EXPONENTIAL STABILITY OF THE QUASIGEOSTROPHIC EQUATION UNDER RANDOM PERTURBATIONS

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Abstract. The quasigeostrophic model describes large scale and relatively slow fluid motion in geophysical flows. We investigate the quasigeostrophic model under random forcing and random boundary conditions. We first transform the model into a partial differential equation with random coefficients. Then we show that, under suitable conditions on the random forcing, random boundary conditions, viscosity, Ekman constant and Coriolis parameter, all quasigeostrophic motion approach a unique stationary state exponentially fast. This stationary state corresponds to a unique invariant Dirac measure.

1. Introduction

The quasigeostrophic (QG) model is a simplified geophysical fluid model at asymptotically high rotation rate or at small Rossby number. It is derived as an approximation of the rotating shallow water equations by a conventional asymptotic expansion for small Rossby number [13]. The lowest order approximation gives the barotropic QG equation, which is also the conservation law for the zero–th order potential vorticity. Warn et al. [19] and Vallis [21] emphasize that this asymptotic expansion is generally secular for all but the simplest flows and propose a modified asymptotic method, which involves expanding only the fast modes. The barotropic QG equation also emerges at the lowest order in this modified expansion.

Moreover, it has recently been shown [18, 3, 7] that quasigeostrophy is a valid approximation of the rotating shallow water equations in the limit of zero Rossby number, i.e., for asymptotically high rotation rate. The three-dimensional baroclinic quasigeostrophic flow model can be derived similarly; see, for example, [13, 7, 3, 9, 6].

We consider the barotropic quasigeostrophic flow model [13, 14, 12]

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi + \dot{W}_2 \tag{1}$$

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on a rectangle $D = (0,1) \times (0,1) \subset \mathbb{R}^2$, where $\psi(x,y,t)$ is the stream function, $\beta \geq 0$ the meridional gradient of the Coriolis parameter, $\nu > 0$ the viscous dissipation constant, $r > 0$ the Ekman dissipation constant, $\dot{W}_2$ the noise due to wind forcing, and J the Jacobian operator, which is defined by $J(f,g) = f_x g_y - f_y g_x$. Equation (1) can be rewritten in terms of the vorticity $q = \Delta\psi$ as

$$q_t + J(\psi, q) + \beta\psi_x = \nu\Delta q - rq + \dot{W}_2 \ , \tag{2}$$

which is usually supplemented with boundary conditions

$$\psi(x,y,t) = 0 \quad \text{on } \partial D \ , \tag{3}$$

$$\frac{\partial}{\partial n} q(x,y,t) = \dot{W}_1 \quad \text{on } \partial D \ , \tag{4}$$

along with an appropriate initial condition

$$q(x,y,0) = q_0(x,y) \ , \tag{5}$$

where n is the unit outward normal vector on the boundary ∂D, W_1 is a *temporally two–sided* Wiener process with values in the function space U. The boundary condition (3) means no normal flow can pass through the boundary. The other boundary condition (4) says that $q = \Delta\psi$ has zero mean but with fluctuations, and thus might be called a random slip boundary condition [14]. As discussed in Pedlosky's book ([14], page 34), the boundary condition $\Delta\psi = 0$ may be an appropriate slip boundary condition for the large scale quasigeostrophic motion. Boundary conditions for the quasigeostrophic model are not quite well understood, since this model describes large scale flows while boundary conditions also involve small scale motions. For this reason we believe that, under the random media or random wind forcing conditions, a random slip condition may be more appropriate than the usual slip boundary conditions [14] for the deterministic quasigeostrophic model. We also note that the Neumann form for the boundary condition on q is mathematical convenience.

In this article we treat the quasigeostrophic flow model with both random forcing and random boundary condition as a mathematical random dynamical system [1]. Our aim is to show that there exists a random steady state under a particular choice of parameter values. This random steady state is a statistically stationary solution towards which any other solution trajectory tends as $t \to \infty$. Our mathematical approach is to formulate the random quasigeostrophic flow model as a stochastic evolution equation with structural similarities to the Navier–Stokes equation and then to show that it generates a random dynamical system for which there exists an attracting random fixed point.

2. Preliminaries

Following Arnold [1] we will model noise in an abstract random dynamical system on a state space H by a metric dynamical system $(\theta, \mathcal{P})$ on a probability space $\mathcal{P}$ $= (\Omega, \mathcal{F}, \mathbb{P})$. A *metric dynamical system* consists of a group $\{\theta_t\}_{t\in\mathbb{R}}$ of operators θ_t $: \Omega \mapsto \Omega$, i.e., satisfying

$$\theta_0 = id_\Omega, \qquad \theta_{s+t} = \theta_s \circ \theta_t \quad \text{for all } s, t \in \mathbb{R},$$

such that the mapping $(t, \omega) \mapsto \theta_t\omega$ from $\mathbb{R}\times\Omega$ into Ω is $(\mathcal{B}(\mathbb{R})\times\mathcal{F}, \mathcal{F})$–measurable and the probability measure $\mathbb{P}$ is ergodic (hence invariant) with respect to the flow θ.

