EIGENVALUE EQUATION FOR THE ENTROPY OF 2–D MARKOV SHIFTS

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Abstract. We find conditions ensuring that the topological entropy of a two-dimensional Markov shifts is an eigenvalue, provided the associated eigenvalue problem has a positive solution. This problem is in general a nonlinear one. We construct explicit examples where our conditions hold, so that the entropy is an eigenvalue associated to a linear problem.

1. Introduction

Contrary to the one–dimensional case, the topological entropy cannot be computed for general multi–dimensional symbolic systems. There are, nevertheless, specific situations where this quantity can actually be computed. It is worth to mention the work of Lind et al. [4], where they establish a relationship between the topological entropy and the so called Malher measure, in the case of subshifts with algebraic structure. The present work is of a more elementary nature. We establish sufficient conditions under which the topological entropy is an eigenvalue associated to a (generally nonlinear) eigenvalue problem. In the same spirit, we can mention the paper of Juang et al. [3], where similar result are obtained for matrix subshifts.

The paper is organized as follows. The next section contains all definitions and known results used in the rest of the paper. Section 3 is devoted to the main results and examples, while in Section 4 contains some final comments.

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2. Preliminaries

2.1. **Markov Shifts.** The configurations of a two–dimensional Markov shift are two–dimensional infinite arrays of elements from a finite alphabet, which we identify with $\mathbb{Z}_p \equiv \{0, 1, \ldots, p-1\}$. Given $\mathcal{P} \subset \mathbb{Z}_p^{\{1,2\}\times\{1,2\}}$, the *Markov shift* it determines is

$$X_{\mathcal{P}} \equiv \left\{ x \in \mathbb{Z}_p^{\mathbb{Z}\times\mathbb{Z}} \; : \; x|_{\{\mathbf{n}(1),\mathbf{n}(1)+1\}\times\{\mathbf{n}(2),\mathbf{n}(2)+1\}} \in \mathcal{P}, \; \forall\, \mathbf{n} \in \mathbb{Z}\times\mathbb{Z} \right\}. \tag{1}$$

More general two–dimensional subshifts are defined in a similar way, using a collection of allowable patters of arbitrary but finite size. When this collection if finite, we have a subshift of finite type. Any finite collection of finite size patterns can be transformed into a finite collection of 2×2–patterns by a standard re–coding procedure (see [7] for details). Thus, any subshift of finite type is equivalent to a Markov shift.

For each $\mathbf{n} \in \mathbb{N}$, let us denote $\mathbb{Z}_p^{\{1,\ldots,\mathbf{n}(1)\}\times\{1,\ldots,\mathbf{n}(2)\}}$ by $\mathbb{Z}_p^{\mathbf{n}}$. An array $\phi \in \mathbb{Z}_p^{\mathbf{n}}$ is $\mathcal{P}$*–admissible* if $\phi|_{\{m,m+1\}\times\{m',m'+1\}} \in \mathcal{P}$ for all $1 \le m < \mathbf{n}(1)$ and $1 \le m' < \mathbf{n}(2)$.

Hence, ϕ is $\mathcal{P}$–admissible if all its 2×2 sub–arrays belong to $\mathcal{P}$. All arrays in $\mathbb{Z}_p^{(1,2)} \bigcup \mathbb{Z}_p^{(2,1)}$ are considered $\mathcal{P}$–admissible.

The Markov shift $X_{\mathcal{P}}$ is *fully extensible* if for all $n \in \mathbb{N}$ and each $\mathcal{P}$–admissible array $\phi \in \mathbb{Z}_p^{\mathbf{n}}$, there exists a configuration $x^\phi \in X_{\mathcal{P}}$ satisfying

$$x^\phi|_{\{1,\ldots,\mathbf{n}(1)\}\times\{1,\ldots,\mathbf{n}(2)\}} = \phi.$$

The dynamics on $X_{\mathcal{P}}$ is generated by the group of shifts $\{\sigma^{\mathbf{n}} :\ \mathbf{n} \in \mathbb{Z}\times\mathbb{Z}\}$, acting on $X_{\mathcal{P}}$ as follows:

$$(\sigma^{\mathbf{n}} x)(\mathbf{m}) = x(\mathbf{m}+\mathbf{n}),\ \forall\ \mathbf{m},\mathbf{n} \in \mathbb{Z}\times\mathbb{Z}. \tag{2}$$

This group is isomorphic to $\mathbb{Z}\times\mathbb{Z}$.

We will consider Markov shifts satisfying the following transitivity property. We say that $X_{\mathcal{P}}$ is *horizontally transitive* if there exists $x \in X_{\mathcal{P}}$ such that for each $\mathbf{n} \in \mathbb{N}$, the set $\left\{\left(\sigma^{(m,0)}x\right)|_{\{1,\ldots,\mathbf{n}(1)\}\times\{1,\ldots,\mathbf{n}(2)\}} :\ m \in \mathbb{Z},\ \mathbf{n} \in \mathbb{Z}\times\mathbb{Z}\right\}$ contains every $\mathcal{P}$–admissible array.

