositions 6.4 and 6.10. $\square$

7.2 Invariant manifolds in the outer domains in the general case: proof of Theorems 4.5, 4.6, 4.7 and 4.8

We devote this section to prove the existence of the invariant manifolds in the outer domains, in general case, that is assuming that $p_0(u)$ can vanish. We do it in four different sections, devoted to prove Theorems 4.5, 4.6, 4.7 and 4.8

7.2.1 The variational equation along the separatrix

In order to prove the existence of the perturbed stable and unstable invariant manifolds in certain domains, we will need to consider a real-analytic fundamental matrix solution of the variational equations along the unperturbed separatrix

$$\dot{\xi} = A(u)\xi, \tag{181}$$

where

$$A(u) = \begin{pmatrix} 0 & 1 \\ -V''(q_0(u)) & 0 \end{pmatrix} \tag{182}$$

and $(q_0(u), p_0(u))$ is the parametrization of the unperturbed separatrix given in Hypothesis **HP2**.

It is a well known fact that the derivative of the parametrization of the separatrix, that is $(p_0(u), \dot{p}_0(u))$ (recall that $\dot{q}_0(u) = p_0(u)$), is a solution of (181). A second independent solution can be given by $(\zeta(u), \dot{\zeta}(u))$, where

$$\zeta(u) = p_0(u) \int_{u_0}^{u} \frac{1}{p_0^2(v)}\, dv, \tag{183}$$

where $u_0 \in \mathbb{R}$ is such that $p_0(u_0) \neq 0$. We consider then the following fundamental matrix

$$\Phi(u) = \begin{pmatrix} p_0(u) & \zeta(u) \\ \dot{p}_0(u) & \dot{\zeta}(u) \end{pmatrix}. \tag{184}$$

Remark 7.9. *Let us point out that the function ζ defined in* (183) *is well defined and analytic even if $p_0(u)$ can vanish for some $u \in \mathbb{C}$ and* a priori *it could seem that the integral depends on the path of integration.*

However, since $\ddot{p}_0(u) = -V''(q_0(u))p_0(u)$, one can see that the Taylor expansion around any zero $u^* \in \mathbb{C}$ of $p_0(u)$ is of the form

$$p_0(u) = \dot{p}_0(u^*)(u - u^*) + \mathcal{O}(u - u^*)^3$$

(observe that $\dot{p}_0(u^*) \neq 0$)) and then, the residue of the integrand appearing in the definition of ζ in (183) *is zero. Finally, let us observe that even if the integral might be divergent if one takes u^* as the upper limit of integration,* $\lim_{u \to u^*} \zeta(u) = -1/p_0(u^*)$.

7.2.2 Proof of Theorem 4.5

In this section we prove the existence of a change of variables which allow us to obtain a parametrization of the invariant manifolds which satisfies equation (44) from the parametrization obtained in Theorem 4.3.

Applying $\mathcal{L}_\varepsilon = \varepsilon^{-1}\partial_\tau + \partial_v$ to both components of equation (44), it is straightforward to see that (53) satisfies these equations provided $\mathcal{U}^u$ satisfies

$$\mathcal{L}_\varepsilon h = M(v + h(v,\tau), \tau), \tag{185}$$

where

$$M(u,\tau) = \frac{1}{p_0^2(u)}\partial_u T_1(u,\tau) + \frac{\mu\varepsilon^\eta}{p_0(u)}\partial_p \widehat{H}_1\left(q_0(u), p_0(u) + \frac{1}{p_0(u)}\partial_u T_1(u,\tau), \tau\right), \tag{186}$$

$\widehat{H}_1$ is the Hamiltonian defined in (34) and T_1 is the function obtained in Proposition 6.4.

Splitting the right hand side of equation (185) into constant, linear and higher order terms in h, it can be rewritten as

$$\mathcal{L}_\varepsilon h = M(h), \tag{187}$$

where

$$M(h) = M(v,\tau) + (N_1(v,\tau) + N_2(v,\tau))\, h + R(h,v,\tau) \tag{188}$$

and

$$N_1(v,\tau) = \mu\varepsilon^\eta \partial_v \left[\frac{1}{p_0(v)} \partial_p \widehat{H}_1^1 \left(q_0(v), p_0(v), \tau\right)\right] \tag{189}$$

$$N_2(v,\tau) = \partial_v M(v,\tau) - N_1(v,\tau) \tag{190}$$

$$R(h,v,\tau) = M(v+h,\tau) - \partial_v M(v,\tau)h - M(v,\tau), \tag{191}$$

where $\widehat{H}_1^1$ and $M(v,\tau)$ are defined in (35) and (186) respectively.

We now define appropriate Banach spaces. For analytic functions $h : I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma \to \mathbb{C}$, where $I^u_{\rho_3,\rho_4}$ is the domain defined in (52), we define the Fourier norm

$$\|h\|_\sigma = \sum_{k\in\mathbb{Z}} \left\|h^{[k]}\right\|_\infty e^{|k|\sigma},$$

where $\|\cdot\|_\infty$ is the classical supremum norm in $I^u_{\rho_3,\rho_4}$. We consider the following function space

$$\mathcal{A}_\sigma = \left\{h : I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma \to \mathbb{C};\ \text{real-analytic}, \|h\|_\sigma < \infty\right\} \tag{192}$$

which is straightforward to see that is a Banach algebra.

Throughout this section we will need to solve equations of the form $\mathcal{L}_\varepsilon h = g$, where $\mathcal{L}_\varepsilon$ is the differential operator defined in (45). We take the operator $\mathcal{G}_\varepsilon$ defined in (168) as right inverse of $\mathcal{L}_\varepsilon$. In Section 7.1.1 it was applied to functions belonging to $\mathcal{E}_{\nu,\rho,\delta,\sigma}$ (see (167)) but it is clear that it can also be applied to functions in $\mathcal{A}_\sigma$ if we take v_1 and $\overline{v}_1$ the vertices of the domain $I^u_{\rho_3,\rho_4}$ and also $-\rho_4$ (see Figure 6).

Lemma 7.10. *The operator $\mathcal{G}_\varepsilon$ in* (168) *satisfies the following properties.*

- *$\mathcal{G}_\varepsilon$ is linear from $\mathcal{A}_\sigma$ to itself and satisfies $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon = \mathrm{Id}$.*
- *If $h \in \mathcal{A}_\sigma$, then*
$$\|\mathcal{G}_\varepsilon(h)\|_\sigma \leq K\|h\|_\sigma.$$
  Furthermore, if $\langle h\rangle = 0$, then
$$\|\mathcal{G}_\varepsilon(h)\|_\sigma \leq K\varepsilon\|h\|_\sigma.$$

Finally, we state a technical lemma which gives some properties of the functions M, N_1, N_2 and R defined in (186), (189), (190) and (191) respectively.

Lemma 7.11. *The functions M, N_1, N_2 and R defined in* (186), (189), (190) *and* (191) *satisfy the following properties:*

1. *$M \in \mathcal{A}_\sigma$ and satisfies*
$$\|M\|_\sigma \leq K|\mu|\varepsilon^\eta, \quad \|\mathcal{G}_\varepsilon(M)\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}. \tag{193}$$
2. *$N_1, N_2 \in \mathcal{A}_\sigma$. Moreover they satisfy $\langle N_1\rangle = 0$ and*
$$\|N_1\|_\sigma \leq K|\mu|\varepsilon^\eta, \quad \|N_2\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}. \tag{194}$$
3. *Let us consider $h_1, h_2 \in B(\nu) \subset \mathcal{A}_\sigma$ with $\nu \ll 1$. Then,*
$$\|R(h_2,v,\tau) - R(h_1,v,\tau)\|_\sigma \leq K\nu\|h_2 - h_1\|_\sigma.$$

Proof. The first bound is straightforward taking into account the bounds for c_{lk} and T_1 obtained in Corollary 5.6 and Propositions 6.4 and 6.10. For the second one, one has to split M as $M = M_1 + M_2$, where

$$M_1(v,\tau) = \mu\varepsilon^\eta \frac{1}{p_0(u)}\partial_p \widehat{H}_1^1(q_0(u), p_0(u), \tau),$$

where $\widehat{H}_1^1$ is the Hamiltonian in (35), and $M_2 = M - M_1$. Since $\langle M_1\rangle = 0$ and satisfies $\|M_1\|_\sigma \leq K|\mu|\varepsilon^\eta$, by Lemma 7.10 we have that $\|\mathcal{G}_\varepsilon(M_1)\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$. On the other hand, by the bound of c_{lk} in Corollary 5.6 and the bound of T_1 given by Proposition 6.4, M_2 satisfies $\|M_2\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$, and therefore $\|\mathcal{G}_\varepsilon(M_2)\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$.

The bounds of N_1, N_2 and R can be obtained analogously taking into account the definition of M in (186) and that R is quadratic in h. ☐

We split Theorem 4.5 in the following proposition and corollary, which are rewritten in terms of the Banach space defined in (192). Theorem 4.5 follows directly from those results.

Proposition 7.12. *Let ρ_1 be the constant considered in Proposition 6.4 and let us consider ρ_3 and ρ_4 such that $\rho_4 > \rho_3 > \rho_1$ and $\varepsilon_0 > 0$ small enough (which might depend on ρ_i, $i = 1,3,4$). Then, for $\varepsilon \in (0, \varepsilon_0)$ there exists a function $\mathcal{U}^u \in \mathcal{A}_\sigma$ defined in $I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$ that satisfies equation (187). Moreover, for $(v,\tau) \in I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$, $v + \mathcal{U}^u(v,\tau) \in D^u_{\infty,\rho_1}$ and there exists a constant $b_3 > 0$ such that*

$$\|\mathcal{U}^u\|_\sigma \leq b_3|\mu|\varepsilon^{\eta+1}.$$

Corollary 7.13. *Let us consider the constants ρ_3 and ρ_4 given by Proposition 7.12 and $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0, \varepsilon_0)$ there exist parameterizations of the invariant manifolds*

$$(Q^u(v,\tau), P^u(v,\tau)) = (q_0(v) + Q_1^u(v,\tau), p_0(v) + P_1^u(v,\tau))$$

which are solution of equation (44). Moreover $(Q_1^u, P_1^u) \in \mathcal{A}_\sigma \times \mathcal{A}_\sigma$ are defined in $I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$ and there exists a constant $b_4 > 0$ such that

$$\begin{aligned}\|Q_1^u\|_\sigma &\leq b_4|\mu|\varepsilon^{\eta+1}\\ \|P_1^u\|_\sigma &\leq b_4|\mu|\varepsilon^{\eta+1}.\end{aligned}$$

The proof of this corollary is a straightforward consequence of Proposition 7.12.

We prove Proposition 7.12 through a fixed point argument. Nevertheless, the operator M in (188) has linear terms in h which are not small when $\eta = 0$. Therefore, we have first to consider a change of variables to obtain a contractive operator. For this purpose, let us consider $\overline{N}_1 = \mathcal{G}_\varepsilon(N_1)$, where $\mathcal{G}_\varepsilon$ is the operator in (168) and N_1 the function in (189). Taking into account that $\langle N_1\rangle = 0$ and applying Lemmas 7.11 and 7.10, we have that

$$\left\|\overline{N}_1\right\|_\sigma = \|\mathcal{G}_\varepsilon(N_1)\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}. \tag{195}$$

Then, we consider the change

$$h = \left(1 + \overline{N}_1\right)\overline{h} \tag{196}$$

which, by (195), is invertible for $(v,\tau) \in I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$. By (187) and (196), $\overline{h}$ is solution of

$$\mathcal{L}_\varepsilon\overline{h} = \widehat{M}(\overline{h}),$$

where

$$\widehat{M}\left(\overline{h}\right)(v,\tau) = \widehat{M}(v,\tau) + \widehat{N}(v,\tau)\overline{h}(v,\tau) + \widehat{R}\left(\overline{h}(v,\tau), v, \tau\right) \tag{197}$$

with

$$\widehat{M}(v,\tau) = \left(1 + \overline{N}_1(v,\tau)\right)^{-1} M(v,\tau) \tag{198}$$

$$\widehat{N}(v,\tau) = \left(1 + \overline{N}_1(v,\tau)\right)^{-1} N_1(v,\tau)\overline{N}_1(v,\tau) + N_2(v,\tau) \tag{199}$$

$$\widehat{R}(\overline{h}, v,\tau) = \left(1 + \overline{N}_1(v,\tau)\right)^{-1} R\left(\left(1 + \overline{N}_1(v,\tau)\right)\overline{h}, v, \tau\right). \tag{200}$$

To find a solution of this equation, we look for a fixed point $\overline{h} \in \mathcal{A}_\sigma$ of the operator

$$\overline{M} = \mathcal{G}_\varepsilon \circ \widehat{M}, \tag{201}$$

where $\mathcal{G}_\varepsilon$ and $\widehat{M}$ are the operators (168) and (197). Then, Proposition 7.12 is a consequence of the following lemma.

Lemma 7.14. *Let us consider $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0, \varepsilon_0)$, there exists a function $\overline{h} \in \mathcal{A}_\sigma$ defined in $I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$, such that it is a fixed point of the operator* (201)*. Moreover, it satisfies*

$$\left\|\overline{h}\right\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$$

and then $u = v + \left(1 + \overline{N}_1(v,\tau)\right)\overline{h}(v,\tau) \in D^u_{\infty,\rho_1}$ for $(v,\tau) \in I^u_{\rho_3,\rho_4} \times \mathbb{T}_\sigma$.

Proof. It is straightforward to see that the operator $\overline{M}$ sends $\mathcal{A}_\sigma$ to itself. We are going to prove that there exists a constant $b_3 > 0$ such that $\overline{M}$ is contractive in $\overline{B}(b_3|\mu|\varepsilon^{\eta+1}) \subset \mathcal{A}_\sigma$.

Let us consider first $\overline{M}(0) = \mathcal{G}_\varepsilon \circ \widehat{M}(0)$. From the definitions of $\widehat{M}$ and $\widehat{M}$ in (197) and (198) respectively, we have that

$$\overline{M}(0) = \mathcal{G}_\varepsilon(\widehat{M}) = \mathcal{G}_\varepsilon\left(\left(1+\overline{N}_1\right)^{-1} M\right) = \mathcal{G}_\varepsilon\left(M\right) - \mathcal{G}_\varepsilon\left(\left(1+\overline{N}_1\right)^{-1}\overline{N}_1 M\right).$$

The first term has already been bounded in Lemma 7.11, and satisfies $\|\mathcal{G}_\varepsilon(M)\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$. For the second one has to take into account Lemma 7.10, and then (195) and Lemma 7.11, to obtain

$$\left\|\mathcal{G}_\varepsilon\left(\left(1+\overline{N}_1\right)^{-1}\overline{N}_1 M\right)\right\|_\sigma \leq K\left\|\overline{N}_1\right\|_\sigma \|M\|_\sigma \leq K|\mu|^2\varepsilon^{2\eta+1}.$$

Therefore, there exists a constant $b_3 > 0$ such that

$$\left\|\overline{M}(0)\right\|_\sigma \leq \frac{b_3}{2}|\mu|\varepsilon^{\eta+1}.$$

Let us consider now $\overline{h}_1, \overline{h}_2 \in \overline{B}(b_3|\mu|\varepsilon^{\eta+1}) \subset \mathcal{A}_\sigma$. Then using the properties of $\mathcal{G}_\varepsilon$ given in Lemma 7.10 and the definition of $\widehat{M}$ in (197),

$$\begin{aligned}\left\|\overline{M}(\overline{h}_2) - \overline{M}(\overline{h}_1)\right\|_\sigma &\leq K\left\|\widehat{M}(\overline{h}_2) - \widehat{M}(\overline{h}_1)\right\|_\sigma \\ &\leq K\left\|\widehat{N}(v,\tau)(\overline{h}_2 - \overline{h}_1) + \widehat{R}(\overline{h}_2, v,\tau) - \widehat{R}(\overline{h}_1, v,\tau)\right\|_\sigma.\end{aligned}$$

Taking into account the definitions of $\widehat{N}$ and $\widehat{R}$ in (199) and (200) and applying Lemma 7.11 and bound (195), one obtains

$$\left\|\overline{M}(\overline{h}_2) - \overline{M}(\overline{h}_1)\right\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}\|\overline{h}_2 - \overline{h}_1\|_\sigma.$$

Therefore, reducing ε if necessary, $\mathrm{Lip}\overline{M} \leq 1/2$ and therefore $\overline{M}$ is contractive from the ball $B(b_3|\mu|\varepsilon^{\eta+1}) \subset \mathcal{A}_\sigma$ into itself and it has a unique fixed point $\overline{h}$. □

Proof of Proposition 7.12. To proof Proposition 7.12 from Lemma 6.6, it is enough to undo the change of variables (196) to obtain $\mathcal{U}^u = \left(1 + \overline{N}_1\right)\overline{h}$. Then, using bound (195) and increasing slightly b_3 if necessary, we obtain the bound for $\mathcal{U}^u$. □

7.2.3 Proof of Theorem 4.6

We prove Theorem 4.6 looking for a solution of (44) through a fixed point argument, taking the parameterizations of the invariant manifolds as perturbations of the parameterizations of the unperturbed separatrix. Since we only deal with the unstable manifold, we omit the superscript u. We consider

$$\begin{pmatrix} Q(v,\tau) \\ P(v,\tau) \end{pmatrix} = \begin{pmatrix} q_0(v) + Q_1(v,\tau) \\ p_0(v) + P_1(v,\tau) \end{pmatrix}$$

and thus we look for (Q_1, P_1) as solutions of

$$(\mathcal{L}_\varepsilon - A(u)) \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = \mathcal{K} \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix}, \tag{202}$$

where $\mathcal{L}_\varepsilon$ is the operator defined in (45), A is the matrix defined in (182),

$$\mathcal{K}(\xi)(u,\tau) = \begin{pmatrix} \mu\varepsilon^\eta \partial_p \widehat{H}_1 \left(q_0(u) + \xi_1, p_0(u) + \xi_2, \tau\right) \\ G(\xi_1)(u,\tau) - \mu\varepsilon^\eta \partial_q \widehat{H}_1 \left(q_0(u) + \xi_1, p_0(u) + \xi_2, \tau\right) \end{pmatrix}$$

and

$$G(\xi_1)(u,\tau) = -\left(V'(x_p(\tau) + q_0(u) + \xi_1) - V'(x_p(\tau)) - V'(q_0(u)) - V''(q_0(u))\xi_1\right), \tag{203}$$

where for shortness we have put ξ_1 and ξ_2 for $\xi_1(u,\tau)$ and $\xi_2(u,\tau)$.

We split $\mathcal{K}$ considering constant and higher order terms in ξ as

$$\mathcal{K}(\xi)(u,\tau) = L(u,\tau) + (M_1(u,\tau) + M_2(u,\tau))\,\xi(u,\tau) + N(\xi)(u,\tau) \tag{204}$$

with

$$L(u,\tau) = \mu\varepsilon^\eta \begin{pmatrix} \partial_p \widehat{H}_1(q_0(u), p_0(u), \tau) \\ -\partial_q \widehat{H}_1(q_0(u), p_0(u), \tau) \end{pmatrix} + \begin{pmatrix} 0 \\ G(0)(u,\tau) \end{pmatrix} \tag{205}$$

$$M_1(u,\tau) = \mu\varepsilon^\eta \begin{pmatrix} \partial_{qp} \widehat{H}_1^1 \left(q_0(u), p_0(u), \tau\right) & \partial_{pp} \widehat{H}_1^1(q_0(u), p_0(u), \tau) \\ -\partial_{qq} \widehat{H}_1^1 \left(q_0(u), p_0(u), \tau\right) & -\partial_{qp} \widehat{H}_1^1(q_0(u), p_0(u), \tau) \end{pmatrix} \tag{206}$$

$$M_2(u,\tau) = \mu\varepsilon^{\eta+1} \begin{pmatrix} \partial_{qp} \widehat{H}_1^2 \left(q_0(u), p_0(u), \tau\right) & \partial_{pp} \widehat{H}_1^2(q_0(u), p_0(u), \tau) \\ -\partial_{qq} \widehat{H}_1^2 \left(q_0(u), p_0(u), \tau\right) & -\partial_{qp} \widehat{H}_1^2(q_0(u), p_0(u), \tau) \end{pmatrix} \tag{207}$$

$$N(\xi)(u,\tau) = L(u,\tau) + (M_1(u,\tau) + M_2(u,\tau))\,\xi(u,\tau) - \mathcal{K}(\xi)(u,\tau). \tag{208}$$

First step is to define the following function space

$$\mathcal{Y}_\sigma = \left\{ h : \widetilde{D}^{\mathrm{out},u}_{\rho,d,\kappa} \times \mathbb{T} \to \mathbb{C};\ \text{real-analytic}, \|h\|_\sigma < \infty \right\},$$

where $\widetilde{D}^{\mathrm{out},u}_{\rho,d,\kappa}$ is the domain defined in (54) and

$$\|h\|_\sigma = \sum_{k\in\mathbb{Z}} \left\| h^{[k]} \right\|_\infty e^{|k|\sigma}, \tag{209}$$

where $\|\cdot\|_\infty$ is the classical supremmum norm. It is a well known fact that this function space is a Banach algebra (see for instance [Sau01]). We also define the product space

$$\mathcal{Y}_\sigma \times \mathcal{Y}_\sigma = \{h = (h_1, h_2) : \widetilde{D}^{\mathrm{out},u}_{\rho,d,\kappa} \times \mathbb{T}_\sigma \to \mathbb{C}^2;\ \text{real-analytic}, \|h\|_\sigma = \|h_1\|_\sigma + \|h_2\|_\sigma < \infty\}. \tag{210}$$

Since we deal with the Banach space $\mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$, it is also useful to consider the norm for 2×2 matrices induced by $\|\cdot\|_\sigma$. Let us consider $B = \left(b^{ij}\right)$ a 2×2 matrix such that $b^{ij} \in \mathcal{Y}_\sigma$. Then, the induced norm with respect to the norm of $\mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$, which we also denote $\|\cdot\|_\sigma$ abusing notation, is given by

$$\|B\|_\sigma = \max_{j=1,2} \left\{ \left\|b^{1j}\right\|_\sigma + \left\|b^{2j}\right\|_\sigma \right\}. \tag{211}$$

The next lemma gives some properties of this induced norm.

Lemma 7.15. *The following statements are satisfied*

1. *If $h \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ and $B = (b^{ij})$ is 2×2 matrix with $b^{ij} \in \mathcal{Y}_\sigma$, then $Bh \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ and*

$$\|Bh\|_\sigma \le \|B\|_\sigma \|h\|_\sigma.$$

2. *If $B_1 = (b_1^{ij})$ and $B_2 = (b_2^{ij})$ are 2×2 matrices which satisfy $b_1^{ij} \in \mathcal{Y}_\sigma$ and $b_2^{ij} \in \mathcal{Y}_\sigma$ respectively, then $B_3 = (b_3^{ij}) = B_1 B_2$ satisfies $b_3^{ij} \in \mathcal{E}_\sigma$ and*

$$\|B_3\|_\sigma \leq \|B_1\|_\sigma \|B_2\|_\sigma.$$

Second step is to look for a right inverse of $\mathcal{L}_\varepsilon - A(u)$, where A is defined in (182). To obtain it we use the operator $\mathcal{G}_\varepsilon$ defined in (168), which is well defined for functions belonging to $\mathcal{Y}_\sigma$, if we take $u_1, \overline{u}_1$ the vertices of the domain $\widetilde{D}^{\mathrm{out},u}_{\rho,d,\kappa}$ defined in (54) (see Figure 7). Recalling that Φ defined in (184) satisfies $\mathcal{L}_\varepsilon \Phi = A\Phi$, we can define a right inverse of $\mathcal{L}_\varepsilon - A(v)$ as

$$\widehat{\mathcal{G}}_\varepsilon(h) = \Phi \mathcal{G}_\varepsilon(\Phi^{-1} h), \text{ for } h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{212}$$

Lemma 7.16. *The operator $\widehat{\mathcal{G}}_\varepsilon$ in (212) satisfies the following properties.*

1. *If $h \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$, then $\widehat{\mathcal{G}}_\varepsilon(h) \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ and*

$$\left\|\widehat{\mathcal{G}}_\varepsilon(h)\right\|_\sigma \leq K\|h\|_\sigma.$$

2. *Furthermore, if $\langle h\rangle = 0$, then*

$$\left\|\widehat{\mathcal{G}}_\varepsilon(h)\right\|_\sigma \leq K\varepsilon\|h\|_\sigma.$$

We rewrite Theorem 4.6 in terms of equation (202) and the Banach spaces defined in (210).