A *random dynamical system* consists of a metric dynamical system $(\theta, \mathcal{P})$ and a cocycle mapping $\varphi : \mathbb{R}\times\Omega\times H \mapsto H$, i.e., satisfying

$$\varphi(0, \omega, \cdot) = \mathrm{id}_H, \quad \varphi(s+t, \omega, \cdot) = \varphi(s, \theta_t, \cdot) \circ \varphi(t, \omega, \cdot) \quad \text{for all } s, t \in \mathbb{R}^+,$$

that is $(\mathcal{B}(\mathbb{R})\otimes\mathcal{F}\otimes\mathcal{B}(H), \mathcal{B}(H))$-measurable. The mapping φ describes the dynamics of the system in the state space H, which will be a separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|\cdot| = \sqrt{(\cdot, \cdot)}$ in this article.

Let X be a random variable defined on $(\theta, \mathcal{P})$ with values in H. By the invariance of $\mathbb{P}$ and the measurability of θ the mapping $(t, \omega) \mapsto X(\theta_t\omega)$ from $\mathbb{R}\times\Omega$ into H is a measurable stationary stochastic process. We will restrict attention here to random variables generating stationary processes that satisfy certain growth conditions. A H–valued random variable X is said to be *tempered* with respect to a θ-invariant set Ω' (of full $\mathbb{P}$-measure) if the mapping $t \mapsto |X(\theta_t\omega)|$ grows at most subexponentially as $t \to \pm\infty$, i.e., for which

$$\lim_{t\to\pm\infty} \frac{\log^+ |X(\theta_t\omega)|}{|t|} = 0.$$

for $\omega \in \Omega'$. Note that the only alternative to this when X is not tempered is

$$\limsup_{t\to\pm\infty} \frac{\log^+ |X(\theta_t\omega)|}{|t|} = +\infty.$$

A *random fixed point* of a random dynamical system φ is a H–valued random variable X^* for which

$$\varphi(t, \omega, X^*(\omega)) = X^*(\theta_t\omega) \qquad \text{for all } t \in \mathbb{R}^+ \tag{6}$$

and for all ω in a θ-invariant set of full $\mathbb{P}$–measure. The dynamics thus follows a stationary regime if we start in $X^*(\omega)$. In particular, the probability distribution of these states is independent of t.

The following theorem is a special case of a random fixed point theorem due to

Theorem 2.1. *Let φ be a random dynamical system with a separable Banach space, $(H,|\cdot|)$ as its state space $(H,|\cdot|)$. Assume that the mapping $x\mapsto$ is continuous for every $t\geq 0$ and $\omega\in\Omega$. In addition, let that $\mathcal{X}=\{\mathcal{X}(\omega)\}_{\omega\in\Omega}$ be a closed random set in H such that*

$$\mathbb{E}\sup_{h_1\neq h_2\in\mathcal{X}(\omega)}\log\frac{|\varphi(1,\omega,h_1)-\varphi(1,\omega,h_2)|}{|h_1-h_2|}<0 \tag{7}$$

and that the real valued random variable defined by $\omega\to\sup_{h\in\mathcal{X}(\omega)}|h|$ is tempered. Moreover, assume that

$$\sup_{t\in[0,1]}\sup_{h_1\neq h_2\in\mathcal{X}(\omega)}\frac{|\varphi(t,\omega,h_1)-\varphi(t,\omega,h_2)|}{|h_1-h_2|} \tag{8}$$

is tempered with respect to $\{\theta_n\}_{n\in\mathbb{Z}}$.
Then there exists a θ-invariant set of full $\mathbb{P}$−measure Ω' and a random variable X^* satisfying (6) on Ω'. Futhermore, X^* is exponentially attracting, i.e.,

$$\lim_{t\to\infty}|\varphi(t,\omega,X(\omega))-X^*(\theta_t\omega)|=0,\qquad a.s.$$

exponentially fast for any measurable selection X of $\mathcal{X}$.

2.1. **Linear stochastic evolution equations.** In the following we will consider the motion relative to a spatially constant flow. By the particular structure of the coefficients this spatially constant flow can by calculated separately. To find this spatially constant flow we have to solve a simpler equation. Henceforth we take for H the space of square integrable functions $L_2(D)$ on the rectangle $D=(0,1)\times(0,1)$ in $\mathbb{R}^2$ fulfilling $\int_D f(x)dD=0$ and denote its norm by $|\cdot|$. We then define V with norm $\|\cdot\|$ to be the Sobolev space of functions contained in H with generalized derivatives of first order belonging to $L_2(D)$, and define $W_2^2(D)$ to be the space of functions with first and second generalized derivatives belonging to $L_2(D)$. In addition, we denote by $U=L_2(\partial D)$ a boundary space associated with square integrable functions on the boundary of D. To use the results of DaPrato and Zabczyk [15] Chapter 13 we use that this perturbation is only defined on one side of the rectangle, say $\{0\}\times(0,1)$. So U consists of L_2-functions on ∂D which are zero outside of $\{0\}\times(0,1)$ with zero average. However, generalizations are possible.