For fully extensible Markov shifts, horizontal transitivity is equivalent to the following property: for all $n \in \mathbb{N}$ and each $\phi, \phi' \in \mathbb{Z}^{(1,n)}$ $\mathcal{P}$–admissible there exists a $\mathcal{P}$–admissible array $\psi \in \mathbb{Z}_p^{(m,n)}$ such that $\psi|_{\{1\}\times\{1,\ldots,n\}} = \phi$ and $\psi|_{\{m\}\times\{1,\ldots,n\}} = \phi'$.

2.2. **Topological Entropy.** It can be shown, following a standard subadditivity argument, that the limit

$$\lim_{k\to\infty} \frac{\log \#\left\{\phi \in \mathbb{Z}_p^{\mathbf{n}_k} :\ \phi \text{ is } \mathcal{P}\text{–admissible}\right\}}{\mathbf{n}_k(1)\times\mathbf{n}_k(2)} \tag{3}$$

exists and has the same value for all sequence $\{\mathbf{n}_k\}_{k=1}^\infty \subset \mathbb{N}\times\mathbb{N}$, such that $\min(\mathbf{n}_k(1),\mathbf{n}_k(2)) \to \infty$ when $k \to \infty$. For $X_{\mathcal{P}}$ fully extensible, this limit coincides with *the topological entropy of* Γ *on* $X_{\mathcal{P}}$, which we denote $h(X_{\mathcal{P}})$ (see [4] for details). The topological entropy measures the exponential growth of distinguishable Γ–orbits.

All the notions introduced in this subsection can be defined in a similar way in any dimension. We will assume that these notions are well known for Markov shifts of dimension one.

2.3. **Transition Matrices.** For each $n \geq 1$, let $M_n : \mathbb{Z}_p^{(1,n)} \times \mathbb{Z}_p^{(1,n)} \to \{0,1\}$ be such that

$$M_n(\phi,\phi') = \begin{cases} 1 & \text{if } [\phi\ \phi'] \text{ is } \mathcal{P} - \text{admissible}, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where $[\phi\ \phi'] \in \mathbb{Z}_p^{(2,n)}$ denotes the concatenation of ϕ and ϕ'. In particular, M_1 is the $p\times p$–matrix all whose entries are one.

Since $(M_n))^{\mathbf{m}-1}(\phi,\phi)$ is the sum over all the chains $\phi \mapsto \phi_1 \mapsto \cdots \mapsto \phi_{m-2} \to \phi'$ of allowed transitions starting with ϕ and ending by ϕ', it follows that

$$\#\left\{\phi \in \mathbb{Z}_p^{\mathbf{m}} :\ \phi \text{ is } \mathcal{P}\text{–admissible}\right\} = \mathbb{1}_n^\dagger M_n^{m-1}\mathbb{1}_n, \tag{5}$$

where $\mathbb{1}_n$ is the all–ones column vector of dimension p^n.

For each $\mathbf{a} \in \mathbb{Z}_p^{(2,1)}$ and each $n \geq 1$ let $M_n^{\mathbf{a}} : \mathbb{Z}_p^{(1,n)} \times \mathbb{Z}_p^{(1,n)} \to \{0,1\}$ be such that

$$M_n^{\mathbf{a}}(\phi, \phi') = \begin{cases} M_1^{\mathbf{a}}(\phi(n), \phi'(n)) & \text{if } [\phi\ \phi'] \text{ is } \mathcal{P}-\text{admissible}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The incidence sub–matrices $M_n^{\mathbf{a}}$ satisfy the relations

$$M_{n+1}^{\mathbf{a}} = [M_1^{\mathbf{a}}(\mathbf{b}) \times M_n^{\mathbf{b}}]_{\mathbf{b}\in\mathbb{Z}_p^{(2,1)}} \tag{7}$$

$$M_{n+1} = [M_n^{\mathbf{a}}]_{\mathbf{a}\in\mathbb{Z}_p^{(2,1)}}. \tag{8}$$

This notation indicates the decomposition of a matrix of dimension p^{n+1} into p^2 sub–matrices of dimension p^n.

Below we will use the *the Kroneker product*. For $A : \mathbb{Z}_p^{(2,1)} \to \mathbb{N}$ and $B : \mathbb{Z}_p^{(1,n)} \times \mathbb{Z}_p^{(1,n)} \to \mathbb{N}$ it is the matrix $A \otimes B : \mathbb{Z}_p^{(1,n+1)} \times \mathbb{Z}_p^{(1,n+1)} \to \mathbb{N}$ such that

$$(A \otimes B)\left(\begin{bmatrix} a \\ \phi \end{bmatrix}, \begin{bmatrix} a' \\ \phi' \end{bmatrix}\right) = A(a,a')B(\phi,\phi'), \tag{9}$$

for all $a, a' \in \mathbb{Z}_p$ and $\phi, \phi' \in \mathbb{Z}_p^{(1,n)}$.