Proposition 7.17. *Let ρ_4 and κ_1 be the constant considered in Proposition 7.12 and 7.4 and let us consider also $d_0 > 0$ and $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0, \varepsilon_0)$ there exist functions $(Q_1, P_1) \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ which satisfy equation (202) and are the analytic continuation of the functions (Q_1, P_1) obtained in Corollary 7.13. Moreover, there exists a constant $b_5 > 0$ such that*

$$\|(Q_1, P_1)\|_\sigma \leq b_5 |\mu| \varepsilon^{\eta+1}.$$

Before proving the proposition, we state and prove the following technical lemma which gives some properties of L, M_1, M_2 and N defined in (205), (206), (207) and (208) respectively.

Lemma 7.18. *The functions L, M_1, M_2 and N defined in (205), (206), (207) and (208) respectively, have the following properties,*

1. *$L \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ and satisfies*

$$\|L\|_\sigma \leq K|\mu|\varepsilon^\eta, \quad \left\|\widehat{\mathcal{G}}_\varepsilon(L)\right\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}.$$

2. *$M_1 = \left(m_1^{ij}\right)$ and $M_2 = \left(m_2^{ij}\right)$ satisfy $m_1^{ij}, m_2^{ij} \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$, $\langle M_1\rangle = 0$ and*

$$\|M_1\|_\sigma \leq K|\mu|\varepsilon^\eta, \quad \|M_2\|_\sigma \leq K|\mu|^2\varepsilon^{2\eta+1}.$$

3. *If $\xi, \xi' \in B(\nu) \subset \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$, then*

$$\|N(\xi') - N(\xi)\|_\sigma \leq K\nu \left\|\xi' - \xi\right\|_\sigma.$$

Proof. For the first statement let us split L as $L = L_1 + L_2 + L_3$ with

$$L_i(u,\tau) = \begin{pmatrix} \mu\varepsilon^{\eta+i-1}\partial_p \widehat{H}_1^i(q_0(u), p_0(u), \tau) \\ -\mu\varepsilon^{\eta+i-1}\partial_q \widehat{H}_1^i(q_0(u), p_0(u), \tau) \end{pmatrix}, \ i = 1,2$$

and

$$L_3(u,\tau) = \left(\begin{array}{c} 0 \\ G(0)(u,\tau) \end{array} \right),$$

where $\widehat{H}_1^1$, $\widehat{H}_1^2$ and G are the functions defined in (35), (37) and (203) respectively. One can easily see that $L_1, L_2 \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$, $\langle L_1 \rangle = 0$ and $\|L_1\|_\sigma \leq K|\mu|\varepsilon^\eta$ and, using Corollary 5.6, also that $\|L_2\|_\sigma \leq K|\mu|^2\varepsilon^{2\eta+1}$. Thus, applying Lemma 7.16 one obtains $\|\widehat{\mathcal{G}}_\varepsilon(L_i)\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$ for $i=1,2$.

To obtain analogous properties for L_3, it is enough to apply Mean Value Theorem to obtain

$$L_3(u,\tau) = \left(\begin{array}{c} 0 \\ -\displaystyle\int_0^1 V'''\left(s_1 x_p(\tau) + s_2 q_0(u)\right) ds_1 ds_2 q_0(u) x_p(\tau) \end{array} \right).$$

Then, $\|L_3\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$. Therefore, applying Lemma 7.16 we have that $\|\widehat{\mathcal{G}}_\varepsilon(L_3)\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$. This finishes the proof of the first statement.

The proof of the other statements is straightforward. $\square$

To prove Proposition 7.17, one has to perform first a change of variables to equation (202) to obtain a contractive operator. In fact, this change is only necessary in the case $\eta = 0$. Let us consider

$$\overline{M}_1 = \left(\overline{m}_1^{ij}\right) \text{ with } \overline{m}_1^{ij} = \mathcal{G}_\varepsilon\left(m_1^{ij}\right), \tag{213}$$

where $\mathcal{G}_\varepsilon$ is the operator defined in (168) and $M_1 = \left(m_1^{ij}\right)$ is the matrix defined in (206). By Lemmas 7.18 and 7.3, one can see that

$$\|\overline{M}_1\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}. \tag{214}$$

We consider the change of variables

$$\xi = \left(\mathrm{Id} + \overline{M}_1\right)\overline{\xi} \tag{215}$$

which is invertible. Using (202) and (215), $\overline{\xi}$ is solution of equation

$$\left(\mathcal{L}_\varepsilon - A(u)\right)\overline{\xi} = \widehat{\mathcal{K}}(\overline{\xi}), \tag{216}$$

where

$$\widehat{\mathcal{K}}(\overline{\xi}) = \widehat{L} + \widehat{M}\overline{\xi} + \widehat{N}\left(\overline{\xi}\right) \tag{217}$$

with

$$\widehat{L} = \left(\mathrm{Id} + \overline{M}_1\right)^{-1} L \tag{218}$$

$$\widehat{M} = \left(\mathrm{Id} + \overline{M}_1\right)^{-1}\left(M_1\overline{M}_1 + A\overline{M}_1 - \overline{M}_1 A + M_2\left(\mathrm{Id} + \overline{M}_1\right)\right) \tag{219}$$

$$\widehat{N}(\overline{\xi}) = \left(\mathrm{Id} + \overline{M}_1\right)^{-1} N\left(\left(\mathrm{Id} + \overline{M}_1\right)\overline{\xi}\right) \tag{220}$$

Since we want to obtain the analytic continuation of the parameterizations of the manifolds obtained in Corollary 7.13, we need to impose *initial conditions*. Nevertheless, since we invert $\mathcal{L}_\varepsilon - A(u)$ by using the operator $\widehat{\mathcal{G}}_\varepsilon$ in (212) which is defined acting on the Fourier coefficients, we need to consider a different initial condition depending on the Fourier coefficient, that is in u_1 or in $\bar{u}_1$ (see Figure 7). Thus, we define the following function

$$\begin{aligned} L_0(v,\tau) = & \sum_{k<0} \Phi(v)\Phi^{-1}(\overline{v}_1)\overline{\xi}^{[k]}\left(\overline{v}_1\right) e^{-ik\varepsilon^{-1}(v-\bar{v}_1)} e^{ik\tau} \\ & + \sum_{k\geq 0} \Phi(v)\Phi^{-1}(v_1)\overline{\xi}^{[k]}\left(v_1\right) e^{-ik\varepsilon^{-1}(v-v_1)} e^{ik\tau} \\ & + \Phi(v)\Phi^{-1}\left(-\rho_4\right)\xi^{[0]}\left(-\rho_4\right). \end{aligned} \tag{221}$$

Recall that $\overline{\xi}(v,\tau)$ is already known for $v = v_1, \overline{v}_1, -\rho_4$ using (215), (213) and Corollary 7.13.

Lemma 7.19. *The function $L_0(u,\tau)$ in* (221) *satisfies de following properties:*

- $(\mathcal{L}_\varepsilon - A(v))\, L_0 = 0$, *where $\mathcal{L}_\varepsilon$ is the operator in* (45).
- $L_0 \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ *and*
$$\|L_0\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}.$$

The function $\overline{\xi}$ satisfies equation (216) and the initial conditions on the Fourier coefficients L_0 in (221) if and only if it is solution of the integral equation

$$\begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = L_0 + \widehat{\mathcal{G}}_\varepsilon \circ \mathcal{K} \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix},$$

where $\widehat{\mathcal{G}}_\varepsilon$ and $\mathcal{K}$ are the operators defined in (212) and (204) respectively. Thus, we look for a fixed point $\xi = (Q_1, P_1) \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ of the operator

$$\overline{\mathcal{K}} = L_0 + \widehat{\mathcal{G}}_\varepsilon \circ \widehat{\mathcal{K}}. \tag{222}$$

Therefore, Proposition 7.17 is a straightforward consequence of the following lemma.

Lemma 7.20. *Let us consider $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0,\varepsilon_0)$, there exists a function $\overline{\xi} \in \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ defined in $\widetilde{D}^{\mathrm{out},u}_{\rho_4,d_0,\kappa_1} \times \mathbb{T}_\sigma$ such that is a fixed point of the operator* (222) *and satisfies*

$$\left\|\overline{\xi}\right\|_\sigma \leq b_5|\mu|\varepsilon^{\eta+1}.$$

for a certain constant $b_5 > 0$ independent of ε and μ. Moreover, $\xi = (\mathrm{Id} + \overline{M}_1)\overline{\xi}$, where $\overline{M}_1$ is the function defined in (213)*, is the analytic continuation of the function $\xi = (Q_1, P_1)$ obtained in Corollary 7.13.*

Proof. To prove the lemma, first we see that there exists a constant $b_5 > 0$ such that the operator $\bar{\mathcal{K}}$ in (222) is contractive from $\overline{B}(b_5|\mu|\varepsilon^{\eta+1}) \subset \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ to itself and thus that it has a fixed point. Then, we will see that $\xi = (\mathrm{Id} + \overline{M}_1)\overline{\xi}$, where $\overline{M}_1$ is the function defined in (213), is the analytic continuation of the parameterizations of the manifolds which have been obtained in Corollary 7.13.

Let us first consider $\overline{\mathcal{K}}(0)$. Using the definitions of $\overline{\mathcal{K}}$, $\widehat{\mathcal{K}}$ and $\widehat{L}$ in (222), (204) and (218), we have that

$$\begin{aligned}\overline{\mathcal{K}}(0) &= L_0 + \widehat{\mathcal{G}}_\varepsilon\left(\widehat{L}\right)\\ &= \overline{L}_0 + \widehat{\mathcal{G}}_\varepsilon(L) + \widehat{\mathcal{G}}_\varepsilon\left(\overline{M}_1\left(\mathrm{Id} + \overline{M}_1\right)^{-1} L\right).\end{aligned}$$

From Lemmas 7.19, 7.16 and 7.18, and applying also the bound of $\overline{M}_1$ in (214), it is straightforward to see that $\|\overline{\mathcal{K}}(0)\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}$, and thus there exists a constant $b_5 > 0$ such that $\|\overline{\mathcal{K}}(0)\|_\sigma \leq b_5|\mu|\varepsilon^{\eta+1}/2$.

Let us consider now $\overline{\xi}^1, \overline{\xi}^2 \in \overline{B}(b_5|\mu|\varepsilon^{\eta+1}) \subset \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$. Then using the definitions of $\overline{\mathcal{K}}$ and $\widehat{\mathcal{K}}$ in (222) and (217), and applying Lemma 7.16,

$$\begin{aligned}\left\|\overline{\mathcal{K}}\left(\overline{\xi}^1\right) - \overline{\mathcal{K}}\left(\overline{\xi}^2\right)\right\|_\sigma &\leq K\left\|\widehat{\mathcal{K}}\left(\overline{\xi}^1\right) - \widehat{\mathcal{K}}\left(\overline{\xi}^2\right)\right\|_\sigma\\ &\leq K\left\|\widehat{M}\left(\overline{\xi}^2 - \overline{\xi}^1\right) + \widehat{N}\left(\overline{\xi}^1\right) - \widehat{N}\left(\overline{\xi}^2\right)\right\|_\sigma.\end{aligned}$$

Then, using the definitions of $\widehat{M}$ and $\widehat{N}$ in (219) and (220) and applying Lemma 7.18 and bound (214), one can see that

$$\left\|\overline{\mathcal{K}}\left(\overline{\xi}^1\right) - \overline{\mathcal{K}}\left(\overline{\xi}^2\right)\right\|_\sigma \leq K|\mu|\varepsilon^{\eta+1}\left\|\overline{\xi}^1 - \overline{\xi}^2\right\|_\sigma.$$

Therefore, reducing ε if necessary, $\mathrm{Lip}\overline{\mathcal{K}} < 1/2$ and then $\overline{\mathcal{K}}$ is contractive from $\overline{B}(b_5|\mu|\varepsilon^{\eta+1}) \subset \mathcal{Y}_\sigma \times \mathcal{Y}_\sigma$ to itself and it has a unique fixed point $\overline{\xi}$.

To prove that $\xi = (\mathrm{Id} + \overline{M}_1)\overline{\xi}$ is the analytic continuation of the function $\xi = (Q_1, P_1)$ obtained in Corollary 7.13, one can proceed as in the proof of Lemma 7.8. □

Proof of Proposition 7.17. It is enough to undo change (215). For the bound of $\xi = (Q_1, P_1)$ it is enough to consider the bound of $\overline{M}_1$ in (214) and the bound of $\overline{\xi}$ in Lemma 7.20 and increase slightly b_5 if necessary. □

7.2.4 Proof of Theorem 4.7

This section is devoted to obtain a parametrization of the invariant manifolds of the form (42) in the domains (26). To this end, we look for changes of variables $v = u + \mathcal{V}^{u,s}(u,\tau)$ which satisfy (56).

Since the proof of Theorem 4.7 is analogous for both invariant manifolds, we only deal with the unstable case and we omit the superscript u to simplify notation.

Splitting $Q(v,\tau) = q_0(v) + Q_1(v,\tau)$, equation (56) reads

$$q_0\left(u + \mathcal{V}(u,\tau)\right) - q_0(u) = -Q_1\left(u + \mathcal{V}(u,\tau),\tau\right).$$

Taking into account that $\dot{q}_0(u) = p_0(u)$, to obtain a solution of this equation is equivalent to obtain a fixed point of the operator

$$\mathcal{N}(h)(u,\tau) = -\frac{1}{p_0(u)}\left(Q_1\left(u + h(u,\tau),\tau\right) + q_0\left(u + h(u,\tau)\right) - q_0(u) - p_0(u)h(u,\tau)\right). \tag{223}$$

Let us consider the function space

$$\mathcal{Q}_{\kappa,d,\sigma} = \left\{h : I^{\mathrm{out},u}_{\kappa,d} \times \mathbb{T}_\sigma \to \mathbb{C};\ \text{real-analytic}, \|h\|_{\kappa,d,\sigma} < \infty\right\}, \tag{224}$$

where $\|\cdot\|_{\kappa,d,\sigma}$ is the Fourier norm defined in (209) but applied to functions defined in $I^{\mathrm{out},u}_{\kappa,d} \times \mathbb{T}_\sigma$.

We split Theorem 4.7 in the following proposition and corollary, which are written in terms of the Banach space defined in (224).

Proposition 7.21. *Let us consider the constant κ_1 given in Proposition 7.17, $d_0 > d_1 > 0$, $\kappa_2 > \kappa_1$ and $\varepsilon_0 > 0$ small enough, which might depend on the previous constants. Then, there exists a constant $b_6 > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$ and κ_1 and κ_2 big enough, the operator $\mathcal{N}$ is contractive from $\overline{B}\left(b_6|\mu|\varepsilon^{\eta+1}\right) \subset \mathcal{Q}_\sigma$ to itself.*

Then, $\mathcal{N}$ has a unique fixed point $\mathcal{V} \in \overline{B}\left(b_6|\mu|\varepsilon^{\eta+1}\right) \subset \mathcal{Q}_\sigma$, which satisfies that $u + \mathcal{V}(u,\tau) \in I^{\mathrm{out},u}_{\kappa_1,d_0}$ for $(u,\tau) \in I^{\mathrm{out},u}_{\kappa_2,d_1} \times \mathbb{T}_\sigma$.

Corollary 7.22. *There exists a function $T : I^{\mathrm{out},u}_{\kappa_2,d_1} \times \mathbb{T}_\sigma \to \mathbb{C}$ such that*

$$\partial_u T(u,\tau) = p_0(u) P(u + \mathcal{V}(u,\tau),\tau),$$

where P and $\mathcal{V}$ are the functions obtained in Theorem 4.6 and Proposition 7.21 respectively, and satisfies equation (41). Moreover, it belongs to $\mathcal{Q}_\sigma$ and satisfies

$$\|\partial_u T - \partial_u T_0\|_{\kappa_2,d_1,\sigma} \leq b_7|\mu|\varepsilon^{\eta+1}.$$

for certain constant $b_7 > 0$.

We devote the rest of this section to prove Proposition 7.21 and Corollary 7.22.

Proof of Proposition 7.21. The operator $\mathcal{N}$ sends $\mathcal{Q}_{\kappa_2,d_1,\sigma}$ to itself. To see that exists a constant $b_6 > 0$ such that $\mathcal{N}$ is contractive in $\overline{B}\left(b_6|\mu|\varepsilon^{\eta+1}\right) \subset \mathcal{Q}_{\kappa_2,d_1,\sigma}$, we first consider $\mathcal{N}(0)$. By Proposition 7.17, there exists a constant $b_6 > 0$ such that

$$\|\mathcal{N}(0)\|_{\kappa_2,d_1,\sigma} = \left\|p_0^{-1}(v)Q_1(v,\tau)\right\|_{\kappa_2,d_1,\sigma} \leq \frac{b_6}{2}|\mu|\varepsilon^{\eta+1}.$$

To see that $\mathcal{N}$ is contractive, let us consider $h_1, h_2 \in \overline{B}\left(b_6|\mu|\varepsilon^{\eta+1}\right) \subset \mathcal{Q}_{\kappa_2,d_1,\sigma}$. By Proposition 7.17, we know that $Q_1(u,\tau)$ is defined in $I^{\mathrm{out},u}_{\kappa_1,d_0}$ and satisfies $\|Q_1\|_{\kappa_1,d_0,\sigma} \leq K|\mu|\varepsilon^{\eta+1}$ in this domain. Applying Cauchy estimates in the nested domains $I^{\mathrm{out},u}_{2\kappa_1,d_0/2} \subset I^{\mathrm{out},u}_{\kappa_1,d_0}$, one has that

$$\|\partial_v Q_1\|_{2\kappa_1,d_0/2,\sigma} \leq \frac{K}{\kappa_1}\mu\varepsilon^\eta.$$

Then, defining $h^s(v,\tau)=sh_2(v,\tau)+(1-s)h_1(v,\tau)$ for $s\in(0,1)$, using the mean value theorem, increasing κ_1 if necessary and taking $\kappa_2>2\kappa_1$,

$$\begin{aligned}\left\|\mathcal{N}(h_2)-\mathcal{N}(h_1)\right\|_{\kappa_2,d_1,\sigma}\leq&\left\|p_0^{-1}(v)\int_0^1\left(\partial_u Q_1\left(v+h^s,\tau\right)+p_0(v+h^s)-p_0(v)\right)ds\right\|_{\kappa_2,d_1,\sigma}\\&\cdot\left\|h_2-h_1\right\|_{\kappa_2,d_1,\sigma}\\\leq&\frac{K|\mu|\varepsilon^\eta}{\kappa_1}\left\|h_2-h_1\right\|_{\kappa_2,d_1,\sigma}\\\leq&\frac{1}{2}\left\|h_2-h_1\right\|_\sigma.\end{aligned}$$

Then, $\mathcal{N}:\overline{B}\left(b_6|\mu|\varepsilon^{\eta+1}\right)\to\overline{B}\left(b_6|\mu|\varepsilon^{\eta+1}\right)\subset\mathcal{Q}_{\kappa_2,d_1,\sigma}$ and is contractive. Therefore, it has a unique fixed point which satisfies the properties stated in Proposition 7.21. ☐

Proof of Corollary 7.22. Proposition 7.21, gives a parametrization of form

$$(q,p)=(Q(u+\mathcal{V}(u,\tau),\tau),P(u+\mathcal{V}(u,\tau),\tau))=(q_0(u),P(u+\mathcal{V}(u,\tau),\tau)).$$

We want to have a parametrization of the form (42), where T is a function which satisfies (41). To recover this function it is enough to point out that, since we want it to be solution of (41), we know its gradient

$$(\partial_u T(u,\tau),\partial_\tau T(u,\tau))=\left(p_0(u)P(u+\mathcal{V}(u,\tau),\tau),-\varepsilon\overline{H}(u,p_0(u)P(u+\mathcal{V}(u,\tau),\tau),\tau\right).$$

Then, it is enough to check the compatibility condition

$$\partial_\tau\left[p_0(u)P(u+\mathcal{V}(u,\tau),\tau)\right]=-\partial_u\left[\varepsilon\overline{H}\left(u,p_0(u)P(u+\mathcal{V}(u,\tau),\tau),\tau\right)\right].\tag{225}$$

Differentiating equation (56), one has that $\mathcal{V}$ satisfies

$$\begin{aligned}\partial_v Q(u+\mathcal{V}(u,\tau),\tau)\left(1+\partial_u\mathcal{V}(u,\tau)\right)&=p_0(u)\\\partial_v Q(u+\mathcal{V}(u,\tau),\tau)\partial_\tau\mathcal{V}(u,\tau)+\partial_\tau Q(u+\mathcal{V}(u,\tau),\tau)&=0\end{aligned}$$

Then, using this equalities and equation (44), one can prove (225).

Finally, recalling that $\partial_u T_0(u)=p_0^2(u)$ and $P(v,\tau)=p_0(v)+P_1(v,\tau)$ and applying Proposition 7.17 and the mean value theorem,

$$\begin{aligned}\left\|\partial_u T-\partial_u T_0\right\|_{\kappa_2,d_1,\sigma}&\leq\left\|p_0(u)\left(P_1(u+\mathcal{V}(u,\tau),\tau)+p_0(u+\mathcal{V}(u,\tau))-p_0(u)\right)\right\|_\sigma\\&\leq b_7|\mu|\varepsilon^{\eta+1}.\end{aligned}$$

☐

7.2.5 Proof of Theorem 4.8

The proof of Theorem 4.8 follows the same steps as the proof of Theorem 4.4. For this reason, in this section we only explain which are the main differences.

First, let us point out that the operator $\mathcal{G}_\varepsilon$ defined in (168) can be also applied to functions defined in $D^u_{\kappa_3,d_2}\times\mathbb{T}_\sigma$ if one takes as $u_1,\bar{u}_1$ the vertices of $D^u_{\kappa_3,d_2}$ (see Figure 2) and as ρ the left endpoint of the interval $D^u_{\kappa_3,d_2}\cap\mathbb{R}$. Now the paths of integration cannot be straight lines. Nevertheless, it is easy to see that $\mathcal{G}_\varepsilon$ satisfies the same properties as the ones stated in Lemma 7.3 but applied to functions defined in the new domain.

Then, if one considers Banach spaces analogous to $\mathcal{E}_{\nu,\sigma}$, with $\nu>0$, given in (167), for functions defined in $D^u_{\kappa_3,d_2}\times\mathbb{T}_\sigma$, one can prove the Proposition 7.4, but looking for the function T_1 as the analytic continuation of the function obtained Corollary 7.22 instead of the function T_1 obtained in Proposition 6.4 and Proposition 6.10.

The rest of the prove follows the same lines as the proof of Proposition 7.4.

7.3 The first asymptotic term of the invariant manifolds near the singularities for the case $\ell = 2r$

In the case $\eta = 0$ and $\ell - 2r = 0$, we need a better knowledge of the first asymptotic terms of the invariant manifolds close the singularities of the unperturbed separatrix $u = \pm ia$. In the next result, we obtain them for the unstable invariant manifold close to $u = ia$. The other cases can be done analogously.

