THE SYMMETRY OF
OPTIMALLY DENSE PACKINGS

by

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Abstract

This is a slightly expanded version of a talk given at the János Bolyai Conference on Hyperbolic Geometry, held in Budapest in July, 2002. The general subject of the talk was the densest packings of simple bodies, for instance spheres or polyhedra, in Euclidean or hyperbolic spaces, and describes recent joint work with Lewis Bowen. One of the main points was to report on our solution of the old problem of treating optimally dense packings of bodies in hyperbolic spaces. The other was to describe the general connection between aperiodicity and nonuniqueness in problems of optimal density.

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I. Packings of Euclidean space

For motivation we begin with packings of regular pentagons in the Euclidean plane, $\mathbb{E}^2$. First we recall that by a "packing" p of pentagons in a square C, we mean a collection of congruent copies, pent_j, of such a body, all contained in C and with pairwise disjoint interiors. By the "density" of such a packing p we mean:

$$\frac{\sum_j \mathrm{volume}(\mathrm{pent}_j)}{\mathrm{volume}(C)}. \tag{1}$$

It is clear that for given square C there exists some maximum possible value of this density, over all p. We are however more interested in an optimum density in regions of infinite volume rather than C, and therefore we need a more sophisticated definition.

To analyze the optimum density in the whole of $\mathbb{E}^2$ we proceed as follows. First, for each square C and packing p of the whole plane, we consider the relative density

$$d_C(p) \equiv \frac{\sum_j \mathrm{volume}(\mathrm{pent}_j \cap C)}{\mathrm{volume}(C)} \tag{2}$$

and then obtain a density for p as

$$d(p) \equiv \lim_C \frac{\sum_j \mathrm{volume}(\mathrm{pent}_j \cap C)}{\mathrm{volume}(C)}, \tag{3}$$

in which we allow C to grow so as to contain every point of $\mathbb{E}^2$.

It is not hard to construct packings p for which this limiting density $d(p)$ does not exist, for instance by constructing p to have arbitrarily large regions empty of pentagons, so that the relative density oscillates instead of having a limit. This is an essential feature of analyzing density in spaces of infinite volume. Density is inherently a *global* quantity, and fundamentally requires a formula somewhat like (3) for its definition [FeK]. As a consequence we are trying to optimize $d(p)$ over packings p even though $d(p)$ is undefined for some p. However this does not prevent us from showing the existence of a convincing optimal density for our pentagon problem, for instance as follows.

First consider a sequence of squares C_n with sides of length n. It is easy to show the existence of packings p_n of $\mathbb{E}^2$ which achieve a maximum for the relative density $d_{C_n}(\cdot)$. We then trim p_n by removing all pentagons from it which have nonempty intersection with the complement of C_n, and "periodize" the result by appropriate translations, obtaining a packing $\tilde{p}_n$ invariant under two perpendicular translations of length n. For $n >> 1$ the density of $\tilde{p}_n$ is still reasonably high relative to any square of edge length n since the only relevant loss is those pentagons lying on the boundary of C_n (and its translates), and the volume of these is negligible for purposes of density, for large n. It is then easy to show that the density of $\tilde{p}_n$ has a well defined limit as $n \to \infty$, and it is reasonable to accept this limit as the optimum density of regular pentagons in $\mathbb{E}^2$. And finally, it is also easy to show the existence of a packing which has this value as a well defined density, in the sense of (3).

The above technique allows us not only to prove the *existence* of an optimal density (for regular pentagons or other bodies) but even to estimate its value – though not to actually determine the optimum value. In fact it is difficult to determine the optimal density for most simple bodies. One of the first interesting examples was that of unit disks in $\mathbb{E}^2$. The history of this, culiminating in the fully acceptable proof of L. Fejes Toth in 1940, is interesting; see [Rog], [Fe5].

