LARGE DEVIATIONS AND VARIATIONAL PRINCIPLE FOR HARMONIC CRYSTALS

P. CAPUTO AND J.-D. DEUSCHEL

ABSTRACT. We consider massless Gaussian fields with covariance related to the Green function of a long range random walk on $\mathbb{Z}^d$. These are viewed as Gibbs measures for a linear-quadratic interaction. We establish thermodynamic identities and prove a version of Gibbs' variational principle, showing that translation invariant Gibbs measures are characterized as minimizers of the relative entropy density. We then study the large deviations of the empirical field of a Gibbs measure. We show that a weak large deviation principle holds at the volume order, with rate given by the relative entropy density.

1. INTRODUCTION

Connections between equilibrium statistical mechanics and the theory of large deviations have been known for long time, [24]. For lattice random fields with bounded interactions, estimates of the Donsker-Varadhan type [13] have been established for the empirical field of a Gibbs measure, [7, 29, 18, 21], showing that probabilities of large fluctuations of the system in the thermodynamical limit are exponentially small in the volume with rate given by a relative entropy functional. The latter can be expressed as a difference of free energies and a classical variational principle applies, giving a complete characterization of equilibrium measures, i.e. translation invariant Gibbs measures, as minima of relative entropy. The presence or absence of a phase transition in the system is then explicitly reflected in the large deviation picture, in the sense that if there exists more than one translation invariant Gibbs state, then large fluctuations can occur with probabilities decaying slower than exponentially in the volume of the system. Moreover, the interplay between large deviations and the variational principle has found interesting applications in testing the Gibbsian property of certain probability measures [16].

The case of unbounded spin systems is far less investigated, the reason being that in the noncompact setting, even existence of thermodynamics becomes nontrivial. Existence problems can be solved by making strong assumptions on the interactions such as superstability, [26]. Assuming superstability and other regularity properties, a version of the variational principle was discussed in [23]. Since there are many simple and basic models - e.g. the massless free field in dimension $d \geqslant 3$ - which violate superstability, it is interesting to see how far the theory can be pushed without these assumptions. In the present work we make a first attempt in this direction, in the simplified context of Gaussian random fields.

Date: May 11, 1999.

We consider an unbounded spin system on $\mathbb{Z}^d$, $d \geqslant 1$, with linear-quadratic pair potential, also known as the harmonic crystal. The formal Hamiltonian is given by

$$\mathcal{H}(\phi) \;=\; \sum_{x,y} J(x,y)\,\phi_x\phi_y\,, \tag{1.1}$$

where $\phi_x \in \mathbb{R}$, $x \in \mathbb{Z}^d$, and the couplings $J(x,y)$ are symmetric and translation invariant. The linearity of the model makes it possible to give a complete description of the Gibbs measures associated to (1.1) - provided they exist - in terms of Gaussian fields, with covariance given by the Green function $G = J^{-1}$, [12]. It follows that, in order to have existence, minimal assumptions on J are

$$\hat{J} \geqslant 0\,, \qquad \text{and} \qquad \int \frac{d\theta}{\hat{J}(\theta)} \;<\; \infty\,,$$

with $\hat{J}$ denoting the Fourier transform of J, as a function on $(-\pi,\pi]^d$ ($\hat{J}^{-1}$ plays the role of a spectral density). We say that the harmonic crystal (1.1) is in a *massless regime* if $\hat{J}$ vanishes at least at one point. On the other hand, if $\hat{J}$ has no zeros the model is much simpler: the spectral density $\hat{J}^{-1}$ is bounded and we have weakly dependent Gaussian fields, with summable covariances. Such fields satisfy hypercontractive estimates which allow to use standard tools to study large deviations, [6, 11]. When $\hat{J}$ has no zeros we therefore speak of an *hypercontractive regime.* In a sense which is made precise in the next sections, in the hypercontractive regime we have uniqueness of the Gibbs measure, while in the massless regime we have a phase transition.

We restrict our attention to couplings J of the following form. Let $p(x,y)$ be the transition function of a symmetric, homogeneous random walk on $\mathbb{Z}^d$. We consider

$$J(x,y) \;=\; \beta \begin{cases} 1 & \text{if } x = y \\ -\eta_{x-y}\,p(x,y) & \text{if } x \neq y \end{cases} \tag{1.2}$$

with $\beta > 0$ and $\eta \in \{-1,+1\}^{\mathbb{Z}^d}$. The parameter β plays no role (there is no "temperature") and it will be set equal to one. The vector η allows us to treat the case of non ferromagnetic couplings.

If we set $\eta \equiv 1$ the model is ferromagnetic, and the Hamiltonian (1.1) takes the form

$$\mathcal{H}(\phi) \;=\; \frac{1}{2}\sum_{x,y} p(x,y)\,(\phi_x - \phi_y)^2\,. \tag{1.3}$$

The system (1.3) can be interpreted as a model for an harmonic interface. The global shift symmetry of the field ϕ is spontaneously broken as soon as Gibbs measures exist, yielding a phase transition regime. As a consequence, Gibbs measures will not exist for short-range models in dimension $d = 1, 2$, cf. [19]. We ensure existence by requiring the random walk p to be *transient.* In particular, low-dimensional models can be treated if the random walk takes sufficiently long jumps. The basic example of a gradient type model is the massless free field, i.e. when p is the transition of the simple random walk on $\mathbb{Z}^d$, $d \geqslant 3$. For this model and generally when p is the transition of a finite range transient random walk large deviations estimates were obtained in [1]. In contrast to [1], we consider here the more delicate case of non-local interaction as well.

If $\eta \not\equiv 1$, the model is not ferromagnetic and monotonicity tools such as FKG inequalities are no longer available. Moreover, there is no more global shift symmetry to be broken but still phase transition can occur, if the Fourier transform $\hat{J}$ of J has zeros. In order to obtain a massless regime we require the constants η to have a suitable oscillating structure.

We now turn to a brief description of the main results of this paper. We refer to the relevant sections for precise statements and proofs. In section 2, after introducing the models and reviewing some known material, we prove a new result concerning the structure of translation invariant Gibbs measures. It states that for ferromagnetic models generated by an irreducible random walk, translation invariant Gibbs measures cannot be realized as mixtures of non-translation invariant extremal Gibbs measures. Or, in other words, the set of translation invariant Gibbs measures is a face of the convex set of Gibbs measures. See remark 3 for further discussion of this topic.

In section 3 we introduce thermodynamics. We prove existence of relative entropy density and establish the corresponding variational principle. In particular, we show that translation invariant tempered Gibbs states are precisely the minima (zeros) of the relative entropy density. In this way we extend previous results of Künsch [22, 23]. To prove that any minimizer must be Gibbsian we make use of a new approximation which enables us to go beyond the standard quasilocality assumption (see [30, 20]). The main tool in this section is an expansion in terms of random walks for the Green function G associated to J, closely related to the expansions introduced in [4].

Our results concerning large deviations are discussed in section 4. The *empirical field* on a cubic box Λ_N of side N is the random probability measure defined by

$$\tilde{R}_N(\phi) \;=\; \frac{1}{N^d} \sum_{x\in\Lambda_N} \delta_{\theta_x \phi}\,, \qquad \phi \in \Omega\,. \tag{1.4}$$

where δ_ξ stands for the Dirac measure at $\xi \in \Omega \equiv \mathbb{R}^{\mathbb{Z}^d}$, and $\theta_x\phi$ denotes the shifted configuration $\phi_{\cdot+x}$. Let P denote the centered Gaussian field with covariance given by G. Since P is ergodic with respect to lattice shifts,

$$\tilde{R}_N \;\Rightarrow\; P\,, \quad P-\text{a.s.} \;\; \text{as} \;\; N \to \infty\,,$$

in the sense of weak convergence. In particular, for any open set Γ of probability measures on Ω such that $P \notin \Gamma$, we have $P(\tilde{R}_N \in \Gamma) \to 0$, as $N \to \infty$. We show that these probabilities are exponentially decaying in the volume N^d, with rate given by the relative entropy density. Since we are interested in the massless regime, classical large deviations results for stationary Gaussian processes based on the continuity of the spectral density (cf. e.g. [14, 3]) cannot be used. Following an idea introduced in [1], the exponential bounds are first established for the hypercontractive regime and then lifted to the general case. In contrast to the hypercontractive regime, in the phase transition regime the upper estimates must be restricted to certain compact sets of well behaved measures. In this sense we speak of a *weak* large deviation principle.

In view of the variational principle one expects *critical* large deviations in the phase transition regime. In particular, if Γ is an open set such that $P \notin \Gamma$ but Γ contains some other translation invariant Gibbs measure, then $P(\tilde{R}_N \in \Gamma)$ decays slower than exponentially in the volume. This is indeed what has been described in detail in [1] for finite range harmonic crystals. We discuss critical large deviations for a class of long range models of the form (1.3) in a forthcoming paper, see also our remarks after Theorem 4.1.

Finally, the appendix contains an expansion for Green functions in terms of random walks, and an estimate relevant for the hypercontractive regime.

Aknowledgements. The authors thank Yvan Velenik and Aernout van Enter for stimulating conversations.

2. The Models

2.1. **Preliminaries.** Let $\mathbb{Z}^d$, be the d-dimensional integer lattice. For $x \in \mathbb{Z}^d$, $|x|$ denotes the euclidean norm. Let $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, equipped with the product topology, and denote by $\mathcal{F}$ the corresponding Borel field. Probability measures on $(\Omega, \mathcal{F})$ are denoted by $\mathcal{M}_1(\Omega)$. The set $\mathcal{M}_1(\Omega)$ is equipped with the weak topology, i.e. the topology induced by convergence against bounded continuous functions on Ω. For each finite subset Λ of $\mathbb{Z}^d$, denoted $\Lambda \subset\subset \mathbb{Z}^d$, $\mathcal{F}_\Lambda$ is the σ-field generated by the canonical projections $\pi_x(\phi) = \phi_x$, $x \in \Lambda$, $\phi \in \Omega$, and ϕ_Λ denotes the restriction of $\phi \in \Omega$ to Λ. We also set $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$. A function $f : \Omega \to \mathbb{R}$ is called *local* if there exists a $\Lambda \subset\subset \mathbb{Z}^d$ such that f is $\mathcal{F}_\Lambda$-measurable. The smallest such Λ is called the support of f and is denoted S_f. Lattice shifts $\{\theta_y\}$ are defined by $\pi_x \circ \theta_y = \pi_{x+y}$, $x, y \in \mathbb{Z}^d$. Elements of $\mathcal{M}_1(\Omega)$ invariant under transformations induced by shifts are denoted $\mathcal{M}_1^S(\Omega)$. The set of shift invariant measures $\mathcal{M}_1^S(\Omega)$ is a closed convex subset of $\mathcal{M}_1(\Omega)$. A measure $\mu \in \mathcal{M}_1^S(\Omega)$ is called ergodic if $\mu(A) = 0$ or 1 for any event $A \in \mathcal{F}$ such that $\theta_x A = A$ for all $x \in \mathbb{Z}^d$.

2.2. **The interaction.** Let $p : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$ be a symmetric, homogeneous function

$$p(x, y) \;=\; p(0, y - x) \;=\; p(y, x)\,, \quad x, y \in \mathbb{Z}^d\,.$$

We make the following basic assumptions.

- p is the transition function of a random walk on $\mathbb{Z}^d$, i.e.

$$\sum_{y \in \mathbb{Z}^d} p(x, y) \;=\; 1\,, \qquad x \in \mathbb{Z}^d\,. \tag{2.1}$$

- p is *transient*, i.e.

$$\sigma^2 \;=\; \sum_{n=0}^{\infty} p^n(0, 0) \;<\; \infty\,, \tag{2.2}$$

  where p^n denotes the $n-$th iterate of p.

The model is specified by the *potential*

$$J(x, y) \;=\; \delta(x, y) \;-\; Q(x, y)\,, \tag{2.3}$$

where $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise, and we have defined

$$Q(x, y) \;=\; \eta_{x-y}\, p(x, y)\,, \tag{2.4}$$

with η an arbitrary symmetric vector with values in $\{-1, +1\}$

$$\eta \in \{-1, +1\}^{\mathbb{Z}^d}\,, \quad \eta_x = \eta_{-x} \quad \forall x \in \mathbb{Z}^d\,.$$

We also use the notation $J = 1\!\!1 - Q$. We will be working in this setting for the rest of the paper, with the assumptions (2.1) and (2.2). In section 4, the following mild decay condition will be considered:

- there exists $\beta > 0$, such that

$$\lim_{|x|\to\infty} |x|^{d+\beta}\, p(0,x) \;=\; 0\,. \tag{2.5}$$

We emphasize that our results up to section 4 hold without any hypothesis on the decay of p. Thanks to the transience condition (2.2), the Green function

$$G(x,y) \;=\; \sum_{n=0}^{\infty} Q^n(x,y) \tag{2.6}$$

is well defined, cf. [32]. Let $\hat{J}$ denote the Fourier transform of J,

$$\hat{J}(\theta) \;=\; \sum_{x\in\mathbb{Z}^d} J(0,x)\,\mathrm{e}^{i\,x\cdot\theta}\,, \qquad \theta\in(-\pi,\pi]^d\,.$$

It is easily checked that (2.1) implies $\hat{J}\geqslant 0$, and (2.2) implies that $f\equiv\hat{J}^{-1}$ is integrable, with

$$G(x,y) \;=\; \frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} f(\theta)\,\mathrm{e}^{i\,(y-x)\cdot\theta}\,d\theta\,. \tag{2.7}$$

Remark 1. Let us define

$$\mathcal{N} = \{\theta\in(-\pi,\pi]^d \;:\; \hat{J}(\theta)=0\}\,, \tag{2.8}$$

the set of zeros of $\hat{J}$. If $\eta\equiv 1$ in (2.4) then clearly $0\in\mathcal{N}$. Moreover, it is known that $\mathcal{N}=\{0\}$ if and only if p is *irreducible*, cf. [32]. By irreducibility we mean that for any $x,y\in\mathbb{Z}^d$ there exists an integer $n>0$ such that $p^n(x,y)>0$. In the non ferromagnetic case $\eta\not\equiv 1$ it could happen that $\mathcal{N}=\emptyset$. As discussed in the introduction, this reduces to an hypercontractive regime, with $G(x,y)$ absolutely summable. One should therefore keep in mind that the interesting cases are the massless models, obtained by choosing η in such a way that $\mathcal{N}\neq\emptyset$. In section 2.5 we discuss some examples.