Let Δ be the Laplacian operator on D. The boundary value problem

$$-\nu\Delta u=f,\qquad \frac{\partial}{\partial n}u=g\quad\text{on }\partial D$$

with $f\in H$, $g\in L_2(U)$ and $\nu>0$, has a unique solution $u=\tilde{G}(f,g)$. The solution operator $\tilde{G}:H\times U\to W_2^2(D)$ is a bounded linear operator, i.e., there exists a constant c_G such that

$$\|\tilde{G}(f,g)\|_{W_2^2(D)}\leq c_G\left(|f|+\|g\|_U\right).$$

Similarly we can consider the same equation but with the homogeneous Dirichlet boundary condition

$$-\Delta u = f, \qquad u|_{\partial D} = 0. \tag{9}$$

The solution operator G for this boundary value problem satisfies

$$\|G(f)\|_{W_2^2(D)} \le c_G|f|.$$

In the following we will denote by $C_{G,x}$the constant which estimates $G(f)_x$ with respect to $L_2(D)$-norm of f.
In order to introduce a *white noise* on the boundary ∂D, we consider a *temporally two–sided* Wiener process W_1 with values in U and denote by $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ the filtration corresponding to this Wiener process, where, roughly speaking, $\mathcal{F}_t$ is generated by the noise sample paths between times $-\infty$ and t. The associated Wiener measure P is defined on the σ-algebra of the canonical sample space Ω that consists of continuous functions ω from $\mathbb{R}$ into a phase space of the noise satisfying $\omega(0) = 0$. We will assume that the covariance operator Q^{W_1} with respect to this measure satisfies $\mathrm{tr}_U Q^{W_1} < \infty$ and define the Wiener shift by

$$W_1(\theta_t\omega, \cdot) = W_1(\omega, t + \cdot) - W_1(\omega, t).$$

The measure P is ergodic with respect to the flow θ of the metric dynamical system formed by the Wiener shift.
Since the solution operator $\tilde{G}(0,\cdot)$ above is a linear and bounded operator, the process $t \mapsto \widetilde{W_1}(\omega, t) := \tilde{G}(0, W_1(\omega, t))$ also defines a Wiener process with trajectories in the Sobolev space $W_2^2(D)$ for which the bounded covariance operator $\widetilde{Q}$ $:= \tilde{G} \circ Q^{W_1} \circ \tilde{G}^*$ has a finite trace with respect to $W_2^2(D)$.

The operator $-\nu\Delta$ with the vanishing Neumann boundary condition can be extended to an operator A defined on $D(A) = W_2^2(D)$ with the vanishing Neumann boundary condition. The space H has a complete orthonormal base, consisting of eigenvectors $e_1, e_2, \cdots$ with corresponding eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$ for the operator A. By the particular choice of H we know that $\lambda_1 > 0$ and that A is coercive. Let the semigroup $\{S(t)\}_{t\ge 0}$ on H be the solution operator (indexed by t) of the initial-boundary value problem

$$\frac{du}{dt} - \nu\Delta u = 0, \qquad u(0) = u_0 \in H, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D.$$

with $\nu > 0$. This semigroup has the generator $-A$.
We consider an expression of the form

$$z(\omega, t) = S(t)z_0 + \int_0^t AS(t-\tau)d\widetilde{W_1}(\omega, \tau), \tag{10}$$

as the solution of the linear stochastic evolution equation

$$\frac{dz}{dt} - \nu\Delta\, z = 0, \qquad z(0) = z_0 \in H, \qquad \frac{\partial}{\partial n} z(t)|_{\partial D} = \dot{W}_1(t); \tag{11}$$

see ([15] Section 13.2). The expression (10) is meaningful if, for example,

$$\int_0^t \| AS(t-\tau)GQ^{W_1\,\frac{1}{2}}\|^2_{\mathcal{L}^2(U,H)} d\tau < \infty, \quad \text{for all } 0 \le t < +\infty.$$

Since by the invariance of the increments of the Wiener process, we then have

$$\mathbb{E}\left| A\int_{-\infty}^0 S(-\tau)d\widetilde{W_1}\right|^2 \le \sum_{i=0}^{\infty} e^{-2\lambda_1 i}\mathbb{E}\left| A\int_{-1}^0 S(\tau)d\widetilde{W_1}\right|^2 < \infty.$$