Of particular relevance here are the optimal packings of $\mathbb{E}^2$ by congruent copies of the two bodies in Figure 1, known as the kite and dart, introduced by Roger Penrose in 1977 [Gar]. It is possible to construct tilings of the plane with these bodies (see Figure 2), which are evidently the densest packings. (See [Rad] for a general introduction to the mathematics of these sorts of tilings.) The relevant point however is that *every* such optimal packing/tiling of kites and darts has "low symmetry": the symmetry group of such a packing does not have a compact fundamental domain in $\mathbb{E}^2$. (A packing is called "periodic" (resp. "nonperiodic") if it has (resp. does not have) a symmetry group with compact fundamental domain, and we say an optimal packing problem is "aperiodic" if *all* its optimally dense packings are nonperiodic.)

One of our main points is that aperiodicity is strongly connected to the uniqueness of the packing problem.

Theorem 1. If there is only one optimally dense packing of $\mathbb{E}^d$ or $\mathbb{H}^d$, up to congruence, by congruent copies of bodies from some fixed, finite collection, then that packing must have a symmetry group with compact fundamental domain.

(We sketch the proof later, after introducing some notation.) So we may conclude for instance that there are many kite and dart tilings, that is, many equivalence classes modulo congruence. Another interesting feature of the kite and dart tilings, besides their low symmetry, is that they are all "locally identical": every bounded region of one such tiling has a congruent copy in every other such tiling. So the nonuniqueness in this case cannot be seen locally – it is an essentially global feature. Of course the optimally dense packing for a given collection of bodies may fail to be unique up to congruence in a simpler way: for unit spheres in $\mathbb{E}^3$ this follows from an accidental degeneracy wherein optimal packings can have different bounded regions.

We summarize some of the above as follows. If we consider the optimization problem in which we seek to optimize the density of packings of $\mathbb{E}^d$ by congruent copies bodies from some some fixed, finite collection, we see from the above that there always *exists* an optimally dense packing, but that the solution may not be unique (up to congruence) — not just because of accidental (local) degeneracy as in sphere packings of $\mathbb{E}^3$, but more fundamentally (globally), as in the kite and dart tilings. We will see later that other considerations suggest a small modification of the framework of this problem, which may eliminate the sort of nonuniquess associated with aperiodicity.

II. Packings of hyperbolic space

Roger Penrose also introduced another example of interest here, in 1978 [Pen]. Congruent copies of the body shown in Figure 3 can tile the hyperbolic upper-half plane, as in Figure 4, but *only nonperiodically*, the latter following from an elegant argument. If a tiling by copies of that body had a symmetry group with compact fundamental domain, that domain would have to contain as many dents as bumps, since they would be paired up. But the body under consideration has two (inward, triangular) dents and one (outward, triangular) bump, and since the compact domain can only contain finitely many bodies, there is an automatic imbalance! (This example led to the following interesting works: [BlW], [Moz], [MaM], [Goo].)

We are bringing up this example in the context of optimally dense packings in part to show a connection with the packings of congruent disks published a few years earlier by Karoly Böröczky [Bo1]; see Figure 5. Those disk packings had been influential in convincing the discrete geometry community that *there could not in fact be a consistent theory of densest packings in hyperbolic spaces*; see [Fe1-5], [Bo1-2], [BoF], [FeK], [FKK], [Kup]. Basically, there have been attempts for at least 50 years to deal with the global notion of density in hyperbolic space, in particular for packings of congruent spheres, and the effort has essentially died out, and turned to various local substitutes. We sketch the two-part argument against a theory of optimal density as follows.

First we note that the technique used above, to prove the *existence* of optimally dense packings of Euclidean spaces, does not have a simple analog for hyperbolic spaces; the weak point is where the bodies straddling the boundary of the square C_n are thrown out. In hyperbolic spaces the role of the C_n could naturally be played by compact fundamental domains, but the problem is that in a hyperbolic space a large fundamental domain has a finite fraction of its volume near its boundary, so the bodies straddling the boundary would not be negligible.

Of course this was only one approach to proving existence of optimally dense packings. However this boundary phenomenon underscores the intrinsic difficulty of proving that limits of the form (3) would exist for any but the simplest sorts of packings.