This proposition will be used later in Section 9.

For real analytic functions $h : D^u_{\kappa_3,d_2} \times \mathbb{T}_\sigma \to \mathbb{C}$, 2π- periodic in τ we define the Fourier norm

$$\|h\|_{\nu,\sigma} = \sum_{k\in\mathbb{Z}} \sup_{(u,\tau)\in D^u_{\kappa_3,d_2}\times\mathbb{T}_\sigma} |(u^2+a^2)^\nu h^{[k]}(u)|e^{|k|\sigma}$$

being, as usual, $h^{[k]}$ the k-Fourier coefficient of h.

Proposition 7.23. *Let us assume $\ell - 2r = 0$, and let us consider the functions Q_j and F_j defined in* (73) *and* (74) *respectively (see also Remark 4.14) and the constant C_+ given in* (12) *and* (13).

Then, there exists a real analytic function $\xi : D^u_{\kappa_3,d_2} \times \mathbb{T}_\sigma \to \mathbb{C}$, 2π- periodic in τ satisfying that:

$$\|\xi\|_{2r+1-1/q,\sigma} \leq K|\mu|\varepsilon^{\eta+1},$$

where $r = \alpha/\beta$ has been defined in Hypothesis ***HP2*** *and, for $(u,\tau) \in D^u_{\kappa_3,c_2} \times \mathbb{T}_\sigma$, the functions T^u obtained respectively in Proposition 7.4 (case $\alpha_0(u) \neq 0$) and Proposition 7.17 (general case), are such that*

$$\left\| \partial_u T_1(u,\tau) - \frac{2r\mu\varepsilon^{\eta+1}C_+^2}{(u-ia)^{2r+1}} \left(F_0(\tau) + \mu\langle Q_0F_1\rangle\right) + \xi(u,\tau) \right\|_{2r+2,\sigma} \leq K|\mu|\varepsilon^{\eta+2}. \tag{226}$$

Proof. We prove Proposition 7.23 in the polynomial case. Taking into account Remark 4.14, the proof of the trigonometric case is completely analogous.

We only deal with the case $p_0(u) \neq 0$ being the other case analogous. For that reason we will only take into account the previous results in this case. In fact we will see that Proposition 7.23 is also valid for $(u,\tau) \in D^{out,u}_{\rho_1',\kappa_0'}$ where ρ_1' and κ_0' are the constants for which Proposition 7.4 holds.

We first obtain the asymptotic expansion for the function $\partial_v \widehat{T}_1(v,\tau)$ obtained in Proposition 7.4, which is defined for $(v,\tau) \in D^{\text{out},u}_{\rho_1',\kappa_0'} \times \mathbb{T}_\sigma$ and then we use the change variables $v = u + h(u,\tau)$ defined in Lemma 7.6.

To obtain the asymptotic expansion, we split $\partial_v \widehat{T}_1$ in several parts taking into account that $\partial_v \widehat{T}_1$ is fixed point a of the operator $\mathcal{J}$ in (180) and that we know explicitly $\mathcal{J}(0)$. We use the functions A_i defined in (152), (153), (154) respectively, the change of variables g obtained in Lemma 7.6 and the operator $\mathcal{J}$ in (180). We take

$$\partial_v \widehat{T}_1 = \sum_{i=1}^{7} D_i(v,\tau)$$

with

$$D_1(v,\tau) = A_0(v,\tau) \tag{227}$$
$$D_2(v,\tau) = \mathcal{G}_\varepsilon\left(\partial_v A_1(v+g(v,\tau),\tau)\right) \tag{228}$$
$$D_3(v,\tau) = \mathcal{G}_\varepsilon\left(\partial_v A_2(v,\tau)\right) \tag{229}$$
$$D_4(v,\tau) = \mathcal{G}_\varepsilon\left(\partial_v\left[\partial_v A_2(v,\tau)g(v,\tau)\right]\right) \tag{230}$$
$$D_5(v,\tau) = \mathcal{G}_\varepsilon\left(\partial_v\left[A_2(v+g(v,\tau),\tau) - \partial_v A_2(v,\tau)g(v,\tau) - A_2(v,\tau)\right]\right) \tag{231}$$
$$D_6(v,\tau) = \mathcal{G}_\varepsilon\left(\partial_v\left[A_3(v+g(v,\tau),\tau)\right]\right) \tag{232}$$
$$D_7(v,\tau) = \mathcal{J}\left(\partial_v \widehat{T}_1\right)(v,\tau) - \mathcal{J}(0)(v,\tau). \tag{233}$$

Let us point out that the sum of the first six terms is $\mathcal{J}(0)$. We bound each term. For the second to fifth terms, we follow the proof of Lemma 7.5, where the functions A_1, A_2 and A_3 have been bounded.

To bound (227), it is enough to recall that, by (179), $D_1 \in \mathcal{E}_{0,\rho_2,\kappa_1\varepsilon,\sigma} \subset \mathcal{E}_{2r+1-1/\beta,\rho_2,\kappa_1,\sigma}$, to obtain

$$\|D_1\|_{2r+1-\frac{1}{\beta},\sigma} \leq \|D_1\|_{0,\sigma} \leq K|\mu|\varepsilon^{\eta+1}.$$

To bound (228), we apply the bound of A_1 obtained in (172) and use $r \geq 1$ to see that $D_2 \in \mathcal{E}_{r+1,\sigma} \subset \mathcal{E}_{2r+1-1/\beta,\sigma}$ and

$$\|D_2\|_{2r+1-\frac{1}{\beta},\sigma} \leq \|D_2\|_{r+1,\sigma} \leq K|\mu|\varepsilon^{\eta+1}.$$

Since $\langle A_2 \rangle = 0$, we can define a function $\overline{A}_2$ such that $\partial_\tau \overline{A}_2 = A_2$ and $\langle \overline{A}_2 \rangle = 0$. Moreover, one can write

$$\begin{aligned} D_3 &= \mathcal{G}_\varepsilon(\partial_v A_2) = \mathcal{G}_\varepsilon\left(\partial^2_{\tau v}\overline{A}_2\right) \\ &= \varepsilon\mathcal{G}_\varepsilon\left(\mathcal{L}_\varepsilon\left(\partial_v\overline{A}_2\right)\right) - \varepsilon\mathcal{G}_\varepsilon\left(\partial_v^2\overline{A}_2\right). \end{aligned}$$

Then, using the definition of $\mathcal{G}_\varepsilon$ in (168) and applying Lemma 7.3, one can see that there exists a function $\widetilde{\xi}_3 \in \mathcal{E}_{0,\sigma} \subset \mathcal{E}_{2r+1-1/\beta,\sigma}$, which satisfies,

$$\|\widetilde{\xi}_3\|_{2r+1-\frac{1}{\beta},\sigma} \leq K\|\widetilde{\xi}_3\|_{0,\sigma} \leq K|\mu|\varepsilon^{\eta+1},$$

such that

$$\left\|D_3 - \varepsilon\partial_v\overline{A}_2 - \widetilde{\xi}_3\right\|_{2r+2,\sigma} \leq K|\mu|\varepsilon^{\eta+2}.$$

Moreover, recalling the definition of A_2 in (153) and defining functions $\overline{a}_{kl}$ such that

$$\partial_\tau\overline{a}_{kl} = 0 \quad \text{and} \quad \langle\overline{a}_{kl}\rangle = 0 \tag{234}$$

we have that

$$\partial_v\overline{A}_2(v,\tau) = -\mu\sum_{2\leq k+l\leq N}\overline{a}_{kl}(\tau)\partial_v\left(q_0(v)^k p_0(v)^l\right).$$

Then, recalling the definition of the functions Q_j and F_j in (73) and (74) and the constant C_+ in (12), $\partial_v\overline{A}_2$ satisfies

$$\varepsilon\partial_v\overline{A}_2(v,\tau) = \frac{2r\mu\varepsilon^{\eta+1}C_+^2F_0(\tau)}{(v-ia)^{2r+1}} + \mathcal{O}\left(\frac{\mu\varepsilon^{\eta+1}}{(v-ia)^{2r+1-\frac{1}{\beta}}}\right).$$

Therefore, there exists $\xi_3 \in \mathcal{E}_{2r+1-1/\beta,\rho_2,\kappa_1\varepsilon,\sigma}$ satisfying

$$\|\xi_3\|_{2r+1-\frac{1}{\beta},\sigma} \leq K|\mu|\varepsilon^{\eta+1},$$

such that

$$\left\|D_3(v,\tau) - \frac{2r\mu\varepsilon^{\eta+1}C_+^2F_0(\tau)}{(v-ia)^{2r+1}} - \xi_3(v,\tau)\right\|_{2r+2,\sigma} \leq K|\mu|\varepsilon^{\eta+2}.$$

To bound (230), we first subtract its averaged term. Then, using Lemma 7.6 to bound g and $\partial_v g$, Lemma 7.18 to bound the first and second derivatives of A_2 and Lemma 7.3, we obtain

$$\|D_4 - \mathcal{G}_\varepsilon\left(\partial_v\langle\partial_v A_2\cdot g\rangle\right)\|_{2r+2,\sigma} \leq K|\mu|^2\varepsilon^{\eta+2}.$$

On the other hand, using the definition of $\mathcal{G}_\varepsilon$ in (168)

$$\mathcal{G}_\varepsilon\left(\partial_v\langle\partial_v A_2\cdot g\rangle\right)(v) = \langle\partial_v A_2\cdot g\rangle(v) - \langle\partial_v A_2\cdot g\rangle(-\rho_1').$$

To obtain its leading term, first we look for the first order of the function g given in (173). Using the definition of B_1 in (144), the functions (234), the bounds of $\partial_v B_1$ in (151) and Lemma 7.3, we have that

$$\left\|g(v,\tau) - \mu\varepsilon^{\eta+1}\sum_{\substack{2\leq k+l\leq N\\ l\geq 1}} l\overline{a}_{kl}(\tau)q_0(v)^k p_0(v)^{l-2}\right\|_{1,\sigma} \leq K|\mu|\varepsilon^{\eta+2}. \tag{235}$$

Then, using the functions Q_j and F_j defined in (73) and (74) respectively, and taking into account the definition of A_2 in (153), there exists a function $\xi_4 \in \mathcal{E}_{2r+1-1/\beta,\rho_2,\kappa_1\varepsilon,\sigma}$ satisfying

$$\|\xi_4\|_{2r+1-\frac{1}{\beta},\sigma} \leq K|\mu|\varepsilon^{\eta+1},$$

such that

$$\mathcal{G}_\varepsilon\left(\partial_v\langle\partial_v A_2 \cdot g\rangle\right) = \frac{2r\mu^2\varepsilon^{2\eta+1}C_+^2\langle Q_0F_1\rangle}{(v-ia)^{2r+1}} + \xi_4(u,\tau).$$

Therefore, one can see that

$$\left\|D_4(v,\tau) - \frac{2r\mu^2\varepsilon^{2\eta+1}C_+^2\langle Q_0F_1\rangle}{(v-ia)^{2r+1}} - \xi_4(u,\tau)\right\|_{2r+2,\sigma} \leq K|\mu|^2\varepsilon^{2\eta+2}.$$

For (231), it is enough to apply Lemmas 7.3 and 7.6, the definition of A_2 and the mean value theorem, to obtain

$$\|D_5\|_{2r+2,\sigma} \leq K|\mu|^3\varepsilon^{3\eta+2}.$$

To bound (232), let us recall the definitions of A_3 and $\widehat{H}_1^2$ in (154) and (37). Then, it is enough to apply Lemma 7.3, to obtain

$$\|D_6\|_{2r+1-\frac{1}{\beta},\sigma} \leq \|D_6\|_{2r,\sigma} \leq K|\mu|\varepsilon^{\eta+1}.$$

Finally, for (233), it is enough to take into account the definitions of $\mathcal{J}$ and $\widehat{\mathcal{F}}$ in (180) and (158) and apply Lemmas 7.3, 7.5 and 7.8, which give,

$$\begin{aligned}
\left\|\mathcal{J}\left(\partial_v\widehat{T}_1\right) - \mathcal{J}(0)\right\|_{2r+2,\sigma} &\leq \left\|\widehat{\mathcal{F}}\left(\partial_v\widehat{T}_1\right) - \widehat{\mathcal{F}}(0)\right\|_{2r+2,\sigma} \\
&\leq \left\|\widehat{B}\cdot\partial_v\widehat{T}_1 + \widehat{C}\left(\partial_v\widehat{T}_1, v,\tau\right) - \widehat{C}(0,v,\tau)\right\|_{2r+2,\sigma} \\
&\leq K|\mu|\varepsilon^{\eta+1}\left\|\partial_v\widehat{T}_1\right\|_{2r+1,\sigma} \leq K|\mu|^2\varepsilon^{2\eta+2}.
\end{aligned}$$

Considering all the bounds of D_i, we define

$$\xi(u,\tau) = D_1(u,\tau) + D_2(u,\tau) + \xi_3(u,\tau) + \xi_4(u,\tau) + D_6(u,\tau)$$

Then, $\xi \in \mathcal{E}_{2r+1-1/\beta,\sigma}$ satisfying

$$\|\xi\|_{2r+1-\frac{1}{\beta},\sigma} \leq K|\mu|\varepsilon^{\eta+1},$$

and then we have

$$\left\|\partial_v\widehat{T}_1(v,\tau) - \frac{2r\mu\varepsilon^{\eta+1}C_+^2}{(v-ia)^{2r+1}}\left(F_0(\tau) + \mu\langle Q_0F_1\rangle\right) - \xi(u,\tau)\right\|_{2r+2,\sigma} \leq K|\mu|\varepsilon^{\eta+2}. \tag{236}$$

To finish the proof of Proposition 7.23, one has to consider the change of variables $v = u + h(u,\tau)$ defined in Lemma 7.6 to obtain

$$\partial_u T_1(u,\tau) = (1+\partial_u h(u,\tau))^{-1}\partial_v\widehat{T}_1(u+h(u,\tau),\tau).$$

Then, the bounds of h and $\partial_u h$ in Lemma 7.6 and (236), finish the proof of the proposition. □

8 Approximation of the invariant manifolds in the inner domains.

8.1 Case $\ell < 2r$: proof of Proposition 4.10

We prove the results stated in Proposition 4.10 concerning the unstable manifold. The proof of the results concerning the stable one follows the same lines. To obtain the bound of $\partial_u T_1^u(u,\tau) - \partial_u \mathcal{T}_0^u(u,\tau)$, we first bound $\partial_v \widehat{T}_1^u(v,\tau) - \partial_v \mathcal{T}_0^u(v,\tau)$ where $\widehat{T}_1^u$ is the function obtained in Theorems 4.4 and 4.8, which is defined for $(v,\tau) \in D^u_{\kappa_3,d_2} \times \mathbb{T}_\sigma$, and $\mathcal{T}_0^u$ is the function defined in (57). Then, we will use the change of variables $v = u + h(u,\tau)$ defined in Lemma 7.6 to obtain the bound stated in Proposition 4.10.

Let us define first v_3 and v_4 the leftmost and rightmost vertices of the inner domain $D^{\mathrm{in},+,u}_{\kappa_3,c_1}$ (see Figure 5). Then, we can define the operator

$$\widetilde{\mathcal{G}}_\varepsilon(h)(v,\tau) = \sum_{k\in\mathbb{Z}} \widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) e^{ik\tau}, \tag{237}$$

where its Fourier coefficients are given by

$$\begin{aligned}
\widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) &= \int_{v_3}^{v} e^{ik\varepsilon^{-1}(t-v)} h^{[k]}(t)\, dt && \text{for } k>0\\
\widetilde{\mathcal{G}}_\varepsilon(h)^{[0]}(v) &= \int_{v_4}^{v} h^{[0]}(t)\, dt &&\\
\widetilde{\mathcal{G}}_\varepsilon(h)^{[k]}(v) &= \int_{v_4}^{v} e^{ik\varepsilon^{-1}(t-v)} h^{[k]}(t)\, dt && \text{for } k<0.
\end{aligned}$$

It can be easily seen that this operator satisfies analogous properties to the ones satisfied by the operator $\mathcal{G}_\varepsilon$ defined in (168), which are given in Lemma 7.3. Let us consider also the Fourier expansions

$$h_1(v,\tau) = H_1(q_0(v), p_0(v), \tau) = \sum_{k\in\mathbb{Z}} H_1^{[k]}(v) e^{ik\tau} \quad \text{and} \quad \widehat{A}(v,\tau) = A(v+g(v,\tau),\tau) = \sum_{k\in\mathbb{Z}} \widehat{A}^{[k]}(v) e^{ik\tau}$$

where H_1 is the function defined in (8) and (9), A is the function defined in (143) and g has been given in Lemma 7.6.

First, we observe that, since $\partial_v \widehat{T}_1 = \mathcal{J}(\partial_v \widehat{T}_1)$ where the operator $\mathcal{J}$ is defined in (180),

$$\partial_v \widehat{T}_1(v,\tau) = \widetilde{\mathcal{G}}_\varepsilon(\partial_v A)(v,\tau) + \sum_{i=1}^{4} N_i(v,\tau)$$

with:

$$N_1(v,\tau) = A_0(v,\tau) \tag{238}$$

$$N_2(v,\tau) = \mathcal{J}\left(\partial_v \widehat{T}_1\right)(v,\tau) - \mathcal{J}(0)(v,\tau). \tag{239}$$

$$N_3(v,\tau) = -\widetilde{\mathcal{G}}_\varepsilon(\partial_v \widehat{A})(v,\tau) + \mathcal{G}_\varepsilon(\partial_v \widehat{A})(v,\tau) \tag{240}$$

$$N_4(v,\tau) = \widetilde{\mathcal{G}}_\varepsilon(\partial_v \widehat{A})(v,\tau) - \widetilde{\mathcal{G}}_\varepsilon(\partial_v A)(v,\tau). \tag{241}$$

Second we split $\partial_v \mathcal{T}_0^u$ as:

$$\partial_v \mathcal{T}_0^u = -\mu\varepsilon^\eta \widetilde{\mathcal{G}}_\varepsilon(\partial_v h_1)(v,\tau) - N_5$$

where

$$\begin{aligned}
N_5(v,\tau) \quad = \quad & \mu\varepsilon^\eta \sum_{k>0} \int_{-\infty}^{v_3} e^{ik\varepsilon^{-1}(t-v)} \partial_v H_1^{[k]}(t) dt\\
& \mu\varepsilon^\eta \sum_{k\le 0} \int_{-\infty}^{v_4} e^{ik\varepsilon^{-1}(t-v)} \partial_v H_1^{[k]}(t) dt.
\end{aligned} \tag{242}$$

Finally, we use the definition of A in (143) and $\widehat{H}_1$ in (34), and the fact that, as the periodic orbit does not depend on v,

$$\partial_v \left(V(x_p(\tau)) + H_1(x_p(\tau), y_p(\tau), \tau) \right) = 0$$

to obtain

$$\widetilde{\mathcal{G}}_\varepsilon(\partial_v A)(v,\tau)+\mu\varepsilon^\eta\widetilde{\mathcal{G}}_\varepsilon(\partial_v h_1)(v,\tau) = -y_p(\tau)p_0(u)+x_p(\tau)\dot{p}_0(u) \tag{243}$$

$$+N_6+N_7+N_8 \tag{244}$$

with

$$N_6 = -\mu\varepsilon^\eta\widetilde{\mathcal{G}}_\varepsilon\partial_v\Big(H_1(q_0(v)+x_p(\tau),p_0(v)+y_p(\tau),\tau)-H_1(q_0(v),p_0(v),\tau)\Big) \tag{245}$$

$$N_7 = -\widetilde{\mathcal{G}}_\varepsilon\partial_v\Big(V(q_0(u)+x_p(\tau))-V(q_0(u))-V'(q_0(u))x_p(\tau)\Big). \tag{246}$$

$$\begin{aligned} N_8 &= \widetilde{\mathcal{G}}_\varepsilon\partial_v\Big(-V'(q_0(u))x_p(\tau)+V'(x_p(\tau))q_0(u) \\ &+ \mu\varepsilon^\eta\left(q_0(u)\partial_x H_1(x_p(\tau),y_p(\tau),\tau)+p_0(u)\partial_y H_1(x_p(\tau),y_p(\tau),\tau)\right)\Big) \\ &+ y_p(\tau)p_0(u)-x_p(\tau)\dot{p}_0(u). \end{aligned} \tag{247}$$

Finally we obtain:

$$\partial_v\widehat{T}_1(v,\tau)-\partial_v\mathcal{T}_0^u = -y_p(\tau)p_0(u)+x_p(\tau)\dot{p}_0(u)+\sum_{i=1}^{8}N_i(v,\tau).$$

Now, we proceed to bound $N_1,\dots,N_8$.

To bound N_1 in (238), it is enough to recall that, by (179), $N_1\in\mathcal{E}_{0,\rho_1',\kappa_0',\sigma}$ and

$$\|N_1\|_{0,\sigma}\le K|\mu|\varepsilon^{\eta+1}.$$

For N_2 in (239), it is enough to consider the bound of $\partial_v\widehat{T}_1$ given in Proposition 7.4 and the Lipschitz constant of the operator $\mathcal{J}$ in (180) restricted to the ball $\overline{B}(|\mu|\varepsilon^{\eta+1})\subset\mathcal{E}_{\ell+1,\rho_1',\kappa_0',\sigma}$, which has been obtained in the proof of Lemma 7.8. Then,

$$\begin{aligned}\|N_2\|_{0,\sigma} &\le K\frac{\varepsilon^{-(\ell+1)}}{(\kappa_0')^{\ell+1}}\|N_2\|_{\ell+1,\sigma}\\ &\le K|\mu|\varepsilon^{-(\ell+1)+\eta+1-\max\{0,\ell-2r+1\}}\left\|\partial_v\widehat{T}_1\right\|_{\ell+1,\sigma}\\ &\le K|\mu|^2\varepsilon^{2\eta-\ell+1-\max\{0,\ell-2r+1\}}.\end{aligned}$$

To bound N_3 in (240) we observe that $\langle N_3\rangle=0$ and

$$\begin{aligned} N_3^{[k]}(v) &= e^{ik\varepsilon^{-1}(v_3-v)}\int_{u_1}^{v_3}e^{ik\varepsilon^{-1}(t-v_3)}\left(\partial_v\widehat{A}^{[k]}\right)(t)dt \qquad \text{for}\quad k>0\\ N_3^{[0]}(v) &= \widehat{A}^{[0]}(v)-\widehat{A}^{[0]}(v_4)\\ N_3^{[k]}(v) &= e^{ik\varepsilon^{-1}(v_4-v)}\int_{\bar{u}_1}^{v_4}e^{ik\varepsilon^{-1}(t-v_4)}\left(\partial_v\widehat{A}^{[k]}\right)(t)dt \qquad \text{for}\quad k<0\end{aligned}$$

Taking into account that the operator $\widetilde{\mathcal{G}}_\varepsilon$ satifies also the properties of the operator $\mathcal{G}_\varepsilon$ given in Lemma 7.3, and using the bounds of g and $\partial_v A$ given in Lemmas 7.6 and 7.5 respectively, we obtain the following bounds. For $k\neq 0$,

$$\begin{aligned}\left\|N_3^{[k]}\right\|_{0,\sigma} &\le \left\|\widetilde{\mathcal{G}}_\varepsilon\left(\partial_v\widehat{A}^{[k]}(v)e^{ik\tau}\right)\right\|_{0,\sigma}\\ &\le K\varepsilon\left\|\partial_v\widehat{A}^{[k]}(v)e^{ik\tau}\right\|_{0,\sigma}\\ &\le K\varepsilon^{1-(\ell+1)\gamma}\left\|\partial_v\widehat{A}^{[k]}(v)e^{ik\tau}\right\|_{\ell+1,\sigma}\\ &\le K|\mu|\varepsilon^{\eta+1-(\ell+1)\gamma}.\end{aligned}$$

For $k=0$, we have that

$$\begin{aligned}\|N_3^{[0]}\|_{0,\sigma} &\leq K\left\|\widehat{A}^{[0]}\right\|_{0,\sigma}\\ &\leq K\varepsilon^{-\ell\gamma}\left\|\widehat{A}^{[0]}\right\|_{\ell,\sigma}\leq K|\mu|\varepsilon^{\eta-\ell\gamma}\end{aligned}$$

Finally, note that in the case $\ell=0$, we have that the change g obtained in Lemma 7.6 satisfies $g=0$. Then $\widehat{A}=A$, that implies $\langle\widehat{A}\rangle=0$. Therefore when $\ell=0$ we have that $N_3^{[0]}=0$. Taking this fact into account, we can bound N_3 by

$$\|N_3\|_{0,\sigma}\leq K|\mu|\varepsilon^{\eta-\ell+\nu_2^*}$$

where

$$\nu_2^*=\begin{cases}\ell(1-\gamma) & \text{if } \ell>0\\ 1-\gamma & \text{if } \ell=0\end{cases}.$$

For N_4 in (241), one has to consider the bound of $\partial_v A$ given in Lemma 7.5 and the bound of g restricted to the inner domain given in Corollary 7.7. Then, using again the bounds analogous to the ones given in Lemma 7.3, but to the operator $\widetilde{\mathcal{G}}_\varepsilon$

$$\|N_4\|_{0,\sigma}\leq K\|\widehat{A}-A\|_{0,\sigma}\leq K\|\partial_v A\|_{0,\sigma}\|g\|_{0,\sigma}\leq K|\mu|^2\varepsilon^{2\eta-\ell+\nu_1^*}$$

with ν_1^* is defined in Corollary 7.7.