Let $\Lambda\subset\subset\mathbb{Z}^d$. We denote by J_Λ and Q_Λ the restriction to Λ of J and Q respectively. Then $J_\Lambda=\mathbb{1}_\Lambda-Q_\Lambda$ is a positive definite matrix and $\Gamma_\Lambda\equiv J_\Lambda^{-1}$ is given by

$$\Gamma_\Lambda(x,y) \;=\; \sum_{n=0}^{\infty} Q_\Lambda^n(x,y)\,, \qquad x,y\in\Lambda\,. \tag{2.9}$$

Let G_Λ be the restriction of G to Λ. Note that G_Λ is positive definite. In the appendix, cf. Lemma A.1, we show that G_Λ has an expansion analogous to (2.9) where Q_Λ must be replaced by an effective coupling matrix $\tilde{Q}_\Lambda$. This gives

$$G_\Lambda^{-1} \;=\; \mathbb{1}_\Lambda \;-\; \tilde{Q}_\Lambda \tag{2.10}$$

The matrix G_Λ can be seen as the Green function of J on Λ with free boundary conditions, while Γ_Λ is associated to Dirichlet boundary conditions. We call P_Λ the centered Gaussian measure on $\mathbb{R}^\Lambda$ with covariance G_Λ, so that

$$\frac{dP_\Lambda}{d\phi_\Lambda}(\phi) \;=\; (2\pi)^{-|\Lambda|/2}\,(\det G_\Lambda)^{-1/2}\,\exp\big(-\frac{1}{2}\,\langle\phi,G_\Lambda^{-1}\phi\rangle\big)\,.$$

Here $d\phi_\Lambda=\prod_{x\in\Lambda}d\phi_x$ denotes Lebesgue measure, $\langle,\rangle$ is the usual scalar product, and $|\Lambda|$ is the cardinality of the set Λ. The measures $\{P_\Lambda,\,\Lambda\subset\subset\mathbb{Z}^d\}$ form a consistent family and there exist a unique measure $P\in\mathcal{M}_1(\Omega)$ such that its marginal on Λ is given by P_Λ. We also refer to P as the centered Gaussian field on $\mathbb{Z}^d$, with covariance G. From (2.7) and

Riemann-Lebesgue lemma, we have $G(x,y) \to 0$, as $|x-y| \to \infty$. This implies P is trivial on tail events, cf. e.g. [20], chapter 7. In particular, P is ergodic w.r.t. lattice shifts.

2.3. **Gibbs Measures.** Let

$$\Omega_J \;=\; \Big\{\phi \in \Omega : \sum_{y\in\mathbb{Z}^d} |J(x,y)\phi_y| \;<\; \infty\,,\ \text{for all } x \in \mathbb{Z}^d\Big\}\,. \tag{2.11}$$

Note that Ω_J is a tail event. Fix $\Lambda \subset\subset \mathbb{Z}^d$ and a boundary condition $\tau \in \Omega_J$. The Hamiltonian is defined by

$$H^\tau_\Lambda(\phi) \;=\; \frac{1}{2}\sum_{x,y\in\Lambda} J(x,y)\phi_x\phi_y \;+\; \sum_{x\in\Lambda, y\notin\Lambda} J(x,y)\phi_x\tau_y\,. \tag{2.12}$$

We define *specifications* $\gamma^\tau_\Lambda \in \mathcal{M}_1(\Omega)$ by

$$\gamma^\tau_\Lambda(d\phi) = \begin{cases} (Z^\tau_\Lambda)^{-1}\,\exp\left(-H^\tau_\Lambda(\phi)\right)d\phi_\Lambda & \text{if } \phi_{\Lambda^c} = \tau_{\Lambda^c} \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

with

$$Z^\tau_\Lambda \;=\; \int \mathrm{e}^{-H^\tau_\Lambda(\phi)}\,d\phi_\Lambda \tag{2.14}$$

Then γ^τ_Λ is the Gaussian measure on $(\Omega,\mathcal{F})$ with mean

$$m^{\tau,\Lambda}_x \equiv \gamma^\tau_\Lambda(\phi_x) = \begin{cases} -\sum_{y\in\Lambda}\sum_{z\notin\Lambda}\Gamma_\Lambda(x,y)J(y,z)\tau_z & \text{if } x\in\Lambda \\ \tau_x & \text{if } x\notin\Lambda \end{cases} \tag{2.15}$$

and covariance (independent of $\tau\in\Omega_J$)

$$\gamma^\tau_\Lambda(\phi_x\phi_y) \;-\; \gamma^\tau_\Lambda(\phi_x)\gamma^\tau_\Lambda(\phi_y) \;=\; \begin{cases} \Gamma_\Lambda(x,y) & \text{if } x,y\in\Lambda \\ 0 & \text{otherwise} \end{cases} \tag{2.16}$$

The partition function (2.14) is explicitly given by

$$Z^\tau_\Lambda \;=\; (2\pi)^{|\Lambda|/2}\,(\det\Gamma_\Lambda)^{1/2}\,\exp\big(\frac{1}{2}\,\langle m^{\tau,\Lambda},\Gamma_\Lambda^{-1}m^{\tau,\Lambda}\rangle\,\big)\,.$$

Given a local bounded function g, we denote by $\gamma_\Lambda(g)$ the $\mathcal{F}_{\Lambda^c}$-measurable function

$$\Omega_J \;\ni\; \tau \;\to\; \gamma^\tau_\Lambda(g) \;=\; \int \gamma^\tau_\Lambda(d\phi)\,g(\phi)\,.$$

A measure $\mu\in\mathcal{M}_1(\Omega)$ is said to be a *Gibbs measure* relative to the potential J if

$$\mu(\Omega_J) \;=\; 1 \tag{2.17}$$

and

$$\mu(\gamma_\Lambda(g)) \;=\; \mu(g)\,, \tag{2.18}$$

for any local bounded function g, for any $\Lambda\subset\subset\mathbb{Z}^d$. Let $\mathcal{G}=\mathcal{G}_J$ be the set of Gibbs measures relative to J. We now recall a basic result about the structure of $\mathcal{G}$. Let $\sigma\in\Omega$ and denote by γ^σ the translate of P by the configuration σ, i.e. γ^σ is the Gaussian field with mean $\gamma(\phi_x)=\sigma_x$, $x\in\mathbb{Z}^d$, and covariance G. Let $\mathcal{H}_J$ denote the set of J-*harmonic* configurations

$$\mathcal{H}_J \;=\; \{\phi\in\Omega:\ \sum_{y\in\mathbb{Z}^d} J(x,y)\phi_y \;=\; 0,\ \text{for all } x\in\mathbb{Z}^d\}\,.$$

Let $\mathcal{G}_{\text{ext}}$ denote the set of extremal points of the convex set $\mathcal{G}$. A result of Rozanov [31] and Dobrushin [12] (see also Theorems (13.24) and (13.36) of [20]) shows that

$$\mathcal{G}_{\text{ext}} \;=\; \{\gamma^{\sigma},\;\; \sigma \in \mathcal{H}_J\}\,. \tag{2.19}$$

Remark 2. The transience assumption (2.2) guarantees existence of Gibbs measures, i.e. $\mathcal{G} \neq \emptyset$. We remark that if p has finite range, the transience hypothesis forces $d \geqslant 3$, cf. e.g. [32]. Moreover, condition (2.17) is automatically satisfied. It is then useful to restrict the attention to measures on smaller subsets of Ω. One can introduce sets of tempered configurations $\overline{\Omega}$, and define *tempered Gibbs measures* as the set of $\mu \in \mathcal{G}$ such that $\mu(\overline{\Omega}) = 1$. A natural choice is e.g. $\overline{\Omega} = \mathcal{S}'(\mathbb{Z}^d)$, where $\mathcal{S}'(\mathbb{Z}^d)$ denotes the space of configurations which increase at most polynomially at infinity.

Remark 3. A general result (see [20], Theorem (14.17)) states that a shift invariant Gibbs measure has a unique decomposition in terms of ergodic Gibbs measures. But an ergodic Gibbs measure need not be extremal in $\mathcal{G}$ in general (an ergodic measure is not necessarily tail trivial). Indeed there are many well known examples (see e.g. [16], section 2) of shift invariant Gibbs measures which can be written as mixtures of non shift invariant elements of $\mathcal{G}_{\text{ext}}$. In Theorem 2.1 below we prove that for ferromagnetic models generated by an *irreducible* random walk this cannot happen, at least if we restrict to tempered measures. In other words all ergodic Gibbs measures are extremal in $\mathcal{G}$ and "extremal shift invariant Gibbs = shift invariant extremal Gibbs". We also give an example of models where this is violated. In particular these examples imply that in our setting, assuming a massless regime, all ergodic Gibbs measures are extremal in $\mathcal{G}$ if and only if $\eta \equiv 1$ and p is irreducible.

2.4. **Ferromagnetic models.** The ferromagnetic case is obtained if $\eta \equiv 1$ in (2.4), i.e. $Q = p$, or equivalently J is the generator of a (continuous time) random walk on $\mathbb{Z}^d$. Since $\sum_y J(x,y) = 0$ for any x, any constant configuration $\sigma_x \equiv c$ is in $\mathcal{H}_J$ and the global shift symmetry $\phi \to \phi + c$, is spontaneously broken. In particular the set $\mathcal{G}$ contains uncountably many measures.

A classical example is the *massless free field.* It is obtained when p is the simple random walk

$$p(x,y) = \begin{cases} 1/2d & \text{if } |x-y| = 1 \\ 0 & \text{otherwise} \end{cases}$$

i.e. $J = -\Delta$, if Δ is the discrete Laplace operator on $\mathbb{Z}^d$. Here, as in other finite range models one considers tempered Gibbs measures, defined as in remark 2 above, [12]. Large deviations for finite range harmonic crystal were considered in [1].

As an example of a non local interaction we shall consider the following model, to which we refer as the *α-stable* crystal. For $\alpha \in (0, 2 \wedge d)$, let p be such that

$$\lim_{|x|\to\infty} |x|^{d+\alpha}\, p(0,x) \;=\; c_{\alpha}\,. \tag{2.20}$$

for some $c_\alpha \in (0,\infty)$. The random walk will then be in the domain of attraction of a *stable law* (see [27] for some interesting properties of stable random walks). In this case the condition (2.17) forces all Gibbs measures to be tempered, with no additional requirement.

In ferromagnetic models the covariance matrices $\Gamma_\Lambda, G_\Lambda$, $\Lambda \subset\subset \mathbb{Z}^d$ admit a simple interpretation in terms of the random walk. We see from (2.9) that Γ_Λ is the Green function of the random walk with transition p which is "killed" upon exiting the box Λ. On the other hand G_Λ is the Green function of the random walk "embedded" in Λ, see the remark after Lemma A.1. One can use these facts to give the following representation for the mean values (2.15). If $\mathbb{P}_x$ denotes the probability for the discrete time random walk $\xi_0, \xi_1 \dots$ defined by p, with start in $\xi_0 = x \in \Lambda$, and if t_Λ denotes the first hitting time for $\mathbb{Z}^d \setminus \Lambda$, we have

$$\gamma_\Lambda^\tau(\phi_x) \;=\; \sum_{y \notin \Lambda} \mathbb{P}_x\big(\xi_{t_\Lambda} = y\big)\, \tau_y\,, \qquad \tau \in \Omega_J\,.$$

This type of formulas represent only a special case of the so called *random walk representation*, which can be generalized to anharmonic crystals with convex pair interaction, cf. e.g. [10] for a recent account.

We turn to the result announced in remark 3 above.

Theorem 2.1. *Let μ be a shift invariant Gibbs measure for $J = \mathbb{I} - p$ with p irreducible, and assume μ has finite first moments, i.e. $\mu(|\phi_0|) < \infty$. Then μ is a mixture of shift invariant elements of $\mathcal{G}_{\rm ext}$. In particular, there exists a probability measure ρ_0 on $\mathbb{R}$ with finite first moment and*

$$\mu \;=\; \int_{-\infty}^{\infty} \gamma_t\, \rho_0(dt)\,, \tag{2.21}$$

where $\gamma_t = \gamma^\sigma$ with $\sigma_x \equiv t \in \mathbb{R}$.

Proof. Let μ be given as in the hypothesis, and let $\rho \in \mathcal{M}_1(\Omega)$ be its extremal decomposition in $\mathcal{G}$, i.e.

$$\mu \;=\; \int_\Omega \gamma^\sigma\, \rho(d\sigma)\,. \tag{2.22}$$

In particular, cf. (2.19), $\rho(\mathcal{H}_J) = 1$. Moreover, ρ is shift invariant and has finite first moments. We want to show that ρ is actually concentrated on the constant elements of $\mathcal{H}_J$. This would reduce the integral on Ω in (2.22) to the integral on $\mathbb{R}$ in (2.21), thereby proving the theorem. It is sufficient to show that for any $x, y \in \mathbb{Z}^d$, for any $\beta > 0$,

$$\rho\big\{|\phi_x - \phi_y| > \beta\big\} \;=\; 0\,. \tag{2.23}$$

In order to prove (2.23) we may proceed as follows. Observe that $\phi \in \mathcal{H}_J$ is equivalent to $p_t\phi = \phi$ for all $t \geqslant 0$, where

$$p_t(x,y) \;=\; \mathrm{e}^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!}\, p^n(x,y)$$

is the continuous time kernel on $\mathbb{Z}^d$ with Fourier transform $\exp(-t\hat{J})$. Since $\rho(\mathcal{H}_J) = 1$ we can replace ϕ in (2.23) by $p_t\phi$, and by Chebyshev's inequality and the shift invariance of ρ,

$$\rho\big\{|\phi_x - \phi_y| > \beta\big\} \;\leqslant\; \beta^{-1}\, \rho(|\phi_0|) \sum_{z \in \mathbb{Z}^d} |p_t(x,z) - p_t(y,z)|\,.$$

Since t is arbitrary, (2.23) follows from

$$\lim_{t\to\infty} \sum_{z\in\mathbb{Z}^d} |p_t(x,z) - p_t(y,z)| \;=\; 0\,. \tag{2.24}$$

If p has finite range and is irreducible, (2.24) follows form the standard local limit theorem estimates, just as in the case of the simple random walk, cf. e.g. [25], Corollary 1.2.3. For a general irreducible random walk one can use the following argument, cf. [8]. Let $K > 0$ be a large number such that $\delta \in (0,1)$ if

$$\delta \;=\; \sum_{x:\,|x|\,\leqslant\, K} p(0,x)\,.$$

Define two homogeneous transitions $r(x,y)$ and $s(x,y)$, by

$$r(x,y) \;=\; \delta^{-1}\, p(x,y)\, 1_{\{|x-y|\,\leqslant\, K\}}\,, \quad s(x,y) \;=\; (1-\delta)^{-1}\, p(x,y)\, 1_{\{|x-y|>K\}}\,.$$

Notice that since p is irreducible, K can be chosen in such a way that r is irreducible. If r_t and s_t are the corresponding continuous time kernels, then it is easy, using e.g. the Fourier transform, to see that

$$p_t(x,z) \;=\; \sum_{z'\in\mathbb{Z}^d} r_{\delta t}(x,z')\, s_{(1-\delta)t}(z',z)\,.$$

We have

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d} |p_t(x,z) - p_t(y,z)| \;&\leqslant\; \sum_{z,z'\in\mathbb{Z}^d} |r_{\delta t}(x,z') - r_{\delta t}(y,z')|\, s_{(1-\delta)t}(z',z)\\ &=\; \sum_{z'\in\mathbb{Z}^d} |r_{\delta t}(x,z') - r_{\delta t}(y,z')| \end{aligned}\tag{2.25}$$

Now (2.24) follows from (2.25) since r has finite range. □

2.5. **Non ferromagnetic models.** Here we consider some examples of potentials J satisfying conditions (2.1) and (2.2) but such that

$$\sum_{x\in\mathbb{Z}^d} J(0,x) \;\neq\; 0 \tag{2.26}$$

Suppose η in (2.4) is such that $\mathcal{N} = \emptyset$, cf. (2.8). Then one can show (see e.g. [20], Corollary (13.40)) that the unique element of $\mathcal{G}$ with uniformly bounded first moments is P. Moreover P is hypercontractive in this case, cf. appendix B. If p has long range, e.g. p satisfies (2.20) for some $\alpha > 0$, one can actually show that $\mathcal{G} = \{P\}$. This is due to the fact that condition (2.17) forces all Gibbs measures to be strongly tempered, cf. [12, 20].