The other part of the argument, based on Böröczky's packing, is as follows. In Figure 6 the packing is displayed together with a copy of Penrose's tiling – a tiling by congruent bodies. Consider the two regions in dark outline in Figure 7, each made from three copies of the Penrose tile (and therefore the two regions have the same volume). From each of these we could, in an obvious way, make congruent copies to produce a tiling of the hyperbolic plane. But each of these tilings then suggests an obvious value for the density of the packing: the relative density in each of its tiles of the disks contained in the tile. But this would suggest one density twice the value of the other!

A similar inconsistency can be demonstrated between the densities based on Voronoi tilings and Dirichlet tilings associated with Böröczky's packing.

In summary, one part of the difficulty of dealing with optimally dense packings in hyperbolic space has been proving the existence of limiting densities of the form (3) for nonperiodic packings, and the other part was the inconsistencies that arise when trying to avoid the limit definition of density by appealing to densities relative to associated tilings.

The other of our main goals here, besides the connection between aperiodicity and nonuniqueness, is to show how a standard part of mathematics, ergodic theory, can in fact be used to prove the existence of limiting densities of the form (3) for complicated (nonperiodic) packings, enough to produce a useful theory of optimally dense packings. We now outline this approach, for the simple case of packings of hyperbolic space, $\mathbb{H}^d$, by balls of fixed radius R; see [BR1] and [BR2] for details.

Consider the space X_R of all possible "relatively dense" packings of hyperbolic space by balls of fixed radius R, and put a metric topology on X_R such that convergence of a sequence of packings corresponds to uniform convergence on compact subsets of hyperbolic space. (A packing of R-balls is relatively dense if every sphere of radius R intersects a ball in the packing.) Such a metric makes X_R compact, and makes continuous the natural action on X_R of the group $\mathcal{G}^d$ of rigid motions of hyperbolic space. We then consider Borel probability measures on X_R which are invariant under $\mathcal{G}^d$.

As examples of such measures, one can identify the orbit $O(p)$ of a periodic packing p with the quotient of $\mathcal{G}^d$ by the symmetry group of p, and thus project Haar measure on the $\mathcal{G}^d$ to $O(p)$. This idea is easily exploited to prove Theorem 1, first for cofinite symmetry groups and then using results on complete saturation [Bow] to handle the cocompact case.

We define the density $d(\mu)$ of each invariant measure μ on X_R as $\mu(A)$, where A is the following set of packings:

$$A \equiv \{p \in X_R \,|\, \text{the origin of } \mathbb{H}^d \text{ is in a ball in } p\}. \tag{4}$$

(It is easy to see from the invariance of μ that this definition is independent of the choice of origin.) We may now introduce the key notion of optimal density.

Definition 1. A probability measure $\bar{\mu}$ on the space X_R of packings, ergodic under rigid motions, is "optimally dense" if $d(\bar{\mu}) = \sup_\mu d(\mu) = \sup_\mu \mu(A)$; the value $d(\bar{\mu})$ is the "optimal density" for packing balls of radius R.

(An invariant measure μ is "ergodic" if it cannot be expressed as an average: $\mu = a_1\mu_1 + a_2\mu_2$, with a_1, $a_2 > 0$ and μ_1, μ_2 invariant.) It is easy to show the existence of such optimal measures. The terminology is then justified – that is, related to the density of packings in the sense of (3) – by recent ergodic theorems of Nevo et al [Nev], [NeS], as follows.

Theorem 2 (Nevo et al). Let μ be a Borel probability measure on the compact metric space X, ergodic with respect to an action of the isometry group $\mathcal{G}^d$ of $\mathbb{H}^d$. For any open subset A of X,

$$\begin{aligned} \mu(A) &= \int_X \chi_A(q)\, d\mu(q) \\ &= \lim_{k\to\infty} \frac{1}{\nu[\mathcal{G}^d(k)]} \int_{\mathcal{G}^d(k)} \chi_A[g(p)]\, d\nu(g) \end{aligned} \tag{5}$$

for μ-a.e. p, where χ_A is the indicator function for A, ν is Haar measure on $\mathcal{G}^d$ and:

$$\mathcal{G}^d(k) = \{g \in \mathcal{G}^d \,:\, m_{\mathbb{H}}[g(O), O] < k\}, \tag{6}$$

where $m_{\mathbb{H}}$ is the metric on, and O the origin in, $\mathbb{H}^d$.