As we pointed out before, the topological entropy $h(X_{\mathcal{P}})$ is independent of the sequence $\{\mathbf{n}_k\}_{k=1}^{\infty}$ used in the limiting process defining it. In the case of a fully extensible Markov shift $X_{\mathcal{P}}$, as a direct consequence of this fact, and taking into account equation (5), we obtain

$$h(X_{\mathcal{P}}) = \lim_{n\to\infty} \frac{1}{n} \lim_{m\to\infty} \frac{\log\left(\mathbb{I}^{\dagger} M_n^{m-1} \mathbb{I}\right)}{m}.$$

If in addition we assume that $X_{\mathcal{P}}$ is horizontally transitive, then each one of the matrices M_n has to be irreducible (see [1] for details). Then Perron–Frobenius Theory applies, and we obtain

$$h(X_{\mathcal{P}}) = \lim_{n\to\infty} \frac{1}{n} \log \rho(M_n), \tag{10}$$

where $\rho(M)$ denotes the spectral radius of the matrix M. For non–negative irreducible matrices it coincides with the maximal eigenvalue.

2.4. **Thermodynamical Formalism.** Let $X \subset \mathbb{Z}_p^{\mathbb{N}}$ be a one–dimensional Markov shift, and $f : X \to \mathbb{R}$ Hölder continuous. It is proven in [2] that the limit

$$P_X(f) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{\substack{\phi\in\mathbb{Z}_p^{n+1} \\ X-\text{admissible}}} \exp\left(\sum_{j=0}^{n} f(\sigma^j x_\phi)\right)\right)$$

exists, for any $x_\phi \in X$ such that $x|_{\{0,1,\ldots,n\}} = \phi$. The quantity $P_X(f)$ is the *topological pressure of the potential* f. Associated to the potential f, there exists a unique σ–invariant probability measure μ_f characterized by the following inequality. There exists a constant $C \geq 1$ such that

$$C^{-1} \leq \frac{\mu_f\{x \in X :\ x|_{\{0,1,\ldots,n\}} = \phi\}}{\exp\left(\sum_{j=0}^{n} \psi(\sigma^j x_\phi) - nP_X(\psi)\right)} \leq C,$$

for each X–admissible $\phi \in \mathbb{Z}_p^{n+1}$. The measure μ_f is the Gibbs measure associated to the potential f. Two potentials f and f' are *cohomologous* if $\mu_f = \mu_{f'}$.

A relation between the topological pressure and the Gibbs measure associated to a given potential, is given by the *variational principle*

$$P_X(f) \geq \mu(f) + h(\mu) \quad \forall \mu \quad \sigma\text{–invariant}, \tag{11}$$

where $h(\mu)$ denotes the entropy of the measure (see [2]). For X topologically transitive, i. e., when its transition matrix is irreducible, the equality holds only for $\mu = \mu_f$.

3. Entropy as an Eigenvalue

3.1. **Eigenvalues' Recursion.** Denote by $\mathbb{R}_+^{(p,p)}$ the set of positive matrices of dimension $p \times p$, and consider the transformation $R : \mathbb{R}_+^{(p,p)} \to \mathbb{R}_+^{(p,p)}$ such that

$$RM = [\rho(M_1^{\mathbf{a}}) * M]_{\mathbf{a} \in \mathbb{Z}_p^{(2,1)}}, \tag{12}$$

where $*$ stands for the coordinate-wise multiplication.

When for each $n \in \mathbb{N}$ all the transition matrices $M_n^{\mathbf{a}}$ commute, the transformation just defined can be used to recursively compute $\rho(M_n)$ for each $n \in \mathbb{N}$.

Lemma 1. *Assume that the matrices $M_n^{\mathbf{a}}$ commute for each $n \in \mathbb{N}$. Then, $\rho(M_{n+1}) = \rho(R^n\ \mathbb{1}\mathbb{1}^\dagger)$ and $\rho(M_n^{\mathbf{a}}) = (R^n\ \mathbb{1}\mathbb{1}^\dagger)(\mathbf{a})$ for all $n \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}_p^{(2,1)}$.*

Proof. We use the following fact. Suppose that $M = [N^{\mathbf{a}}]_{\mathbf{a} \in \mathbb{Z}_p^{(2,1)}}$ and that the submatrices $N^{\mathbf{a}}$ commute, then $\rho(M) = \rho([\rho(N^{\mathbf{a}})]_{\mathbf{a} \in \mathbb{Z}_p^{(2,1)}})$.