More interesting is the case $\mathcal{N} \neq \emptyset$. Let $\theta_0 \in \mathcal{N}$ (by symmetry, this implies $-\theta_0 \in \mathcal{N}$). Then it is easy to see that there exists $m \in \mathbb{Z}^d$ such that $\theta_0 = \pi(m_1,\dots,m_d)$, and the coefficients η_x have the form

$$\eta_x = \cos\big(\pi \sum_{i=1}^d m_i\, x_i\big)\,,$$

for any $x \in \mathbb{Z}^d$ such that $p(0,x) > 0$. Moreover, all oscillating configurations

$$\sigma_x = \alpha\, \cos\left(\theta_0 \cdot x\right)\,, \quad x \in \mathbb{Z}^d, \alpha \in \mathbb{R},$$

are elements of $\mathcal{H}_J$. We now give an example of translation invariant Gibbs measures which are not mixtures of translation invariant extremal Gibbs states. Such cases were first pointed out by Rozanov, [31]. Since (2.26) implies that P is the unique shift invariant element of $\mathcal{G}_{\text{ext}}$, it will be enough to exhibit a shift invariant $\mu \in \mathcal{G}$ with $\mu \neq P$. The following example shows much more, namely that for any $L > 0$, we can can find $\mu \in \mathcal{G} \cap \mathcal{M}_1^S(\Omega)$ with $\mu(\phi_0^2) > L$. Let $T > 0$ and let ρ_T be the centered Gaussian field with spectral density

$$\hat{\rho}_T \;=\; \frac{T}{2}\left(\delta_{\theta_0} + \delta_{-\theta_0}\right),$$

or, in other words ρ_T is the centered Gaussian field with covariance

$$\rho_T(\sigma_x, \sigma_y) = T\cos\theta_0 \cdot (x-y)$$

(cf. e.g. [20], Proposition (13.A7) to check that ρ_T is well defined). Then, for any $T > 0$,

$$\mu_T \;=\; \int \gamma^\sigma \rho_T(d\sigma) \tag{2.27}$$

is a centered Gaussian field, since it is a convolution of centered Gaussian measures. Moreover it is translation invariant, since its covariance

$$G_T(x,y) = G(x,y) + T\,\cos\theta_0 \cdot (x-y)\,,$$

is homogeneous on $\mathbb{Z}^d$. To show that $\mu_T \in \mathcal{G}$ we have to show $\rho_T(\mathcal{H}_J) = 1$. But this follows from

$$\rho_T\big(|\sum_{y\in\mathbb{Z}^d} J(x,y)\sigma_y|^2\big) \;=\; T\sum_{y,z\in\mathbb{Z}^d} J(x,y)J(x,z)\cos\theta_0\cdot(y-z) \;=\; T\,\hat{J}(\theta_0)^2 \;=\; 0\,,$$

for any $x \in \mathbb{Z}^d$. We remark that in the case of Theorem 2.1 the measure μ_T can be realized only for $\theta_0 = 0$, cf. remark 1, and ρ_T is concentrated on the constant configurations. On the other hand if $\eta \equiv 1$ but p is not irreducible, there exists $0 \neq \theta_0 \in \mathcal{N}$ and we see that the measure (2.27) violates the claim of Theorem 2.1.

3. Thermodynamics and Variational Principle

In this section we prove a version of Gibbs' variational principle. We first prove a basic thermodynamic identity (cf. Proposition 3.1) expressing relative entropy density as a difference of free energies. In the case of bounded interactions, this is a classical result which goes back to Föllmer's paper [17]. For Gaussian random fields the identity was proven by Künsch under a strong assumption on the decay of the potential, cf. [22], Theorem 2.11. We show that in our setting the result is true with no assumption on the decay of J, i.e. only assuming (2.1) and (2.2).

3.1. **Thermodynamics.** For any $N \in \mathbb{Z}_+$ let

$$\Lambda_N = [-N/2, N/2)^d \cap \mathbb{Z}^d\,.$$

We will always assume N is an even integer, so that Λ_N has side N and $|\Lambda_N| = N^d$. We define the sets of translation invariant measures

$$\mathcal{K}_L \;=\; \big\{\mu \in \mathcal{M}_1^S(\Omega),\quad \mu(\omega_0^2) \leqslant L\big\}, \qquad \mathcal{K}_\infty \;=\; \bigcup_{L\,\geqslant\,0} \mathcal{K}_L \tag{3.1}$$

A tightness argument shows that $\mathcal{K}_L$ is compact in the weak topology for any $L \geqslant 0$.

Given two measures μ, ν, let μ_N, ν_N denote their marginals on Λ_N. The *relative entropy* of μ with respect to ν on Λ_N is defined by

$$H_N(\mu|\nu) \;=\; \begin{cases} \int d\mu_N \log \dfrac{d\mu_N}{d\nu_N} & \text{if } \mu_N \ll \nu_N \\ +\infty & \text{otherwise} \end{cases} \tag{3.2}$$

Let also $H_N(\mu)$ denote the relative entropy of μ with respect to Lebesgue measure on Λ_N. The *entropy density* of $\mu \in \mathcal{M}_1^S(\Omega)$ is defined as

$$s(\mu) \;=\; -\lim_{N\to\infty} \frac{1}{N^d}\, H_N(\mu) \tag{3.3}$$

A general subadditivity argument shows that this limit always exists, although it may be equal to $-\infty$. Moreover, s is *affine* (see e.g. chapter 15 of [20]). The *internal energy density* of $\mu \in \mathcal{K}_\infty$ is defined as

$$e(\mu) \;=\; \frac{1}{2} \lim_{N\to\infty} \frac{1}{N^d} \sum_{x,y\in\Lambda_N} J(x,y)\mu(\phi_x\phi_y)\,. \tag{3.4}$$

The ergodic theorem implies (cf. [22], Theorem 2.2) that for $\mu \in \mathcal{K}_\infty$ the limit exists and equals

$$e(\mu) \;=\; \frac{1}{2} \sum_{y\in\mathbb{Z}^d} J(0,y)\mu(\phi_0\phi_y)\,. \tag{3.5}$$

The *pressure* is defined by

$$p \;=\; \lim_{N\to\infty} \frac{1}{N^d}\;\log Z^0_{\Lambda_N}\,, \tag{3.6}$$

where $Z^0_{\Lambda_N}$ is given by (2.14) with $\tau = 0$. By Theorem 2.5 in [22] the above limit exists and equals

$$p \;=\; \frac{1}{2}\log(2\pi) \;+\; \frac{1}{2}\frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} \log f(\theta)d\theta \tag{3.7}$$

where $f = \hat{J}^{-1}$ is the spectral density of P (cf. (2.7)). We define the *relative entropy density* of μ w.r.t. ν as

$$h(\mu|\nu) \;=\; \lim_{N\to\infty} \frac{1}{N^d}\, H_N(\mu|\nu)\,. \tag{3.8}$$

Proposition 3.1. *The function $h(\,\cdot\,|P)$ is well defined on $\mathcal{M}_1^S(\Omega)$, and*

$$h(\mu|P) \;=\; \begin{cases} +\infty & \text{if } \mu \in \mathcal{M}_1^S(\Omega)\setminus\mathcal{K}_\infty \\ e(\mu) \,-\, s(\mu) \,+\, p & \text{if } \mu \in \mathcal{K}_\infty \end{cases} \tag{3.9}$$

Moreover, $h(\,\cdot\,|P)$ is affine on $\mathcal{K}_\infty$.

Proof. Let $\mu \in \mathcal{M}_1(\Omega)$. We check the first line of (3.9). An alternative expression for H_N is given by (cf. e.g. Lemma (3.2.13) of [11])

$$H_N(\mu|P) \;=\; \sup_\psi \left\{\mu(\psi) \,-\, \log P(\mathrm{e}^\psi)\right\} \tag{3.10}$$

where the supremum is taken over bounded $\mathcal{F}_{\Lambda_N}-$measurable functions. Let $\alpha \in (0, \sigma^2/2)$, so that $\log P(\mathrm{e}^{\alpha\phi_0^2}) \equiv c(\alpha) < \infty$. Then choosing $\psi_T(\phi) = \alpha(\phi_0^2 \wedge T)$, $T > 0$ in (3.10) gives

$$H_N(\mu|P) \;\geqslant\; \alpha\mu(\phi_0^2\wedge T) \,-\, c(\alpha)\,.$$

Letting $T \to \infty$, this implies $H_N(\mu|P) \geqslant \alpha\mu(\phi_0^2) - c(\alpha)$. Hence, $h(\mu|P) = \infty$ if $\mu \notin \mathcal{K}_\infty$.

We turn to the proof of the identity in the second line of (3.9). We may assume that μ_N has a density, otherwise the identity holds trivially, since $s(\mu) = -\infty$ in this case. Let ψ_N be the density of P_N, and call ψ_N^0 the density of $\gamma_{\Lambda_N}^0$, cf. (2.13). We have

$$H_N(\mu|P) \;=\; H_N(\mu) \;-\; \mu(\log\psi_N^0) \;+\; \mu(\log\psi_N^0\psi_N^{-1})$$

Since

$$\lim_{N\to\infty} \frac{1}{N^d}\,\mu(\log\psi_N^0) \;=\; -\,p \;-\; e(\mu)\,,$$

we need to prove

$$\lim_{N\to\infty} \frac{1}{N^d}\,\mu(\log\psi_N^0\psi_N^{-1}) \;=\; 0\,. \tag{3.11}$$

Recall that

$$\psi_N^0\psi_N^{-1}(\phi) \;=\; \det\big(G_N\Gamma_N^{-1}\big)^{1/2}\,\exp\Big(-\frac{1}{2}\,\langle\phi,(\Gamma_N^{-1}-G_N^{-1})\phi\rangle\Big)\,,$$

where $G_N \equiv G_{\Lambda_N}$ and $\Gamma_N \equiv \Gamma_{\Lambda_N}$. The argument of Theorem 2.5 in [22], shows that

$$\lim_{N\to\infty} \frac{1}{N^d}\,\log\det\big(G_N\Gamma_N^{-1}\big) \;=\; 0\,.$$

We are left with

$$\lim_{N\to\infty} \frac{1}{N^d}\,\mu\big[\,\langle\phi,(\Gamma_N^{-1}-G_N^{-1})\phi\rangle\,\big] \;=\; 0\,. \tag{3.12}$$

Let $x, y \in \Lambda_N$. Using the expansion of Lemma A.1, we have

$$\Gamma_N^{-1}(x,y) \;-\; G_N^{-1}(x,y) \;=\; \sum_{n=1}^{\infty}\sum_{z_1\notin\Lambda_N}\cdots\sum_{z_n\notin\Lambda_N} Q(x,z_1)\cdots Q(z_n,y)$$

Since $\mu \in \mathcal{K}_\infty$, by Schwarz' inequality,

$$\begin{aligned}\big|\mu\big[\,\langle\phi,(\Gamma_N^{-1}-G_N^{-1})\phi\rangle\,\big]\big| \;&\leqslant\; \sum_{x,y\in\Lambda_N}\big|\Gamma_N^{-1}(x,y)-G_N^{-1}(x,y)\big|\,|\mu(\phi_x\phi_y)| \\ &\leqslant\; \mu(\phi_0^2)\sum_{x,y\in\Lambda_N}\sum_{n=1}^{\infty}\sum_{z_1\notin\Lambda_N}\cdots\sum_{z_n\notin\Lambda_N} p(x,z_1)\cdots p(z_n,y)\,.\end{aligned} \tag{3.13}$$

Let $\xi_0, \xi_1, \dots$ be the random walk with transition p. We call $\mathbb{P}_x$ the probability measure associated to the paths with $\xi_0 = x$, $x \in \Lambda_N$. Let

$$t_N \;=\; \inf\{\,n \geqslant 1\,,\;\; \xi_n \in \Lambda_N\,\}\,.$$

After summing over $y \in \Lambda_N$, (3.13) can be written as

$$\mu(\phi_0^2)\sum_{x\in\Lambda_N}\sum_{n=1}^{\infty}\mathbb{P}_x\big(\xi_1\notin\Lambda_N;\; t_N = n+1\big) \;\leqslant\; \mu(\phi_0^2)\sum_{x\in\Lambda_N}\mathbb{P}_x\big(\xi_1\notin\Lambda_N\big)\,.$$

We then have

$$\big|\mu\big[\,\langle\phi,(\Gamma_N^{-1}-G_N^{-1})\phi\rangle\,\big]\big| \;\leqslant\; \mu(\phi_0^2)\sum_{x\in\Lambda_N}\sum_{z\notin\Lambda_N} p(x,z)\,.$$

Now a simple argument using just absolute summability of $p(0,y)$ shows that the above expression is $o(N^d)$. This implies (3.11) and (3.9) follows. Finally, the affinity of $h(\,\cdot\,|P)$ on $\mathcal{K}_\infty$ follows from (3.9) and the affinity of s. ☐

3.2. **Variational Principle.** In classical statistical mechanics Gibbs' variational principle can be stated as follows. A measure $\mu \in \mathcal{M}_1^S(\Omega)$ is a minimum for the *free energy*

$$f(\mu) \;=\; e(\mu) \;-\; s(\mu)$$

if and only if μ is a translation invariant Gibbs state. In view of Proposition 3.1 this can be turned into a statement about relative entropy density. Let $\mathcal{G}_2^S = \mathcal{G} \cap \mathcal{K}_\infty$, the set of translation invariant Gibbs states with finite second moment.