We use this ergodic theorem as follows. With A as in (4), the theorem shows the existence of a limiting density, in the sense of (3), (with expanding spheres instead of squares) for μ-a.e. packing p. (We improved Nevo's theorem slightly in [BR2] to obtain existence relative to any "origin" O.)

Now that we have a mechanism to prove the existence of limiting densities in packings, we define "optimally dense packings" as those packings which reproduce an optimally dense measure.

Definition 2. A packing $p \in X_R$ is "optimally dense" if for some optimally dense measure $\bar{\mu}$, and all continuous functions f on X_R:

$$\int_{X_R} f(q)\, d\bar{\mu}(q) = \lim_{k\to\infty} \frac{1}{\nu[\mathcal{G}^d(k)]} \int_{\mathcal{G}^d(k)} f[g(p)]\, d\nu(g). \tag{7}$$

It follows easily from the above that for every optimally dense $\bar{\mu}$, $\bar{\mu}$-almost every packing is optimally dense (and in particular optimally dense packings exist!)

From the next result we see that optimally dense packings need not be periodic.

Theorem 3. For most $R > 0$ (all but countably many), the densest packing of $\mathbb{H}^d$ by spheres of radius R is not unique (up to rigid motion) – in fact for most fixed radii R the sphere packing problem in $\mathbb{H}^d$ is aperiodic.

III. Conclusion

It is appropriate to step back and see what this formalism, introduced to solve the old problem of densest packings of bodies in hyperbolic spaces, has to say about the general problem, for Euclidean as well as hyperbolic spaces. (Our definitions of optimal density are easily shown to agree with the standard ones for packings in Euclidean space – using Birkhoff's pointwise ergodic theorem instead of Nevo's.) As we saw, in Euclidean spaces the phenomenon of aperiodicity, as exemplified for instance by the Penrose kite and dart tilings, could be understood as a certain (global) form of nonuniqueness, up to congruence, of the optimally dense packings of some set of bodies. For natural reasons, aperiodicity was first noted when the bodies were polyhedra and the densest packings were tilings. (See [Sch], which was not sufficiently appreciated when first published.) We now see that in hyperbolic space this same phenomenon already appears in the simpler setting of densest packings of congruent spheres. And the ergodic theory formalism, which we introduced to overcome the conceptual difficulties of densest packings in hyperbolic space, also suggests that the nonuniqueness associated with aperiodicity could be eliminated by reformulating the optimization problem as having its solutions be invariant measures, rather than packings which reproduce such measures. Indeed, it is reasonable to conjecture that, with an appropriate notion of genericity, the problem of optimally dense packings of $\mathbb{E}^d$ or $\mathbb{H}^d$, by copies of a generic finite set of bodies, has a unique invariant measure as solution.

IV. References

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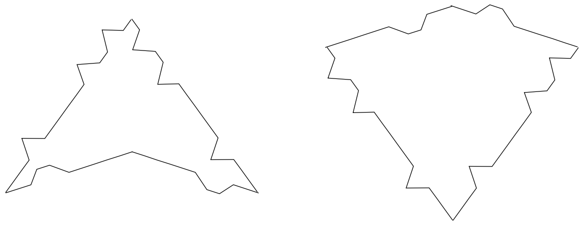