The random variable z_{W_1} defined by

$$z_{W_1}(\omega) := A\int_{-\infty}^0 S(-\tau)d\widetilde{W_1}(\omega,\tau)$$

is thus well defined and has finite second moment with respect to the norm of H. Moreover, we have formally

$$\begin{aligned}
S(t)z_{W_1}(\omega) \quad &+ \quad A\int_0^t S(t-\tau)d\widetilde{W_1}(\omega,\tau) \\
&= \quad S(t)A\int_{-\infty}^0 S(-\tau)d\widetilde{W_1}(\omega,\tau) + A\int_0^t S(t-\tau)d\widetilde{W_1}(\omega,\tau) \\
&= \quad A\int_{-\infty}^0 S(t-\tau)d\widetilde{W_1}(\omega,\tau) + A\int_0^t S(t-\tau)d\widetilde{W_1}(\omega,\tau) \\
&= \quad A\int_{-\infty}^t S(t-\tau)d\widetilde{W_1}(\omega,\tau) \\
&= \quad A\int_{-\infty}^0 S(-\tau)d\widetilde{W_1}(\theta_t\omega,\tau) = z_{W_1}(\theta_t\omega),
\end{aligned}$$

so the stationary process

$$t \mapsto z(\theta_t\omega) = A\int_{-\infty}^0 S(-\tau)d\widetilde{W_1}(\theta_t\omega,\tau), \quad t \in \mathbb{R}$$

solves the boundary value problem (11).
Since $\mathbb{E}|z_{W_1}|^2$ is finite, we can apply the Burkholder inequality to obtain

$$\mathbb{E}\sup_{t\in[0,1]} |z_{W_1}(\theta_t\omega)|^2 < \infty,$$

and it then follows from the Birkhoff Ergodic Theorem [15] that

$$\lim_{i\to\pm\infty} \frac{\sup_{\tau\in[0,1]} |z_{W_1}(\theta_{i+\tau}\omega)|^2}{i} = 0$$

on a θ-invariant subset of Ω of full P–measure. Hence

$$\lim_{t\to\pm\infty} \frac{|z_{W_1}(\theta_t\omega)|^2}{t} = 0$$

on a θ-invariant subset of Ω of full P–measure, i.e., $|z_{W_1}|$ is tempered. Note that similar techniques can be used to show that z_{W_1} is defined on a θ–invariant set of full measure.

Finally, equations for the generalized spatial derivatives of z_{W_1} can be investigated if we suppose that the covariance Q^{W_1} is sufficiently regular. Conditions are formulated in DaPrato and Zabczyk [15] Theorem 13.3.1. In particular, ∇z_{W_1} is well defined and tempered.

3. Transformation of the quasigeostrophic equation

We return to the QG vorticity equation (2), in which we now write $\Delta\psi$ for the vorticity. That is, we consider

$$\frac{d\Delta\psi}{dt} + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi + \dot{W}_2(\omega, t), \tag{12}$$

with a non zero boundary condition

$$\frac{\partial}{\partial n}\Delta\psi(t) = \dot{W}_1(\omega, t), \quad \psi = 0 \quad \text{on } \partial D, \tag{13}$$

that involves a white noise $\dot{W}_1$ on the boundary ∂D as described in the previous section. In addition, the wind forcing white noise $\dot{W}_2$ is based on a temporally two–sided noise adapted Wiener process W_2 with values in V and covariance Q^{W_2} such that $\mathrm{tr}_V Q^{W_2} < \infty$. In particular, W_1 and W_2 are assumed to be independent.
We can now define a metric dynamical system with the properties of our white noise terms. For Ω we choose an appropriate subset of the function space $C_0(\mathbb{R}, U) \times C_0(\mathbb{R}, V)$ with the usual Borel σ–algebra of a Fréchet space, i.e., an element ω is a continuous path from $\mathbb{R}$ into (U, V) with $\omega(0) = 0$. Then we take $\mathbb{P} = P^{W_1} \otimes P^{W_2}$ to be the product measure of the Wiener measures corresponding to W_1 and W_2, which is ergodic since both P^{W_1} and P^{W_2} are ergodic. The flow θ on Ω is defined in terms of shift operators θ applied to the sample paths of W_1 and W_2.

The above QG equation has structural similarities to equations of Navier–Stokes type. To be able to adapt well known results of such equations, we need to replace these boundary conditions by zero boundary conditions, which is possible with particular types of stationary transformations; see Crauel and Flandoli [5] or Brannan, Duan and Wanner [4], or in a more general context Keller and Schmalfuss [11] or Imkeller and Schmalfuss [10]. In particular, we transform (12) into

$$\begin{aligned} &\frac{d}{dt}u + J(G(u), u) + \beta G(u)_x = \nu\Delta u - r\,u + \dot{W}_2(\omega, t) \\ &\frac{\partial}{\partial n}u(t)| = \dot{W}_1(\omega, t) \quad \text{on } \partial D \end{aligned} \tag{14}$$

where G is the solution operator of the boundary value problem (cf. (9))

$$\Delta\psi = u, \qquad \psi|_{\partial D} = 0,$$

i.e., with the solution $\psi = G(u)$. We consider equation (14) as an evolution equation on the triple $V \subset H \subset V'$ where V' is the dual space of V.