Theorem 3.2. *Let $\mu \in \mathcal{M}_1^S(\Omega)$. Then,*

$$\mu \in \mathcal{G}_2^S \quad \Longleftrightarrow \quad h(\mu|P) \;=\; 0 \tag{3.14}$$

Proof. Let $\mu \in \mathcal{G}_2^S$, and denote by $\rho \in \mathcal{M}_1(\Omega)$ its extremal decomposition, i.e. (cf. (2.19))

$$\mu \;=\; \int \gamma^\sigma \, \rho(d\sigma)\,, \qquad \rho(\mathcal{H}_J) = 1\,.$$

Note that $\mu \in \mathcal{K}_\infty$ implies $\rho \in \mathcal{K}_\infty$. By convexity

$$H_N(\mu|P) \;\leqslant\; \int H_N(\gamma^\sigma|P)\, \rho(d\sigma)\,. \tag{3.15}$$

Moreover, by explicit computation,

$$H_N(\gamma^\sigma|P) \;=\; \frac{1}{2}\, \left\langle \sigma, G_N^{-1}\sigma \right\rangle\,.$$

For $\sigma \in \mathcal{H}_J$ we can write

$$\left\langle \sigma, G_N^{-1}\sigma \right\rangle \;=\; \left\langle \sigma, (G_N^{-1} - J_N)\sigma \right\rangle \;-\; \sum_{x\in\Lambda_N} \sum_{y\notin\Lambda_N} J(x,y)\sigma_x\sigma_y\,,$$

Here $J_N = \Gamma_N^{-1}$ is the restriction of J to Λ_N. We can now use the argument of Proposition 3.1, cf. (3.12), and the fact that ρ is concentrated on $\mathcal{H}_J$ to obtain

$$\lim_{N\to\infty} \frac{1}{N^d}\, \rho\big[\left\langle \sigma, G_N^{-1}\sigma \right\rangle\big] \;=\; 0\,.$$

From (3.15) we see that

$$h(\mu|P) \;=\; \lim_{N\to\infty} \frac{1}{N^d}\, H_N(\mu|P) \;=\; 0\,.$$

We now turn to the more delicate part of (3.14). Namely, suppose $\mu \in \mathcal{M}_1^S(\Omega)$, and

$$h(\mu|P) \;=\; 0\,. \tag{3.16}$$

We are going to prove that μ satisfies (2.17) and the DLR equations (2.18). Note that Proposition 3.1 implies that μ has finite second moments, and therefore (2.17) follows from

$$\mu\Big(\sum_{y\in\mathbb{Z}^d} |J(x,y)|\,|\phi_y|\Big) \;=\; \mu(|\phi_0|) \sum_{y\in\mathbb{Z}^d} |J(x,y)| \;<\; \infty\,, \qquad x \in \mathbb{Z}^d\,.$$

Let g be a local, bounded continuous function, and let $\Lambda \subset\subset \mathbb{Z}^d$. We will show

$$\mu(\gamma_\Lambda(g)) \;=\; \mu(g)\,. \tag{3.17}$$

Let $V \subset\subset \mathbb{Z}^d$ be a large set containing both Λ and the support of g. We write

$$f_V = \frac{d\mu_V}{dP_V}$$

for the density of μ w.r.t. P on V. Then, since $\mu(g) = P(f_V g)$, we have

$$|\mu(\gamma_\Lambda(g)) - \mu(g)| \leqslant |\mu(\gamma_\Lambda(g)) - P(f_V\gamma_\Lambda(g))| + |P(f_V\gamma_\Lambda(g)) - P(f_V g)|. \tag{3.18}$$

To prove (3.17) we show that both terms in the r.h.s. of (3.18) can be made arbitrarily small by choosing V large enough. To be more precise, (3.17) is a consequence of the next two lemmas. Note that Lemma 3.3 becomes trivial if p has finite range.

Lemma 3.3. *For any $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{Z}_+$ such that for any $V \supset \Lambda_N$*

$$|\mu(\gamma_\Lambda(g)) - P(f_V\gamma_\Lambda(g))| < \varepsilon \tag{3.19}$$

Lemma 3.4. *For any $\varepsilon > 0$ and for all $N \in \mathbb{Z}_+$, there exists $V = V(\varepsilon, N) \subset\subset \mathbb{Z}^d$, such that $V \supset \Lambda_N$, and*

$$|P(f_V\gamma_\Lambda(g)) - P(f_V g)| < \varepsilon \tag{3.20}$$

Proof of Lemma 3.3. Notice that

$$P(f_V\gamma_\Lambda(g)) = P(f_V P(\gamma_\Lambda(g)|\mathcal{F}_V)) = \mu(P(\gamma_\Lambda(g)|\mathcal{F}_V)). \tag{3.21}$$

Define

$$\gamma^\tau_{\Lambda,V}(g) = \gamma_\Lambda^{\tau_V 0_{V^c}}(g)\,,$$

i.e. we put 0-boundary conditions outside of V. Note that $\gamma_{\Lambda,V}(g)$ is now a $\mathcal{F}_V$-measurable function. Let us fix a sequence $\{V_N\}$, with $V_N \supset \Lambda_N$, $N \in \mathbb{Z}_+$. We define

$$\varphi_N = \gamma_\Lambda(g) - \gamma_{\Lambda,V_N}(g)\,.$$

Using (3.21) we see that for any N

$$|\mu(\gamma_\Lambda(g)) - P(f_{V_N}\gamma_\Lambda(g))| \leqslant \mu(|\varphi_N|) + \mu(P(|\varphi_N|\,|\mathcal{F}_{V_N}))\,.$$

To prove (3.19) it is then sufficient to show

$$\lim_{N\to\infty} \mu(|\varphi_N|) = 0 \tag{3.22}$$

and

$$\lim_{N\to\infty} \mu(P(|\varphi_N|\,|\mathcal{F}_{V_N})) = 0\,. \tag{3.23}$$

We may assume g to be a bounded Lipshitz-continuous local function. For such a function, define

$$\|g\|_{\mathrm{L}} = \sup_{x\in\mathbb{Z}^d} \sup \left\{ \frac{|g(\sigma) - g(\eta)|}{|\sigma_x - \eta_x|}\,,\ \ \sigma,\eta \in \Omega\ :\ \sigma_y = \eta_y,\ \forall\, y \neq x \right\}$$

From (2.15) and (2.16) we know that the measure $\gamma_\Lambda^{\tau_{V_N} 0_{V_N^c}}$ can be seen as the measure γ^τ_Λ shifted by the mean values vector $\xi^N(\tau)$, given by

$$\xi^N_x(\tau) \equiv \gamma_\Lambda^{\tau_{V_N} 0_{V_N^c}}(\phi_x) - \gamma^\tau_\Lambda(\phi_x) = \sum_{y\in\Lambda}\sum_{z\notin V_N} \Gamma_\Lambda(x,y)J(y,z)\tau_z$$

We then have

$$
\begin{aligned}
|\varphi_N(\tau)| \;&\leqslant\; \int \gamma_\Lambda^\tau(d\phi)\, |g(\phi+\xi^N(\tau)) - g(\phi)| \\
&\leqslant\; \|g\|_{\mathrm{L}} \sum_{x\in\Lambda} |\xi_x^N(\tau)| \;\leqslant\; |\Lambda|^2\, \|g\|_{\mathrm{L}} \sup_{x,y\in\Lambda} |\Gamma_\Lambda(x,y)| \sup_{y\in\Lambda} \sum_{z\notin V_N} |J(y,z)|\, |\tau_z|\,. \quad (3.24)
\end{aligned}
$$

Let us fix

$$
C(g,\Lambda) = |\Lambda|^2\, \|g\|_{\mathrm{L}} \sup_{x,y\in\Lambda} |\Gamma_\Lambda(x,y)|\,.
$$

Using the shift invariance of μ and the fact that $\mu \in \mathcal{K}_\infty$, we have

$$
\begin{aligned}
\mu(|\varphi_N|) \;&\leqslant\; C(g,\Lambda) \sup_{y\in\Lambda} \sum_{z\notin V_N} |J(y,z)|\, \mu(|\phi_z|) \\
&=\; C(g,\Lambda)\, \mu(|\phi_0|) \sup_{y\in\Lambda} \sum_{z\notin V_N} |J(y,z)|\,. \quad (3.25)
\end{aligned}
$$

Now, the r.h.s. of (3.25) vanishes as $N\to\infty$, and (3.22) follows.

We turn to the proof of (3.23). From (3.24) we have

$$
\mu(P(|\varphi_N|\,|\mathcal{F}_{V_N})) \;\leqslant\; C(g,\Lambda) \sup_{y\in\Lambda} \sum_{z\notin V_N} |J(y,z)|\, \mu(P(|\phi_z|\,|\mathcal{F}_{V_N}))\,. \quad (3.26)
$$

Let us fix $\tau\in\Omega_J$. Then $P(\cdot|\tau_{V_N})$ is the Gaussian field on $\Omega_{V_N^{\mathrm{c}}}$ with mean

$$
P(\phi_z|\tau_{V_N}) = -\sum_{y\in V_N^{\mathrm{c}}} \sum_{x\in V_N} \Gamma_{V_N^{\mathrm{c}}}(z,y) J(y,x)\tau_x\,, \qquad z\in\mathbb{Z}^d\setminus V_N\,, \quad (3.27)
$$

and covariance

$$
P(\phi_z\phi'_z|\tau_{V_N}) - P(\phi_z|\tau_{V_N})P(\phi'_z|\tau_{V_N}) = \Gamma_{V_N^{\mathrm{c}}}(z,z')\,, \qquad z,z'\in\mathbb{Z}^d\setminus V_N\,, \quad (3.28)
$$

where

$$
\Gamma_{V_N^{\mathrm{c}}}(z,z') = \sum_{n=0}^{\infty} Q^n_{V_N^{\mathrm{c}}}(z,z')\,,
$$

if $Q_{V_N^{\mathrm{c}}}$ is the function given by

$$
Q_{V_N^{\mathrm{c}}}(x,y) = \begin{cases} Q(x,y) & \text{if } x,y\in\mathbb{Z}^d\setminus V_N \\ 0 & \text{otherwise} \end{cases}
$$

Note that (3.27) and (3.28) can be obtained as a limiting form of (2.15) and (2.16). For ferromagnetic models (3.27) and (3.28) assume a simple probabilistic meaning in terms of the random walk killed as it enters the set V_N (cf. discussion in section 2.4). Replacing $Q(x,y)$ by its absolute value $p(x,y)$ we can reduce ourselves to the ferromagnetic case. Let $\mathbb{P}_z$, $z\in\mathbb{Z}^d\setminus V_N$ denote the probability associated to the random walk $\xi_0=z,\xi_1,\xi_2,\dots$ with transition p. Let t_N be the first hitting time of the set V_N. We have

$$
\begin{aligned}
P(\phi_z^2\,|\tau_{V_N}) \;&=\; \Gamma_{V_N^{\mathrm{c}}}(z,z) + \Big(\sum_{y\in V_N^{\mathrm{c}}}\sum_{x\in V_N} \Gamma_{V_N^{\mathrm{c}}}(z,y)J(y,x)\tau_x\Big)^2 \\
&\leqslant\; \sigma^2 + \sum_{x,x'\in V_N} \mathbb{P}_z(\xi_{t_N}=x)\mathbb{P}_z(\xi_{t_N}=x')|\tau_x\tau_{x'}|\,,
\end{aligned}
$$

where we have used (2.2) to bound $\Gamma_{V_N^c}(z,z)$. We now use Schwarz' inequality and the shift invariance of μ to write

$$\mu\big(P(\phi_z^2\,|\mathcal{F}_{V_N})\big) \leqslant \sigma^2 + \sum_{x,x'\in V_N} \mathbb{P}_z(\xi_{t_N}=x)\mathbb{P}_z(\xi_{t_N}=x')\mu(|\tau_x\tau_{x'}|) \leqslant \sigma^2+\mu(\tau_0^2)\,.$$

Therefore, by Schwarz' inequality

$$\mu\big(P(|\phi_z|\,|\mathcal{F}_{V_N})\big) \leqslant \mu\big(P(\phi_z^2\,|\mathcal{F}_{V_N})\big)^{1/2} \leqslant \big(\sigma^2+\mu(\tau_0^2)\big)^{1/2}\,. \tag{3.29}$$

Plugging (3.29) into (3.26) shows that

$$\mu(P(|\varphi_N|\,|\mathcal{F}_{V_N})) \leqslant C(g,\Lambda)\,\big(\sigma^2+\mu(\tau_0^2)\big)^{1/2}\sup_{y\in\Lambda}\sum_{z\notin V_N}|J(y,z)| \;\to\; 0\,,\quad N\to\infty\,.$$

This completes the proof of Lemma 3.3. □

Proof of Lemma 3.4. Since $f_{V\setminus\Lambda}=d\mu_{V\setminus\Lambda}/dP_{V\setminus\Lambda}$ is $\mathcal{F}_{\Lambda^c}-$measurable, the DLR property of P implies

$$P(f_{V\setminus\Lambda}\gamma_\Lambda(g)) \;=\; P(f_{V\setminus\Lambda}g)\,,$$

and we can write

$$|P(f_V\gamma_\Lambda(g)) \,-\, P(f_Vg)| \;\leqslant\; 2\,\|g\|_\infty\,P(|f_{V\setminus\Lambda}\,-\,f_V|)\,.$$

The Lemma now follows from a classical argument (cf. [30, 20]). Precisely, following [20], Step 2 in the proof of Theorem (15.37), one shows that (3.16) implies that for any $\varepsilon>0$ and for all $N\in\mathbb{Z}_+$, there exists $V=V(\varepsilon,N)\subset\subset\mathbb{Z}^d$, such that $V\supset\Lambda_N$, and

$$P(|f_V\,-\,f_{V\setminus\Lambda}|) \;<\; \varepsilon \tag{3.30}$$

This completes the proof of Lemma 3.4. □

4. Large Deviations

In this section we discuss volume order large deviations of the empirical field with respect to the measure P. From now on, in addition to the usual assumptions (2.1) and (2.2), we assume that the coupling coefficients satisfy the decay condition (2.5). We also assume $\mathcal{N}\neq\emptyset$, i.e. we focus on the massless regime. The (periodized) *empirical field* R_N is defined by

$$R_N(\phi) \;=\; \frac{1}{N^d}\sum_{x\in\Lambda_N}\delta_{\theta_x\,\phi_N}\,, \tag{4.1}$$

where δ_ξ stands for the Dirac measure at $\xi\in\Omega$, $\theta_x\phi$ denotes the shifted configuration $\phi_{\cdot+x}$, and ϕ_N denotes the configuration on Λ_N periodized to the whole of $\mathbb{Z}^d$. With this definition $R_N(\phi)\in\mathcal{M}_1^S(\Omega)$ for all $\phi\in\Omega$, and the map $\phi\to R_N(\phi)$ is $\mathcal{F}_{\Lambda_N}$-measurable. It is well known, cf. e.g. [21], that as far as the large deviation behaviour is concerned (4.1) is equivalent to (1.4). Let

$$\mathcal{I}(\mu) \;=\; \begin{cases} +\infty & \mu\in\mathcal{M}_1(\Omega)\setminus\mathcal{M}_1^S(\Omega)\\ h(\mu|P) & \mu\in\mathcal{M}_1^S(\Omega)\end{cases} \tag{4.2}$$

The main result in this section states that R_N satisfies a weak large deviation principle with rate function $\mathcal{I}$. Recall the definition (3.1) of the compact sets $\mathcal{K}_L\subset\mathcal{M}_1^S(\Omega)$.