Figure 1. The kite and dart tiles

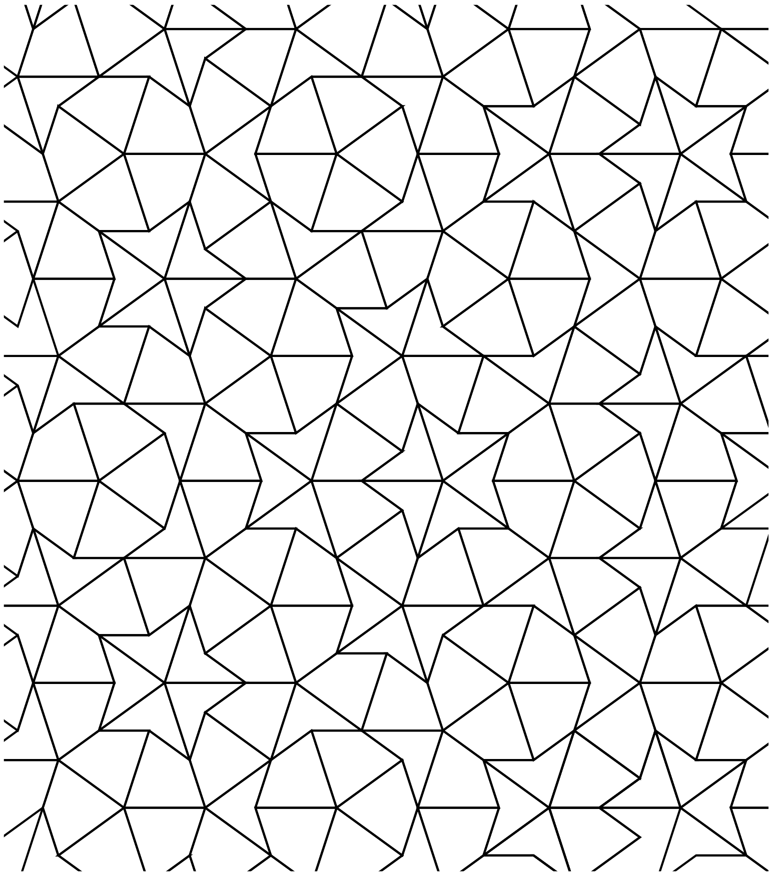

Figure 2. A kite and dart tiling of the Euclidean plane