For N_5 in (242), it is enough to take into account that $\langle h_1\rangle=0$, that h_1 has a ramified point of order ℓ at $u=ia$ and that both v_3 and v_4 satisfy $|v_i-ia|=\mathcal{O}(\varepsilon^\gamma)$, $i=3,4$. Then, bounding the integrals as in Lemma 6.2 and 6.8, one has that

$$\|N_5\|_{0,\sigma}\leq K|\mu|\varepsilon^{\eta+1}\|\partial_v h_1\|_{0,\sigma}\leq K|\mu|\varepsilon^{\eta+1-\gamma(\ell+1)}.$$

To bound N_6 in (245) we fist use the mean value theorem to obtain

$$\|H_1(q_0(v)+x_p(\tau),p_0(v)+y_p(\tau),\tau)-H_1(q_0(v),p_0(v),\tau)\|_{0,\sigma}\leq|\mu|\varepsilon^{\eta-\ell+r}.$$

Then, using that $\widetilde{\mathcal{G}}_\varepsilon$ has similar properties to the ones given in Lemma 7.3 for the operator $\mathcal{G}_\varepsilon$ we obtain

$$\|N_6\|_{0,\sigma}\leq K|\mu|^2\varepsilon^{2\eta-\ell+r}.$$

The bound for N_7 in (246) comes from applying the mean bound theorem to the function

$$V(q_0(u)+x_p(\tau))-V(q_0(u))-V'(q_0(u))x_p(\tau)$$

and using that $V''(q_0(u))$ has a pole of second order, the bound of the periodic orbit and the properties of $\widetilde{\mathcal{G}}_\varepsilon$. Then, we obtain

$$\|N_7\|_{0,\sigma}\leq K\|V(q_0(u)+x_p(\tau))-V(q_0(u))-V'(q_0(u))x_p(\tau)\|_{0,\sigma}\leq K|\mu|^2\varepsilon^{2\eta}=K|\mu|^2\varepsilon^{(\eta-\ell)+(\eta+\ell)}.$$

To bound N_8 in (247), we write it as

$$N_8=\widetilde{\mathcal{G}}_\varepsilon\partial_v\left(N_8^0\right)+y_p(\tau)p_0(u)-x_p(\tau)\dot{p}_0(u)$$

with

$$\begin{aligned}N_8^0(v,\tau)=&-V'(q_0(u))x_p(\tau)+V'(x_p(\tau))q_0(u)\\ &+\mu\varepsilon^\eta\left(q_0(u)\partial_x H_1(x_p(\tau),y_p(\tau),\tau)+p_0(u)\partial_y H_1(x_p(\tau),y_p(\tau),\tau)\right).\end{aligned}$$

Using that $-V'(q_0(u))=\dot{p}_0(u)$, $\dot{q}_0(u)=p_0(u)$ and that the periodic orbit satisfies equations (31), one has

$$\begin{aligned}N_8^0(v,\tau)=&\dot{p}_0(u)x_p(\tau)-\varepsilon^{-1}\partial_\tau y_p(\tau)q_0(u)-p_0(u)y_p(\tau)+\varepsilon^{-1}\partial_\tau x_p(\tau)p_0(u)\\ =&-\mathcal{L}_\varepsilon(y_p(\tau)q_0(u))+\mathcal{L}_\varepsilon(x_p(\tau)p_0(u))\end{aligned}$$

Therefore N_8 can be written as

$$\begin{aligned}N_8 =&\widetilde{\mathcal{G}}_\varepsilon\partial_v\mathcal{L}_\varepsilon\left(-y_p(\tau)q_0(u)+x_p(\tau)p_0(u)\right)\\&+y_p(\tau)p_0(u)-x_p(\tau)\dot{p}_0(u)\\=&\widetilde{\mathcal{G}}_\varepsilon\mathcal{L}_\varepsilon\left(-y_p(\tau)p_0(u)+x_p(\tau)\dot{p}_0(u)\right)\\&-\left(-y_p(\tau)p_0(u)+x_p(\tau)\dot{p}_0(u)\right).\end{aligned}$$

Then, using that $\widetilde{\mathcal{G}}_\varepsilon$ satisfies an analogous property to the one given for $\mathcal{G}_\varepsilon$ in the last item of Lemma 7.3:

$$\|N_8\|_{0,\sigma}\leq K|\mu|\varepsilon^{\eta+1-(r+1)\gamma}$$

Now, choosing γ such that

$$1-(r+1)\gamma>-\ell$$

that is,

$$\gamma<\frac{\ell+1}{r+1}$$

and considering all the bounds of N_i and taking

$$\nu^*=\min\left\{\nu_2^*,\nu_1^*,1-\max\{0,\ell-2r+1\},r,\ell,\ell+1-(r+1)\gamma\right\},$$

we obtain

$$\left\|\partial_v\widehat{T}_1(v,\tau)-\partial_v\mathcal{T}_0(v,\tau)\right\|_{0,\sigma}\leq K|\mu|\varepsilon^{\eta-\ell+\nu^*}.$$

To finish the proof of Proposition 4.10, it is enough to consider the change of variables $v=u+h(u,\tau)$ defined in Lemma 7.6 and its bounds restricted to the inner domains given in Corollary 7.7.

8.2 Case $\ell\geq 2r$: proof of Theorem 4.16

This section is devoted to obtain good approximations of the invariant manifolds in the *inner domains* defined in (30) for the case $\ell\geq 2r$.

First in Section 8.2.1 we define the Banach spaces that will be used in the forthcoming sections and we state some technical lemmas. In Section 8.2.2 we prove Theorem 4.16.

8.2.1 Banach spaces and technical lemmas

We start by defining some norms. Given $\nu\in\mathbb{R}$ and an analytic function $h:\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}\to\mathbb{C}$, where $\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}$ is the domain defined in (30), we consider

$$\|h\|_{\nu,\kappa,c}=\sup_{z\in\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}}\left|z^\nu h(z)\right|.$$

Then, for analytic functions $h:\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}\times\mathbb{T}_\sigma\to\mathbb{C}$ which are 2π-periodic in τ, we define the corresponding Fourier norm

$$\|h\|_{\nu,\kappa,c,\sigma}=\sum_{k\in\mathbb{Z}}\|h^{[k]}\|_{\nu,\kappa,c}e^{|k|\sigma}$$

and the function space

$$\mathcal{Z}_{\nu,\kappa,c,\sigma}=\left\{h:\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}\times\mathbb{T}_\sigma\to\mathbb{C};\ \text{analytic},\|h\|_{\nu,\kappa,c,\sigma}<\infty\right\}\tag{248}$$

which can be checked that is a Banach space for any $\nu\in\mathbb{R}$.

If there is no danger of confusion about the definition domain $\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}$ we will denote

$$\|\cdot\|_{\nu,\sigma}=\|\cdot\|_{\nu,\kappa,c,\sigma}\quad\text{and}\quad\mathcal{Z}_{\nu,\sigma}=\mathcal{Z}_{\nu,\kappa,c,\sigma}.$$

The next lemma gives some properties of these Banach spaces.

Lemma 8.1. *Let us consider $c, \kappa > 0$.*

1. *If $\nu_1 \le \nu_2$, $\mathcal{Z}_{\nu_2,\sigma} \subset \mathcal{Z}_{\nu_1,\sigma}$. Moreover,*
$$\|h\|_{\nu_2,\sigma} \le \frac{K}{\kappa^{\nu_2-\nu_1}}\|h\|_{\nu_1,\sigma}.$$
2. *If $h \in \mathcal{Z}_{\nu_1,\sigma}$ and $g \in \mathcal{Z}_{\nu_2,\sigma}$, the product $hg \in \mathcal{Z}_{\nu_1+\nu_2,\sigma}$ and*
$$\|hg\|_{\nu_1+\nu_2,\sigma} \le \|h\|_{\nu_1,\sigma}\|g\|_{\nu_2,\sigma}.$$
3. *Let us consider $h \in \mathcal{Z}_{\nu,\kappa,c,\sigma}$ and $\widehat{c} < c$, then, $\partial_x h \in \mathcal{X}_{\nu,2\kappa,\widehat{c},\sigma}$ and*
$$\|\partial_x h\|_{\nu,2\kappa,\widehat{c},\sigma} \le \frac{K}{\kappa}\|h\|_{\nu,\kappa,c,\sigma}.$$

Throughout this section we are going to solve equations of the form $\mathcal{L}h = g$ and $\mathcal{L}h = \partial_z g$, where
$$\mathcal{L} = \partial_z + \partial_\tau. \tag{249}$$
To solve these equations we consider operators $\mathcal{G}$ and $\overline{\mathcal{G}}$, which are defined "acting on the Fourier coefficients".

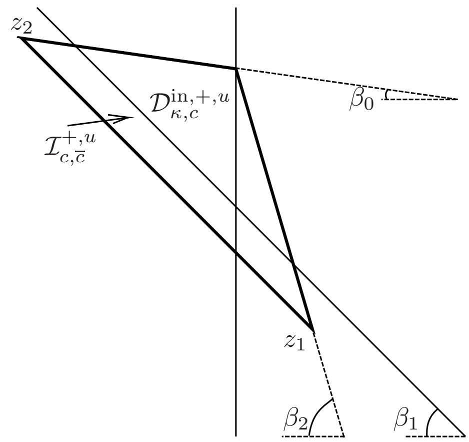

Figure 11: The *inner domain* $\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}$ defined in (64) and the transition domain $\mathcal{I}^{+,u}_{c,\overline{c}}$ defined in (253).

Let us consider z_1 and z_2 the vertices of the inner domain $\mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}$ (see Figure 11). As we have done in Section 7.2.2 to invert the operator $\mathcal{L}_\varepsilon = \varepsilon^{-1}\partial_\tau + \partial_v$, we invert $\mathcal{L}$ integrating from z_1 or z_2 depending on the harmonic.

We define the operators
$$\mathcal{G}(h)(z,\tau) = \sum_{k\in\mathbb{Z}} \mathcal{G}(h)^{[k]}(z)e^{ik\tau}, \tag{250}$$
where the Fourier coefficients are given by
$$\mathcal{G}(h)^{[k]}(z) = \int_{z_1}^{z} e^{-ik(z-s)}h^{[k]}(s)\,ds \qquad \text{for } k<0$$
$$\mathcal{G}(h)^{[k]}(z) = \int_{z_2}^{z} e^{-ik(z-s)}h^{[k]}(s)\,ds \qquad \text{for } k\ge 0$$

and

$$\overline{\mathcal{G}}(h)(z,\tau) = \sum_{k\in\mathbb{Z}} \overline{\mathcal{G}}(h)^{[k]}(z)e^{ik\tau}, \tag{251}$$

where its Fourier coefficients are given by

$$\begin{aligned}
&\overline{\mathcal{G}}(h)^{[k]}(z) = h^{[k]}(z) - e^{-ik(z-z_1)}h^{[k]}(z_1) - ik\int_{z_1}^{z} e^{-ik(z-s)}h^{[k]}(s)\,ds && \text{for } k<0\\
&\overline{\mathcal{G}}(h)^{[0]}(z) = h^{[0]}(z) - h^{[0]}(z_2)\\
&\overline{\mathcal{G}}(h)^{[k]}(z) = h^{[k]}(z) - e^{-ik(z-z_2)}h^{[k]}(z_2) - ik\int_{z_2}^{z} e^{-ik(z-s)}h^{[k]}(s)\,ds && \text{for } k>0.
\end{aligned}$$

The next lemma gives some properties of these operators. Its proof is analogous to the one of Lemma 5.5 in [GOS10].

Lemma 8.2. *Let us consider $\kappa, c, \nu > 0$ and $\gamma \in (0,1)$. Then,*

1. *The operator $\mathcal{G} : \mathcal{Z}_{\nu+1,\sigma} \to \mathcal{Z}_{\nu,\sigma}$ is well defined. Moreover, if $h \in \mathcal{Z}_{\nu+1,\sigma}$,*

$$\|\mathcal{G}(h)\|_{\nu,\sigma} \le K\|h\|_{\nu+1,\sigma}.$$

2. *The operator $\mathcal{G} : \mathcal{Z}_{\nu,\sigma} \to \mathcal{Z}_{\nu,\sigma}$ is well defined. Moreover, if $h \in \mathcal{Z}_{\nu,\sigma}$,*

$$\|\mathcal{G}(h)\|_{\nu,\sigma} \le K\varepsilon^{\gamma-1}\|h\|_{\nu,\sigma}.$$

3. *The operator $\overline{\mathcal{G}} : \mathcal{Z}_{\nu,\sigma} \to \mathcal{Z}_{\nu,\sigma}$ is well defined. Moreover, if $h \in \mathcal{Z}_{\nu,\sigma}$,*

$$\left\|\overline{\mathcal{G}}(h)\right\|_{\nu,\sigma} \le K\|h\|_{\nu,\sigma}.$$

8.2.2 Proof of Theorem 4.16

We rewrite Theorem 4.16 in terms of the Banach space (248).

Proposition 8.3. *Let us consider $\gamma \in (0,\gamma_2)$, where*

$$\gamma_2 = \frac{\beta(\ell-2r+1)}{\beta(\ell-2r+1)+1}, \tag{252}$$

$c_1 > 0$, $\varepsilon_0 > 0$ small enough and $\kappa_6 > \max\{\kappa_3,\kappa_5\}$ big enough, where κ_5 are the constants defined in Theorems 4.8 and 4.12 respectively. Let,

$$\varphi = \psi^u - \psi_0^u,$$

where ψ^u is the function in (61)and ψ_0 is the function obtained in Theorem 4.12. Then, for $\varepsilon \in (0,\varepsilon_0)$, we have $\varphi \in \mathcal{Z}_{2r-\frac{1}{\beta},\kappa_6,c_1,\sigma}$ and there exists a constant $b_{10} > 0$ such that

$$\|\partial_z\varphi\|_{2r-\frac{1}{\beta},\kappa_6,c_1,\sigma} \le b_{10}\varepsilon^{\frac{1}{\beta}},$$

where $r = \alpha/\beta$ has been defined in (12).

Remark 8.4. *We emphasize that Proposition 8.3 implies straightforwardly Theorem 4.16. Indeed, we observe that the only restriction is about the range of values of $\gamma \in (0,\gamma_2)$. Let us denote by D_γ^{in} the inner domain defined by γ. It is clear that, if $\gamma \ge \gamma_2 > \gamma_1$, then $D_\gamma^{\mathrm{in}} \subset D_{\gamma_1}^{\mathrm{in}}$ and henceforth the result holds also for values of $\gamma \ge \gamma_2$.*

We need to impose this condition about γ just for technical reasons.

In the proof of this proposition we will refer several times to the bounds given in Theorem 4.12. In fact, we need these bounds expressed in terms of the Fourier norm, which are given in Proposition 4.8 of [Bal06], instead of the ones given in this theorem, which use the classical supremmum norm.

Let us point out that using the bounds of Proposition 4.8 of [Bal06] and Corollary 7.22 leads to a bound of $\partial_z\varphi$ of order 1 with respect to ε. Nevertheless, this bound is too rough to prove later the asymptotic formula for the splitting of separatrices and therefore we will need the improved estimates given in Proposition 8.3.

The proof of Proposition 8.3 goes as follows. First in Section 8.2.2 we obtain a (non-homogeneous) linear partial differential equation satisfied by $\varphi = \psi - \psi_0$. Then, in Section 8.2.2, we obtain quantitative estimates of $\partial_z\varphi$ in the *transition domain* $\mathcal{I}^{+,u}_{c,\overline{c}}$ defined as

$$\mathcal{I}^{\pm,u}_{c,\overline{c}} = \left\{ z \in \mathbb{C}; ia + \varepsilon z \in D^{\mathrm{out},u}_{\rho_2,\overline{c}\varepsilon^\gamma} \cap D^{\mathrm{in},\pm,u}_{\kappa,c} \right\}, \tag{253}$$

where $* = u, s$ (see Figure 11), which allow us to obtain an integral equation satisfied by $\partial_z\varphi$. Finally, in Sections 8.2.2 and 8.2.2 we obtain the improved bound for $\partial_z\varphi$ for the cases $\ell - 2r > 0$ and $\ell - 2r = 0$ respectively, proving Proposition 8.3.

The Hamilton-Jacobi equation First we look for the equation satisfied by

$$\varphi = \psi - \psi_0. \tag{254}$$

Subtracting the Hamilton-Jacobi equations (62) and (65), one obtains

$$\partial_\tau\varphi + \mathcal{H}(\partial_z\psi_0 + \partial_z\varphi, z, \tau) - \mathcal{H}_0(\partial_z\psi_0, z, \tau) = 0.$$

Taking into account that we already know the existence of φ, we know that it is also solution of

$$\mathcal{L}\varphi = \mathcal{W}(\partial_z\varphi, z, \tau), \tag{255}$$

where $\mathcal{L}$ is the operator defined in (249) and

$$\mathcal{W}(w, z, \tau) = -L(z, \tau) - \left(Q_1(\tau)\frac{\hat{\mu}}{z^{\ell-2r}} + M(z, \tau) \right) w, \tag{256}$$

where Q_1 is the function defined in (73) and

$$L(z, \tau) = \mathcal{H}(\partial_z\psi_0, z, \tau) - \mathcal{H}_0(\partial_z\psi_0, z, \tau) \tag{257}$$

$$M(z, \tau) = \int_0^1 \partial_w\mathcal{H}\left(\partial_z\psi_0(z, \tau) + s\partial_z\varphi(z, \tau), z, \tau\right) ds - 1 - Q_1(\tau)\frac{\hat{\mu}}{z^{\ell-2r}}, \tag{258}$$

where $\mathcal{H}$ and $\mathcal{H}_0$ are the Hamiltonian defined in (63) and (68) respectively. Even if M depends on φ, since its existence is already known, M can be seen as a function depending on the variables z and τ, and then equation (255) can be seen as a linear equation. This fact simplifies considerably the obtention of the estimates for φ.

Let us point out that the term $\hat{\mu}Q_1(\tau)z^{-(\ell-2r)}$ in (256) behaves completely different in the cases $\ell - 2r > 0$ and $\ell - 2r = 0$, since in the first case is small for $z \in \mathcal{D}^{\mathrm{in},+,u}_{\kappa,c}$ and in the second is not. For this reason, we split the proof of Proposition 8.3 in these two cases.

Finally in this section, we state the following lemma, which gives some properties of the functions involved in equation (255).

Lemma 8.5. *Let us consider any $\kappa \geq \kappa_5$ and any $c > 0$. The functions L and M defined in* (257) *and* (258) *respectively, satisfy the following properties.*

1. *$L \in \mathcal{Z}_{2r-\frac{1}{\beta},\kappa,c,\sigma}$ and satisfies*

$$\|L\|_{2r-\frac{1}{\beta},\kappa,c,\sigma} \leq K\varepsilon^{\frac{1}{\beta}}.$$

2. $M \in \mathcal{Z}_{0,\kappa,c,\sigma}$ *and satisfies*

$$\|M\|_{0,\kappa,c,\sigma} \leq \frac{K}{\kappa^{\ell-2r+1}}.$$

Proof. We prove the lemma in the polynomial case. In the trigonometric case can be done analogously taking into account Remark and 4.14.

First we bound L. Using the definitions of $\mathcal{H}$, $\overline{H}$, $\widehat{H}$ and $\mathcal{H}_0$ in (63), (40), (33) and (68) respectively, we split it as $L = L_1 + L_2 + L_3 + L_4$ with

$$\begin{aligned}
L_1(z,\tau) =& \frac{1}{2}\left(\frac{C_+^2}{\varepsilon^{2r}p_0^2(ia+\varepsilon z)} - z^{2r}\right)(\partial_z\psi_0)^2 \\
L_2(z,\tau) =& \frac{\varepsilon^{2r}}{C_+^2}\left(V(q_0\left(ia+\varepsilon z\right) + x_p(\tau)) - V\left(x_p(\tau)\right) - V'\left(x_p(\tau)\right)q_0(ia+\varepsilon z)\right) \\
& - \frac{1}{2z^{2r}} \\
L_3(z,\tau) =& \frac{\hat{\mu}\varepsilon^{\ell}}{C_+^2}\widehat{H}_1^1\left(q_0(ia+\varepsilon z), C_+^2\varepsilon^{-2r}\partial_z\psi_0(z,\tau),\tau\right) \\
& - \frac{\hat{\mu}}{z^{\ell}}\sum_{(r-1)k+rl=\ell} a_{kl}(\tau)\frac{C_+^{k+l-2}}{(1-r)^k}\left(z^{2r}\partial_z\psi_0(z,\tau)\right)^l \\
L_4(z,\tau) =& \frac{\hat{\mu}\varepsilon^{\ell+1}}{C_+^2}\widehat{H}_1^2\left(q_0(ia+\varepsilon z), C_+^2\varepsilon^{-2r}\partial_z\psi_0(z,\tau)\right).
\end{aligned}$$

Taking into account the properties of $p_0(u)$ in (12) and Theorem 4.12, one can see that

$$\|L_1\|_{2r-\frac{1}{\beta},\kappa,c,\sigma} \leq K\varepsilon^{\frac{1}{\beta}}.$$

For L_2 one has to take into account that $V(q_0(u)) = -p_0^2(u)/2$, use (15) and the bound of $x_p(\tau)$ in Proposition 5.5. Then, one can obtain

$$\|L\|_{2r-\frac{1}{\beta},\kappa,c,\sigma} \leq K\varepsilon^{\frac{1}{\beta}}.$$

To bound the third term, using the definition of $\widehat{H}_1^1$ in (35) and also (12), one can rewrite it as

$$\begin{aligned}
L_3(z,\tau) =& \hat{\mu}\varepsilon^{\ell-(r-1)k-rl}\sum a_{kl}(\tau)\frac{C_+^{k+l-2}}{(1-r)^k}\left(\frac{1}{z^{r-1}} + \mathcal{O}\left(\frac{\varepsilon^{\frac{1}{\beta}}}{z^{r-1-\frac{1}{\beta}}}\right)\right)^k\left(z^r\partial_z\psi\right)^l \\
& - \frac{\hat{\mu}}{z^{\ell}}\sum_{(r-1)k+rl=\ell} a_{kl}(\tau)\frac{C_+^{k+l-2}}{(1-r)^k}\left(z^{2r}\partial_z\psi_0(z,\tau)\right)^l.
\end{aligned}$$

Then, it is easy to see that $L_3 \in \mathcal{Z}_{\ell-\frac{1}{\beta},\kappa,c,\sigma} \subset \mathcal{Z}_{2r-\frac{1}{\beta},\kappa,c,\sigma}$ and

$$\|L_3\|_{2r-\frac{1}{\beta},\kappa,c,\sigma} \leq K\|L_3\|_{\ell-\frac{1}{\beta},\kappa,c,\sigma} \leq K\varepsilon^{\frac{1}{\beta}}.$$

The bound of L_4 is straightforward.