Theorem 4.1. *For any $L>0$, the restriction of $\mathcal{I}$ to $\mathcal{K}_L$ is lower semi-continuous. Moreover, the following bounds hold:*

- *For any closed set $F \subset \mathcal{M}_1(\Omega)$, for any $L > 0$,*

$$\limsup_{N\to\infty} \frac{1}{N^d} \log P\big(R_N \in F \cap \mathcal{K}_L\big) \leqslant - \inf_{\mu \in F\cap\mathcal{K}_L} \mathcal{I}(\mu)\,. \tag{4.3}$$

- *For any open set $G \subset \mathcal{M}_1(\Omega)$,*

$$\liminf_{N\to\infty} \frac{1}{N^d} \log P\big(R_N \in G\big) \geqslant - \inf_{\mu \in G} \mathcal{I}(\mu)\,. \tag{4.4}$$

Before starting the proof of Theorem 4.1 we make some remarks.

Remark 1. Since the upper bound (4.3) is restricted to compact sets we speak of a *weak* large deviation principle. The standard way to remove this restriction would be to prove exponential tightness in the form

$$\lim_{L\to\infty} \limsup_{N\to\infty} \frac{1}{N^d} \log P\big(R_N \notin \mathcal{K}_L\big) = -\infty\,.$$

But this is false in the massless regime. Indeed, for fixed $L > 0$, the measure μ_T defined by (2.27) is not in $\mathcal{K}_L$, for any $T > L - \sigma^2$. But $\mu_T \in \mathcal{G}_2^S$, and therefore the lower bound (4.4) together with Theorem 3.2 implies

$$\liminf_{N\to\infty} \frac{1}{N^d} \log P\big(R_N \notin \mathcal{K}_L\big) \geqslant - h(\mu_T|P) = 0\,.$$

It is also easy to see that, in contrast to the hypercontractive case discussed in Theorem 4.2 below, $\mathcal{I}$ cannot be lower semi-continuous on the whole of $\mathcal{M}_1(\Omega)$. Indeed, consider e.g. mixtures of the type (2.21). Let μ_n, $n \in \mathbb{Z}_+$ be the measures obtained with such mixtures when $\rho_0^n(dt) = \psi_n(t)dt$ with densities $\psi_n \in L^2(\mathbb{R})$ such that ψ_n converges in $L^1(\mathbb{R})$ to some $\psi \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$. Clearly $\mu_n \to \mu$ weakly, as $n \to \infty$, where μ is given by (2.21) with $\rho_0(dt) = \psi(t)dt$. On the other hand $\mathcal{I}(\mu_n) = 0$ for all n but since $\mu \notin \mathcal{K}_\infty$, we have $\mathcal{I}(\mu) = \infty$.

Remark 2. In view of the variational principle proven in Theorem 3.2, we have

$$\mathcal{I}(\mu) = 0 \iff \mu \in \mathcal{G}_2^S\,.$$

In particular we expect deviations of order $o(N^d)$ in the class of shift invariant Gibbs states. We then speak of *critical* large deviations. One can investigate this issue at the level of the empirical mean

$$M_N(\phi) = \frac{1}{N^d} \sum_{x\in\Lambda_N} \phi_x\,.$$

With respect to P, M_N is a Gaussian variable with mean zero and variance

$$\sigma_N^2 = P(M_N^2) = \frac{1}{N^{2d}} \sum_{x,y\in\Lambda_N} G(x,y)$$

It is easy to check

$$N^d\sigma_N^2 \to f(0) = \sum_{x\in\mathbb{Z}^d} G(x,y)\,, \quad N\to\infty\,.$$

This implies that, for any $\beta > 0$

$$\lim_{N\to\infty} \frac{1}{N^d} \log P\big(M_N \geqslant \beta\big) = -\frac{\beta^2}{2f(0)}\,.$$

In other words, if the model is not ferromagnetic, i.e. $f(0) < \infty$, the event $M_N \geqslant \beta$ cannot be realized in the class $\mathcal{G}_2^S$. For ferromagnetic models, $f(0) = +\infty$ and the event $M_N \geqslant \beta$ can always be produced within the class $\mathcal{G}_2^S$ (the latter contains all the fields γ^σ with σ a constant). For finite range ferromagnetic models with irreducible random walks, the full picture of critical large deviations has been obtained in the paper [1]. Since a classical invariance principle applies, one has here the order N^{d-2}, for $d \geqslant 3$. In particular one can prove that exponential tightness holds at this order. It is possible to extend this analysis of critical large deviations to the $\alpha-$stable models, cf. (2.20), for any $d \geqslant 1$. The correct order here is $N^{d-\alpha}$, due to the different scaling. This will be the subject of a separate paper, [5].

Remark 3. The proof of the upper bound in theorem 4.1 is based on the argument introduced in [1] for the case of finite range ferromagnetic interaction. The main tool in extending this to our general setting is the expansion proved in the appendix, Lemma A.1, which is heavily used for comparisons of different Gaussian fields. The main line of the proof can be roughly described as follows. We perturb the original measure P by removing the singularity of its spectral density, cf. (4.5). This defines a new Gaussian field, P^ε, with bounded spectral density. The new field is hypercontractive and satisfies a strong large deviation principle, cf. Theorem 4.2. This is a multidimensional version of the sort of results proven in section 5.4 of [11], see also [6]. Hypercontractivity is obtained via the classical theorem of Nelson [28], cf. Lemma B.2 in the appendix. We then show that the bounds satisfied by P^ε are stable (on the volume scale) when the singularity is restored, cf. Theorem 4.3. This is the most delicate part of the proof. In particular, the arguments of [1] do not apply and finer estimates are needed, cf. Lemma 4.6.

Remark 4. The lower bound in theorem 4.1 can be proven along the line of the classical Donsker-Varadhan argument [13]. This is done at the end of this section. Our proof of Theorem 4.1 also shows, however, how the lower bound for P can be in principle derived from a lower bound on P^ε.

Remark 5. As a final remark we observe that the proof of Theorem 4.1 carries through with minor modifications if we replace the infinite volume measure P by $P_N^0 = \gamma_{\Lambda_N}^0$, i.e. the Gibbs measure on Λ_N with zero boundary condition. In particular the statement of Theorem 4.1 holds as it is when P is replaced by P_N^0. Indeed, this amounts to replace the covariance G_N by Γ_N. As (the proof of) Proposition 3.1 shows, this cannot affect thermodynamics, and in general will not affect any volume order asymptotic. Of course one cannot expect this to be true for critical large deviations, where boundary effects play an important role, cf. [5].

4.1. **Proof of Theorem 4.1.** For each $\varepsilon > 0$, we define the measure $P^\varepsilon \in \mathcal{M}_1(\Omega)$, as the centered Gaussian field on $\mathbb{Z}^d$ with spectral density (cf. (2.6)

$$f^\varepsilon \;=\; (\hat{J} \,+\, \varepsilon)^{-1} \;=\; \frac{f}{1 \,+\, \varepsilon\, f}\,. \tag{4.5}$$

Let $h(\mu|P^\varepsilon)$ be the relative entropy density associated to the measure P^ε, cf. (3.8), and define

$$\mathcal{I}_\varepsilon(\mu) \;=\; \begin{cases} +\infty & \mu \in \mathcal{M}_1(\Omega) \setminus \mathcal{M}_1^S(\Omega) \\ h(\mu|P^\varepsilon) & \mu \in \mathcal{M}_1^S(\Omega) \end{cases} \tag{4.6}$$

Theorem 4.2. *The function $\mathcal{I}_\varepsilon : \mathcal{M}_1(\Omega) \to [0,\infty]$ is lower semi-continuous and its level sets*

$$\{\mu \in \mathcal{M}_1(\Omega) : \mathcal{I}(\mu) \leqslant L\}, \quad L \geqslant 0$$

are compact. For each $\mu \in \mathcal{K}_\infty$ it satisfies

$$\mathcal{I}_\varepsilon(\mu) = \mathcal{I}(\mu) + \frac{\varepsilon}{2}\mu(\phi_0^2) + \phi(\varepsilon), \quad \mu \in \mathcal{K}_\infty, \tag{4.7}$$

where $\phi(\varepsilon)$ is given by

$$\phi(\varepsilon) = -\frac{1}{2}\frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} \log(1 + \varepsilon f(\theta))d\theta. \tag{4.8}$$

Moreover,

- *For any closed set $F \subset \mathcal{M}_1(\Omega)$,*

$$\lim_{N\to\infty} \frac{1}{N^d} \log P^\varepsilon\big(R_N \in F\big) \leqslant - \inf_{\mu\in F} \mathcal{I}_\varepsilon(\mu), \tag{4.9}$$

- *For any open set $G \subset \mathcal{M}_1(\Omega)$,*

$$\lim_{N\to\infty} \frac{1}{N^d} \log P^\varepsilon\big(R_N \in G\big) \geqslant - \inf_{\mu\in G} \mathcal{I}_\varepsilon(\mu). \tag{4.10}$$

The proof of Theorem 4.2 is postponed until the end of the section.

Observe that the semi-continuity of $\mathcal{I}$ on each $\mathcal{K}_L$ is a consequence of (4.7) and the semi-continuity of $\mathcal{I}_\varepsilon$. We turn to the proof of the bounds (4.3) and (4.4). Let $\varepsilon > 0$ be fixed, and choose an integer N. We define a new measure $P^{N,\varepsilon} \in \mathcal{M}_1(\Omega)$ by

$$\frac{dP^{N,\varepsilon}}{dP}(\phi) = F_{N,\varepsilon}(\phi) \equiv \frac{1}{Z_{N,\varepsilon}} \exp\Big(-\frac{\varepsilon}{2}\sum_{x\in\Lambda_N}\phi_x^2\Big). \tag{4.11}$$

The normalizing factor $Z_{N,\varepsilon}$ can be computed

$$Z_{N,\varepsilon} = P\Big[\exp\Big(-\frac{\varepsilon}{2}\sum_{x\in\Lambda_N}\phi_x^2\Big)\Big] = \det\big(1\!\!1_N + \varepsilon G_N\big)^{-1/2}.$$

In particular, the argument of Theorem 2.5 in [22] shows that

$$\lim_{N\to\infty}\frac{1}{N^d}\log Z_{N,\varepsilon} = \phi(\varepsilon). \tag{4.12}$$

where $\phi(\varepsilon)$ is given in (4.8).

The key intermediate step in deriving the bounds of Theorem 4.1 is given by the following result.

Theorem 4.3. *For each closed set $F \subset \mathcal{M}_1(\Omega)$,*

$$\limsup_{N\to\infty}\frac{1}{N^d}\log P^{N,\varepsilon}\big(R_N \in F\big) \leqslant -\inf_{\mu\in F}\mathcal{I}_\varepsilon(\mu), \tag{4.13}$$

and, for each open set $G \subset \mathcal{M}_1(\Omega)$,

$$\liminf_{N\to\infty}\frac{1}{N^d}\log P^{N,\varepsilon}\big(R_N \in G\big) \geqslant -\inf_{\mu\in G}\mathcal{I}_\varepsilon(\mu). \tag{4.14}$$

Let us assume, in fact, the validity of Theorem 4.3. Let $L > 0$ be fixed. For each closed set $F \subset \mathcal{M}_1(\Omega)$, setting $F_L \equiv F \cap \mathcal{K}_L$, we have

$$\begin{aligned} P\big(R_N \in F_L\big) &= P^{N,\varepsilon}\big(F_{N,\varepsilon}^{-1}\,;\, R_N \in F_L\big) \\ &\leqslant Z_{N,\varepsilon}\, \exp\left(\varepsilon\, L N^d/2\right) P^{N,\varepsilon}\big(R_N \in F_L\big)\,, \end{aligned} \tag{4.15}$$

where we have used the fact that if $R_N(\phi) \in \mathcal{K}_L$, then

$$\sum_{x\in\Lambda_N} \phi_x^2 \;\leqslant\; L\, N^d\,.$$

On the other hand, for any open set $G \subset \mathcal{M}_1(\Omega)$, we have

$$\begin{aligned} P\big(R_N \in G\big) &= P^{N,\varepsilon}\big(F_{N,\varepsilon}^{-1}\,;\, R_N \in G\big) \\ &\geqslant Z_{N,\varepsilon}\, P^{N,\varepsilon}\big(R_N \in G\big)\,. \end{aligned} \tag{4.16}$$

Let us prove the upper bound (4.3). From (4.15) and (4.13) we have

$$\limsup_{N\to\infty} \frac{1}{N^d}\, \log P\big(R_N \in F_L\big) \;\leqslant\; -\inf_{\mu\in F_L} \mathcal{I}_\varepsilon(\mu) \,+\, \phi(\varepsilon) \,+\, \frac{1}{2}\,\varepsilon\, L\,.$$

Using (4.7) we have

$$\limsup_{N\to\infty} \frac{1}{N^d}\, \log P\big(R_N \in F_L\big) \;\leqslant\; -\inf_{\mu\in F_L} \mathcal{I}(\mu) \,+\, \frac{1}{2}\,\varepsilon\, L\,. \tag{4.17}$$

Then (4.3) follows from (4.17) letting $\varepsilon \to 0$. To prove the lower bound (4.4), it is sufficient to show that for each $\mu \in G$,

$$\liminf_{N\to\infty} \frac{1}{N^d}\, \log P\big(R_N \in G\big) \;\geqslant\; -\,\mathcal{I}(\mu)\,. \tag{4.18}$$

We may assume $\mu \in \mathcal{K}_\infty$. From (4.16) and (4.14), we have

$$\liminf_{N\to\infty} \frac{1}{N^d}\, \log P\big(R_N \in G\big) \;\geqslant\; -\,\mathcal{I}_\varepsilon(\mu) \,+\, \phi(\varepsilon)\,.$$

Using (4.7) this implies

$$\liminf_{N\to\infty} \frac{1}{N^d}\, \log P\big(R_N \in G\big) \;\geqslant\; -\,\mathcal{I}(\mu) \,-\, \frac{1}{2}\,\varepsilon\mu(\phi_0^2)\,.$$

Letting $\varepsilon \to 0$ we obtain (4.18). This completes the proof of Theorem 4.1. ☐

4.2. **Proof of Theorem 4.3.** Before proving the bounds (4.13) and (4.14), we shall derive some useful facts about the measures P^ε and $P^{N,\varepsilon}$. Let Q be given by (2.4). For $x, y \in \mathbb{Z}^d$, we define the modified couplings

$$Q^\varepsilon(x,y) \;=\; \frac{Q(x,y)}{(1+\varepsilon)}\,, \qquad Q^{\varepsilon,N}(x,y) \;=\; \begin{cases} Q^\varepsilon(x,y) & \text{if } x \in \Lambda_N \\ Q(x,y) & \text{if } x \notin \Lambda_N \end{cases} \tag{4.19}$$