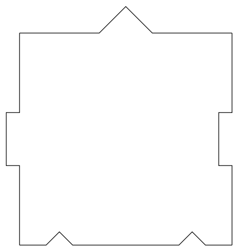

Figure 3. The hyperbolic Penrose tile

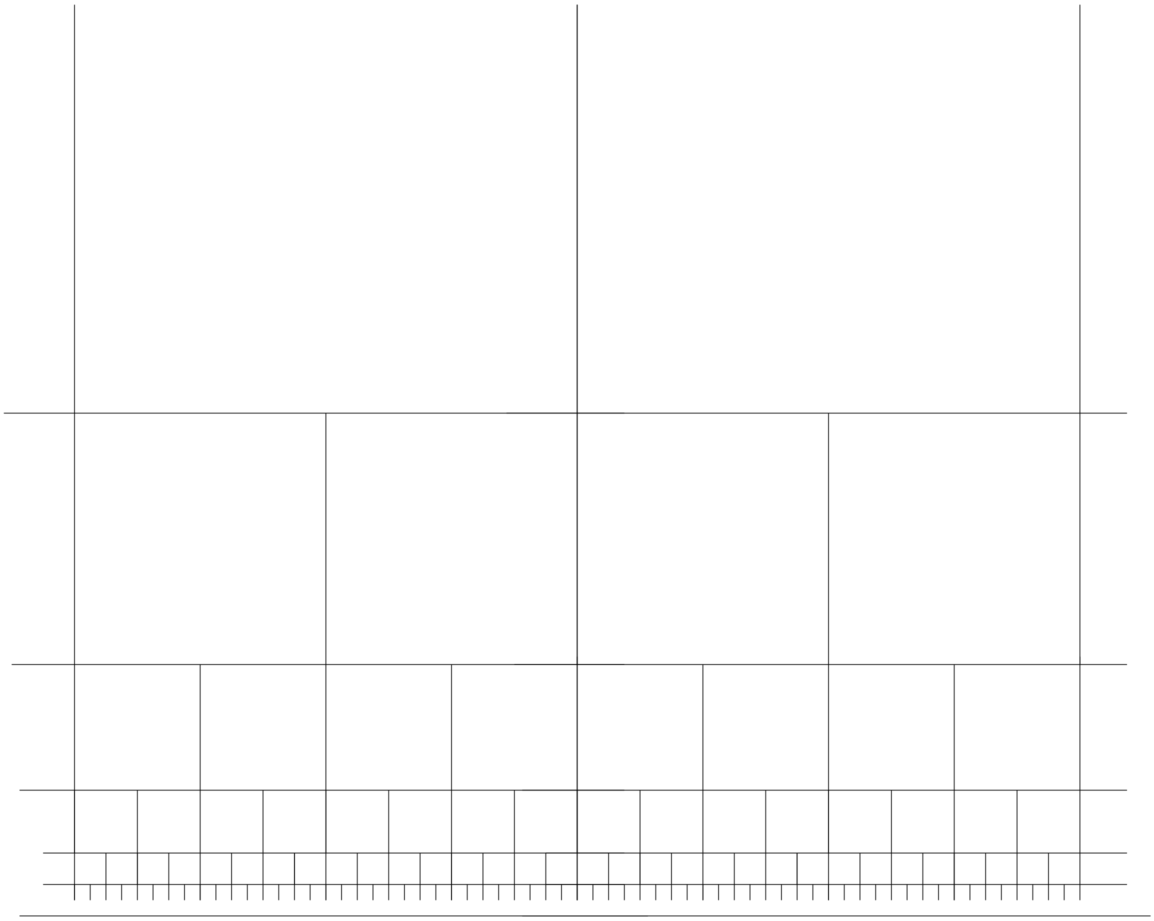

Figure 4. A hyperbolic Penrose tiling