For the bound of M, we split it as $M = M_1 + M_2 + M_3$ with

$$\begin{aligned}
M_1(z,\tau) &= \partial_w\mathcal{H}_0\left(\partial_z\psi_0, z,\tau\right) - Q_1(\tau)\frac{\hat{\mu}}{z^{\ell-2r}} - 1 \\
M_2(z,\tau) &= \int_0^1\left(\partial_w\mathcal{H}_0\left(\partial_z\psi_0 + s\partial_z\varphi, z,\tau\right) - \partial_w\mathcal{H}_0\left(\partial_z\psi_0, z,\tau\right)\right)ds \\
M_3(z,\tau) &= \int_0^1\left(\partial_w\mathcal{H}\left(\partial_z\psi_0 + s\partial_z\varphi, z,\tau\right) - \partial_w\mathcal{H}_0\left(\partial_z\psi_0 + s\partial_z\varphi, z,\tau\right)\right)ds
\end{aligned}$$

and we bound each term.

Taking into account the definitions of $\mathcal{H}_0$ and Q_j in (68) and (73) respectively, and the properties of ψ_0 given by Theorem 4.12, one can see that $M_1 \in \mathcal{Z}_{\ell-2r+1,\kappa,c,\sigma}$ and $\|M_1\|_{\ell-2r+1,\kappa,c,\sigma} \leq K$, which implies

$$\|M_1\|_{0,\kappa,c,\sigma} \leq \frac{K}{\kappa^{\ell-2r+1}}.$$

For the second term, let us recall that, using the definition of T_0 in (51), by Theorems 4.4 (see also Section 7.2.5) and 4.12, we have an *a priori* estimate for $\partial_z\varphi$,

$$\|\partial_z\varphi\|_{\ell+1,\kappa,c,\sigma} \leq K.$$

Then, it is enough to apply again the mean value theorem and the bounds of ψ_0 in Theorem 4.12 to obtain

$$\|M_2\|_{0,\kappa,c,\sigma} \leq \frac{K}{\kappa^{\ell-2r+1}}.$$

For M_3, it is enough to proceed as in the bound for L to obtain

$$\|M_3\|_{0,\kappa,c,\sigma} \leq K\varepsilon^{\frac{\gamma}{\beta}}.$$

□

The initial condition in the transition domains To obtain better estimates of $\partial_z\varphi$ we use an integral equation. To obtain it from (255) we need initial conditions. Therefore, we take constants $c_1 < c_0' < c_0$ an we look for them in the transition domains $\mathcal{I}^{+,u}_{c_0,c_0'} \times \mathbb{T}_\sigma$, defined in (253) (see also Figure 11). In this domain, the next lemma gives sharp estimates of the function $\partial_z\varphi$. We abuse notation and we use the norms defined in Section 7.2.4, even if here the suprema are taken in $\mathcal{I}^{+,u}_{c_0,c_0'}$.

Lemma 8.6. *Let us consider $\gamma \in (0,\gamma_2)$, where γ_2 is defined in* (252), *and $\varepsilon_0 > 0$ small enough. Then, for $\varepsilon \in (0,\varepsilon_0)$, the function $\partial_z\varphi$ restricted to $\mathcal{I}^{+,u}_{c_0,c_0'}$ satisfies*

$$\|\partial_z\varphi\|_{0,\sigma} \leq K\varepsilon^{2r(1-\gamma)+\frac{\gamma}{\beta}}.$$

Proof. Considering the functions $T = T_0 + T_1$, obtained in Proposition 7.4 (see also Section 7.2.5), and

$$\psi_0(z,\tau) = -\frac{1}{(2r-1)z^{2r-1}} + \hat{\mu}\overline{\psi}_0(z,\tau) + K,$$

obtained in Theorem 4.12, and recalling that $\partial_u T_0(u) = p_0^2(u)$, we split $\partial_z\varphi$ as

$$\begin{aligned}\partial_z\varphi(z,\tau) =&\partial_z\psi(z,\tau) - \partial_z\psi_0(z,\tau)\\ =&\varepsilon^{2r}C_+^2\Big(\partial_u T(\varepsilon z + ia,\tau) - \partial_u T_0(\varepsilon z + ia)\Big)\\ &+\left(\varepsilon^{2r}C_+^2 p_0^2(\varepsilon z + ia) - \frac{1}{z^{2r}}\right) - \hat{\mu}\partial_z\overline{\psi}_0(z,\tau).\end{aligned}$$

We bound each term. For the first term it is enough to apply the result obtained in Proposition 7.4 to obtain

$$\left\|\varepsilon^{2r}C_+^2\Big(\partial_u T(\varepsilon z + ia,\tau) - \partial_u T_0(\varepsilon z + ia)\Big)\right\|_{0,\sigma} \leq K\varepsilon^{(1-\gamma)(\ell+1)}.$$

Then, since $\gamma \in (0,\gamma_2)$, $(\ell+1)(1-\gamma) \geq 2r(1-\gamma) + \frac{\gamma}{\beta}$, we obtain the desired bound. For the second term we use (12). Finally, the bound of the third term is a direct consequence of Proposition 4.8 of [Bal06]. This proposition states the same results of Theorem 4.12 but bounds $\overline{\psi}_0(z,\tau)$ using Fourier norms instead of using classical supremmum norms. □

The fixed point equation for $\ell-2r>0$ In this section we prove Proposition 8.3 under the hypothesis $\ell-2r>0$. Let us define $\phi=\partial_z\varphi$, which, using (255), is solution of

$$(\mathcal{L}\phi)(z,\tau)=\partial_z\left[\mathcal{W}(\phi(z,\tau),z,\tau)\right], \tag{259}$$

where $\mathcal{L}=\partial_\tau+\partial_z$ and $\mathcal{W}$ is the operator defined in (256). We use this equation to obtain bounds for ϕ.

To invert the operator $\mathcal{L}=\partial_\tau+\partial_z$, we consider the operator $\overline{\mathcal{G}}$ defined in (251). Since the operator $\overline{\mathcal{G}}$ is defined acting on the Fourier harmonics, we impose a different initial condition for each. Recall that for the negative harmonics we integrate from $z_1\in\mathcal{D}^{u,+}_{\kappa_5',c_0}$ and for the positive and zero harmonics from $z_2\in\mathcal{D}^{u,+}_{\kappa_5',c_0}$ (see Figure 11) for a fixed $\kappa_5'>\kappa_5$. Then, we define the function

$$W_0(z,\tau)=\sum_{k<0}\partial_z\varphi^{[k]}(z_1)e^{-ik(z-z_1)}e^{ik\tau}+\sum_{k\geq 0}\partial_z\varphi^{[k]}(z_2)e^{-ik(z-z_2)}e^{ik\tau}, \tag{260}$$

where $\partial_z\varphi$ is the function bounded in Lemma 8.6. The next lemma, whose proof is straightforward, gives some properties of this function.

Lemma 8.7. *The function W_0 defined in* (260) *satisfies:*

1. *$\mathcal{L}W_0=0$, where $\mathcal{L}=\partial_\tau+\partial_x$.*
2. *$W_0\in\mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$ and*
$$\|W_0\|_{2r-\frac{1}{\beta},\sigma}\leq K\varepsilon^{\frac{1}{\beta}}.$$

Then, the function ϕ is a solution of the integral equation

$$\phi=W_0+\overline{\mathcal{G}}\circ\mathcal{W}(\phi).$$

We use a fixed point argument to obtain good estimates of ϕ. We study $\phi\in\mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$ as a fixed point of the operator

$$\overline{\mathcal{W}}=W_0+\overline{\mathcal{G}}\circ\mathcal{W}. \tag{261}$$

Lemma 8.8. *Let us consider $\gamma\in(0,\gamma_2)$, ε_0 small enough and $\kappa_5'>\kappa_5$ big enough. Then, for $\varepsilon\in(0,\varepsilon_0)$, the operator $\overline{\mathcal{W}}$ is contractive from $\mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$ to itself.*

Then, there exists a constant $b_{10}>0$ such that ϕ, the unique fixed point of $\overline{\mathcal{W}}$, satisfies

$$\|\phi\|_{2r-\frac{1}{\beta},\sigma}\leq b_{10}\varepsilon^{\frac{1}{\beta}}.$$

Proof. $\overline{\mathcal{W}}$ sends $\mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$ to itself. To see that $\overline{\mathcal{W}}$ is contractive from $\mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$ to itself, let us consider $\phi_1,\phi_2\in\mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$. Then, applying Lemmas 8.2 and 8.5 and the definition of $\mathcal{W}$ in (256), and increasing $\kappa_5'>0$ if necessary,

$$\begin{aligned}\left\|\overline{\mathcal{W}}(\phi_2)-\overline{\mathcal{W}}(\phi_1)\right\|_{2r-\frac{1}{\beta},\sigma}&\leq K\left\|\mathcal{W}(\phi_2)-\mathcal{W}(\phi_1)\right\|_{2r-\frac{1}{\beta},\sigma}\\&\leq K\left\|\left(Q_1(\tau)\frac{\hat{\mu}}{z^{\ell-2r}}+M(z,\tau)\right)\cdot(\phi_2-\phi_1)\right\|_{2r-\frac{1}{\beta},\sigma}\\&\leq\frac{K}{(\kappa_5')^{\ell-2r}}\left\|\phi_2-\phi_1\right\|_{2r-\frac{1}{\beta},\sigma}\\&\leq\frac{1}{2}\left\|\phi_2-\phi_1\right\|_{2r-\frac{1}{\beta},\sigma}.\end{aligned}$$

Then $\overline{\mathcal{W}}$ is contractive from $\mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$ to itself, and then it has a unique fixed point ϕ.

To obtain a bound for ϕ, it is enough to take into account that $\|\phi\|_{2r-\frac{1}{\beta},\sigma} \leq 2\|\overline{\mathcal{W}}(0)\|_{2r-\frac{1}{\beta},\sigma}$. By the definition of $\overline{\mathcal{W}}$ in (261), we have that $\overline{\mathcal{W}}(0) = W_0 + \overline{\mathcal{G}}(L)$. Then, applying Lemmas 8.2, 8.5 and 8.6, there exists a constant $b_{10} > 0$ such that,

$$\left\|\overline{\mathcal{W}}(0)\right\|_{2r-\frac{1}{\beta},\sigma} \leq \|W_0\|_{2r-\frac{1}{\beta},\sigma} + \left\|\overline{\mathcal{G}}\,(L)\right\|_{2r-\frac{1}{\beta},\sigma} \leq \frac{b_{10}}{2}\varepsilon^{\frac{1}{\beta}}.$$

Let us point out that since the fixed point of $\overline{\mathcal{W}}$ is unique in $\mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$, the obtained function ϕ must coincide with $\phi = \psi^u - \psi_0^u$, where ψ^u is the function defined in (61) and ψ_0^u is the one given in Theorem 4.12. □

The fixed point equation for $\ell - 2r = 0$ We devote this section to prove Proposition 8.3 under the hypothesis $\ell - 2r = 0$. Now, the term $\hat{\mu}Q_1(\tau)z^{-(\ell-2r)} = \hat{\mu}Q_1(\tau)$ in $\mathcal{W}$ (see (256)) is not small. Then, following [Bal06], the first step is to perform the change of variables

$$z = x + \hat{\mu}F_1(\tau), \tag{262}$$

where F_1 is the function defined in (74). Then, we define

$$\widehat{\varphi}(x,\tau) = \varphi\left(x + \hat{\mu}F_1(\tau), \tau\right),$$

which satisfies equation

$$\mathcal{L}\widehat{\varphi} = \widehat{\mathcal{W}}(\partial_x\widehat{\varphi}, x, \tau), \tag{263}$$

with

$$\widehat{\mathcal{W}}(w,x,\tau) = L(x + \hat{\mu}F_1(\tau), \tau) + M(x + \hat{\mu}F_1(\tau), \tau)w. \tag{264}$$

We study this equation through a fixed point argument, as we have done in Section 8.2.2. Then, we define $\widehat{\phi} = \partial_x\widehat{\varphi}$, which is a solution of

$$\mathcal{L}\widehat{\phi} = \partial_x\left[\widehat{\mathcal{W}}(\partial_x\widehat{\phi}, x, \tau)\right].$$

Let us take $c_0'' \in (c_0', c_0)$ and $\kappa_5'' > \kappa_5$. Then, we look for $\widehat{\phi}$ defined for $(x,\tau) \in \mathcal{D}^{\mathrm{in},+,u}_{\kappa_5'',c_0''} \times \mathbb{T}_\sigma$.

To invert the operator $\mathcal{L} = \partial_\tau + \partial_x$, we consider the operator $\overline{\mathcal{G}}$ defined in (251) and initial conditions as we have done in Section 8.2.2. Thus, we define

$$\begin{aligned}\widehat{W}_0(x,\tau) = &\sum_{k<0} \partial_z\varphi^{[k]}(x_1 + \hat{\mu}F_1(\tau))e^{-ik(x-x_1)}e^{ik\tau}\\ &+ \sum_{k\geq 0} \partial_z\varphi^{[k]}(x_2 + \hat{\mu}F_1(\tau))e^{-ik(x-x_2)}e^{ik\tau},\end{aligned} \tag{265}$$

where x_1 and x_2 are the vertices of $\mathcal{D}^{\mathrm{in},+,u}_{\kappa_5'',c_0''}$. Since $c_0'' \in (c_0', c_0)$, $x_1, x_2 \in \mathcal{I}^{+,u}_{c_0,c_0'}$ and then $\partial_z\varphi$ is already defined in $x_i + \mu F_1(\tau), i = 1, 2$ and moreover, we can use the bounds in Lemma 8.6. Then, it is straightforward to see that $\widehat{W}_0$ satisfies the same properties as the function W_0 given in Lemma 8.7.

The function $\widehat{\phi}$ is a solution of the integral equation

$$\widehat{\phi} = \widehat{W}_0 + \overline{\mathcal{G}} \circ \widehat{\mathcal{W}}(\widehat{\phi}).$$

We study $\widehat{\phi} \in \mathcal{Z}_{2r-\frac{1}{\beta},\sigma}$ as a fixed point of the operator

$$\widetilde{\mathcal{W}} = \widehat{W}_0 + \overline{\mathcal{G}} \circ \widehat{\mathcal{W}}. \tag{266}$$

Lemma 8.9. *Let us consider $\gamma \in (0, \gamma_2)$, $\varepsilon_0 > 0$ small enough and $\kappa_5'' > \kappa_5$ big enough. Then, for $\varepsilon \in (0, \varepsilon_0)$, the operator $\widetilde{\mathcal{W}}$ is contractive from $\mathcal{Z}_{2r-\frac{1}{\beta},\kappa_5'',c_0'',\sigma}$ to itself.*

Then, there exists a constant $b_{10} > 0$ such that $\widehat{\phi}$, the unique fixed point of $\widetilde{\mathcal{W}}$, satisfies

$$\|\widehat{\phi}\|_{2r-\frac{1}{\beta},\kappa_5'',c_0'',\sigma} \leq b_{10}\varepsilon^{\frac{1}{\beta}}.$$

Proof. The proof of this lemma is completely analogous to the proof of Lemma 8.8. The only fact that one has to take into account is that the functions $L(x+\hat{\mu}F_1(\tau),\tau)$ and $M(x+\hat{\mu}F_1(\tau),\tau)$ satisfy the same properties as $L(z,\tau)$ and $M(z,\tau)$, which are given in Lemma 8.5. $\square$

To prove Proposition 8.3 for $\ell-2r=0$, it is enough to undo the change of variables (262). Then, taking $\phi(z,\tau)=\widehat{\phi}(x-\hat{\mu}F_1(\tau),\tau)$, we recover $\partial_z\varphi$ which is defined for $(z,\tau)\in D^{\mathrm{in},+,u}_{\kappa_6,c_1}\times\mathbb{T}_\sigma$, where $c_1<c_0''$ and $\kappa_6>\kappa_5''$.

9 An injective solution of the partial differential equation $\widetilde{\mathcal{L}}_\varepsilon\xi=0$

In this section we prove the existence and provide useful properties of a solution ξ_0 of the equation $\widetilde{\mathcal{L}}_\varepsilon\xi=0$ (see (77)) of the form

$$\xi_0(u,\tau)=\varepsilon^{-1}u-\tau+\mathcal{C}(u,\tau).$$

The function $\mathcal{C}$ must satisfy

$$\mathcal{L}_\varepsilon\mathcal{C}(u,\tau)=\mathcal{F}\left(\mathcal{C}\right)(u,\tau), \tag{267}$$

where $\mathcal{L}_\varepsilon$ is the operator in (45),

$$\mathcal{F}(\mathcal{C})(u,\tau)=-\varepsilon^{-1}G(u,\tau)-G(u,\tau)\partial_u\mathcal{C}(u,\tau) \tag{268}$$

and G is the function defined in (79) (case $\ell-2r<0$) and (97) (case $\ell-2r\geq 0$). We devote the rest of the section to obtain a solution of this equation in both cases.

9.1 Banach spaces and technical lemmas

This section is devoted to define the Banach spaces and to state some technical lemmas which will be used in Sections 9.2 and 9.3.

We start by defining some norms. Given $\nu\geq 0$ and an analytic function $h:R_{\kappa,d}\to\mathbb{C}$, where $R_{\kappa,d}$ is the domain defined in (27), we consider

$$\|h\|_{\nu,\kappa,d}=\sup_{u\in R_{\kappa,d}}\left|\left(u^2+a^2\right)^\nu h(u)\right|$$
$$\|h\|_{\ln,\kappa,d}=\sup_{u\in R_{\kappa,d}}\left|\ln^{-1}\left|u^2+a^2\right|\cdot h(u)\right|.$$

Moreover for analytic functions $h:R_{\kappa,d}\times\mathbb{T}_\sigma\to\mathbb{C}$ which are 2π-periodic in τ, we consider the corresponding Fourier norms

$$\|h\|_{\nu,\kappa,d,\sigma}=\sum_{k\in\mathbb{Z}}\left\|h^{[k]}\right\|_{\nu,\kappa,d}e^{|k|\sigma}$$
$$\|h\|_{\ln,\kappa,d,\sigma}=\sum_{k\in\mathbb{Z}}\left\|h^{[k]}\right\|_{\ln,\kappa,d}e^{|k|\sigma}.$$

We consider, thus, the following function spaces

$$\begin{aligned}\mathcal{X}_{\nu,\kappa,d,\sigma}&=\{h:R_{\kappa,d}\times\mathbb{T}_\sigma\to\mathbb{C};\ \text{real-analytic},\|h\|_{\nu,\kappa,d,\sigma}<\infty\}\\ \mathcal{X}_{\ln,\kappa,d,\sigma}&=\{h:R_{\kappa,d}\times\mathbb{T}_\sigma\to\mathbb{C};\ \text{real-analytic},\|h\|_{\ln,\kappa,d,\sigma}<\infty\},\end{aligned} \tag{269}$$

which can be checked that are a Banach spaces.

If there is no danger of confusion about the definition domain $R_{\kappa,d}$ we will denote

$$\|\cdot\|_{\nu,\sigma}=\|\cdot\|_{\nu,\kappa,d,\sigma}\quad\text{ and }\quad\mathcal{X}_{\nu,\sigma}=\mathcal{X}_{\nu,\kappa,d,\sigma}.$$

In the next lemma, we state some properties of these Banach spaces.

Lemma 9.1. *The following statements hold:*

1. *If $\nu_1 \geq \nu_2 \geq 0$, $\mathcal{X}_{\nu_1,\sigma} \subset \mathcal{X}_{\nu_2,\sigma}$ and moreover if $h \in \mathcal{X}_{\nu_1,\sigma}$,*
$$\|h\|_{\nu_2,\sigma} \leq K(\kappa\varepsilon)^{\nu_2-\nu_1}\|h\|_{\nu_1,\sigma}.$$
2. *If $0 \leq \nu_1 \leq \nu_2$, $\mathcal{X}_{\nu_1,\sigma} \subset \mathcal{X}_{\nu_2,\sigma}$ and moreover if $h \in \mathcal{X}_{\nu_1,\sigma}$,*
$$\|h\|_{\nu_2,\sigma} \leq K\|h\|_{\nu_1,\sigma}.$$
3. *If $h \in \mathcal{X}_{\nu_1,\sigma}$ and $g \in \mathcal{X}_{\nu_2,\sigma}$, then $hg \in \mathcal{X}_{\nu_1+\nu_2,\sigma}$ and*
$$\|hg\|_{\nu_1+\nu_2,\sigma} \leq \|h\|_{\nu_1,\sigma}\|g\|_{\nu_2,\sigma}.$$
4. *Let us consider $d > d' > 0$, such that $d - d'$ has a positive lower bound independent of ε, and $h \in \mathcal{X}_{\nu,\kappa,d,\sigma}$. Then, $\partial_u h \in \mathcal{X}_{\nu,2\kappa,d',\sigma}$ and satisfies*
$$\|\partial_u h\|_{\nu,2\kappa,d',\sigma} \leq \frac{K}{\kappa\varepsilon}\|h\|_{\nu,\kappa,d,\sigma}.$$

Throughout this section we are going to solve equations of the form $\mathcal{L}_\varepsilon h = g$, where $\mathcal{L}_\varepsilon$ is the operator defined in (45). To find a right-inverse of this operator in $R_{\kappa,d}$ let us consider $u_1 = i(a - \kappa\varepsilon)$ and u_0 the left endpoint of $R_{\kappa,d} \cap \mathbb{R}$. Then, we define the operator $\mathcal{G}_\varepsilon$ as

$$\mathcal{G}_\varepsilon(h)(u,\tau) = \sum_{k\in\mathbb{Z}} \mathcal{G}_\varepsilon(h)^{[k]}(u)e^{ik\tau}, \tag{270}$$

where its Fourier coefficients are given by

$$\begin{aligned}
\mathcal{G}_\varepsilon(h)^{[k]}(u) &= \int_{-u_1}^{u} e^{ik\varepsilon^{-1}(v-u)}h^{[k]}(v)\,dv && \text{if } k < 0\\
\mathcal{G}_\varepsilon(h)^{[0]}(u) &= \int_{u_0}^{u} h^{[0]}(v)\,dv\\
\mathcal{G}_\varepsilon(h)^{[k]}(u) &= -\int_{u}^{u_1} e^{ik\varepsilon^{-1}(v-u)}h^{[k]}(v)\,dv && \text{if } k > 0.
\end{aligned}$$

where we make the integrals along any path contained in $R_{\kappa,d}$.

Let us point that we will apply this operator to functions defined in $R_{\kappa,d} \times \mathbb{T}_\sigma$ with different values of κ and d and then the definition of $\mathcal{G}_\varepsilon$ depends on the domain.