By hypothesis the p^2 submatrices of M_2 commute, then

$$\rho(M_2) = \rho([\rho(M_1^{\mathbf{a}})]_{\mathbf{a} \in \mathbb{Z}_p^{(2,1)}}) = R\ \mathbb{1}\mathbb{1}^\dagger.$$

Note that $(R\ \mathbb{1}\mathbb{1}^\dagger)(\mathbf{a}) = \rho(M_1^{\mathbf{a}})$ for each $\mathbf{a} \in \mathbb{Z}_p^{(2,1)}$. Suppose that for $n \geq 2$ we have $\rho(M_n^{\mathbf{a}}) = (R^n U)(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{Z}_p^{(2,1)}$. Since the matrices $M_n^{\mathbf{b}}$ commute, we have

$$\begin{aligned} \rho(M_{n+1}^{\mathbf{a}}) &= \rho(M_1^{\mathbf{a}}(\mathbf{b}) \times \rho([M_n^{\mathbf{b}}]_{\mathbf{b} \in \mathbb{Z}_p^{(2,1)}})) \\ &= \rho(M_1^{\mathbf{a}} * (R^n\ \mathbb{1}\mathbb{1}^\dagger)) = (R^{n+1}\ \mathbb{1}\mathbb{1}^\dagger)(\mathbf{a}). \end{aligned}$$

We have proved by that $\rho(M_n^{\mathbf{a}}) = (R^n\ \mathbb{1}\mathbb{1}^\dagger)(\mathbf{a})$ for all $n \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}_p^{(2,1)}$. Finally, since $M_{n+1} = [M_n^{\mathbf{a}}]_{\mathbf{a} \in \mathbb{Z}_p^{(2,1)}}$ and the sub–matrices commute, the result follows. □

Denote by $\Delta^{(p,p)} \subset \mathbb{R}_+^{(p,p)}$ the set all $p \times p$ matrices with positive entries and maximal eigenvalue equals to 1, and define the normalized transformation

$$F : \Delta^{(p,p)} \to \Delta^{(p,p)}, \quad FM \equiv \frac{1}{\rho(RM)}\ RM. \tag{13}$$

Let us remark that for a fully extensible and horizontally transitive Markov shift,

$$h(X_{\mathcal{P}}) = \lim_{n \to \infty} \log(\rho(M_{n+1})/\rho(M_n)),$$

whenever this limit exists. In this case, if the sub–matrices $M_n^{\mathbf{a}}$ commute for each $n \in \mathbb{N}$, we have

$$h(X_{\mathcal{P}}) = \lim_{n\to\infty} \log\left(\frac{\rho(R^{n+1}\ 1\!\!\mathrm{I}\!\mathrm{I}^{\dagger})}{\rho(R^{n}\ 1\!\!\mathrm{I}\!\mathrm{I}^{\dagger})}\right) = \lim_{n\to\infty} \log(R\ F^{n}\ 1\!\!\mathrm{I}\!\mathrm{I}^{\dagger}). \tag{14}$$

Below we give conditions ensuring that the limit $F^n\ 1\!\!\mathrm{I}\!\mathrm{I}^{\dagger}$ exists. For this we use the *projective distance* in $\Delta^{(p,p)}$ defined by

$$d(M, M') = \max_{\mathbf{a}\in\mathbb{Z}_p^{(2,1)}} \left\{\log\left(\frac{M(\mathbf{a})}{M'(\mathbf{a})}\right)\right\} - \min_{\mathbf{a}\in\mathbb{Z}_p^{(2,1)}} \left\{\log\left(\frac{M(\mathbf{a})}{M'(\mathbf{a})}\right)\right\}. \tag{15}$$

If the limit $M_f = \lim_{n\to\infty} F^n\ 1\!\!\mathrm{I}\!\mathrm{I}^{\dagger}$ exists, then it has to be a fixed point for F, i. e., $FM_f = M_f$. Thus, under suitable conditions, the computation of the topological entropy reduces to the computation of a fixed point.

The next theorem gives sufficient conditions ensuring this convergence.

Lemma 2. *Let $R : \mathbb{R}_+^{(p,p)} \to \mathbb{R}_+^{(p,p)}$ and $F : \Delta^{(p,p)} \to \Delta^{(p,p)}$ be defined as before. Suppose that $M_1^{\mathbf{a}}$ is irreducible for each $\mathbf{a} \in \mathbb{Z}_p^{(2,1)}$. Suppose also that*

a) *$M_1^{\mathbf{a}} * M_1^{\mathbf{a}'} \neq \mathbf{0}$ for each $\mathbf{a}, \mathbf{a}' \in \mathbb{Z}_p^{(2,1)}$,*
b) *there exists $M_f \in \Delta^{(p,p)}$ such that $F\ M_f = M_f$.*

Under these conditions, $\lim_{n\to\infty} F^n\ M = M_f$ *for all $M \in \Delta^{(p,p)}$.*

Proof. The irreducibility the matrices $M_1^{\mathbf{a}}$ ensures that both R and F are well defined. We start the proof by showing that the R is a (non–uniform) contraction with respect to the projective distance, i. e. $d(F\ M, F\ M') < d(M, M')$, whenever $M \neq M'$. For this we will use the results and notions developed in subsection 2.4.