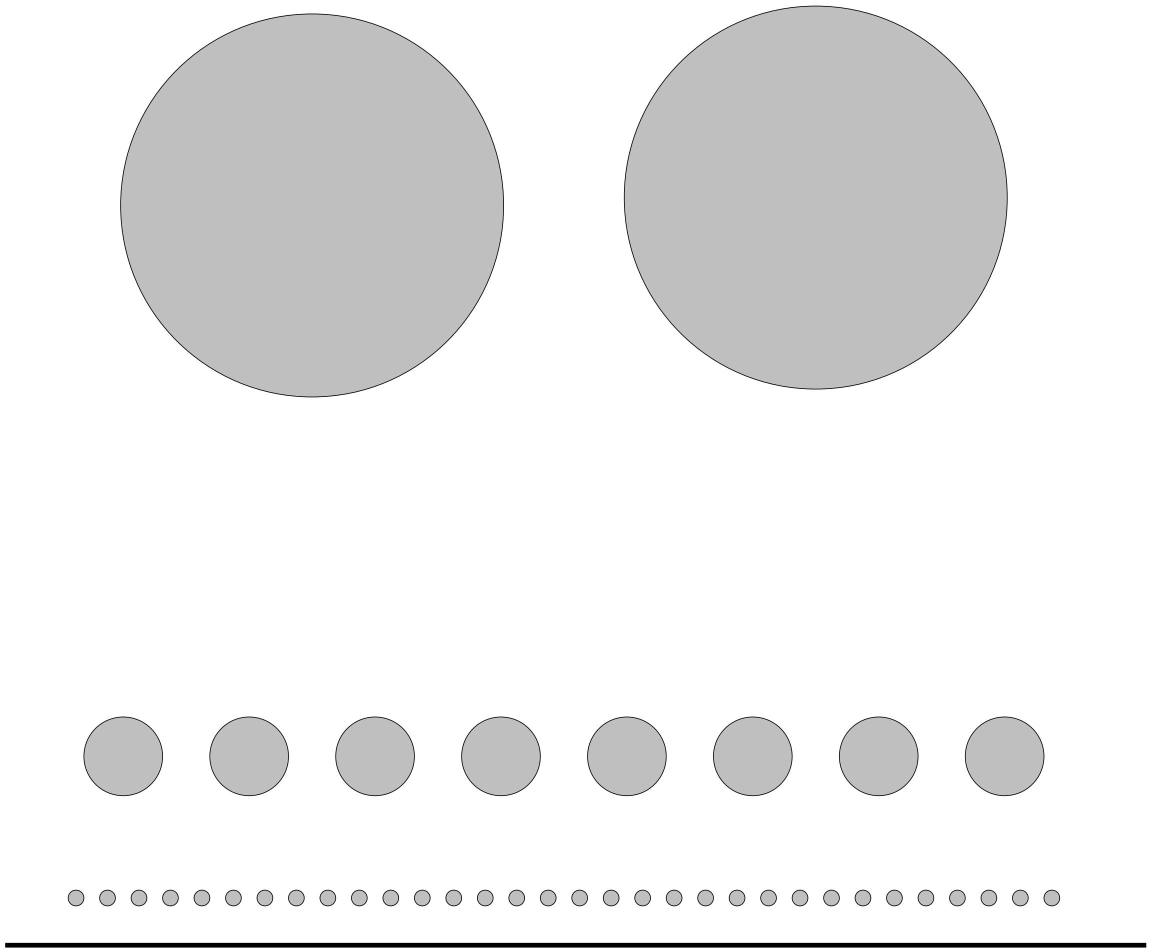

Figure 5. The Böröczky disk packing of the hyperbolic plane

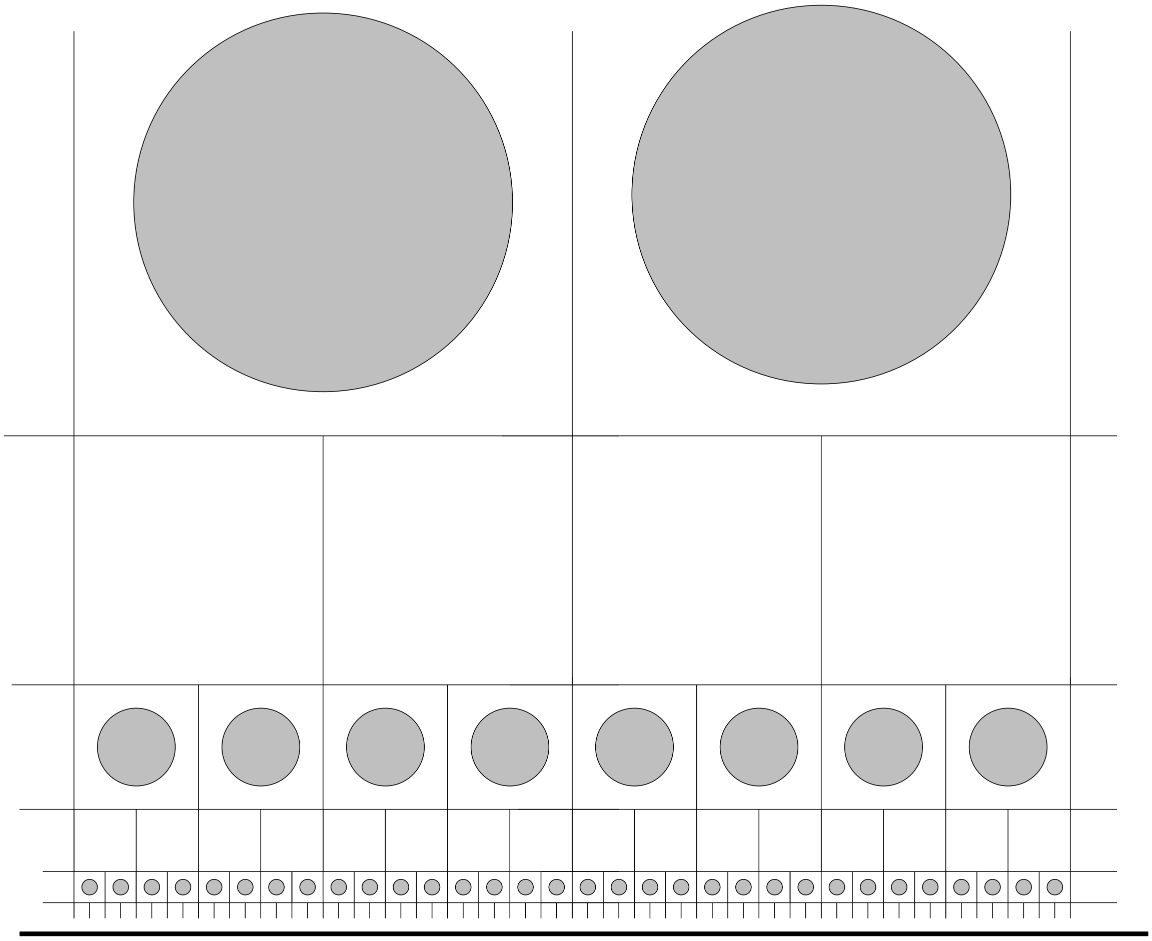