Lemma 9.2. *The operator $\mathcal{G}_\varepsilon$ in (270) satisfies the following properties.*

1. *If $h \in \mathcal{X}_{\nu,\sigma}$ for some $\nu \geq 0$, then $\mathcal{G}_\varepsilon(h) \in \mathcal{X}_{\nu,\sigma}$ and*
$$\|\mathcal{G}_\varepsilon(h)\|_{\nu,\sigma} \leq K\|h\|_{\nu,\sigma}.$$
Furthermore, if $\langle h\rangle = 0$,
$$\|\mathcal{G}_\varepsilon(h)\|_{\nu,\sigma} \leq K\varepsilon\,\|h\|_{\nu,\sigma}\,.$$
2. *If $h \in \mathcal{X}_{\nu,\sigma}$ for some $\nu > 1$, then $\mathcal{G}_\varepsilon(h) \in \mathcal{X}_{\nu-1,\sigma}$ and*
$$\|\mathcal{G}_\varepsilon(h)\|_{\nu-1,\sigma} \leq K\|h\|_{\nu,\sigma}.$$
3. *If $h \in \mathcal{X}_{\nu,\sigma}$ for some $\nu \in (0,1)$, then $\mathcal{G}_\varepsilon(h) \in \mathcal{X}_{0,\sigma}$ and*
$$\|\mathcal{G}_\varepsilon(h)\|_{0,\sigma} \leq K\|h\|_{\nu,\sigma}.$$

4. *If $h \in \mathcal{X}_{1,\sigma}$, then $\mathcal{G}_\varepsilon(h) \in \mathcal{X}_{\ln,\sigma}$ and*

$$\|\mathcal{G}_\varepsilon(h)\|_{\ln,\sigma} \leq K\|h\|_{1,\sigma}.$$

5. *If $h \in \mathcal{X}_{\nu,\sigma}$ for some $\nu \geq 0$, then $\mathcal{G}_\varepsilon(\partial_u h) \in \mathcal{X}_{\nu,\sigma}$ and*

$$\|\mathcal{G}_\varepsilon(\partial_u h)\|_{\nu,\sigma} \leq K\|h\|_{\nu,\sigma}.$$

6. *If $h \in \mathcal{X}_{\nu,\sigma}$ for some $\nu \geq 0$, then $\partial_u \mathcal{G}_\varepsilon(h) \in \mathcal{X}_{\nu,\sigma}$ and*

$$\|\partial_u \mathcal{G}_\varepsilon(h)\|_{\nu,\sigma} \leq K\|h\|_{\nu,\sigma}.$$

7. *If $h \in \mathcal{X}_{\nu,\sigma}$ for some $\nu \geq 0$, $\mathcal{L}_\varepsilon \circ \mathcal{G}_\varepsilon(h) = h$ and*

$$\mathcal{G}_\varepsilon \circ \mathcal{L}_\varepsilon(h)(v,\tau) = h(v,\tau) - \sum_{k<0} e^{ik\varepsilon^{-1}(-u_1-u)} h^{[k]}(-u_1) - h^{[0]}(u_0) - \sum_{k>0} e^{ik\varepsilon^{-1}(u_1-u)} h^{[k]}(u_1)$$

Proof. The first four statements are straightforward. For the fifth one, one has to integrate by parts and for the sixth one has to apply Leibnitz rule. □

9.2 Case $\ell < 2r$: proof of Theorem 4.17 and Proposition 4.18

9.2.1 Proof of Theorem 4.17

Theorem 4.17 is a straightforward consequence of the following proposition.

Proposition 9.3. *Let us consider the constants $d_2 > 0$ and $\kappa_3 > 0$ defined in Theorem 4.8, $d_3 < d_2$, $\varepsilon_0 > 0$ small enough and $\kappa_7 > \kappa_3$ big enough, which might depend on the previous constants. Then, for $\varepsilon \in (0, \varepsilon_0)$ and any $\kappa \geq \kappa_7$ such that $\varepsilon\kappa < a$, there exists a function $\mathcal{C} : R_{\kappa,d_3} \times \mathbb{T}_\sigma \to \mathbb{C}$ that satisfies equation* (267).

Moreover,

$$(\xi_0(u,\tau),\tau) = \left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau),\tau\right)$$

is injective and there exists a constant $b_{11} > 0$ independent of ε, μ and κ such that

$$\|\mathcal{C}\|_{0,\sigma} \leq b_{11}|\mu|\varepsilon^\eta$$
$$\|\partial_u \mathcal{C}\|_{0,\sigma} \leq b_{11}\kappa^{-1}|\mu|\varepsilon^{\eta-1}.$$

To prove this proposition, first we split G in several parts. Recall that, since $\ell - 2r < 0$, the perturbation $\widehat{H}_1$ in (34) is a polynomial of degree one in p. Then, G can be split as $G = G_1 + G_2 + G_3$ with

$$G_1(u,\tau) = \mu\varepsilon^\eta p_0(u)^{-1}\partial_p \widehat{H}_1^1(q_0(u), p_0(u), \tau) \tag{271}$$
$$G_2(u,\tau) = \mu\varepsilon^{\eta+1} p_0(u)^{-1}\partial_p \widehat{H}_1^2(q_0(u), p_0(u), \tau) \tag{272}$$
$$G_3(u,\tau) = \frac{\partial_u T_1^s(u,\tau) + \partial_u T_1^u(u,\tau)}{2p_0^2(u)}. \tag{273}$$

Next lemma gives several properties of these functions.

Lemma 9.4. *Let us consider any $\kappa > \kappa_3$ and $d < d_2$, where κ_3 and d_2 are the constants given in Theorem 4.8. Then, the functions G_1, G_2 and G_3 defined in* (271), (272) *and* (273) *respectively, have the following properties.*

1. *$G_1 \in \mathcal{X}_{0,\sigma}$ and it satisfies $\langle G_1 \rangle = 0$ and*

$$\|G_1\|_{0,\sigma} \leq K|\mu|\varepsilon^\eta$$
$$\|\partial_v G_1\|_{\max\{\ell-2r+1,0\},\sigma} \leq K|\mu|\varepsilon^\eta.$$

2. $G_2 \in \mathcal{X}_{0,\sigma}$ *and it satisfies*

$$\|G_2\|_{0,\sigma} \leq K|\mu|\varepsilon^{\eta+1}.$$

3. $G_3 \in \mathcal{X}_{\max\{\ell-2r+1,0\},\sigma}$ *and it satisfies*

$$\|G_3\|_{\max\{\ell-2r+1,0\},\sigma} \leq K|\mu|\varepsilon^{\eta+1}.$$

Proof. The proof of the statements about G_1 and G_2 are straightforward, taking into account for G_2 the bounds obtained in Corollary 5.6. For G_3, one has to take into account the bounds for T_1^u obtained in Proposition 7.4 and the analogous bounds that T_1^s satisfies. $\square$

To prove Proposition 9.3, we first perform a change of variables which reduces the linear terms of equation (267).

Lemma 9.5. *Let us consider* $\kappa_7 > \kappa_3' > \kappa_3$ *and* $d_3 < d_2' < d_2$*. Then, for* $\varepsilon > 0$ *small enough, there exists a function* g *which is solution of the equation*

$$\mathcal{L}_\varepsilon g(v,\tau) = G_1(v,\tau),$$

where G_1 *is the function defined in* (271)*. Moreover, it satisfies that*

$$\|g\|_{0,\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^{\eta+1}, \quad \|\partial_v g\|_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^{\eta+1}$$

and that $u = v + g(v,\tau) \in R_{\kappa_3,d_2}$ *for* $(v,\tau) \in R_{\kappa_3',d_2'} \times \mathbb{T}_\sigma$*.*

Furthermore, the change $(u,\tau) = (v + g(v,\tau),\tau)$ *is invertible and its inverse is of the form* $(v,\tau) = (u + h(u,\tau),\tau)$*. The function* h *is defined in the domain* $R_{\kappa_7,d_3} \times \mathbb{T}_\sigma$ *and it satisfies*

$$\|h\|_{0,\kappa_7,d_3,\sigma} \leq K|\mu|\varepsilon^{\eta+1}$$

and that $u + h(u,\tau) \in R_{\kappa_3',d_2'}$ *for* $(u,\tau) \in R_{\kappa_7,d_3} \times \mathbb{T}_\sigma$*.*

Furthermore, we need precise bounds of both functions g and h restricted to the inner domain $D^{\mathrm{in},+,u}_{\kappa_7,c}$ defined in (30). These bounds are given in next corollary, whose proof is straightforward. We abuse notation and we use the norms defined in Section 9.1 for functions restricted to the inner domain.

Corollary 9.6. *Let* $c_1 > 0$ *be the constant defined in Corollary 7.7 and let us consider also* $c_2 > c_1$*. Then, the functions* g *ad* h *obtained in Lemma 9.5 restricted to the inner domains* $D^{\mathrm{in},+,u}_{\kappa_3',c_1}$ *and* $D^{\mathrm{in},+,u}_{\kappa_7,c_2}$ *respectively satisfy the following bounds*

$$\|g\|_{0,\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^{\eta+1+(2r-\ell)\gamma} \quad \text{and } \|h\|_{0,\kappa_7,d_3,\sigma} \leq K|\mu|\varepsilon^{\eta+1+(2r-\ell)\gamma}.$$

Proof of Lemma 9.5. From Lemma 9.4, $\langle G_1\rangle = 0$ and then we can define a function $\overline{G}_1$ such that

$$\partial_\tau \overline{G}_1 = G_1 \quad \text{and} \quad \langle \overline{G}_1\rangle = 0, \tag{274}$$

which satisfies

$$\begin{aligned} \left\|\overline{G}_1\right\|_{0,\kappa_3',d_2',\sigma} &\leq K|\mu|\varepsilon^{\eta} \\ \left\|\partial_v\overline{G}_1\right\|_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma} &\leq K|\mu|\varepsilon^{\eta}. \end{aligned} \tag{275}$$

Then, we can define g as

$$g(v,\tau) = \varepsilon\overline{G}_1(v,\tau) - \varepsilon\mathcal{G}_\varepsilon\left(\partial_v\overline{G}_1\right)(v,\tau), \tag{276}$$

where $\mathcal{G}_\varepsilon$ is the operator defined in (270) adapted to the domain $R_{\kappa_3',d_2'} \times \mathbb{T}_\sigma$.

Finally, applying Lemma 9.4 and 9.2, one obtains the bounds for g and $\partial_v g$. The other statements are straightforward. $\square$

We perform the change of variables $u = v + g(v,\tau)$ given in Lemma 9.5 to equation (268) and we obtain

$$\mathcal{L}_\varepsilon \widehat{\mathcal{C}} = \widehat{\mathcal{F}}\left(\widehat{\mathcal{C}}\right), \tag{277}$$

where $\widehat{\mathcal{C}}$ is the unknown

$$\widehat{\mathcal{C}}(v,\tau) = \mathcal{C}(v + g(v,\tau),\tau) \tag{278}$$

and

$$\widehat{\mathcal{F}}(h) = M(v,\tau) + N(v,\tau)\partial_v h \tag{279}$$

with

$$M(v,\tau) = -\varepsilon^{-1} G\left(v + g(v,\tau),\tau\right) \tag{280}$$

$$N(v,\tau) = -\frac{G\left(v + g(v,\tau),\tau\right) - G_1(v,\tau)}{1 + \partial_v g(v,\tau)}. \tag{281}$$

Next lemma gives some properties of these functions

Lemma 9.7. *The functions M and N defined in* (280) *and* (281) *satisfy the following properties.*

- $\mathcal{G}_\varepsilon(M) \in \mathcal{X}_{0,\kappa_3',d_2',\sigma}$ *and it satisfies*

$$\|\mathcal{G}_\varepsilon(M)\|_{0,\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^\eta.$$

- $\langle M\rangle \in \mathcal{X}_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma}$ *and it satisfies*

$$\|\langle M\rangle\|_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^\eta.$$

- $\partial_v M \in \mathcal{X}_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma}$ *and it satisfies*

$$\|\partial_v M\|_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^{\eta-1}.$$

- *Restricted to* $\left(D^{\mathrm{in},+,u}_{\kappa,c_1} \cap D^{\mathrm{in},+,s}_{\kappa,c_1}\right) \times \mathbb{T}_\sigma$, *the function M satisfies*

$$\|M\|_{0,\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^{\eta+\nu-1}$$
$$\|\langle M\rangle\|_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^\eta$$

  where

$$\nu = \min\{1 - \max\{\ell - 2r + 1, 0\}, (2r-\ell)\gamma\}. \tag{282}$$

- $N \in \mathcal{X}_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma}$ *and it satisfies*

$$\|N\|_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma} \leq K|\mu|\varepsilon^{\eta+1}$$
$$\|\partial_v N\|_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma} \leq K\frac{|\mu|\varepsilon^\eta}{\kappa_3'}.$$

Proof. We split M as $M = M_1 + M_2$ with

$$M_1(v,\tau) = -\varepsilon^{-1} G_1(v,\tau)$$
$$M_2(v,\tau) = -\varepsilon^{-1}\left(G_1\left(v + g(v,\tau),\tau\right) - G_1(v,\tau) + G_2\left(v + g(v,\tau),\tau\right) + G_3\left(v + g(v,\tau),\tau\right)\right).$$

Then, for the first statement it is enough to use the properties of the functions G_1, G_2 and G_3 given by Lemma 9.4 and apply also Lemmas 9.2, 9.1 and 9.5. For the second and the third one has to apply again Lemmas 9.4, 9.1 and 9.5, taking also into account for the second that $\langle M_1\rangle = 0$. Besides, these lemmas, for the fourth statement, one has to consider also the bound of the change g in the inner domain, which is given in Corollary 9.6. For the last statement, it is enough to apply again Lemmas 9.4, 9.1 and 9.5. $\square$

With the bounds obtained in Lemma 9.7, we can look for a solution of equation (277) through a fixed point argument. For that purpose, we define the operator

$$\widetilde{\mathcal{F}} = \mathcal{G}_\varepsilon \circ \widehat{\mathcal{F}} \tag{283}$$

where $\mathcal{G}_\varepsilon$ and $\widehat{\mathcal{F}}$ are the operators defined in (270) and (279) respectively. For convenience, we rewrite $\widehat{\mathcal{F}}$ as

$$\widehat{\mathcal{F}}(h)(u,\tau) = M(u,\tau) + \partial_v\left(N(v,\tau)h(v,\tau)\right) + \partial_v N(v,\tau)h(v,\tau). \tag{284}$$

Lemma 9.8. *Let us consider $\varepsilon_0 > 0$ small enough and $\kappa_3' > \kappa_3$ big enough. Then, the operator $\widetilde{\mathcal{F}}$ defined in* (283) *is contractive from $\mathcal{X}_{0,\kappa_3',d_2',\sigma}$ to itself.*

Thus, it has a unique fixed point, which moreover satisfies

$$\begin{aligned}\left\|\widehat{\mathcal{C}}\right\|_{0,\kappa_3',d_2',\sigma} &\leq K|\mu|\varepsilon^\eta\\ \left\|\partial_v\widehat{\mathcal{C}}\right\|_{0,\kappa_3',d_2',\sigma} &\leq K\frac{|\mu|\varepsilon^{\eta-1}}{\kappa_3'}.\end{aligned}$$

Proof. To see that $\widetilde{\mathcal{F}}$ is contractive, let us consider $h_1, h_2 \in \mathcal{X}_{0,\kappa_3',d_2',\sigma}$. Then, recalling the definition of $\widetilde{\mathcal{F}}$ and $\widehat{\mathcal{F}}$ in (283) and (284) respectively and applying Lemmas 9.2, 9.1 and 9.7,

$$\begin{aligned}\left\|\widetilde{\mathcal{F}}(h_2) - \widetilde{\mathcal{F}}(h_1)\right\|_{0,\kappa_3',d_2',\sigma} &\leq \left\|\mathcal{G}_\varepsilon\partial_v\left(N\cdot(h_2-h_1)\right)\right\|_{0,\kappa_3',d_2',\sigma} + \left\|\mathcal{G}_\varepsilon\left(\partial_v N\cdot(h_2-h_1)\right)\right\|_{0,\kappa_3',d_2',\sigma}\\ &\leq \|N\|_{0,\kappa_3',d_2',\sigma}\|h_2-h_1\|_{0,\kappa_3',d_2',\sigma} + \|\partial_v N\|_{\max\{\ell-2r+1,0\},\kappa_3',d_2',\sigma}\|h_2-h_1\|_{0,\kappa_6',d_2',\sigma}\\ &\leq \frac{K|\mu|\varepsilon^\eta}{\kappa_3'}\|h_2-h_1\|_{0,\kappa_3',d_2',\sigma}.\end{aligned}$$

Then, increasing κ_3' if necessary, $\widetilde{\mathcal{F}}$ is contractive from $\mathcal{X}_{0,\kappa_3',d_2',\sigma}$ and then it has a unique fixed point.

To obtain a bound for the fixed point $\widehat{\mathcal{C}}$, it is enough to recall that

$$\left\|\widehat{\mathcal{C}}\right\|_{0,\kappa_3',d_2',\sigma} \leq 2\left\|\widetilde{\mathcal{F}}(0)\right\|_{0,\kappa_3',d_2',\sigma}.$$

By the definition of $\widetilde{\mathcal{F}}$ in (283), $\widetilde{\mathcal{F}}(0) = \mathcal{G}_\varepsilon(M)$. Then, applying Lemma 9.7, we obtain the bound for $\widehat{\mathcal{C}}$. For the bound of $\partial_v\widehat{\mathcal{C}}$ it is enough to reduce slightly the domain and apply fourth statement of Lemma 9.1. □

Proof of Proposition 9.3. To recover $\mathcal{C}$ from $\widehat{\mathcal{C}}$ it is enough to consider the change of variables $v = u + h(u,\tau)$ obtained in Lemma 9.5, which is defined for $(u,\tau) \in R_{\kappa_7,d_3}\times\mathbb{T}_\sigma$ with $\kappa_7 > \kappa_3'$ and $d_3 < d_2'$. Applying this change, one obtains $\mathcal{C}$ which satisfies the bounds of $\mathcal{C}$ and $\partial_u\mathcal{C}$ stated in Proposition 9.3. To check that $(\xi_0(u,\tau),\tau)$ is injective, it is enough to see that for $(u_1,\tau),(u_2,\tau)\in R_{\kappa_7,d_3}\times\mathbb{T}_\sigma$,

$$\varepsilon^{-1}u_2 - \tau + \mathcal{C}(u_2,\tau) = \varepsilon^{-1}u_1 - \tau + \mathcal{C}(u_1,\tau)$$

implies $u_2 = u_1$. To prove this fact, it is enough to take into account the just obtained bound of $\partial_u\mathcal{C}$, which gives

$$\begin{aligned}|u_2-u_1| &= \varepsilon\left|\mathcal{C}(u_2,\tau)-\mathcal{C}(u_1,\tau)\right|\\ &\leq \frac{K|\mu|\varepsilon^\eta}{\kappa_7}|u_2-u_1|.\end{aligned}$$

Then, increasing κ_7 if necessary, one can see that $u_2 = u_1$. □

9.2.2 Proof of Proposition 4.18

To prove Proposition 4.18 it is enough to study the first asymptotic terms of the function $\widehat{\mathcal{C}}$ obtained in Lemma 9.5. For that purpose, we define

$$\widetilde{M}(v,\tau) = M(v,\tau) - \langle M\rangle(v) \tag{285}$$

and we split $\widehat{\mathcal{C}}$ as $\widehat{\mathcal{C}} = E_1 + E_2 + E_3$ with

$$E_1(v) = \mathcal{G}_\varepsilon\left(\langle M\rangle\right)(v) \tag{286}$$

$$E_2(v,\tau) = \mathcal{G}_\varepsilon\left(\widetilde{M}\right)(v,\tau) \tag{287}$$

$$E_3(v,\tau) = \widetilde{\mathcal{F}}\left(\widehat{\mathcal{C}}\right) - \widetilde{\mathcal{F}}(0) \tag{288}$$

Let us point out that the sum of the first two terms corresponds to $\widetilde{\mathcal{F}}(0)$. We study each term separately. We abuse notation and we use the same norms as in the previous section but now for functions defined in $\left(D^{\mathrm{in},+,u}_{\kappa,c_1} \cap D^{\mathrm{in},+,s}_{\kappa,c_1}\right) \times \mathbb{T}_\sigma$.

For E_1, using the definition of $\mathcal{G}_\varepsilon$ in (270), one has that

$$E_1(v,\tau) = \int_{v_0}^{v} \langle M\rangle(w)dw$$

and then, if we consider $v_1 = i(a - \kappa_3'\varepsilon)$ the upper vertex of the domain R_{κ_3',d_3} (see Figure 3), we can define

$$C(\mu,\varepsilon) = \int_{v_0}^{v_1} \langle M\rangle(w)dw, \tag{289}$$

which by Lemmas 9.2 and 9.7 satisfies

$$\|C(\mu,\varepsilon)\|_{0,\sigma} \le K|\mu|\varepsilon^\eta.$$

Then

$$\|E_1 - C(\mu,\varepsilon)\|_{0,\sigma} \le K|\mu|\varepsilon^{\eta+(2r-\ell)\gamma}.$$

To bound E_2 defined in (287), we first recall that $\langle\widetilde{M}\rangle = 0$. Then we can define a function $\overline{M}$ such that

$$\partial_\tau \overline{M} = \widetilde{M} \quad \text{and} \quad \langle\overline{M}\rangle = 0,$$

which satisfies that for $(v,\tau) \in \left(D^{\mathrm{in},+,u}_{\kappa,c_1} \cap D^{\mathrm{in},+,s}_{\kappa,c_1}\right) \times \mathbb{T}_\sigma$,

$$\left\|\overline{M}\right\|_{0,\sigma} \le K|\mu|\varepsilon^{\eta+\nu-1}$$

where ν is the constant defined in (282). Then, we can write E_2 as

$$E_2 = \varepsilon\mathcal{G}_\varepsilon \circ \mathcal{L}_\varepsilon(\overline{M}) - \varepsilon\mathcal{G}_\varepsilon\left(\partial_v \overline{M}\right).$$

and therefore, by Lemma 9.2,

$$\|E_2\|_{0,\sigma} \le K|\mu|\varepsilon^{\eta+\nu}$$

For E_3 in (288), it is enough to consider the bound of the Lipschitz constant of the operator $\widetilde{\mathcal{F}}$ given in the proof of Lemma 9.8, which gives

$$\|E_3\|_{0,\sigma} \le K\frac{|\mu|\varepsilon^{2\eta}}{\kappa_3'}.$$

Thus, we have that

$$\left\|\widehat{\mathcal{C}} - C(\mu,\varepsilon)\right\|_{0,\sigma} \le K\frac{|\mu|\varepsilon^{\eta}}{\kappa_3'}.$$

To finish the proof of Proposition 4.18, it is enough to consider the change of variables $v = u + h(u,\tau)$ obtained in Lemma 9.5. Since h restricted to the inner domains satisfies the bounds given in Corollary 9.6, this change of variables does not change the asymptotic first order of $\mathcal{C}$.

9.3 Case $\ell \geq 2r$: Proof of Theorem 4.21 and Proposition 4.22

9.3.1 Proof of Theorem 4.21

This section is devoted to prove Theorem 4.21. This theorem is a straightforward consequence of the following proposition.

Proposition 9.9. *Let us consider the constants $d_2 > 0$ and $\kappa_6 > 0$ defined in Theorem 4.8 and Proposition 8.3, $d_3 < d_2$, $\varepsilon_0 > 0$ small enough and $\kappa_8 > \kappa_6$ big enough, which might depend on the previous constants. Then, for $\varepsilon \in (0, \varepsilon_0)$ and any $\kappa \geq \kappa_8$ such that $\varepsilon\kappa < a$, there exists a function $\mathcal{C} : R_{\kappa,d_3} \times \mathbb{T}_\sigma \to \mathbb{C}$ that satisfies equation* (267).