To each $\mathbf{a} \in \mathbb{Z}_p^{(2,1)}$ we associate the one–dimensional Markov shift

$$X^{\mathbf{a}} = \{x \in \mathbb{Z}_p^{\mathbb{N}} :\ M_1^{\mathbf{a}}(x(n), x(m+1)) = 1\ \forall n \in \mathbb{N}\} \subset \mathbb{Z}_p^{\mathbb{N}},$$

and to each positive matrix $M \in \Delta^{(p,p)}$, the potential $f_M : \mathbb{Z}_p^{\mathbb{N}} \to \mathbb{R}$ such that $f_M(x) = \log M(x(0), x(1))$.

The projective distance extends to a pseudo–metric in $\mathbb{R}_+^{(p,p)}$ satisfying

$$d(M, M') = d(\lambda M, \lambda' M')\ \ \forall\ \lambda, \lambda' \in \mathbb{R}^+,$$

implying that $d(F\ M, F\ M') = d(R\ M, R\ M')$. From this and the fact that $(R\ M)(\mathbf{a}) = \exp\left(P_{X^{\mathbf{a}}}(f_M)\right)$, we obtain

$$d(F\ M, F\ M') = \max_{\mathbf{a}} \left\{P_{X^{\mathbf{a}}}(f_M) - P_{X^{\mathbf{a}}}(f_{M'})\right\} - \min_{\mathbf{a}} \left\{P_{X^{\mathbf{a}}}(f_M) - P_{X^{\mathbf{a}}}(f_{M'})\right\}.$$

Let us denote by $\mu_M^{\mathbf{a}}$ (resp. $\mu_{M'}^{\mathbf{a}}$) the Gibbs measure associated to the potential f_M (resp. $f_{M'}$), restricted to $X^{\mathbf{a}}$. The variational principle for the topological pressure implies

$$\mu_{M'}^{\mathbf{a}}(f_{M'} - f_M) \leq P_{X^{\mathbf{a}}}(f_M) - P_{X^{\mathbf{a}}}(f_{M'}) \leq \mu_M^{\mathbf{a}}(f_M - f_{M'}). \tag{16}$$

Let $\mathbf{b},\mathbf{b}' \in \mathbb{Z}_p^{(2,1)}$ be such that

$$\begin{aligned}
\min_{\mathbf{a}\in\mathbb{Z}_p^{(2,1)}} \{P_{X^{\mathbf{a}}}(f_M) - P_{X^{\mathbf{a}}}(f_{M'})\} &= P_{X^{\mathbf{b}}}(f_M) - P_{X^{\mathbf{b}}}(f_{M'}),\\
\max_{\mathbf{a}\in\mathbb{Z}_p^{(2,1)}} \{P_{X^{\mathbf{a}}}(f_M) - P_{X^{\mathbf{a}}}(f_{M'})\} &= P_{X^{\mathbf{b}'}}(f_M) - P_{X^{\mathbf{b}'}}(f_{M'}).
\end{aligned}$$

Then

$$\begin{aligned}
d(F\ M, F\ M') &\leq \mu_M^{\mathbf{b}}(f_M - f_{M'}) - \mu_{M'}^{\mathbf{b}'}(f_M - f'_M)\\
&\leq \max_{\mathbf{a}\in\mathbb{Z}_p^{(2,1)}}\{f_M(\mathbf{a}) - f_{M'}(\mathbf{a})\} - \min_{\mathbf{a}\in\mathbb{Z}_p^{(2,1)}}\{f_M(\mathbf{a}) - f_{M'}(\mathbf{a})\}\\
&= d(M, M').
\end{aligned}$$

The first inequality is strict unless f_M and $f_{M'}$ are cohomologous in both $X^{\mathbf{b}}$ and $X^{\mathbf{b}'}$. The second inequality is strict unless $(f_M - f_{M'})$ is constant on $X^{\mathbf{b}}$ and equals its maximal value, while on $X^{\mathbf{b}'}$ is constant and equals its minimal value.

Hypothesis a) implies that there are points $x \in X^{\mathbf{b}}$ and $y \in X^{\mathbf{b}'}$ such that $(x(1), x(2)) = (y(1), y(2))$, therefore $(f_M - f_{M'})(x) = (f_M - f_{M'})(y)$. Hence, even if f_M and $f_{M'}$ are cohomologous in both $X^{\mathbf{b}}$ and $X^{\mathbf{b}'}$, the second inequality above has to be strict for all $M \neq M'$.