Lemma 4.4. *Let G^ε be the covariance of the Gaussian field P^ε. Then*

$$G^\varepsilon \;=\; (1+\varepsilon)^{-1} \sum_{n=0}^{\infty} (Q^\varepsilon)^n\,. \tag{4.20}$$

In particular, the marginal of P^ε on each $\Lambda \subset\subset \mathbb{Z}^d$ is given by the centered Gaussian measure with covariance

$$G^\varepsilon_\Lambda \;=\; (1+\varepsilon)^{-1}\sum_{n=0}^{\infty}(\tilde{Q}^\varepsilon_\Lambda)^n\,, \tag{4.21}$$

where, for $x,y \in \Lambda$

$$\tilde{Q}^\varepsilon_\Lambda(x,y) \;=\; Q^\varepsilon(x,y) \,+\, \sum_{l=1}^{\infty}\sum_{z_1\notin\Lambda}\cdots\sum_{z_l\notin\Lambda} Q^\varepsilon(x,z_1)\cdots Q^\varepsilon(z_l,y)\,. \tag{4.22}$$

Proof. The covariance G^ε is, by definition,

$$G^\varepsilon(x,y) \;=\; \frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d} f^\varepsilon(\theta)\,\mathrm{e}^{i\,(y-x)\cdot\theta}\,d\theta\,.$$

From (4.5) we see that f^ε is the Fourier transform of $(1+\varepsilon)^{-1}(\mathbb{I}-Q^\varepsilon)^{-1}$. This implies (4.20). The rest is a straightforward consequence of Lemma A.1 applied to (4.20). ☐

Lemma 4.5. *Let $\Lambda \subset \Lambda_N$. Then the marginal of $P^{N,\varepsilon}$ on Λ is the centered Gaussian measure with covariance*

$$G^{\varepsilon,N}_\Lambda \;=\; (1+\varepsilon)^{-1}\sum_{n=0}^{\infty}(\tilde{Q}^{\varepsilon,N}_\Lambda)^n\,, \tag{4.23}$$

where, for $x,y \in \Lambda$

$$\tilde{Q}^{\varepsilon,N}_\Lambda(x,y) \;=\; Q^\varepsilon(x,y) \,+\, \sum_{l=1}^{\infty}\sum_{z_1\notin\Lambda}\cdots\sum_{z_l\notin\Lambda} Q^{\varepsilon,N}(x,z_1)\cdots Q^{\varepsilon,N}(z_l,y)\,. \tag{4.24}$$

Proof. We first observe that the marginal of $P^{N,\varepsilon}$ on Λ_N is the centered Gaussian measure with covariance

$$G^{\varepsilon,N} \;=\; \big(G_N^{-1} \,+\, \varepsilon\mathbb{I}_N\big)^{-1}\,. \tag{4.25}$$

From Lemma A.1 we have $G_N^{-1} = \mathbb{I}_N - \tilde{Q}_N$ and we can write

$$G^{\varepsilon,N} \;=\; (1+\varepsilon)^{-1}\sum_{n=0}^{\infty}(1+\varepsilon)^{-n}\tilde{Q}_N^n\,. \tag{4.26}$$

The marginal of $P^{N,\varepsilon}$ on $\Lambda \subset \Lambda_N$ is then the centered Gaussian measure with covariance $\{G^{\varepsilon,N}(x,y)\}_{x,y\in\Lambda}$. By applying Lemma A.1 to (4.26), it is straightforward to check that the effective couplings in (4.23) are given precisely by (4.24). ☐

Remark 6. In the ferromagnetic case, the couplings Q^ε, cf. (4.19), define a discrete time random walk on $\mathbb{Z}^d$ with constant *killing*. The probability to die at each step is $\varepsilon/(1+\varepsilon)$. Similarly, the couplings $Q^{\varepsilon,N}$ define a random walk for which killing occurs only *inside* Λ_N. Moreover, the covariance G^ε can be seen as the Green function associated to the random walk with constant killing probability $\varepsilon/(1+\varepsilon)$, with killing starting at time 0, cf. (4.20).

We turn to the proof of the upper bound (4.13). We use here the *projective limit* approach. Let $M \in \mathbb{Z}_+$ and let $\pi_M : \Omega \to \Omega_M$ be the canonical projection $\phi \to \phi_{\Lambda_M}$. We will also use the notation $\mu \circ \pi_M^{-1}$ to denote the marginal μ_M on Λ_M of a measure $\mu \in \mathcal{M}_1(\Omega)$. The Dawson-Gärtner theorem (see e.g. Theorem 4.6.1 in [9]) shows that in order to establish the upper bound (4.13) it is enough to show, for each $M \in \mathbb{Z}_+$, for each closed set $F_M \subset \mathcal{M}_1(\Omega_M)$,

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N^d} \log P^{N,\varepsilon}\big(R_N \circ \pi_M^{-1} \in F_M\big) \\ \leqslant -\inf\big\{\mathcal{I}_\varepsilon(\mu)\,,\ \ \mu \in \mathcal{M}_1(\Omega):\ \ \mu_M \in F_M\big\}\,.\end{aligned} \tag{4.27}$$

Let us fix $M \in \mathbb{Z}^+$ and denote by ρ_M a metric on $\mathcal{M}_1(\Omega_M)$, compatible with the weak topology and such that ρ_M is dominated by the total variation norm (cf. e.g. [15]). For a given $\delta \in (0,1)$, define

$$R_{N,\delta}(\phi) \;=\; \frac{1}{|\Lambda_{N,\delta}|}\sum_{x\in\Lambda_{N,\delta}} \delta_{\theta_x\phi_N}\,,$$

where

$$\Lambda_{N,\delta} \;\equiv\; \big[(1-\delta)N/2,(1-\delta)N/2\big]^d \cap \mathbb{Z}^d.$$

It is easy to check that there exist constants $\delta_0(d) > 0$ and $c(d) < \infty$, only depending on d, such that for any bounded measurable function f, for any $N \in \mathbb{Z}^+$,

$$\sup_{\phi\in\Omega}\big|\,R_N(\phi)\,(f) \;-\; R_{N,\delta}(\phi)\,(f)\,\big| \;<\; c(d)\,\|f\|_\infty\,\delta$$

provided $\delta < \delta_0(d)$. This implies

$$\sup_{\phi\in\Omega}\,\rho_M\big(\,R_N(\phi),R_{N,\delta}(\phi)\,\big) \;<\; c(d)\,\delta \tag{4.28}$$

for $\delta < \delta_0(d)$, for integers $N > M$. Now observe that, if e.g. $N > 2M/\delta$, the map

$$\phi \;\to\; R_{N,\delta}(\phi)\circ\pi_M^{-1}$$

is $\mathcal{F}_{\Lambda_{N,\delta/2}}$-measurable. Let us define the density

$$\phi_{N,\delta} \;\equiv\; \frac{dP^{N,\varepsilon}_{N,\delta/2}}{dP^{\varepsilon}_{N,\delta/2}}\,, \tag{4.29}$$

where $P^{N,\varepsilon}_{N,\delta/2}$ and $P^{\varepsilon}_{N,\delta/2}$ are the marginals of $P^{N,\varepsilon}$ and P^ε respectively, on the box $\Lambda_{N,\delta/2}$. We define the set

$$F^{(\delta)} \;=\; \Big\{\nu \in \mathcal{M}_1(\Omega_M) \quad \rho_M(\nu,F_M) \;\leqslant\; c(d)\delta\,\Big\}\,.$$

Then, by (4.28),

$$\begin{aligned}P^{N,\varepsilon}\big(R_N\circ\pi_M^{-1}\in F_M\big) \;&\leqslant\; P^{N,\varepsilon}\big(R_{N,\delta}\circ\pi_M^{-1}\in F^{(\delta)}\,\big)\\ &=\; P^{\varepsilon}\big(\,\phi_{N,\delta}\,;\;R_{N,\delta}\circ\pi_M^{-1}\in F^{(\delta)}\,\big)\,.\end{aligned}$$

For each $1 < p < \infty$, $q = p/(p-1)$, Hölder's inequality yields

$$\log P^{N,\varepsilon}\big(R_N\circ\pi_M^{-1}\in F_M\big) \;\leqslant\; \frac{1}{p}\log P^{\varepsilon}\big(\phi_{N,\delta}^p\big) \;+\; \frac{1}{q}\log P^{\varepsilon}\big(R_N\circ\pi_M^{-1}\in F^{(2\delta)}\,\big)\,, \tag{4.30}$$

where we have used (4.28) again. We shall need the following key approximation result.

Lemma 4.6. *Let $\phi_{N,\delta}$ be the density defined in (4.29). For all $p \geqslant 1$, $\varepsilon > 0$ and $\delta > 0$, we have*

$$\lim_{N\to\infty}\frac{1}{N^d}\,\log P^{\varepsilon}\big(\phi_{N,\delta}^p\big) \;=\; \lim_{N\to\infty}\frac{1}{N^d}\,\log P^{N,\varepsilon}\big(\phi_{N,\delta}^{-p}\big) \;=\; 0. \tag{4.31}$$

Let us show that Lemma 4.6 allows to prove the desired estimate. Suppose (4.31) holds. Then (4.30) together with the upper bound of Theorem (4.2) imply

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N^d} \log P^{N,\varepsilon}\big(R_N \circ \pi_M^{-1} \in F_M\big) \\ \leqslant \; - \frac{1}{q} \inf \big\{\mathcal{I}_\varepsilon(\mu)\,, \;\; \mu \in \mathcal{M}_1(\Omega) \; : \;\; \mu \circ \pi_M^{-1} \in F^{(2\delta)} \big\}\,.\end{aligned}$$

This implies (4.27) since q and δ are arbitrarily close to 1 and 0 respectively. The proof of (4.13) is complete.

Before proving Lemma 4.6, we derive the lower bound (4.14). Let us fix an open set $G_M \subset \mathcal{M}_1(\Omega_M)$, for a given $M \in \mathbb{Z}^+$. Given $\delta > 0$ define

$$G^{(\delta)} \;=\; \big\{\mu \in \mathcal{M}_1(\Omega_M) \quad \rho_M(\mu, G_M^c) \;\geqslant\; c(d)\delta\big\}\,.$$

Then, by (4.28)

$$\begin{aligned}P^{N,\varepsilon}\big(R_N \circ \pi_M^{-1} \in G_M\big) \;&\geqslant\; P^{N,\varepsilon}\big(R_{N,\delta} \circ \pi_M^{-1} \in G^{(\delta)}\,\big) \\ &=\; P^{\varepsilon}\big(\,\phi_{N,\delta}\,;\; R_{N,\delta} \circ \pi_M^{-1} \in G^{(\delta)}\,\big)\,.\end{aligned}$$

For any $1 < p < \infty$, $p/q = p - 1$, Hölder's inequality yields

$$\begin{aligned}&\log P^{N,\varepsilon}\big(R_N \circ \pi_M^{-1} \in G_M\big) \\ &\qquad \geqslant\; - \frac{q}{p}\log P^{N,\varepsilon}\big(\phi_{N,\delta}^{-p}\big) \;+\; q \log P^{\varepsilon}\big(R_N \circ \pi_M^{-1} \in G^{(2\delta)}\big)\,.\end{aligned} \tag{4.32}$$

From Lemma 4.6, the arbitrariness of q and δ, and from the lower bound of Theorem 4.2, we have

$$\liminf_{N\to\infty} \frac{1}{N^d} \log P^{N,\varepsilon}\big(R_N \circ \pi_M^{-1} \in G_M\big) \;\geqslant\; -\, \mathcal{I}_\varepsilon(\mu)\,,$$

for any $\mu \in \mathcal{M}_1(\Omega)$ such that $\mu_M \in G_M$. This implies (4.14). The proof of Theorem 4.3 is completed. □

4.3. **Proof of Lemma 4.6.** We fix $\varepsilon > 0$ and $\delta > 0$. Let $G^{N,\varepsilon}$ be the covariance function of the field $P^{N,\varepsilon}$, cf. (4.25). Let us define the matrices

$$C_N \;=\; \big\{G^{N,\varepsilon}(x,y)\big\}_{x,y\in\Lambda_{N,\delta/2}}\,, \qquad D_N \;=\; \big\{G^{\varepsilon}(x,y)\big\}_{x,y\in\Lambda_{N,\delta/2}}\,.$$

Then (4.29) can be written

$$\phi_{N,\delta}(\sigma) \;=\; \det\big(D_N\, C_N^{-1}\big)^{\frac{1}{2}} \;\exp\Big(-\frac{1}{2}\,\big\langle\sigma, (C_N^{-1} \;-\; D_N^{-1})\sigma\big\rangle\Big)\,. \tag{4.33}$$

An explicit computation shows that

$$P^{\varepsilon}\big(\phi_{N,\delta}^{p}\big) \;=\; \det\big(\mathbb{1}_N + B_N\big)^{p/2} \,\det\big(\mathbb{1}_N + p\,B_N\big)^{-\frac{1}{2}}\,, \tag{4.34}$$

where we have defined

$$B_N \;=\; D_N\, C_N^{-1} \;-\; \mathbb{1}_N\,,$$

with $\mathbb{1}_N$ denoting the unit diagonal matrix on $\Lambda_{N,\delta/2}$. Moreover, setting $q = p/(p-1)$,

$$P^{N,\varepsilon}\big(\phi_{N,\delta}^{-p}\big) \;=\; \det\big(\mathbb{1}_N + B_N\big)^{-p/2q} \,\det\Big(\mathbb{1}_N - \frac{p}{2q}\,B_N\Big)^{-\frac{1}{2}}\,. \tag{4.35}$$

Let us denote by

$$\{\lambda_j(N)\,,\;\; j=1,\dots,|\Lambda_{N,\delta}|\}$$

the set of eigenvalues of the matrix B_N. Observe that, by elementary matrix calculus, we have

$$\sup_j |\lambda_j(N)| \;\leqslant\; \|B_N\| \;\equiv\; \sup_{x\in\Lambda_{N,\delta}} \sum_{y\in\Lambda_{N,\delta}} |B_N(x,y)|\,. \tag{4.36}$$

From (4.34),

$$\begin{aligned}\log P^{\varepsilon}\big(\phi^{p}_{N,\delta}\big) \;&=\; \frac{p}{2}\sum_i \log\left(1+\lambda_i\right) \;-\; \frac{1}{2}\sum_i \log\left(1+p\,\lambda_i\right)\\ &\leqslant\; \frac{p}{2}|\Lambda_{N,\delta}|\,\log\left(1+\|B_N\|\right) \;-\; \frac{1}{2}|\Lambda_{N,\delta}|\,\log\left(1-p\,\|B_N\|\right).\end{aligned}$$