Moreover,

$$(\xi_0(u,\tau), \tau) = \left(\varepsilon^{-1}u - \tau + \mathcal{C}(u,\tau)\right)$$

is injective and there exists a constant $b_{15} > 0$ such that

- *If $\ell - 2r > 0$,*

$$\begin{aligned}\|\mathcal{C}\|_{\ell-2r,\sigma} &\leq b_{15}|\hat{\mu}|\varepsilon^{\ell-2r}\\ \|\partial_u\mathcal{C}\|_{\ell-2r,\sigma} &\leq b_{15}\kappa^{-1}|\hat{\mu}|\varepsilon^{\ell-2r-1}.\end{aligned}$$

- *If $\ell - 2r = 0$,*

$$\begin{aligned}\|\mathcal{C}\|_{\ln,\sigma} &\leq b_{15}|\hat{\mu}|\\ \|\partial_u\mathcal{C}\|_{1,\sigma} &\leq b_{15}|\hat{\mu}|.\end{aligned}$$

We split the proof into two cases. First in Section 9.3.1 we prove it under the hypothesis $\ell - 2r > 0$. Then, in Section 9.3.1 we give a technical lemma, which is used in Section 9.3.1 to deal with the case $\ell - 2r = 0$.

Nevertheless we need to state some useful properties of function G defined in (97).

Properties of the function G Let us split the function G in (97) as $G = G_1 + G_2 + G_3 + G_4$ with

$$G_1(u,\tau) = \hat{\mu}\varepsilon^{\ell-2r}p_0(u)^{-1}\partial_p\widehat{H}_1^1(q_0(u), p_0(u), \tau) \tag{290}$$

$$G_2(u,\tau) = \hat{\mu}\varepsilon^{\ell-2r+1}p_0(u)^{-1}\partial_p\widehat{H}_1^2(q_0(u), p_0(u), \tau) \tag{291}$$

$$G_3(u,\tau) = \frac{1}{2}\left(1 + \hat{\mu}\varepsilon^{\ell-2r}\partial_p^2\widehat{H}_1^1(q_0(u), p_0(u), \tau)\right)\frac{\partial_u T_1^s(u,\tau) + \partial_u T_1^u(u,\tau)}{p_0^2(u)} \tag{292}$$

$$G_4(u,\tau) = G(u,\tau) - G_1(u,\tau) - G_2(u,\tau) - G_3(u,\tau), \tag{293}$$

where $\widehat{H}_1^1$ and $\widehat{H}_1^2$ are the functions defined in (35) and (37). The next lemma gives properties of these functions.

Lemma 9.10. *Let us consider any $\kappa > \kappa_6$ and $d < d_2$, where κ_6 an d_0 are the constants defined in Theorems 8.3 and 4.7. Then, the functions G_i, $i = 1, 2, 3, 4$ defined in* (290), (291), (292) *and* (293) *respectively, have the following properties.*

1. *$G_1 \in \mathcal{X}_{\ell-2r,\sigma}$ and it satisfies $\langle G_1 \rangle = 0$ and*

$$\|G_1\|_{\ell-2r,\sigma} \leq K|\hat{\mu}|\varepsilon^{\ell-2r}.$$

Moreover

- *If $\ell - 2r > 0$, $\partial_u G_1 \in \mathcal{X}_{\ell-2r+1,\sigma}$ and it satisfies*

$$\|\partial_u G_1\|_{\ell-2r+1,\sigma} \leq K|\hat{\mu}|\varepsilon^{\ell-2r}.$$

- *If* $\ell-2r=0$, $\partial_u G_1 \in \mathcal{X}_{1-\frac{1}{\beta},\sigma}$ *and it satisfies*

$$\|\partial_u G_1\|_{1-\frac{1}{\beta},\sigma} \leq K|\hat{\mu}|.$$

2. $G_2 \in \mathcal{X}_{\ell-2r,\sigma}$ *and it satisfies*
$$\|G_2\|_{\ell-2r,\sigma} \leq K|\hat{\mu}|^2\varepsilon^{2(\ell-2r)+1}.$$
Moreover

- *If* $\ell-2r>0$, $\partial_u G_2 \in \mathcal{X}_{\ell-2r+1,\sigma}$ *and it satisfies*
$$\|\partial_u G_2\|_{\ell-2r+1,\sigma} \leq K|\hat{\mu}|^2\varepsilon^{2(\ell-2r)+1}.$$
- *If* $\ell-2r=0$, $\partial_u G_2 \in \mathcal{X}_{1-\frac{1}{\beta},\sigma}$ *and it satisfies*
$$\|\partial_u G_2\|_{1-\frac{1}{\beta},\sigma} \leq K|\hat{\mu}|^2\varepsilon.$$

3. $G_3 \in \mathcal{X}_{\ell-2r+1,\sigma}$ *and satisfies*
$$\|G_3\|_{\ell-2r+1,\sigma} \leq K|\hat{\mu}|\varepsilon^{\ell-2r+1}.$$
Moreover,

- *If* $\ell-2r>0$, $\partial_u G_3 \in \mathcal{X}_{\ell-2r+1,\sigma}$ *and it satisfies*
$$\|\partial_u G_3\|_{\ell-2r+1,\sigma} \leq K\kappa^{-1}|\hat{\mu}|\varepsilon^{\ell-2r}.$$
- *If* $\ell-2r=0$, $\partial_u G_3 \in \mathcal{X}_{2,\sigma}$ *and it satisfies*
$$\|\partial_u G_3\|_{2,\sigma} \leq K|\hat{\mu}|\varepsilon.$$

4. $G_4, \partial_u G_4 \in \mathcal{X}_{3(\ell-2r)+2,\sigma}$ *and satisfy*
$$\begin{aligned}\|G_4\|_{3(\ell-2r)+2,\sigma} &\leq K|\hat{\mu}|^3\varepsilon^{3(\ell-2r)+2}\\ \|\partial_u G_4\|_{3(\ell-2r)+2,\sigma} &\leq K\kappa^{-1}|\hat{\mu}|^3\varepsilon^{3(\ell-2r)+1}.\end{aligned}$$

Proof. The proof of the statements about G_1 and G_2 are straightforward, taking into account, for G_2, the bounds obtained in Corollary 5.6. For G_3, one has to take into account the bounds for T_1^u obtained in Proposition 7.4 and Corollary 7.22 and the analogous bounds that T_1^s satisfies. To obtain the bound for its derivative, one can apply the fourth statement of Lemma 9.1. Analogously, one can obtain the bounds for G_4 and $\partial_u G_4$. □

Case $\ell-2r>0$ To prove Proposition 9.9 for $\ell-2r>0$, we look for $\mathcal{C}$ as a fixed point of the operator
$$\overline{\mathcal{F}} = \mathcal{G}_\varepsilon \circ \mathcal{F}, \tag{294}$$
where $\mathcal{G}_\varepsilon$ and $\mathcal{F}$ are the operators defined in (270) and (268) respectively. For convenience, we rewrite $\mathcal{F}$ as
$$\mathcal{F}(\mathcal{C})(u,\tau) = -\varepsilon^{-1}G(u,\tau) - \partial_u\left(G(u,\tau)\mathcal{C}(u,\tau)\right) + \partial_u G(u,\tau)\mathcal{C}(u,\tau). \tag{295}$$
Then Proposition 9.9 is a consequence of the following lemma.

Lemma 9.11. *Let us consider $\varepsilon_0>0$ small enough and $\kappa_8>\kappa_6$ big enough. Then, for $\varepsilon\in(0,\varepsilon_0)$ and any $\kappa\geq\kappa_8$ such that $\varepsilon\kappa<a$, the operator $\overline{\mathcal{F}}$ defined in (294) is contractive from $\mathcal{X}_{\ell-2r,\sigma}$ to itself.*

Then, it has a unique fixed point $\mathcal{C}\in\mathcal{X}_{\ell-2r,\sigma}$, which moreover satisfies

$$\begin{aligned}\|\mathcal{C}\|_{\ell-2r,\sigma} &\leq K|\hat{\mu}|\varepsilon^{\ell-2r}\\ \|\partial_u\mathcal{C}\|_{\ell-2r,\sigma} &\leq K\kappa^{-1}|\hat{\mu}|\varepsilon^{\ell-2r-1}.\end{aligned}$$

Before proving Lemma 9.11, we state the following technical lemma about the properties of the function G defined in (97).

Lemma 9.12. *Let us assume $\ell - 2r > 0$. Then, the function G defined in* (97) *has the following properties:*

1. *$G \in \mathcal{X}_{\ell-2r,\sigma}$ and satisfies*
$$\|G\|_{\ell-2r,\sigma} \leq K|\hat{\mu}|\varepsilon^{\ell-2r}.$$

2. *$\partial_u G \in \mathcal{X}_{\ell-2r+1,\sigma}$ and satisfies*
$$\|\partial_u G\|_{\ell-2r+1,\sigma} \leq K|\hat{\mu}|\varepsilon^{\ell-2r}.$$

3. *$\mathcal{G}_\varepsilon(G) \in \mathcal{X}_{\ell-2r,\sigma}$ and satisfies*
$$\|\mathcal{G}_\varepsilon(G)\|_{\ell-2r,\sigma} \leq K|\hat{\mu}|\varepsilon^{\ell-2r+1}.$$

Proof. The bounds for G and $\partial_u G$ are a direct consequence of Lemma 9.10. To obtain the bound for $\mathcal{G}_\varepsilon(G)$, it is enough to apply Lemma 9.2 and to take into account that $\langle G_1 \rangle = 0$. □

Using the bounds given in this lemma, we can prove Lemma 9.11.

Proof of Lemma 9.11. To see that $\overline{\mathcal{F}}$ is contractive, let us consider $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{X}_{\ell-2r,\sigma}$. Then, recalling the definition of $\mathcal{F}$ in (295) and applying Lemmas 9.2, 9.1 and 9.12,

$$\begin{aligned}
\left\|\overline{\mathcal{F}}(\mathcal{C}_2) - \overline{\mathcal{F}}(\mathcal{C}_1)\right\|_{\ell-2r,\sigma} &\leq \|\mathcal{G}_\varepsilon\left(\partial_u\left(G \cdot (\mathcal{C}_2 - \mathcal{C}_1)\right)\right)\|_{\ell-2r,\sigma} + \|\mathcal{G}_\varepsilon\left(\partial_u G \cdot (\mathcal{C}_2 - \mathcal{C}_1)\right)\|_{\ell-2r,\sigma} \\
&\leq K\,\|G\|_{0,\sigma}\,\|\mathcal{C}_2 - \mathcal{C}_1\|_{\ell-2r,\sigma} + K\,\|\partial_u G\|_{1,\sigma}\,\|\mathcal{C}_2 - \mathcal{C}_1\|_{\ell-2r,\sigma} \\
&\leq \frac{K|\hat{\mu}|}{\kappa_7^{\ell-2r}}\,\|\mathcal{C}_2 - \mathcal{C}_1\|_{\ell-2r,\sigma}.
\end{aligned}$$

Then, increasing κ_8 if necessary, $\overline{\mathcal{F}}$ is contractive from $\mathcal{X}_{\ell-2r,\sigma}$ to itself, and then it has a unique fixed point $\mathcal{C} \in \mathcal{X}_{\ell-2r,\sigma}$.

To obtain a bound for the fixed point $\mathcal{C}$ it is enough to recall that

$$\|\mathcal{C}\|_{\ell-2r,\sigma} \leq 2\left\|\overline{\mathcal{F}}(0)\right\|_{\ell-2r,\sigma}.$$

By the definition of $\overline{\mathcal{F}}$ in (294), $\overline{\mathcal{F}}(0) = -\varepsilon^{-1}\mathcal{G}_\varepsilon(G)$. Then, applying Lemma 9.12, we obtain the bound for $\mathcal{C}$. Finally, to obtain the bound for $\partial_u\mathcal{C}$ it is enough to reduce slightly the domain and apply the fourth statement of Lemma 9.1. □

Proof of Proposition 9.9 for $\ell - 2r > 0$. To prove Proposition 9.9 from Lemma 9.11, it only remains to check that $(\xi_0(u,\tau),\tau)$ is injective in $R_{\kappa,d_3} \times \mathbb{T}_\sigma$. We prove this fact as in the proof of Proposition 9.3, that is, we check that if

$$\varepsilon^{-1}u_2 - \tau + \mathcal{C}(u_2,\tau) = \varepsilon^{-1}u_1 - \tau + \mathcal{C}(u_1,\tau)$$

for $(u_1,\tau),(u_2,\tau) \in R_{\kappa,d_3} \times \mathbb{T}_\sigma$, then we have that $u_2 = u_1$. To prove this fact, it is enough to take into account the bound of $\partial_u\mathcal{C}$ given in Lemma 9.11, which gives

$$\begin{aligned}
|u_2 - u_1| &= \varepsilon\,|\mathcal{C}(u_2,\tau) - \mathcal{C}(u_1,\tau)| \\
&\leq \frac{K|\hat{\mu}|}{\kappa_8^{\ell-2r+1}}|u_2 - u_1|.
\end{aligned}$$

Then, increasing κ_8 if necessary, one can see that $u_2 = u_1$. □

Case $\ell - 2r = 0$ We devote this section to prove Proposition 9.9 under the hypothesis $\ell - 2r = 0$. Now, as happened in Section 9.2, the linear term G_1 in (290) of $\mathcal{F}$ in (268) is not small. Then, we perform again a change of variables.

Lemma 9.13. *Let us consider $\kappa_8 > \kappa_6' > \kappa_6$ and $d_3 < d_2' < d_2$. Then, for $\varepsilon > 0$ small enough, there exists a function g which is solution of the equation*

$$\mathcal{L}_\varepsilon g(v,\tau) = G_1(v,\tau),$$

where G_1 is the function defined in (290)*. Moreover, it satisfies that*

$$\|g\|_{0,\kappa_6',d_2',\sigma} \le K|\hat\mu|\varepsilon, \quad \|\partial_v g\|_{1-\frac{1}{\beta},\kappa_6',d_2',\sigma} \le K|\hat\mu|\varepsilon$$

and that $u = v + g(v,\tau) \in R_{\kappa_6,d_2}$ for $(v,\tau) \in R_{\kappa_6',d_2'} \times \mathbb{T}_\sigma$.

Furthermore, the change $(u,\tau) = (v + g(v,\tau),\tau)$ is invertible and its inverse is of the form $(v,\tau) = (u + h(u,\tau),\tau)$. The function h is defined in the domain $R_{\kappa_8,d_3} \times \mathbb{T}_\sigma$ and it satisfies

$$\|h\|_{0,\kappa_8,d_3,\sigma} \le K|\hat\mu|\varepsilon$$

and that $u + h(u,\tau) \in R_{\kappa_6',d_2'}$ for $(u,\tau) \in R_{\kappa_8,d_3} \times \mathbb{T}_\sigma$.

Proof. From Lemma 9.10, $\langle G_1\rangle = 0$ and then we can define a function $\overline{G}_1$ such that

$$\partial_\tau \overline{G}_1 = G_1 \quad\text{and}\quad \langle \overline{G}_1\rangle = 0, \tag{296}$$

which satisfies

$$\begin{aligned} \left\|\overline{G}_1\right\|_{0,\kappa_6',d_2',\sigma} &\le K|\hat\mu| \\ \left\|\partial_v\overline{G}_1\right\|_{1-\frac{1}{\beta},\kappa_6',d_2',\sigma} &\le K|\hat\mu|. \end{aligned} \tag{297}$$

Then, we can define g as

$$g(v,\tau) = \varepsilon\overline{G}_1(v,\tau) - \varepsilon\mathcal{G}_\varepsilon\left(\partial_v\overline{G}_1\right)(v,\tau), \tag{298}$$

where $\mathcal{G}_\varepsilon$ is the operator defined in (270) adapted to the domain $R_{\kappa_6',d_2'} \times \mathbb{T}_\sigma$.

Finally, applying Lemma 9.10 and 9.2, one obtains the bounds for g and $\partial_v g$. The other statements are straightforward. ◻

We perform the change of variables $u = v + g(v,\tau)$ given in Lemma 9.13 to equation (268) and we obtain

$$\mathcal{L}_\varepsilon\widehat{\mathcal{C}} = M(v,\tau) + N(v,\tau)\partial_v\widehat{\mathcal{C}}, \tag{299}$$

where $\widehat{\mathcal{C}}$ is the unknown

$$\widehat{\mathcal{C}}(v,\tau) = \mathcal{C}(v + g(v,\tau),\tau) \tag{300}$$

and

$$M(v,\tau) = -\varepsilon^{-1}G\left(v + g(v,\tau),\tau\right) \tag{301}$$

$$N(v,\tau) = -\frac{G\left(v + g(v,\tau),\tau\right) - G_1(v,\tau)}{1 + \partial_v g(v,\tau)}. \tag{302}$$

Moreover, we want to have the first order terms in $\widehat{\mathcal{C}}$, coming from G_1, G_2 and G_3, explicitly. For this purpose, we define

$$\begin{aligned} \widehat{\mathcal{C}}_0(v,\tau) = &-\overline{G}_1(v,\tau) - \varepsilon^{-1}\mathcal{G}_\varepsilon\left(\langle\partial_v G_1 g\rangle\right)(v) \\ &- \varepsilon^{-1}\mathcal{G}_\varepsilon\left(\langle G_2 + G_3\rangle\right)(v), \end{aligned} \tag{303}$$

where $\overline{G}_1$ is the function defined in (296), g is the function given by Lemma 9.13 and G_2 and G_3 are the functions defined in (291) and (292) respectively. The next lemma, whose proof is straightforward applying Lemmas 9.2, 9.10 and 9.13, gives some properties of $\widehat{\mathcal{C}}_0$.

Lemma 9.14. *The function* $\widehat{\mathcal{C}}_0$ *defined in* (303) *satisfies that*

$$\left\|\widehat{\mathcal{C}}_0\right\|_{\ln,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|, \quad \left\|\partial_v\widehat{\mathcal{C}}_0\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|.$$

Then, we define

$$\widehat{\mathcal{C}}_1 = \widehat{\mathcal{C}} - \widehat{\mathcal{C}}_0.$$

Taking into account equation (299), $\widehat{\mathcal{C}}_1$ is a solution of

$$\mathcal{L}_\varepsilon\widehat{\mathcal{C}}_1 = \widehat{\mathcal{F}}\left(\widehat{\mathcal{C}}_1\right), \tag{304}$$

where

$$\widehat{\mathcal{F}}(h) = \widehat{M}(v,\tau) + N(v,\tau)\partial_v h \tag{305}$$

and

$$\widehat{M}(v,\tau) = M(v,\tau) - \mathcal{L}_\varepsilon\widehat{\mathcal{C}}_0 + N(v,\tau)\partial_v\widehat{\mathcal{C}}_0. \tag{306}$$

We obtain $\widehat{\mathcal{C}}_1$ through a fixed point argument. For this purpose we define the operator

$$\widetilde{\mathcal{F}} = \mathcal{G}_\varepsilon \circ \widehat{\mathcal{F}}, \tag{307}$$

where $\widehat{\mathcal{F}}$ and $\mathcal{G}_\varepsilon$ are the operators defined (305) and (270). For convenience, we rewrite it as

$$\widehat{\mathcal{F}}(h)(v,\tau) = \widehat{M}(v,\tau) + \partial_v\left(N(v,\tau)h(v,\tau)\right) - \partial_v N(v,\tau)h(v,\tau). \tag{308}$$

Lemma 9.15. *Let us consider* $\varepsilon_0 > 0$ *small enough and* $\kappa_6' > \kappa_6$ *big enough. Then, the operator* $\widetilde{\mathcal{F}}$ *is contractive from* $\mathcal{X}_{1,\kappa_6',d_2',\sigma}$ *to itself.*

Thus, it has a unique fixed point, which moreover satisfies that

$$\left\|\widehat{\mathcal{C}}_1\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|\varepsilon$$

$$\left\|\partial_v\widehat{\mathcal{C}}_1\right\|_{1,\kappa_6',d_2',\sigma} \leq K\frac{|\hat{\mu}|}{\kappa_6'}.$$

Before proving this lemma, we state the following lemma, whose proof is postponed to the end of this section.

Lemma 9.16. *The functions* $\widehat{M}$ *and* N *defined in* (306) *and* (302) *respectively, satisfy the following properties.*

- $\mathcal{G}_\varepsilon(\widehat{M}) \in \mathcal{X}_{1,\kappa_6',d_2',\sigma}$ *and satisfies*

$$\left\|\mathcal{G}_\varepsilon(\widehat{M})\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|\varepsilon.$$

- $N, \partial_v N \in \mathcal{X}_{1,\kappa_6',d_2',\sigma}$ *and satisfy*

$$\|N\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|\varepsilon$$

$$\|\partial_v N\|_{1,\kappa_6',d_2',\sigma} \leq K\frac{|\hat{\mu}|}{\kappa_6'}.$$

Proof of Lemma 9.15. The operator $\widetilde{\mathcal{F}}$ sends $\mathcal{X}_{1,\kappa_6',d_2',\sigma}$ to itself. To see that it is contractive, let us consider $h_1, h_2 \in \mathcal{X}_{1,\kappa_6',d_2',\sigma}$. Then, recalling the definitions of $\widetilde{\mathcal{F}}$ and $\widehat{\mathcal{F}}$ in (307) and (308) and applying Lemmas 9.2 and 9.16, one can see that

$$\begin{aligned}
\left\|\widetilde{\mathcal{F}}(h_2) - \widetilde{\mathcal{F}}(h_1)\right\|_{1,\kappa_6',d_2',\sigma} &\leq \|\mathcal{G}_\varepsilon\partial_v\left(N\cdot(h_2-h_1)\right)\|_{1,\kappa_6',d_2',\sigma} + \|\mathcal{G}_\varepsilon\left(\partial_v N\cdot(h_2-h_1)\right)\|_{1,\kappa_6',d_2',\sigma} \\
&\leq \|N\|_{0,\kappa_6',d_2',\sigma}\|h_2-h_1\|_{1,\kappa_6',d_2',\sigma} + \|\partial_v N\|_{1,\kappa_6',d_2',\sigma}\|h_2-h_1\|_{1,\kappa_6',d_2',\sigma} \\
&\leq \frac{K|\hat{\mu}|}{\kappa_6'}\|h_2-h_1\|_{1,\kappa_6',d_2',\sigma}.
\end{aligned}$$

and therefore, increasing κ_6' if necessary, $\widetilde{\mathcal{F}}$ is contractive in $\mathcal{X}_{1,\kappa_6',d_2',\sigma}$ and has a unique fixed point $\widehat{\mathcal{C}}_1$. To obtain bounds for $\widehat{\mathcal{C}}_1$ it is enough to recall that

$$\left\|\widehat{\mathcal{C}}_1\right\|_{1,\kappa_6',d_2',\sigma} \leq 2\left\|\widetilde{\mathcal{F}}(0)\right\|_{1,\kappa_6',d_2',\sigma}.$$

By the definition of $\widetilde{\mathcal{F}}$ in (307), $\widetilde{\mathcal{F}}(0) = \mathcal{G}_\varepsilon(\widehat{M})$. Then, it is enough to apply Lemma 9.16 to obtain

$$\left\|\widehat{\mathcal{C}}_1\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|\varepsilon.$$

For the bound of $\partial_v\widehat{\mathcal{C}}_1$ it is enough to apply the fourth statement of Lemma 9.1 and rename κ_6'. □

Proof of Proposition 9.9 for $\ell - 2r = 0$. By Lemmas 9.14 and 9.15, we have that there exists a constant $b_{15} > 0$ such that

$$\left\|\widehat{\mathcal{C}}\right\|_{\ln,\sigma} \leq b_{15}|\hat{\mu}|$$
$$\left\|\partial_v\widehat{\mathcal{C}}\right\|_{1,\sigma} \leq b_{15}|\hat{\mu}|.$$

To recover $\mathcal{C}$ it is enough to consider the change of variables $v = u + h(u,\tau)$ obtained in Lemma 9.13, which is defined for $(u,\tau) \in R_{\kappa_8,d_3} \times \mathbb{T}_\sigma$ with $\kappa_8 > \kappa_6'$ and $d_3 < d_2'$. Applying this change, one obtains $\mathcal{C}$ which satisfies the bounds stated in Proposition 9.9. To check that $(\xi_0(u,\tau),\tau)$ is injective, one can proceed as in the proof of Proposition 9.9 for $\ell - 2r > 0$. Finally let us point out that it is easy to see that this proposition is also satisfied taking any $\kappa \geq \kappa_8$ such that $\varepsilon\kappa < a$. □

It only remains to prove Lemma 9.16.