By hypothesis b), F has a fix point $M_f \in \Delta^{(p,p)}$. For arbitrary $M \in \Delta^{(p,p)}$, let $C_M \equiv \{M' \in \Delta^{(p,p)} : \ d(M', M_f) \leq d(M, M_f)\}$. This set is compact and $F-$invariant. Indeed,

$$d(F\ M', M_f) < d(M', M_f) \leq d(M, M_f)\ \forall M' \in C_M.$$

Since $\{d(F^k\ M, M_f)\}_{k=0}^{\infty}$ is a decreasing sequence, then $r = \lim_{k\to\infty} d(F^k\ M, M_f)$ exists. Furthermore, each one of the matrices $F^k\ M$ belongs to the compact set C_M, then there exists an accumulation point $M^* = \lim_{n\to\infty} F^{k_n}\ M$, which necessarily satisfies $r = d(M^*, M_f)$. If $r > 0$, i. e., if $M^* \neq M_f$, then

$$r > d(F\ M^*, M_f) = \lim_{n\to\infty} d(F^{k_n+1}\ M, M_f) = r,$$

which is impossible. Therefore $r = 0$, and the lemma follows.

3.2. **Classes and Commutativity.** Given $X_{\mathcal{P}}$, the set $\mathbb{Z}_p^{(2,1)}$ can be partitioned in classes $C_1, C_2, \ldots, C_K$, defined according to the equivalent relation $\mathbf{a} \sim \mathbf{a}' \Leftrightarrow M_1^{\mathbf{a}} = M_1^{\mathbf{a}'}$.

The classes are preserved along the recursion (7). Indeed, if $\mathbf{a} \sim \mathbf{a}'$ then

$$\begin{aligned}
M_{n+1}^{\mathbf{a}} &= \left(M_1^{\mathbf{a}}(\mathbf{b}) \times M_n^{\mathbf{b}}\right)_{\mathbf{b}\in\mathbb{Z}_p^{(2,1)}}\\
&= \left(M_1^{\mathbf{a}'}(\mathbf{b}) \times M_n^{\mathbf{b}}\right)_{\mathbf{b}\in\mathbb{Z}_p^{(2,1)}}\\
&= M_n^{\mathbf{a}'},
\end{aligned}$$

for all $\mathbf{n} \in \mathbb{N}$. Hence, to each class we can associate *representative transition matrices* $M_n^{[k]} \equiv M_n^{\mathbf{a}}$ for any $\mathbf{a} \in C_k$.

The representative transition matrices inherit the recurrence relation given in (7), which we rewrite by using the Kroneker product. For this let us define the *linking matrices* $L^{[k,\ell]} : \mathbb{Z}_p^{(2,1)} \to \{0,1\}$ for $k, \ell \in \{1, 2, \ldots, K\}$, such that

$$L^{[k,\ell]}(\mathbf{b}) = M_1^{[k]} * \mathbb{1}_{C_\ell},$$

where $\mathbb{1}_{C_\ell} : \mathbb{Z}_p^{(2,1)} \to \{0,1\}$ denotes the characteristic function of the class C_ℓ. With this we have

$$M_{n+1} = \sum_{k=1}^{K} \mathbb{1}_{C_k} \otimes M_n^{[k]} \tag{17}$$

$$M_{n+1}^{[k]} = \sum_{\ell=1}^{K} L^{[k,\ell]} \otimes M_n^{[\ell]}, \tag{18}$$

for all $k \in \{1, 2, \ldots, K\}$ and $n \in \mathbb{N}$.

Lemma 3. *Suppose that all the $K \times K$ linking matrices commute. Then,*

$$M_n^{[k]} M_n^{[\ell]} = M_n^{[\ell]} M_n^{[k]} \quad \forall\ k, \ell \in \{1, 2, \ldots, K\},\ n \in \mathbb{N},$$

whenever $M_1^{[k]} M_1^{[\ell]} = M_1^{[\ell]} M_1^{[k]}$ for all $k, \ell \in \{1, 2, \ldots, K\}$

Proof. This follows by induction, taking into account that the Kroneker product satisfies

$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB')$$

for all matrices A, A' and B, B' of compatible dimension.

□

3.3. **Main Result.** We will consider a Markov shift $X_{\mathcal{P}}$, which is fully extensible and horizontally transitive. Let $M_1^{[1]}, \ldots, M_1^{[K]}$ be the representative matrices of the transition submatrices $\{M_1^{\mathbf{a}} :\ \mathbf{a} \in \mathbb{Z}_p^{(2,1)}\}$, and $\{L^{[k,\ell]} : , \ell \in \{1, 2, \ldots, K\}\}$ the linking matrices relating them.