A similar computation for (4.35) shows

$$\begin{aligned}&\log P^{N,\varepsilon}\big(\phi^{-p}_{N,\delta}\big)\\ &\quad\leqslant\; -\,\frac{p}{2q}|\Lambda_{N,\delta}|\,\log\left(1+\|B_N\|\right) \;-\; \frac{1}{2}|\Lambda_{N,\delta}|\,\log\left(1+\frac{p}{2q}\,\|B_N\|\right).\end{aligned}$$

In order to prove (4.31) it is then sufficient to show

$$\lim_{N\to\infty} \|B_N\| \;=\; 0\,. \tag{4.37}$$

We have

$$\|B_N\| \;\leqslant\; \|D_N\|\,\|C_N^{-1} \;-\; D_N^{-1}\|\,,$$

where the norm $\|\cdot\|$ is defined as in (4.36). From the absolute summability of G^{ε}, cf. appendix, it follows that for each $\varepsilon>0$

$$\sup_{N\in\mathbb{Z}_+} \|D_N\| \;<\; \infty\,.$$

Then (4.37) will follow from

$$\lim_{N\to\infty} \|C_N^{-1} \;-\; D_N^{-1}\| \;=\; 0\,. \tag{4.38}$$

Thanks to Lemma 4.4 and Lemma 4.5, we can write, for $x,y\in\Lambda_{N,\delta/2}$

$$C_N^{-1}(x,y) \;-\; D_N^{-1}(x,y) \;=\; (1+\varepsilon)\left[\tilde Q^{\varepsilon,N}_{\Lambda_{N,\delta/2}}(x,y) \;-\; \tilde Q^{\varepsilon}_{\Lambda_{N,\delta/2}}(x,y)\right].$$

We now use explicitly the expressions (4.19), (4.22) and (4.24). Given sites

$$z_1\notin\Lambda_{N,\delta/2}\,,\;\dots\;,\;z_l\notin\Lambda_{N,\delta/2}\,,$$

let $k(z_1,\dots,z_l)$ be the number of i's, $i \in \{1,\dots,l\}$ such that $z_i \notin \Lambda_N$. Then, for all $x,y \in \Lambda_{N,\delta/2}$, letting $p^\varepsilon(x,y) = |Q^\varepsilon(x,y)|$,

$$\begin{aligned}
&|C_N^{-1}(x,y) \;-\; D_N^{-1}(x,y)| \\
&\quad\leqslant \sum_{l=1}^{\infty} \sum_{z_1\notin\Lambda_{N,\delta/2}} \cdots \sum_{z_l\notin\Lambda_{N,\delta/2}} p^\varepsilon(x,z_1)\cdots p^\varepsilon(z_l,y)\left[(1+\varepsilon)^{k(z_1,\dots,z_l)} \;-\; 1\right]. \\
&\quad\leqslant 2\sum_{l=1}^{\infty} \sum_{z_1\notin\Lambda_{N,\delta/2}} \cdots \sum_{z_l\notin\Lambda_{N,\delta/2}} p^\varepsilon(x,z_1)\cdots \\
&\qquad\qquad \cdots p^\varepsilon(z_l,y)\,(1+\varepsilon)^{k(z_1,\dots,z_l)}\,\chi(k(z_1,\dots,z_l)\geqslant 1)\,,
\end{aligned} \tag{4.39}$$

where χ stands for the indicator function. Let us consider now the random walk on $\mathbb{Z}^d$ with rates $p^{\varepsilon,N}(x,y) = |Q^{\varepsilon,N}(x,y)|$, i.e. the process with killing occurring only inside Λ_N, cf. the remark after Lemma 4.5. We observe that the quantity appearing in the r.h.s. of (4.39), apart from the factor 2, is the probability that the walker started in $x \in \Lambda_{N,\delta/2}$, arrives eventually in $y \in \Lambda_{N,\delta/2}$ taking only jumps out of $\Lambda_{N,\delta/2}$ and visiting at least once the set $\mathbb{Z}^d \setminus \Lambda_N$. Let $\{\xi_n\}$, $n = 0,1\dots$, $\xi_0 = x$, denote the above described process, and let $\mathbb{P}_x$ be the associated probability measure. Let τ_ε be the time at which $\{\xi_n\}$ is killed. Let also τ_N be the hitting time for $\mathbb{Z}^d \setminus \Lambda_N$, i.e.

$$\tau_N \;=\; \inf\{n \geqslant 1\,,\;\; \xi_n \notin \Lambda_N\}\,.$$

Summing over $y \in \Lambda_{N,\delta/2}$ in (4.39) we can estimate

$$\|D_N^{-1} \;-\; C_N^{-1}\| \;\leqslant\; 2 \sup_{x\in\Lambda_{N,\delta/2}} \mathbb{P}_x\left(\tau_\varepsilon \;>\; \tau_N\right) \tag{4.40}$$

Observe that due to the constant killing inside Λ_N, there exists $m(\varepsilon) > 0$ such that for any N, $x \in \Lambda_{N,\delta/2}$ for any $l > 0$ sufficiently large

$$\mathbb{P}_x\left(\tau_\varepsilon \;>\; l\,;\, \tau_N = l\right) \;\leqslant\; \exp\left(-m(\varepsilon)\,l\right).$$

We then estimate, for any sufficiently large integer n,

$$\begin{aligned}
\mathbb{P}_x\left(\tau_\varepsilon \;>\; \tau_N\right) \;&\leqslant\; \sum_{l=0}^{n} \mathbb{P}_x\left(\tau_N = l\right) \;+\; \sum_{l=n+1}^{\infty} \mathbb{P}_x\left(\tau_\varepsilon > l\,;\; \tau_N = l\right) \\
&\leqslant\; n\,\mathbb{P}_x\left(\tau_N \leqslant n\right) \;+\; \exp\left(-m(\varepsilon)\,n/2\right).
\end{aligned} \tag{4.41}$$

On the other hand, for each fixed $n > 0$,

$$\sup_{x\in\Lambda_{N,\delta/2}} \mathbb{P}_x\left(\tau_N \leqslant n\right) \;\leqslant\; n \sum_{|z|>\delta N/n} p(0,z)\,,$$

is vanishing in the limit $N \to \infty$. Since n is arbitrary in (4.41), this implies (4.38) and the Lemma follows. ☐

4.4. **The hypercontractive regime.** This subsection is devoted to the proof of theorem 4.2. Let the rate functions $\mathcal{I}$ and $\mathcal{I}_\varepsilon$ be defined as in (4.2) and (4.6). Let also $\phi(\varepsilon)$ be given by (4.8).

Lemma 4.7. *For each $\varepsilon > 0$, $\mathcal{I}_\varepsilon$ satisfies the variational principle*

$$\mathcal{I}_\varepsilon(\mu) \;=\; 0 \;\iff\; \mu \;=\; P^\varepsilon\,, \qquad \mu \in \mathcal{M}_1(\Omega) \tag{4.42}$$

and the identity

$$\mathcal{I}_\varepsilon(\mu) \;=\; \mathcal{I}(\mu) \,+\, \frac{\varepsilon}{2}\mu(\phi_0^2) \,+\, \phi(\varepsilon)\,, \quad \mu \in \mathcal{K}_\infty\,. \tag{4.43}$$

Proof. From Proposition 3.1 we have

$$h(\mu|P^\varepsilon) \;=\; e^\varepsilon(\mu) \,-\, s(\mu) \,+\, p^\varepsilon \tag{4.44}$$

where the entropy s is given by (3.3) (independent of ε), and e^ε, p^ε are the internal energy and the pressure associated to the potential $J+\varepsilon\mathbb{I}$. According to (3.7) and (4.5) we see that

$$p^\varepsilon \,-\, p \;=\; \phi(\varepsilon)\,.$$

Then (4.43) follows from the obvious identity (cf. (3.5))

$$e^\varepsilon(\mu) \,-\, e(\mu) \;=\; \frac{\varepsilon}{2}\mu(\phi_0^2)\,.$$

To prove the variational principle we may use theorem 3.2 and the fact that P^ε is the only shift invariant Gibbs measure with finite second moment for the potential $J+\varepsilon\mathbb{I}$, cf. e.g. Corollary (13.40) in [20]. □

Upper bound. Once the Hypercontractivity of P^ε is established, cf. Lemma B.2, the proof of the upper bound in theorem 4.2 follows the standard line, cf. section 5.4 of [11]. We point out the main steps needed.

Step 1. Let $Q \subset\subset \mathbb{Z}^d$ be a fixed cube containing the origin. Using Lemma B.2 one shows that for all bounded measurable function f such that $S_f \subset Q$, the limit

$$\Lambda_Q(f) \;=\; \lim_{N\to\infty} \frac{1}{N^d}\, \log P^\varepsilon\big(\exp \sum_{x\in\Lambda_N} f\circ\theta_x\big) \tag{4.45}$$

exists, cf. appendix. An application of Hölder's inequality shows that Λ_Q is a convex functional. Its *Legendre transform* is the convex function on $\mathcal{M}_1(\Omega_Q)$ given by

$$\Lambda_Q^*(\mu) \;=\; \sup_f\, \big\{\mu(f) \,-\, \Lambda_Q(f)\big\}\,, \tag{4.46}$$

where the supremum is taken over continuous bounded f with $S_f \subset Q$.

Step 2. Following Lemma 5.4.13 and Corollary 5.1.11 in [11], one can check that Λ^* has compact level sets on $\mathcal{M}_1(\Omega_Q)$ and for each closed set $F_Q \subset \mathcal{M}_1(\Omega_Q)$,

$$\limsup_{N\to\infty} \frac{1}{N^d}\, \log P^\varepsilon\big(R_N\circ\pi_Q^{-1} \in F_Q\big) \;\leqslant\; -\inf_{\mu\in F_Q} \Lambda_Q^*(\mu). \tag{4.47}$$

Step 3. Let

$$\Lambda^*(\mu) \;=\; \sup_Q \Lambda_Q^*(\mu\circ\pi_Q^{-1})\,, \qquad \mu \in \mathcal{M}_1(\Omega)\,,$$

where the supremum is taken over all finite cubes containing the origin. By definition, the function Λ^* is non-negative, convex and lower semicontinuous. Moreover, the standard *projective limit* argument (cf. e.g. [9], theorem 4.6.1) shows that Λ^* has compact level sets, and (4.47) can be lifted to

$$\limsup_{N\to\infty} \frac{1}{N^d}\, \log P^\varepsilon\big(R_N \in F\big) \;\leqslant\; -\inf_{\mu\in F} \Lambda^*(\mu). \tag{4.48}$$

for all closed sets $F \subset \mathcal{M}_1(\Omega)$.

Step 4. With the same arguments as in [11], Lemma 5.4.17, one then proves the identity

$$\Lambda^*(\mu) \;=\; \mathcal{I}_\varepsilon(\mu)\,, \qquad \mu \in \mathcal{M}_1(\Omega)\,.$$

From step 3, this implies that $\mathcal{I}_\varepsilon$ has compact level sets and satisfies the upper bound (4.9) of theorem 4.2.

Lower Bound. We remark that a proof of the lower bound in theorem 4.2 using hypercontractivity as in [6, 11] would require a condition stronger than Lemma B.2 (cf. section 5.4 of [11]). Since the proof we give below does not depend on hypercontractivity we may equally well set $\varepsilon = 0$ and consider the original field P, instead of P^ε. We are going to show that for any $\mu \in \mathcal{M}_1^S(\Omega)$ and for any open neighbourhood G of μ,

$$\liminf_{N\to\infty} \frac{1}{N^d} \log P\big(R_N \in G\big) \;\geqslant\; -\; \mathcal{I}(\mu)\,. \tag{4.49}$$

The above statement is equivalent to the lower bound (4.10), if we replace P by P^ε and $\mathcal{I}$ by $\mathcal{I}_\varepsilon$. A classical change of measure argument, [13] or Lemma 5.4.21 in [11], implies that (4.49) is always satisfied if μ is ergodic with respect to lattice shifts. The general case can be reduced to $\mu \in \mathcal{M}_1^S(\Omega)$ of the form

$$\mu \;=\; \sum_{i=1}^n p_i\,\mu_i\,, \qquad \sum_{i=1}^n p_i \;=\; 1, \tag{4.50}$$

with $p_i \geqslant 0$ and μ_i ergodic, for each $i = 1,\ldots,n$. For each N, let $\Lambda^i(N), i = 1,\ldots,n$ be a partition of the unit box $[0,1]^d$ such that

$$\Lambda^i(N) \;=\; \prod_{l=1}^d [a_l^i(N), b_l^i(N)]\,, \quad 0 \leqslant a_l^i(N) < b_l^i(N) \leqslant 1\,,$$
$$p_i(N) \equiv |\Lambda^i(N)| \;\to\; p_i\,, \quad \text{as } \; N \to \infty\,. \tag{4.51}$$

and let $\Lambda_N^i = N\Lambda^i(N)$, so that $p_i(N) = N^{-d}|\Lambda_N^i|$. We write

$$R_N(\phi) \;=\; \sum_{i=1}^n p_i(N)\,R_{N,i} \tag{4.52}$$

where

$$R_{N,i} \;=\; \frac{1}{|\Lambda_N^i|} \sum_{x\in\Lambda_N^i} \delta_{\theta_x\phi_N}\,. \tag{4.53}$$

We also denote $\overline{R}_{N,i}(\phi)$ the $\mathcal{F}_{\Lambda_N^i}$-measurable version of $R_{N,i}$. That is $\overline{R}_{N,i}(\phi)$ is obtained from (4.53) by replacing ϕ_N with $\phi_{N,i}$, the periodized version of ϕ restricted to Λ_N^i. By approximating $R_{N,i}$ with $\overline{R}_{N,i}$ (cf. [11], section 5), and using (4.52) one checks the following statement. For any open set G containing μ, we can find neighbourhoods U_i of μ_i, such that if $\overline{R}_{N,i} \in U_i$, $i = 1,\ldots,n$, then $R_N \in G$, for all sufficiently large N. Consider the product measure

$$\nu^{(N)} \;=\; \big(\otimes_{i=1}^n\; \mu_{N,i}\big) \otimes P_{\Lambda_N^c}\,,$$

where we have introduced the marginals $\mu_{N,i}$ of μ^i on Λ_N^i, and the marginal of P on Λ_N^c. We have

$$\nu^{(N)}\big(R_N \in G\big) \;\geqslant\; \nu^{(N)}\big(\cap_{i=1}^n \{\overline{R}_{N,i} \in U_i\}\big) \;=\; \prod_{i=1}^n \mu^i\big(R_{N,i} \in U_i\big)\,.$$

The ergodic theorem implies

$$\lim_{N\to\infty} \mu^i(\overline{R}_{N,i} \in U_i) \;=\; 1\,,$$

for all $i = 1, \ldots, n$, and therefore

$$\lim_{N\to\infty} \nu^{(N)}\Big(R_N \in G\Big) \;=\; 1\,.$$

Again the argument of [11], Lemma 5.4.21, shows that

$$\liminf_{N\to\infty} \frac{1}{N^d}\,\log P\big(R_N \in G\big) \;\geqslant\; -\,\limsup_{N\to\infty} \frac{1}{N^d}\, H_N(\nu^{(N)}|P).$$