The properties of the nonlinear term of equation (14) follow from those of the bilinear operator $B : L_2(D) \times W_2^1(D) \to V'$ defined by

$$B(v_1, v_2) = J(G(v_1), v_2). \tag{15}$$

Lemma 3.1. *B is a well defined, continuous operator and*

$$i) \qquad \langle B(v_1, v_2), v_3\rangle = -\langle B(v_1, v_3), v_2\rangle,$$

$$ii) \qquad \langle B(v_1, v_2), v_2\rangle = 0.$$

for $v_1 \in L_2(D)$, $v_2 \in W_2^1(D)$, $v_3 \in V$

Proof. There exist positive constants c, c' and c_B such that for any $v_1 \in L_2(D)$, $v_2 \in W_2^1(D) \subset L_4(D)$ and $v_3 \in V \subset L_4(D)$ we have

$$\begin{aligned}
|\langle B(v_1, v_2), v_3\rangle| &= \left|\int_D (G(v_1)_x v_{2y} v_3 - G(v_1)_y v_{2x} v_3) dD\right| \\
&\le c\|\nabla G(v_1)\|_{L_4(D)} |\nabla v_2| \|v_3\|_{L_4(D)} \\
&\le c'\|\nabla G(v_1)\|_{W_2^1(D)} |\nabla v_2| \|v_3\| \\
&\le c_B |v_1| |\nabla v_2| \|v_3\|,
\end{aligned}$$

which implies that B is well defined and continuous.
Property $i)$ follows from the integration by parts formula:

$$\begin{aligned}
&\int_D G(v_1)_x v_{2\,y} v_3 dD - \int_D G(v_1)_y v_{2\,x} v_3 dD \\
&\quad = -\int_D G(v_1)_x v_{3\,y} v_2 dD + \int_D G(v_1)_y v_{3\,x} v_2 dD \\
&\qquad + \int_{\partial D} G(v_1)_x v_2 v_3 \cos(n, y) dS - \int_{\partial D} G(v_1)_y v_2 v_3 \cos(n, x) dS \\
&\quad = -\langle B(v_1, v_3), v_2\rangle
\end{aligned}$$

because the boundary integrals are zero. Indeed, for two sides of ∂D these integrals are zero by the orthogonality of n and the direction of the derivative. For the other both sides the integrals are also zero. For example, for the first integral we have by the properties of G

$$G(v_1)[x, 1] = G(v_1)[x, 0] = 0,$$

hence $G(v_1)_x[x, 1] = G(v_1)_x[x, 0] = 0$.
Property $ii)$ is a consequence of the antisymmetric nature of property $i)$. □

Remark 3.2. If $v_1 \in V$, which one can assume to be the solution of (14), then we can similarly get that $B(v_1, v_1) \in H \subset L_2(D)$. This shows that we can split up the solution of the original equation into a special constant part plus the remaining part. Similarly we get $G(v_1)_x \in H$.

Equation (14) is similar to the equations of the Navier–Stokes type. Indeed, the Laplace operator term in (14) is also present in the Navier–Stokes equations (see Temam [20]), while the bilinear operator B defined by (15) has similar properties (actually, a bit stronger) to the bilinear operator defining the nonlinearity of the 2–dimensional Navier–Stokes equations. It thus follows from the general theory of the stochastic Navier–Stokes equation that (14) has a unique solution, see for instance Schmalfuss [16]. The linear terms ru and $\beta G(u)_x$ appearing in (14) but not in the Navier–Stokes equation are not essential for a proof of existence and uniqueness. See Brannan, Duan and Wanner [4] for another proof of existence and uniqueness based on mild solutions.

4. The stationary solution

We now transform the stochastic evolution equation (14) into a random evolution equation in $V \subset H \subset V'$, i.e., with stationary random coefficients rather than white noise driving or boundary terms. This will make it easier to find a forward invariant random set on which we can verify an appropriate Lipschitz condition. We introduce the random variable

$$z_{W_2}(\omega) = \int_{-\infty}^{0} S(-\tau)dW_2(\omega, \tau) \in V,$$

which we note without proof is a tempered random variable on a θ–invariant set of full measure. We also assume that W_2 (hence Q^{W_2}) is sufficiently regular such that the Neumann boundary condition is fulfilled. Since z_{W_2} fulfills the Neumann boundary condition there is no influence to the boundary condition of (14).
We consider the random evolution equation

$$\begin{aligned} &\frac{d}{dt}z(t,\omega) + B(z,z) + Az + \beta\, G(z)_x + rz \\ &\qquad = -\,B(z, z_{W_1}(\theta_t\omega) + z_{W_2}(\theta_t\omega)) - B(z_{W_1}(\theta_t\omega) + z_{W_2}(\theta_t\omega), z) \\ &\qquad\quad - B(z_{W_1}(\theta_t\omega) + z_{W_2}(\theta_t\omega), z_{W_1}(\theta_t\omega) + z_{W_2}(\theta_t\omega)) \\ &\qquad\quad - \beta G(z_{W_1}(\theta_t\omega) + z_{W_2}(\theta_t\omega))_x - r(z_{W_1}(\theta_t\omega) + z_{W_2}(\theta_t\omega)) \end{aligned} \tag{16}$$

with $z(0) = z_0 \in H$.