If the representative matrices commute, as well as the linking matrices, then, under some additional conditions, the entropy is the solution of an eigenvalue problem.

Theorem 1 (Entropy as an Eigenvalue)**.** *Suppose that the representative matrices commute, as well as the linking matrices. Suppose also that the representative matrices are irreducible and such that $M_1^{[k]} * M_1^{[\ell]} \neq \mathbf{0}$, for all $k, \ell \in \{1, 2, \ldots, K\}$. Then if the system of equations*

$$\lambda \times x_k = \rho \left(\sum_{\ell=1}^{K} \mathbb{1}_{C_\ell} * M_1^{[k]} \right), k = 1, 2, \ldots, K,$$

has a positive solution, then λ is uniquely determined and $\lambda = \exp(h(X_{\mathcal{P}}))$.

Proof. We need $X_{\mathcal{P}}$ to be fully extensible and horizontally transitive in order to relate the entropy to the maximal eigenvalue of the transition matrices M_n. Since the representative matrices commute, as well as the linking matrices, then Lemma 3 holds, and with this (14) applies. On the other hand, the representative matrices are primitive and $M_1^{[k]} * M_1^{[\ell]} \neq \mathbf{0}$ for each k, ℓ, then Lemma (2) holds. Therefore, whenever

$$\lambda \times M(\mathbf{a}) = \rho(M * M_1^{\mathbf{a})}), \mathbf{a} \in \mathbb{Z}_p^{(p,p)}$$

has a positive solution $M \in \mathbb{R}_+^{(p,p)}$, necessarily $\lambda = \exp(h(X_p p))$. Assuming this is true, since the right hand side of the previous equation remains unchanged inside a class C_k, then there is a constant $x_k > 0$ such that $M(\mathbf{a}) = x_k$ for $\mathbf{a} \in C_k$. Hence, we may write $M = \sum_{k=1}^K x_k \mathbb{I}_{C_k}$, and the previous equation becomes

$$\exp(h(X_p p)) \times x_k = \rho \left(\sum_{\ell=1}^K x_\ell \mathbb{I}_{C_\ell} * M_1^{[k]} \right), k = 1, 2, \ldots, K,$$

and so the theorem follows.

□

3.4. **Conditioned Products.** In this subsection we exhibit a class of fully extensible and horizontally transitive Markov shifts, for which all the hypotheses of Theorem 1 hold. Therefore, their entropy is an eigenvalue, provided the associated eigenvalue problem has a positive solution. Further, in the next subsection, we will consider Markov shifts for which this eigenvalue problem is a linear one.

Assume that $N^{[k]} : \mathbb{Z}_p^{(2,1)} \to \{0, 1\}$, with $k \in \{1, 2, \ldots, K\}$, are commuting irreducible matrices such that, considered as characteristic functions on $\mathbb{Z}_p^{(2,1)}$, their associated sets form a partition. Hence we have $\mathbb{Z}_p^{(2,1)} = \dot{\bigcup}_{k=1}^K C_k$, with $C_k \equiv \{\mathbf{a} \in \mathbb{Z}_p^{(2,1)} : \ N^{(k)}(\mathbf{a}) = 1\}$.

To each $k \in \{1, 2, \ldots, K\}$ we associate a nonempty set $S_k \subset \{1, 2, \ldots, K\}$ satisfying $S_k \cap S_\ell \neq \emptyset$ for each $k, \ell \in \{1, 2, \ldots, K\}$.

Now, let us define the set of admissible arrays as

$$\mathcal{P} \equiv \left\{ \left[\begin{array}{c} \mathbf{a} \\ \mathbf{a}' \end{array} \right] \in \mathbb{Z}_p^{(2,2)} : \ N^{(k)}(\mathbf{a}) N^{(\ell)}(\mathbf{a}') = 1 \ \text{ for some } 1 \leq k \leq K \text{ and } \ell \in S_k \right\}.$$

We assume that for each $k \in \{1, 2, \ldots, K\}$, among all the matrices in $\{N^{[\ell]} : \ \ell \in S_k\}$ there are two whose periods are relative primes. This ensures that the Markov shift $X_{\mathcal{P}}$ is horizontally transitive. It is also easily verified that $X_{\mathcal{P}}$ is fully extensible.

The resulting Markov shift may be considered as a conditioned product of one–dimensional Markov shifts. Each matrix $N^{[k]}$, $k \in \{1, 2, \ldots, K\}$ defines a Markov shift $X^{[k]}$. Locally, the horizontal transitions $N^{[k]}$ and $N^{[\ell]}$ may coexist if and only if $\ell \in S_k$. Hence, locally it corresponds to a product $X^{[k]} \times X^{[\ell]}$, with $\ell \in S_k$.