Figure 6. Böröczky's packing of disks, with tiling background

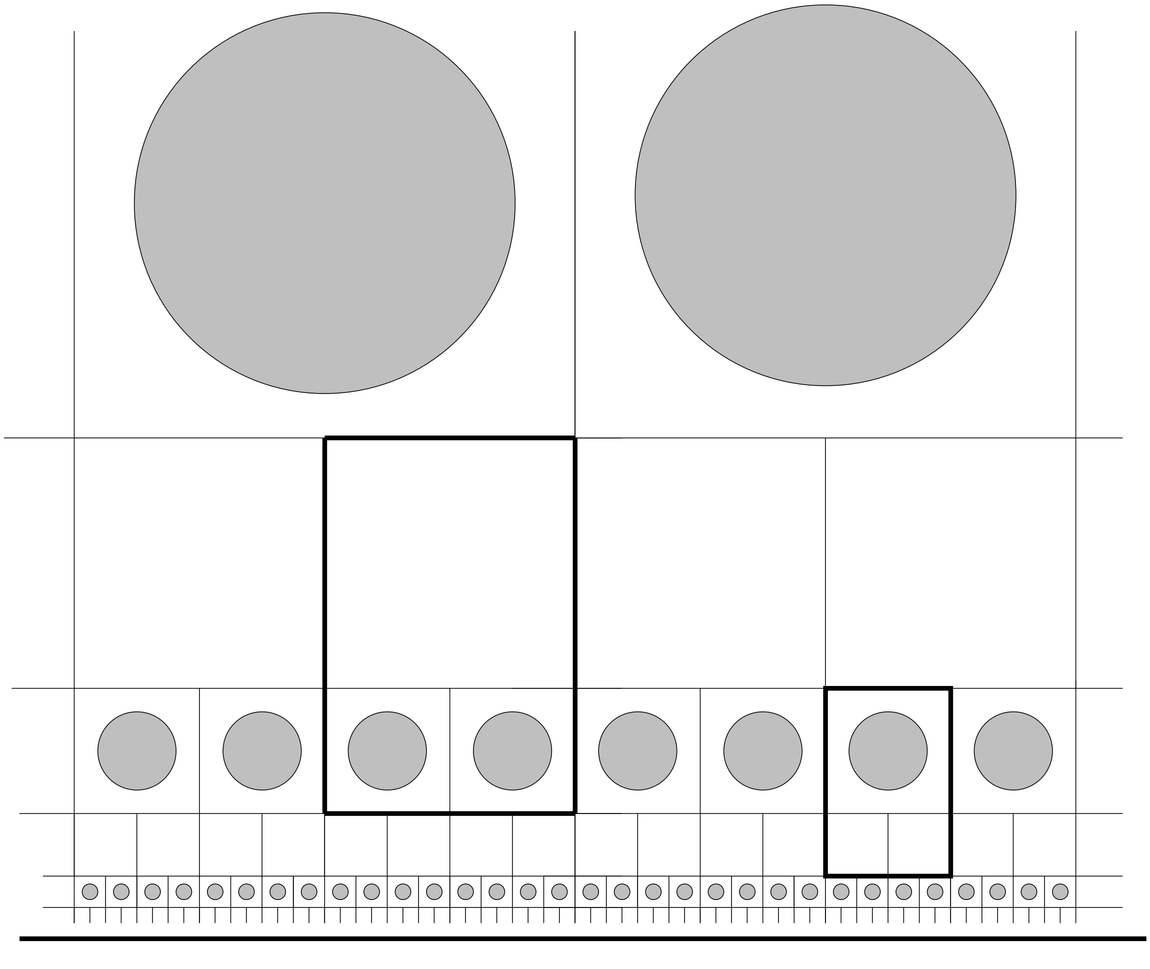

Figure 7. Böröczky's packing with two tiles in dark outline