Lemma 4.1. *The random evolution equation (16) has a unique solution for any initial condition $z_0 \in H$ and this solution defines a random dynamical system with* *respect to the metric dynamical system θ introduced in Section 3 for which the associated cocycle mapping is defined by $(t, \omega, z_0) \mapsto z(t, \omega, z_0)$.*

For the proof of this lemma we can use the fact that equation (16) is quite similar to the Navier–Stokes equation. Although some linear terms are also present, similar a priori estimates can be obtained to those in Temam ([20], Chapter III) or Bensoussan and Temam [2]. Because of the properties of the operator B introduced in the previous section. Moreover, the random terms appearing inside the coefficients of equation (16) are given by stationary processes, so we obtain a random dynamical system, see Arnold ([1], page 58).

Remark 4.2. To see that $t \mapsto z(t, \omega, z_0)$ is continuous for any $z_0 \in H$ and $\omega \in \Omega$ we can use Lemma III.1.2 in [20] since the solution of equation (16) satisfies

$$\int_0^t \|z(\tau, \omega, z_0)\|^2 d\tau < \infty$$

for any $z_0 \in H$. Indeed, by the chain rule,

$$\begin{aligned} |z(t)|^2 + 2\nu \int_0^t \|z(\tau)\|^2 d\tau \quad &\leq \quad |z_0|^2 + 2(\beta c_{G,x} - r) \int_0^t |z(\tau)|^2 d\tau \\ &\quad +2c_B \int_0^t |z(\tau)| \|z(\tau)\| |\nabla z_{W_1}(\theta_\tau \omega) + \nabla z_{W_2}(\theta_\tau \omega)| d\tau \\ &\quad +2 \int_0^t \|f(\tau)\|_{-1} \|z(\tau)\| d\tau, \end{aligned}$$

where f consists of all the terms in (16) that do not contain z. Then, using

$$\begin{aligned} 2|z| \|z\| |\nabla z_{W_1} + \nabla z_{W_2}| d\tau \quad &\leq \quad \frac{\nu}{2} \|z\|^2 + \frac{2}{\nu} |z|^2 |\nabla z_{W_1} + \nabla z_{W_2}|^2, \\ 2\|f\|_{-1} \|z\| \quad &\leq \quad \frac{\nu}{2} \|z\|^2 + \frac{2}{\nu} \|f\|_{-1}^2, \end{aligned}$$

the asserted estimate follows by an application of the Gronwall inequality. Moreover, by the properties of the operators A and B, we also have

$$\int_0^t \|z(\tau, \omega, z_0)\|_{V'}^2 d\tau < \infty, \quad z_0 \in H.$$

We now define random isomorphism $i(\omega) : H \to H$ by

$$i(\omega)a = -(z_{W_1}(\omega) + z_{W_2}(\omega)) + a,$$

for which the inverse isomorphism $i^{-1}(\omega)$ is given by

$$i^{-1}(\omega)a = (z_{W_1}(\omega) + z_{W_2}(\omega)) + a.$$

Note that the random variable $i(\omega)a(\omega)$ is tempered for any tempered $a(\omega)$.

Lemma 4.3. *Let $z(\cdot,\omega,z_0)$ be the solution of (16). Then the process*

$$u(t,\omega,u_0)=i^{-1}(\theta_t\omega)\circ z(t,\omega,i(\omega)\circ u_0)$$

solves (14). In particular, u satisfies the boundary conditions (13).

Proof. The assertion follows by replacing z by $u-z_{W_1}(\theta_t\omega)-z_{W_2}(\theta_t\omega)$. □

We will now check in the following Lemmata that the assumptions of the random fixed point theorem 2.1 are satisfied. First, we show that there exists a tempered random set $\mathcal{X}(\omega)$ of (single valued) random variables that will be mapped into itself.

Lemma 4.4. *Let $\mathcal{X}(\omega)$ be the ball $B(0,\rho(\omega))$ in H with center zero and $\mathcal{F}_0$–measurable radius*

$$\begin{aligned}\rho(\omega) &= \Bigg(\int_{-\infty}^{0}\exp\Bigg((\lambda_1\nu-2\beta C_{G,x}+2r)\tau\\ &\quad+\frac{3c_B^2}{\nu}\int_\tau^0|\nabla z_{W_1}(\theta_s\omega)+\nabla z_{W_2}(\theta_s\omega))|^2ds\Bigg)\cdot R(\theta_\tau\omega)d\tau\Bigg)^{\frac{1}{2}},\end{aligned}$$

where

$$\begin{aligned}R(\omega) &= 3\frac{(C_{G,x}\beta+r)^2}{\nu\lambda_1}|z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega)|^2\\ &+ \frac{3c_B^2}{\nu}|z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega)|^2|\nabla z_{W_1}(\theta_t\omega)+\nabla z_{W_2}(\theta_t\omega)|^2,\end{aligned}$$

and suppose that

$$\lambda_1\nu+2r>2C_{G,x}\beta+\frac{3c_B^2}{\nu}\mathbb{E}|\nabla z_{W_1}+\nabla z_{W_2}|^2,$$

where $\lambda_1>0$ is the first eigenvalue of the operator A. Then the random set $\mathcal{X}$ is forward invariant, i.e.,

$$z(t,\omega,\mathcal{X}(\omega))\subset\mathcal{X}(\theta_t\omega),\qquad t\geq 0.$$