For $X_{\mathcal{P}}$ defined as before we have representative matrices $M_1^{[k]} \equiv \sum_{\ell \in S_k} N^{[\ell]}$, with $k = 1, 2, \ldots, K$. These matrices inherit from $\{N^{[k]} : \ 1 \leq k \leq K\}$ the commutativity. Their irreducibility follows form the irreducibility of $N^{[k]}$ for each k. Furthermore, since $S_k \cap S_{k'}$ for all $k, k' \in \{1, \ldots, K\}$, it follows that $M_1^{[k]} * M_1^{[k']} \neq \mathbf{0}$.

In this example the linking matrices are $L^{[k,\ell]} = \mathbb{I}_{S_k}(\ell) \times N^{[\ell]}$. It follows directly that these matrices commute.

Since all the desired hypotheses hold, we can compute the topological entropy $h(X_{\mathcal{P}})$ of this Markov shift, provided

$$\lambda \times M(\mathbf{a}) = \left[\rho \left(\sum_{k=1}^K N^{[k]}(\mathbf{a}) \left(\sum_{\ell \in S_k} M * N^{[k]} \right) \right) \right]$$

has a solution in $\Delta^{(p,p)}$. For matrices of the kind $M = \sum_{k=1}^{K} x_k N^{[k]}$, this problem reduces to the K–dimensional linear eigenvalue problem for the non–negative matrix

$$S_{\mathcal{P}} : \{1,2,\ldots,K\} \times \{1,2,\ldots,K\} \to \mathbb{R}, \text{ such that } S(k,\ell) = 1\!\!\mathrm{I}_{S_k}(\ell) \times \rho(N^{[k]}).$$

If we further assume that this matrix is irreducible, by the Perron–Frobenius Theorem we have a positive solution. In this case

$$h(X_{\mathcal{P}}) = \log \rho(S_{\mathcal{P}}). \tag{19}$$

3.5. **Examples.** Let $\{P_1, P_2, \ldots, P_M\}$ be a collection of commuting irreducible permutations in $\mathbb{Z}_p$, such that

$$(i \neq j) \Rightarrow (P_i(a) \neq P_j(a)) \ \forall\ a \in \mathbb{Z}_p.$$

To the permutation P_i we associate the transition matrix $Q_i : \mathbb{Z}_p^{(2,1)} \to \{0,1\}$ such that $Q_i(\mathbf{a}) = 1$ if and only if $P_i\mathbf{a}(1) = \mathbf{a}(2)$. Let $\{A_k :\ k = 1,2,\ldots,K < M\}$ be a partition of $\{1,2,\ldots,M\}$, and define

$$N_k = \sum_{i \in A_k} Q_i,\ k = 1,2,\ldots,K-1 \text{ and } N_K = 1\!\!\mathrm{I}1\!\!\mathrm{I}^{\dagger} - \sum_{i\in A_K} Q_i.$$

Finally, let $S : \{1,2,\ldots,K\} \times \{1,2,\ldots,K\} \to \{0,1\}$ be irreducible, one of its columns has all entries equal to 1. With $S_k = \{\ell :\ S(k,\ell) = 1\}$, the matrices $\{N^{[k]} :\ k = 1,\ldots,K\}$ define a conditioned product $X_{\mathcal{P}}$, with entropy

$$h(X_{\mathcal{P}}) = \log \rho(S_{\mathcal{P}}), \quad \text{with} \quad S_{\mathcal{P}}(k,\ell) = \begin{cases} S(k,\ell) \times \#A_\ell & \text{if } \ell < K \\ S(K,\ell) \times (p - \#A_K) & \text{otherwise.} \end{cases}$$

A particular instance of this is the following. Let $P_1, P_2, \ldots, P_M$ be cyclic permutations, and define $N^{[1]} = \sum_{i=1}^{M} Q_i$, with $Q_i : \mathbb{Z}_p^{(2,1)} \to \{0,1\}$ as before. Define $N^{[2]} = 1\!\!\mathrm{I}1\!\!\mathrm{I}^{\dagger} - N^{[1]}$, and

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

These matrices define a conditioned product $X_{\mathcal{P}}$ such that

$$h(X_{\mathcal{P}}) = \log \rho\left(\begin{bmatrix} M & p-M \\ M & 0 \end{bmatrix}\right) = \log\left(\frac{M + \sqrt{4Mp - 3M^2}}{2}\right).$$

4. Final Comments

The class of examples we presented are quite restrictive in nature. It would be interesting to construct examples of other kind satisfying all the conditions of Theorem 1. In particular, it would be interesting to provide examples for which the entropy is a nonlinear eigenvalue. It is worth to mention that the class of examples we showed in the previous section are related, though not of the same kind, to the Matrix subshifts introduced in [5].

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