Proof of Lemma 9.16. We start proving the second statement. Let us split the function N defined in (302) as $N = N_1 + N_2$ with

$$N_1(v,\tau) = -\left(1+\partial_v g(v,\tau)\right)^{-1}\left(G_1(v+g(v,\tau),\tau) - G_1(v,\tau)\right) \tag{309}$$
$$N_2(v,\tau) = -\left(1+\partial_v g(v,\tau)\right)^{-1}\left(G_2(v+g(v,\tau),\tau) + G_3(v+g(v,\tau),\tau) + G_4(v+g(v,\tau),\tau)\right). \tag{310}$$

To bound N_1, we apply Lemmas 9.13 and 9.10 and mean value theorem, obtaining

$$\|N_1\|_{1-\frac{1}{\beta},\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|^2\varepsilon.$$

Applying the same lemmas, one can see that

$$\|N_2\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|\varepsilon.$$

which gives the bound for N. To obtain the bound for $\partial_v N$ it is enough to apply the fourth statement of Lemma 9.1 and rename κ_6'.

For the first statement, taking into account the definitions of $\widehat{M}$ and M in (301) and (306) respectively, and using the functions G_i, $i = 1,2,3,4$ and $\overline{G}_1$ defined in (290), (291), (292), (293) and (296), let us split $\widehat{M}$ as

$$\widehat{M}(v,\tau) = \sum_{i=1}^{6}\widehat{M}_i(v,\tau)$$

with

$$\widehat{M}_1(v,\tau) = \partial_v\overline{G}_1(v,\tau) - \varepsilon^{-1}\left(\partial_v G_1(v,\tau)g(v,\tau) - \langle\partial_v G_1 g\rangle(v)\right) \tag{311}$$
$$\widehat{M}_2(v,\tau) = -\varepsilon^{-1}\left(G_1(v+g(v,\tau),\tau) - G_1(v,\tau) - \partial_v G_1(v,\tau)g(v,\tau)\right) \tag{312}$$
$$\widehat{M}_3(v,\tau) = -\varepsilon^{-1}\left(G_2(v,\tau) + G_3(v,\tau) - \langle G_2 + G_3\rangle(v)\right) \tag{313}$$
$$\widehat{M}_4(v,\tau) = -\varepsilon^{-1}\left(G_2(v+g(v,\tau),\tau) + G_3(v+g(v,\tau),\tau) - G_2(v,\tau) - G_3(v,\tau)\right) \tag{314}$$
$$\widehat{M}_5(v,\tau) = -\varepsilon^{-1}G_4(v+g(v,\tau),\tau) \tag{315}$$
$$\widehat{M}_6(v,\tau) = N(v,\tau)\partial_v\widehat{\mathcal{C}}_0(v,\tau). \tag{316}$$

We bound each term. For the first one, by Lemmas 9.13 and 9.10, we have that $\widehat{M}_1 \in \mathcal{X}_{1-\frac{1}{\beta},\kappa_6',d_2',\sigma} \subset \mathcal{X}_{1,\kappa_6',d_2',\sigma}$. Moreover, taking also into account (297),

$$\left\|\widehat{M}_1\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|.$$

and therefore, since $\langle\widehat{M}_1\rangle = 0$, by Lemma 9.2,

$$\left\|\mathcal{G}_\varepsilon\left(\widehat{M}_1\right)\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|\varepsilon.$$

For the term (312), it is enough to apply Lemmas 9.13 and Taylor's formula to obtain $\widehat{M}_2 \in \mathcal{X}_{2-\frac{1}{\beta},\kappa_6',d_2',\sigma} \subset \mathcal{X}_{2,\kappa_6',d_2',\sigma}$ and

$$\left\|\widehat{M}_2\right\|_{2,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|^3\varepsilon.$$

Then, applying again Lemma 9.2, we have that,

$$\left\|\mathcal{G}_\varepsilon\left(\widehat{M}_2\right)\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|^3\varepsilon.$$

To bound (313), it is enough to Lemma 9.10 to see that $M_3 \in \mathcal{X}_{1,\kappa_6',d_0',\sigma}$ and

$$\left\|\widehat{M}_3\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|$$

which, using that $\langle\widehat{M}_3\rangle = 0$, implies

$$\left\|\mathcal{G}_\varepsilon\left(\widehat{M}_3\right)\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|\varepsilon.$$

Applying mean value theorem, using the definition of G_3 in (292) and Proposition 7.23, and the definition of G_2 in (291), Lemmas 9.13 and 9.10, one can see that $\widehat{M}_4$ in (314) satisfies

$$\left\|\widehat{M}_4\right\|_{2,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|^2\varepsilon$$

and then,

$$\left\|\mathcal{G}_\varepsilon\left(\widehat{M}_4\right)\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|^2\varepsilon.$$

For $\widehat{M}_5$ in (315), it is enough to recall that, by Lemma 9.10 and 9.2,

$$\left\|\widehat{M}_5\right\|_{2,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|^3\varepsilon$$

and

$$\left\|\mathcal{G}_\varepsilon\left(\widehat{M}_5\right)\right\|_{1,\kappa_0',d_2',\sigma} \leq K|\hat{\mu}|^3\varepsilon.$$

Finally, for the last term (316), one has to apply the bound of N already obtained and Lemma 9.14, to see that

$$\left\|\widehat{M}_6\right\|_{2,\kappa_6',d_2',\sigma} \leq \|N\|_{1,\kappa_6',d_0',\sigma}\left\|\partial_v\widehat{\mathcal{C}}_0\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|^2\varepsilon.$$

Then, by Lemma 9.2, we have that,

$$\left\|\mathcal{G}_\varepsilon\left(\widehat{M}_6\right)\right\|_{1,\kappa_6',d_2',\sigma} \leq K|\hat{\mu}|^2\varepsilon.$$

Joining all these bounds, we prove the first statement of Lemma 9.16 □

9.3.2 Proof of Proposition 4.22

To prove Proposition 4.22, it is enough to obtain the first asymptotic terms of the function $\widehat{\mathcal{C}}_0$ obtained in Lemma 9.14. From them, we can deduce the first order terms of $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}_0 + \widehat{\mathcal{C}}_1$, where $\widehat{\mathcal{C}}_1$ is the function bounded in Lemma 9.15, and from them, using (300), the ones of $\mathcal{C}$.

Recall that $\widehat{\mathcal{C}}_0$ has been defined in (303) as $\widehat{\mathcal{C}}_0 = E_1 + E_2 + E_3 + E_4$ with

$$E_1(v,\tau) = -\,\overline{G}_1(v,\tau) \tag{317}$$

$$E_2(v) = -\,\varepsilon^{-1}\mathcal{G}_\varepsilon\left(\langle \partial_v G_1 g\rangle\right)(v) \tag{318}$$

$$E_3(v) = -\,\varepsilon^{-1}\mathcal{G}_\varepsilon\left(\langle G_2\rangle\right)(v) \tag{319}$$

$$E_4(v) = -\,\varepsilon^{-1}\mathcal{G}_\varepsilon\left(\langle G_3\rangle\right)(v), \tag{320}$$

where G_1, G_2, G_3 and $\overline{G}_1$ are the functions defined in (290), (291), (292) and (296) respectively and g is the function given by Lemma 9.13.

We analyze each of the four terms that form $\widehat{\mathcal{C}}_0$ for $(v,\tau) \in \left(D^{\mathrm{in},+,u}_{\kappa_6',c_1} \cap D^{\mathrm{in},+,s}_{\kappa_6',c_1}\right) \times \mathbb{T}_\sigma$. For the first one (317), it is enough to recall that, by definition, the function F_1 defined in (74) satisfies that

$$\hat{\mu} F_1(\tau) = \overline{G}_1(ia,\tau)$$

and therefore,

$$E_1(v,\tau) = -\overline{G}_1(v,\tau) = -\hat{\mu} F_1(\tau) + \mathcal{O}(v-ia)^{\frac{1}{\beta}}.$$

Then, using (98) and that $|v - ia| \leq K\varepsilon^\gamma$,

$$\|E_1 + \mu F_1\|_{1,\sigma} \leq K|\hat{\mu}|\varepsilon.$$

For the second term, let us recall that by (298) and applying Lemma 9.2, we have that the function g, obtained in Lemma 9.13, satisfies

$$\left\| g - \varepsilon\overline{G}_1(v,\tau)\right\|_{1-\frac{1}{\beta},\sigma} \leq K|\hat{\mu}|\varepsilon^2.$$

Then, by Lemma 9.10, one can see that

$$\left\|\partial_v\left(g - \varepsilon\overline{G}_1(v,\tau)\right)\right\|_{2-\frac{2}{\beta},\sigma} \leq K|\mu|^2\varepsilon^2$$

and therefore, using Lemma 9.2,

$$\left\|\varepsilon^{-1}\mathcal{G}_\varepsilon\left(\partial_v\left(g - \varepsilon\overline{G}_1(v,\tau)\right)\right)\right\|_{1,\sigma} \leq K|\hat{\mu}|^2\varepsilon.$$

Now remains to bound, the first order of E_3, which is given by

$$-\mu\int_{v_0}^{v} \langle \partial_v G_1 \overline{G}_1\rangle(w)\, dw,$$

where we recall that $v_0 \in R_{\kappa_6',d_3} \times \mathbb{T}_\sigma$.

Since $\langle \partial_v G_1\overline{G}_1\rangle = \mathcal{O}(v-ia)^{1-\frac{1}{\beta}}$, we can define the constant

$$C_2(\mu) = -\hat{\mu}\int_{v_0}^{ia} \langle \partial_v G_1\overline{G}_1\rangle(w)\, dw$$

and then, using (98), one has that

$$\|E_2 - C_2(\hat{\mu})\|_{1,\sigma} \leq K|\hat{\mu}|^2\varepsilon.$$

For the third term, by the definitions of G_2 in (291) and $\mathcal{G}_\varepsilon$ in (270), we have that

$$\begin{aligned} E_3(v) &= -\hat{\mu}\int_{v_0}^{v} \langle \widehat{H}_1^2\rangle(w)\,dw \\ &= -\hat{\mu}\int_{v_0}^{ia} \langle \widehat{H}_1^2\rangle(w)dw + \mathcal{O}(v-ia)^{\frac{1}{\beta}}. \end{aligned}$$

Then, proceeding as for E_2, we define

$$C_3(\mu,\varepsilon) = -\hat{\mu}\int_{v_0}^{ia} \langle \widehat{H}_1^2\rangle(w)\,dw$$

and using (98), we have that

$$\left\| E_3 - C_3(\mu,\varepsilon)\right\|_{1,\sigma} \leq K|\hat{\mu}|\varepsilon.$$

To bound E_4, using Proposition 7.23, we split G_3 in two parts as $G_3 = G_3^1 + G_3^2$

$$G_3^1(v,\tau) = \left(1+\hat{\mu}\partial_p^2\widehat{H}_1^1\left(q_0(u),p_0(u),\tau\right)\right)p_0(u)^{-2}\left(\frac{2r\hat{\mu}\varepsilon C_+^2}{(v-ia)^{2r+1}}\left(F_0(\tau)+\hat{\mu}\langle Q_0F_1\rangle\right)+\xi(u,\tau)\right)$$

and $G_3^2 = G_3 - G_3^1$. By Proposition 7.23, $\|G_3^2\|_{2,\sigma} \leq K\hat{\mu}\varepsilon^2$ and therefore

$$\left\|\varepsilon^{-1}\mathcal{G}_\varepsilon\left(\langle G_3^2\rangle\right)\right\|_{2,\sigma} \leq K|\hat{\mu}|\varepsilon.$$

For the other term, using the definitions of $\widehat{H}_1^1$, b, Q_j and F_j in (35), (75), (73) and (74), and recalling that by Proposition 7.23, $\xi \in \mathcal{X}_{1-\frac{1}{\beta},\sigma}$, there exist a function $\widehat{\xi} \in \mathcal{X}_{1-\frac{1}{\beta},\sigma}$, such that

$$\left\langle G_3^1\right\rangle(v) = \frac{b\hat{\mu}^2\varepsilon}{v-ia} + \widehat{\xi}(v,\tau).$$

Then, one can see that there exist a constant $C_4(\hat{\mu},\varepsilon)$ satisfying $|C_4(\hat{\mu},\varepsilon)| \leq K|\hat{\mu}|$, such that,

$$\left\|E_4(v) + b\hat{\mu}^2\ln(v-ia) - C_4(\hat{\mu},\varepsilon)\right\|_{1,\sigma} \leq K|\hat{\mu}|\varepsilon.$$

Taking $C = C_2 + C_3 + C_4$ one obtains that

$$\left\|\widehat{\mathcal{C}}(v,\tau) + \hat{\mu}F_1(\tau) - C(\hat{\mu},\varepsilon) + b\hat{\mu}^2\ln(v-ia)\right\|_{1,\sigma} \leq K|\hat{\mu}|\varepsilon.$$

To finish the proof of Proposition 4.22, it is enough to consider the change of variables $v = u + h(u,\tau)$ obtained in Lemma 9.13, which does not change the asymptotic first order of $\mathcal{C}$.

9.4 An asymptotic formula for $C(\mu,\varepsilon)$

When $\eta = 0$, the constants $C(\mu,\varepsilon)$ considered in Theorems 2.5 and 2.6 satisfy that $\lim_{\varepsilon\to 0} C(\mu,\varepsilon) = C_0(\mu)$ for certain function $C_0(\mu)$ analytic in μ. We devote this section to prove this fact for the case $\ell - 2r < 0$. The proof for the case $\ell - 2r = 0$ is completely analagous.

The proof for the case $\ell - 2r < 0$ follows the same lines as the one of Proposition 4.18 in Section 9.2.2 and, therefore, we only sketch it. Recall that throughout this section we assume $\eta = 0$.

We split the constant $C(\mu,\varepsilon)$ as $C(\mu,\varepsilon) = C^1(\mu,\varepsilon) + C^2(\mu,\varepsilon) + C^3(\mu,\varepsilon)$ and we obtain the corresponding first orders in ε, which we call $C_0^i(\mu)$ for $i = 1,2,3$. Then, the function $C_0(\mu)$ will be given by $C_0(\mu) = C_0^1(\mu) + C_0^2(\mu) + C_0^3(\mu)$.

Recall that $C(\mu,\varepsilon)$ has been defined as (289) where v_0 is the left endpoint of $R_{\kappa_3',d_3}\cap\mathbb{R}$, $v_1 = i(a-\kappa_3'\varepsilon)$ is the upper vertex of the domain R_{κ_3',d_3} (see Figure 3) and M is the function defined in (280). To obtain the constants C^i we split M as $M = M^1 + M^2 + M^3$ with

$$M^i(v,\tau) = -\varepsilon^{-1}G_i\left(v+g(v,\tau),\tau\right) \quad \text{for } i = 1,2,3, \tag{321}$$

where G_i, $i=1,2,3$, are the functions defined in (271), (272) and (273) and g is the function obtained in Lemma 9.5. Then,

$$C^i(\mu,\varepsilon)=\int_{v_0}^{v_1}\left\langle M^i\right\rangle(v)dv.$$

To define C_0^1, we expand M^1 with respect to ε. Using the formulas (276) for g and (271) for G_1, one can easily see that for $(v,\tau)\in R_{\kappa_3',d_3}\times\mathbb{T}_\sigma$,

$$M^1(v,\tau)=-\varepsilon^{-1}G_1(v,\tau)-\partial_v G_1(v,\tau)\overline{G}_1(v,\tau)+\mathcal{O}\left(\frac{\mu\varepsilon}{(v-ia)^{\max\{0,2-\nu_1\}}}\right)$$

for certain $\nu_1>0$. Recall that by Lemma 9.4, we have that $\langle G_1\rangle=0$ and therefore this first term does not contribute to $C_1(\mu,\varepsilon)$. The second term, that is $-\partial_v G_1(v,\tau)\overline{G}_1(v,\tau)$, is independent of ε. Moreover, using the properties of G_1 stated in Lemma 9.4, one can see that it can be analytically extended to reach $v=ia$ and that it satisfies

$$-\partial_v G_1(v,\tau)\overline{G}_1(v,\tau)=\mathcal{O}\left(\frac{\mu}{(v-ia)^{\max\{0,1-\nu_1'\}}}\right)$$

for certain $\nu_1'>0$. Therefore, one can define

$$C_0^1(\mu)=-\int_{v_0}^{ia}\left\langle\partial_v G_1(v,\tau)\overline{G}_1(v,\tau)\right\rangle dv, \tag{322}$$

which is a constant independent of ε. Finally it can be easily seen that

$$\left|C^1(\mu,\varepsilon)-C_0^1(\mu)\right|\leq K|\mu|\varepsilon^{\nu_1''} \tag{323}$$

for certain $\nu_1''>0$.

To obtain $C_0^2(\mu)$, let us first point out that, following the proof of Theorem 4.1, one can see that the parameterization of the periodic orbit satisfies

$$(x_p(\tau),y_p(\tau))=\left(\varepsilon x_p^0(\tau),\varepsilon y_p^0(\tau)\right)+\mathcal{O}\left(\mu\varepsilon^2\right), \tag{324}$$

where $\left(x_p^0(\tau),y_p^0(\tau)\right)$ is independent of ε. Using that fact, one can easily deduce that the functions c_{kl} involved in the definition of $\widehat{H}_1^2$ in (37) satisfy

$$c_{kl}(\tau)=c_{kl}^0(\tau)+\mathcal{O}(\mu\varepsilon),$$

for certain functions $c_{kl}^0(\tau)$ independent of ε. Therefore, $\widehat{H}_1^2$ satisfies

$$\widehat{H}_1^2(q,p,\tau)=\varepsilon\widehat{H}_1^{20}(q,p,\tau)+\varepsilon^2\widehat{H}_1^{22}(q,p,\tau), \tag{325}$$

where $\widehat{H}_1^{20}(q,p,\tau)$ is independent of ε. Taking into account the definition of M_2 in (321) and recalling that for $(v,\tau)\in R_{\kappa_3',d_3}\times\mathbb{T}_\sigma$,

$$p_0(v)^{-1}\widehat{H}_1^{20}(q_0(v),p_0(v),\tau)=\mathcal{O}\left((v-ia)^{2r-\ell}\right),$$

we can define

$$C_0^2(\mu)=-\mu\int_{v_0}^{ia}\left\langle p_0(v)^{-1}\widehat{H}_1^{20}(q_0(v),p_0(v),\tau)\right\rangle dv. \tag{326}$$

Then, the constant $C_0^2(\mu)$ is independent of ε. Moreover, using Lemmas 9.2 and 9.4 and 9.5, one can see that

$$\left|C^2(\mu,\varepsilon)-C_0^2(\mu)\right|\leq K|\mu|\varepsilon^{\nu_2} \tag{327}$$

for certain constant $\nu_2>0$.

To obtain $C_3^0(\mu)$ we need a careful study of the function G_3 in (273). To this end, we have to expand asymptotically the functions $\partial_v \widehat{T}_1^{u,s}(v,\tau)$ obtained in Theorems 4.4 and 4.8. To obtain this expansion we consider equation (157) for $(v,\tau) \in R_{\kappa_3',d_3} \times \mathbb{T}_\sigma$.

As a first step we expand the function $A(u,\tau)$ defined in (143). It can be seen that it satisfies

$$A(u,\tau) = A^0(u,\tau) + \varepsilon A^1(u,\tau) + \mathcal{O}\left(\frac{\mu\varepsilon^2}{(v-ia)^\ell}\right)$$

where

$$A_0(u,\tau) = -\mu \widehat{H}_1^1\left(q_0(u), p_0(u), \tau\right) \tag{328}$$

$$A_1(u,\tau) = -\mu \widehat{H}_1^{20}\left(q_0(u), p_0(u), \tau\right) - V'(q_0(u)) x_p^0(\tau) + \lambda^2 x_p^0(\tau) \tag{329}$$

where $\widehat{H}_1^1$, $\widehat{H}_1^{20}$ and x_p^0 are the functions defined in (35), (325) and (324) respectively, and λ is the constant defined in Hypothesis **HP1.1**. Recall that in the parabolic case, we have that $x_p^0(\tau) = 0$. It is clear that both A^0 and A^1 are independent of ε.

From this expansion, one can deduce the expansion of the function $\widehat{A}$ defined in (159). Let us first recall that the change of variables g obtained in Lemma 7.6 can be written as

$$g(v,\tau) = -\varepsilon \overline{B}_1(v,\tau) + \mathcal{O}\left(\frac{\mu\varepsilon^2}{(v-ia)^{\max\{1+\ell-2r,0\}}}\right),$$

where $\overline{B}_1$ is the function defined on the proof of Lemma 7.6, which is independent of ε.

Therefore,

$$\widehat{A}(v,\tau) = \widehat{A}^0(v,\tau) + \varepsilon \widehat{A}^1(v,\tau) + \mathcal{O}\left(\frac{\mu\varepsilon^2}{(v-ia)^{\ell+2+2(\ell-2r)}}\right),$$

with

$$\widehat{A}_0(v,\tau) = A_0(v,\tau)$$
$$\widehat{A}_1(v,\tau) = A_1(v,\tau) - \partial_v A_0(v,\tau)\overline{B}_1(v,\tau).$$

Using this fact and the properties of the functions $\widehat{B}$ and $\widehat{C}$ in (160) and (161), one can see that the functions $\widehat{T}_1^{u,s}(v,\tau)$ obtained in Theorems 4.4 and 4.8 satisfy that

$$\partial_v \widehat{T}_1^{u,s}(v,\tau) = \varepsilon \partial_v \widehat{T}_1^0(v,\tau) + \mathcal{O}\left(\frac{\mu\varepsilon^2}{(v-ia)^{\max\{0,2+\ell-\nu_3\}}}\right)$$

for certain $\nu_3 > 0$. The first order $\partial_v \widehat{T}_1^0(v,\tau)$ is defined by $\partial_v \widehat{T}_1^0(v,\tau) = \partial_v \overline{A}_0(v,\tau) + \langle \widehat{A}_1 \rangle(v)$, where $\overline{A}_0$ is a function satisfying that $\partial_\tau \overline{A}_0 = A_0$ and $\langle A_0 \rangle = 0$. Then, $\partial_v \widehat{T}_1^0(v,\tau)$ is independent of ε and can be analytically extended to reach $v = ia$.

Taking into account the properties of the change g stated in Lemma 7.6, one can see that the function $\partial_u T_1(u,\tau)$ has the same expansion as the function $\partial_v \widehat{T}_1(v,\tau)$.

We can define

$$C_0^3(\mu) = -\int_{v_0}^{ia} \left\langle p_0(v)^{-2} \partial_v T_1^0(v,\tau) \right\rangle dv, \tag{330}$$

which is a constant independent of ε. Doing little effort, it can be seen also that

$$\left| C^3(\mu,\varepsilon) - C_0^3(\mu) \right| \leq K|\mu|\varepsilon^{\nu_3'} \tag{331}$$

for certain $\nu_3' > 0$.

Finally, it is enough to define $C_0(\mu) = C_0^1(\mu) + C_0^2(\mu) + C_0^3(\mu)$ where $C_0^i(\mu)$ are the constants defined in (322), (326) and (330). It is straightforward to see that $C_0(\mu)$ is an entire function. Moreover, by (323), (327) and (331), it is clear that

$$\lim_{\varepsilon \to 0} C(\mu,\varepsilon) = C_0(\mu).$$

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