Proof. We have to estimate $|z|^2$ for which we need the following relations that are a consequence of Lemma 3.1:

$$2\langle B(z,z),z\rangle = 0, \qquad 2\langle Az,z\rangle = 2\nu\|z\|^2 \geq \nu\|z\|^2 + \nu\lambda_1|z|^2,$$

$$2\beta\langle G(z)_x, z\rangle \leq 2\beta C_{G,x}|z|^2, \qquad 2r\langle z,z\rangle = 2r|z|^2,$$

$$\begin{aligned}2|\langle B(z, z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega)),z\rangle| &\leq 2c_B|z||\nabla z_{W_1}(\theta_t\omega)+\nabla z_{W_2}(\theta_t\omega))|\|z\|\\ &\leq \frac{3c_B^2}{\nu}|\nabla z_{W_1}(\theta_t\omega)+\nabla z_{W_2}(\theta_t\omega))|^2|z|^2 + \frac{\nu}{3}\|z\|^2,\end{aligned}$$

$$2\langle B(z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega)),z),z\rangle = 0,$$

$$\begin{aligned}&2|\langle B(z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega), z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega)),z\rangle|\\ &\quad\leq \frac{3c_B^2}{\nu}|z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega)|^2|\nabla z_{W_1}(\theta_t\omega)+\nabla z_{W_2}(\theta_t\omega)|^2 + \frac{\nu}{3}\|z\|^2,\end{aligned}$$

$$\begin{aligned}&2|\langle\beta G(z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega))_x - r\langle z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega),z\rangle|\\ &\quad\leq 3\frac{(C_{G,x}\beta+r)^2}{\nu\lambda_1}|z_{W_1}(\theta_t\omega)+z_{W_2}(\theta_t\omega)|^2 + \frac{\nu\lambda_1}{3}|z|^2.\end{aligned}$$

It can be shown by a comparison argument that $|z(t,\omega,z_0)|^2$ is bounded by a solution of the affine random differential equation:

$$\begin{aligned}&\frac{d\zeta}{dt} + (\lambda_1\nu - 2\beta C_{G,x} + 2r - \frac{3c_B^2}{\nu}(|\nabla z_{W_1}(\theta_t\omega)+\nabla z_{W_2}(\theta_t\omega))|^2))\zeta = R(\theta_t\omega)\\ &\qquad\qquad \zeta(0) = |z_0|^2,\end{aligned} \tag{17}$$

for which the solution is given by a variation of constant formula. A direct calculation verifies that $t \to \rho^2(\theta_t\omega)$ is a solution of equation (17) with initial value $\zeta(0) = \rho^2(\omega)$, which means that ρ^2 is a random fixed point of (17). It thus follows that $z(t,\omega,z_0) \in \mathcal{X}(\theta_t\omega)$ whenever $z_0 \in \mathcal{X}(\omega)$. □

We note that the random variable ρ is tempered (see [17] , page 110), so any selector contained in $\mathcal{X}$ is also tempered.

It remains to check that the contraction condition (7) of the random fixed point theorem holds.

Lemma 4.5. *Suppose that*

$$\begin{aligned}&-\nu\lambda_1 + 2\beta C_{G,x} - 2r + \frac{3c_B^2}{\nu}\mathbb{E}|\nabla z_{W_1} + \nabla z_{W_2}|^2 +\\ &\frac{2c_B^2}{\nu^2}(1+2\lambda_1^{\frac{1}{2}}C_{G,x}\beta)\mathbb{E}\rho^2 + \frac{c_B^2}{\nu}\mathbb{E}\rho^4 + \frac{c_B^2}{\nu}\mathbb{E}|\nabla z_{W_1} + \nabla z_{W_2}|^4 + \frac{2\mathbb{E}R}{\nu} < 0,\end{aligned} \tag{18}$$

where R was defined in Lemma 4.4. Then the contraction condition (7) is fulfilled.

Proof. It follows immediately from Lemma 3.1 with $z_1,\ z_2 \in V$ that

$$\begin{aligned} |\langle B(z_1,z_1)-B(z_2,z_2),z_1-z_2\rangle| &= |\langle B(z_1,z_1),z_2\rangle+\langle B(z_2,z_1),z_2\rangle| \\ &\le |\langle B(z_1-z_2,z_1),z_1-z_2\rangle| \\ &\le c_B|z_1-z_2|\|z_1\|\|z_1-z_2\|. \end{aligned}$$

Set $z_1=z(t,\omega,h_1),\ z_2=z(t,\omega,h_2)\in H$ and $\delta z=z_1-z_2$. By the chain rule we obtain

$$\begin{aligned} |\delta z(1)|^2+2\nu\int_0^1\|\delta z\|^2d\tau &\le |h_1-h_2|^2+\int_0^1(2C_{G,x}\beta|\delta z|^2-2r|\delta z|^2 \\ &\quad +2c_B|\nabla z_{W_1}(\theta_s\omega)+\nabla z_{W_2}(\theta_s\omega))|\|\delta z\||\delta z| \\ &\quad +2c_B\|\delta z\|\|z_1\||\delta z|)ds \end{aligned}$$

for $h_1,\ h_2\in\mathcal{X}(\omega)$. It then follows from

$$2c_B|\nabla z_{W_1}(\theta_s\omega)+\nabla z_{W_2}(\theta_s\omega))|\|\delta z\||\delta z|\le\frac{3c_B^2}{\nu}|\nabla z_{W_1}+\nabla z_{W_2}|^2|\delta z|^2+\frac{\nu}{2}\|\delta z\|^2$$

$$2c_B\|\delta z\|\|z_1\||\delta z|\le\frac{\nu}{2}\|\delta z\|^2+\frac{2c_B^2}{\nu}\|z_1\|^2|\delta z|^2,\qquad \nu\|z\|^2\ge\nu\lambda:1|z|^2$$

that

$$\frac{1}{2}\mathbb{E}\sup_{h_1\neq h_2\in\mathcal{X}(\omega)}\log\frac{|z(1,\omega,h_1)-z(1,\omega,h_2)|^2}{|h_1-h_2|^2}$$

is less than or equal to the expression on the left hand side of inequality (18). Here we have used the fact that

$$\begin{aligned} \frac{2c_B^2}{\nu}\int_0^1\|z_1(t,\omega,z_0)\|^2d\tau &\le \frac{2c_B^2}{\nu^2}\Big(\rho(\omega)^2+(2C_{G,x}\beta-2r)\int_0^1\rho(\theta_\tau\omega)^2d\tau \\ &\quad +\frac{c_B^2}{\nu}\int_0^1\rho(\theta_\tau\omega)^4d\tau \\ &\quad +\frac{c_B^2}{\nu}\int_0^1|\nabla z_{W_1}(\theta_\tau\omega)+\nabla z_{W_2}(\theta_\tau\omega)|^4d\tau+\frac{2\int_0^1R(\theta_\tau\omega)d\tau}{\nu}\Big) \end{aligned}$$

for $z_0\in\mathcal{X}(\omega)$. □

Remark 4.6. Note that the assumption of Lemma 4.5 is sufficient for (4.4).

The crucial point for the assumptions of the last lemma is to show at least for large ν and for small $\mathrm{tr}_U Q^{W_1}$ and $\mathrm{tr}_{W_2}Q^V$ that the random variables ρ^2 and ρ^4 have finite and sufficiently small expectations. In addition, the finiteness of the expectation of these random variables ensures that (8) is satisfied.

Lemma 4.7. *The expectations of ρ^2 and ρ^4 are sufficiently small when ν is sufficiently large and $\mathrm{tr}_U Q^{W_1}$ and $\mathrm{tr}_V Q^{W_2}$ are sufficiently small.*

We give only a brief comment on the proof of this very technical lemma. The essential ingredient is that z_{W_2} and ∇z_{W_1} are Gaußsian processes, so R has finite moments of arbitrary order and

$$E \exp\left(\alpha_1 \int_0^t |\nabla z_{W_1}|^2 d\tau\right) \le e^{\alpha_2 t}, \tag{19}$$

and similarly for ∇z_{W_2}. The constant $\alpha_1 > 0$ depends of the data ν, λ_1, $\cdots$ of the problem. The assertion of the Lemma follows if α_2 is sufficiently small, which can be controlled by the traces of Q^{W_2} and Q^{W_1}. Finally, to obtain a finite dimensional version of the estimate (19) we refer to Hasminskiĭ ([8], page 37, Lemma 7.2), where we need the main assumption

$$E\,\mathrm{tr}_{\mathrm{H}}(\nabla z_{W_1}(\theta_{t_1}\omega) \otimes \nabla z_{W_1}(\theta_{t_2}\omega)) \le c\, e^{-\lambda_1 |t_1 - t_2|}$$

for an appropriate constant c. The variable z_{W_2} can be handled similarly.

Summarising, we have

Theorem 4.8. *Suppose that the assumption of Lemma 4.5 is satisfied and let z^* be the random fixed point of the random dynamical system generated by (16). Then there exists a random fixed point for (14) that attracts the states of the phase space exponentially fast.*

Indeed, the random variable u that generates an exponentially stable stationary solution is given by

$$u^*(\omega) = z^*(\omega) + z_{W_2}(\omega) + z_{W_1}(\omega).$$

5. Discussions

We have shown that, under suitable conditions on the random forcing, random boundary conditions, viscosity, Ekman constant and Coriolis parameter, all quasi-geostrophic motion approach a unique stationary state exponentially fast as time goes to infinity. In deterministic systems a high level of stability is obtained when there is an exponential attractor which attracts trajectories exponentially fast. In some situations this attractor is a single point (point attractor) which describes the laminar behavior of the flow. We are looking for such stability in the case of quasigeostrophic fluid motion under random perturbations. In particular, we find a random attractor which is defined by a single random variable. This random variable attracts all other quasigeostrophic motion exponentially fast. This random variable corresponds to a unique invariant measure, which is the Dirac measure with the random variable as the random mass point. The corresponding stationary Markov measure is the expectation of this random Dirac measure.

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