We may assume $\mu^i \in \mathcal{K}_\infty$, for all $i = 1, \ldots, n$. Since $h(\cdot|P)$ is affine on $\mathcal{K}_\infty$ (see Proposition 3.1) (4.49) will follow from

$$\limsup_{N\to\infty} \frac{1}{N^d}\, H_N(\nu^{(N)}|P) \;\leqslant\; \sum_{i=1}^{n} p_i\, h(\mu^i|P)\,. \tag{4.54}$$

Let φ_N denote the density of $\nu^{(N)}$ w.r.t. P on the set Λ_N. Let also $P^{(N)}$ denote the measure obtained as products of $P_{N,i}$, the marginals of P on $\Lambda_{N,i}$, and let ψ_N denote its density w.r.t. P on Λ_N. Finally let φ^i_N denote the density of μ^i w.r.t. P on $\Lambda_{N,i}$. We have

$$H_N(\nu^{(N)}|P) \;=\; \nu^{(N)}\big(\log \phi_N\big) \;=\; \sum_{i=1}^{n} \mu^i\big(\log \phi^i_N\big) \;+\; \nu^{(N)}\big(\log \psi_N\big)\,. \tag{4.55}$$

By shift invariance

$$\lim_{N\to\infty} \frac{1}{N^d}\;\sum_{i=1}^{n} \mu^i\big(\log \phi^i_N\big) \;=\; \sum_{i=1}^{n} p_i\, h(\mu^i|P)\,.$$

We will prove

$$\limsup_{N\to\infty} \frac{1}{N^d}\, \nu^{(N)}\big(\log \psi_N\big) \;=\; 0\,. \tag{4.56}$$

Observe that by the argument of Proposition 3.1, cf. (3.11), we can replace ψ_N with the density corresponding to 0-boundary condition, ψ^0_N. We have

$$\psi^0_N(\phi) \;=\; \big(\det \Gamma_N\big)^{\frac{1}{2}} \prod_{i=1}^{n} \big(\det \Gamma^i_N\big)^{-\frac{1}{2}}\, \exp\Big(\frac{1}{2}\sum_{i\neq j}\,\sum_{x\in\Lambda_{N,i}}\,\sum_{y\in\Lambda_{N,j}} J(x,y)\,\phi_x\phi_y\Big)\,.$$

where Γ^i_N is the matrix inverse of $\{J(x,y)\}_{x,y\in\Lambda^i_N}$. By theorem 2.5 of [22],

$$\lim_{N\to\infty} \frac{1}{N^d}\, \log\det \Gamma_N \;=\; \lim_{N\to\infty} \frac{1}{|\Lambda^i_N|} \log\det \Gamma^i_N\,, \quad i = 1, \ldots, n\,.$$

It follows, using (4.51), that

$$\lim_{N\to\infty} \frac{1}{N^d}\, \log\Big(\det \Gamma_N \prod_{i=1}^{n} (\det \Gamma^i_N)^{-1}\Big) \;=\; 0\,.$$

Therefore (4.56) follows from

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\Lambda_{N,i}}\,\sum_{y\in\Lambda_{N,j}} J(x,y)\,\nu^{(N)}(\phi_x\phi_y) \;=\; 0\,, \quad \text{if } i \neq j\,.$$

This is easily checked as in Proposition 3.1. We have shown that equality holds in (4.54). The proof of (4.49) is complete. □

Appendix A. The Expansion

The point of this section is to replace (2.6) by an expansion involving only paths belonging to some finite region $\Lambda \subset\subset \mathbb{Z}^d$ rather than the whole lattice. We fix a finite box $\Lambda \subset\subset \mathbb{Z}^d$, and for each integer $n \geqslant 1$, we define the matrix

$$B_n(x,y) \;=\; \sum_{z_1 \notin \Lambda} \cdots \sum_{z_n \notin \Lambda} Q(x,z_1)\cdots Q(z_n,y)\,, \qquad x\,,y\,\in\Lambda$$

Let also $B_0(x,y) = Q(x,y)$. Here Q is given by (2.4).

Lemma A.1. *For each $x, y \in \Lambda$, we have*

$$G(x,y) \;=\; \sum_{n=0}^{\infty} \tilde{Q}_\Lambda^n(x,y)\,, \tag{A.1}$$

where $\tilde{Q}_\Lambda$ is the matrix given by

$$\tilde{Q}_\Lambda(x,y) \;=\; \sum_{l=0}^{\infty} B_l(x,y)\,. \tag{A.2}$$

Remark. In the *ferromagnetic* case $Q = p$, the above Lemma becomes a restatement of the results of [32], chapter 3. In particular, in this case $\tilde{Q}_\Lambda(x,y)$ can be interpreted as the transition function of a 'renormalized' or *embedded* random walk.

Proof. Let $x, y \in \Lambda$ be fixed. Then, from (2.6), we have

$$G(x,y) \;=\; \delta(x,y) \;+\; Q(x,y) \;+\; \sum_{n=2}^{\infty} Q^n(x,y) \tag{A.3}$$

Moreover,

$$\begin{aligned}\sum_{n=2}^{\infty} Q^n(x,y) \;&=\; \sum_{n=1}^{\infty} \sum_{z_1\in\mathbb{Z}^d} \cdots \sum_{z_n\in\mathbb{Z}^d} Q(x,z_1)\cdots Q(z_n,y)\\ &=\; \sum_{n=1}^{\infty} \sum_{\pi\in\{\Lambda,\Lambda^c\}^n} \sum_{z_1\in\pi(1)} \cdots \sum_{z_n\in\pi(n)} Q(x,z_1)\cdots Q(z_n,y)\,,\end{aligned}$$

where we have isolated the contribution of each sequence $\pi \in \{\Lambda, \Lambda^c\}^n$. For integers $n \geqslant 1$ and $k \leqslant n$, define the matrix element $\mathcal{A}(n,k)$ by

$$\mathcal{A}(n,k) \;=\; \sum_{\substack{\pi\in\{\Lambda,\Lambda^c\}^n:\\ \#_\Lambda(\pi)=k}} \sum_{z_1\in\pi(1)} \cdots \sum_{z_n\in\pi(n)} Q(x,z_1)\cdots Q(z_n,y)\,,$$

where we only have summed the contributions of those sequences $\pi \in \{\Lambda, \Lambda^c\}^n$ such that Λ occurs exactly k times ($\#_\Lambda(\pi) = k$). Setting $\mathcal{A}(0,0) = 0$, we have

$$\sum_{n=2}^{\infty} Q^n(x,y) \;=\; \sum_{n=1}^{\infty}\sum_{k=0}^{n} \mathcal{A}(n,k) \;=\; \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \mathcal{A}(n,k)\,. \tag{A.4}$$

Now observe that $\mathcal{A}(n,0) = B_n(x,y)$, for all $n \geqslant 1$. In particular

$$\sum_{n=1}^{\infty} \mathcal{A}(n,0) \;=\; \tilde{Q}_\Lambda(x,y) \;-\; Q(x,y)\,,$$

where $\tilde{Q}_\Lambda$ is defined by (A.2). From (A.3) and (A.4) we have

$$G(x,y) \;=\; \delta(x,y) \;+\; \tilde{Q}_\Lambda(x,y) \;+\; \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \mathcal{A}(n,k)\,.$$

In order to finish the proof, we have to check the identity

$$\sum_{n=k}^{\infty} \mathcal{A}(n,k) \;=\; \tilde{Q}_\Lambda^{k+1}(x,y)\,, \quad k \;\geqslant\; 1\,. \tag{A.5}$$

For fixed integers $n \geqslant k \geqslant 1$, we can write

$$\begin{aligned}\mathcal{A}(n,k) \;=\; &\sum_{1 \,\leqslant\, i_1<\cdots<i_k \,\leqslant\, n}\; \sum_{\xi_1\in\Lambda} \cdots \\ &\cdots \sum_{\xi_k\in\Lambda} B_{i_1-1}(x,\xi_1)\, B_{i_2-i_1-1}(\xi_1,\xi_2)\cdots B_{n-i_k}(\xi_k,y)\,.\end{aligned}$$

Summing over n we obtain

$$\begin{aligned}\sum_{n=k}^{\infty} \mathcal{A}(n,k) \;=\; &\sum_{i_1=1}^{\infty}\;\sum_{i_2=i_1+1}^{\infty} \cdots \sum_{i_k=i_{k-1}+1}^{\infty}\;\sum_{n=i_k}^{\infty}\;\sum_{\xi_1\in\Lambda} \cdots \\ &\cdots \sum_{\xi_k\in\Lambda} B_{i_1-1}(x,\xi_1)\, B_{i_2-i_1-1}(\xi_1,\xi_2)\cdots B_{n-i_k}(\xi_k,y)\,.\end{aligned}$$

The above line can now be written as

$$\tilde{Q}_\Lambda^{k+1}(x,y) \;=\; \sum_{l_1=0}^{\infty} \cdots \sum_{l_{k+1}=0}^{\infty}\;\sum_{\xi_1\in\Lambda} \cdots \sum_{\xi_k\in\Lambda} B_{l_1}(x,\xi_1)\, B_{l_2}(\xi_1,\xi_2)\cdots B_{l_{k+1}}(\xi_k,y)\,.$$

This proves the identity (A.5) and completes the proof of (A.1). □

Appendix B. Hypercontractivity estimates

We start with an application of Nelson's famous result on the hypercontractivity of Gaussian measures. Similar estimates have been obtained in the work [2]. Let P^ε be the centered Gaussian field with spectral density given in (4.5). Also let $\beta > 0$ be as in the decay assumption (2.5). We denote by $\|f\|_p$, $1 \leqslant p < \infty$ the L^p norm of f w.r.t. P^ε.

Lemma B.1. *There exist constants $\alpha = \alpha(\beta) > 0$ and $l_0 > 0$ such that for all $l > l_0$ and for any pair of local functions $f, g \in L^1(P^\varepsilon)$ whose supports satisfy $d(S_f, S_g) > l$,*

$$\|f\,g\|_1 \;\leqslant\; \|f\|_{1+\tau(l)}\, \|g\|_{1+\tau(l)}\,, \tag{B.1}$$

with $\tau(l) \leqslant l^{-\alpha}$.

Proof. Given two configurations $\xi, \eta \in \Omega$ of finite support, we define

$$\langle \xi, \eta \rangle_\varepsilon \;=\; \sum_{x,y\in\mathbb{Z}^d} \xi_x\, G^\varepsilon(x,y)\, \eta_y\,,$$

and let $\|\xi\|_\varepsilon^2 = \langle \xi, \xi \rangle_\varepsilon$. Following [6] we define, for $\Lambda_1, \Lambda_2 \subset\subset \mathbb{Z}^d$

$$\tau(\Lambda_1, \Lambda_2) = \sup\{ \langle \xi, \eta \rangle_\varepsilon \,, \ \|\xi\|_\varepsilon = \|\eta\|_\varepsilon = 1 \,, \\ \xi = 0 \text{ on } \Lambda_1^c \,, \ \eta = 0 \text{ on } \Lambda_2^c \} \,,$$

and set, for any $l > 0$

$$\tau(l) = \sup \{ \tau(\Lambda_1, \Lambda_2) \,; \ \Lambda_1, \Lambda_2 \subset\subset \mathbb{Z}^d, \ d(\Lambda_1, \Lambda_2) > l \} \,. \tag{B.2}$$

We may use now Nelson's estimate (cf. e.g. theorem 5.11 and Lemma 5.12 in [6]), to write

$$\|f\, g\|_1 \leqslant \|f\|_{1+\tau(l)} \, \|g\|_{1+\tau(l)} \,,$$

for any pair of local functions $f, g \in L^1(P^\varepsilon)$ whose supports satisfy $d(S_f, S_g) > l$.
Let us check the decay of $\tau(l)$, as $l \to \infty$. Choose $\Lambda_1, \Lambda_2 \subset\subset \mathbb{Z}^d$ and $\xi, \eta \in \Omega$ such that $\xi = 0$ on Λ_1^c and $\eta = 0$ on Λ_2^c, and $\|\xi\|_\varepsilon = \|\eta\|_\varepsilon = 1$. We write

$$\begin{aligned} | \langle \xi, \eta \rangle_\varepsilon | &\leqslant 2 \sum_{x, y \in \mathbb{Z}^d} (\xi_x^2 + \eta_y^2) \, |G^\varepsilon(x,y)| \\ &\leqslant 4 \sup_{x \in \Lambda_1} \sum_{y \in \Lambda_2} |G^\varepsilon(x,y)| + 4 \sup_{y \in \Lambda_2} \sum_{x \in \Lambda_1} |G^\varepsilon(x,y)| \,, \end{aligned} \tag{B.3}$$

having used the fact that, for any $\xi \in \Omega$ of finite support

$$\sum_{x \in \mathbb{Z}^d} \xi_x^2 \leqslant 2 \, \|\xi\|_\varepsilon^2 \,,$$

which is easily checked using e.g. the Fourier transform (2.7). It is then sufficient to show

$$\sum_{|x| > l} |G^\varepsilon(0,x)| \leqslant l^{-\alpha} \,, \tag{B.4}$$

for some $\alpha > 0$ and for all l sufficiently large. We first observe that $G^\varepsilon(0,x)$ is indeed absolutely summable, cf. (4.20). We can then write

$$\sum_{|x| > l} |G^\varepsilon(0,x)| \leqslant \sum_{|x| > l} p(0,x) + \sum_{n=2}^{\infty} (1+\varepsilon)^{-n} \sum_{|x| > l} p^n(0,x) \,. \tag{B.5}$$

Let β be the constant appearing in (2.5). It is easy to check that the first term above can be bounded by $l^{-\beta}$ as soon as l is sufficiently large. Let $\mathbb{P}_0$ denote the probability measure for the random walk $\{\xi_0, \xi_1, \xi_2, \dots\}$ with transition p and starting point $\xi_0 = 0$. For each $n \geqslant 2$, we have

$$\sum_{|x| \geqslant l} p^n(0,x) = \mathbb{P}_0\big(|\xi_n| \geqslant l\big) \leqslant n \, \mathbb{P}_0\big(|\xi_1| \geqslant l/n\big) \leqslant c_1 \, n^{1+\beta} \, l^{-\beta} \,.$$

where $c_1 = c_1(d)$ is a finite constant (independent of n and l). Therefore there exist constants $c_2 < \infty$ and $m(\varepsilon) > 0$, such that for any integer M,

$$\begin{aligned} \sum_{n=2}^{\infty} (1+\varepsilon)^{-n} \sum_{|x| > l} p^n(0,x) &\leqslant \sum_{n=2}^{M} \sum_{|x| > l} p^n(0,x) + \sum_{n=M+1}^{\infty} (1+\varepsilon)^{-n} \sum_{|x| > l} p^n(0,x) \\ &\leqslant c_2 \, M^{2+\beta} \, l^{-\beta} + \exp(-m(\varepsilon) M) \,. \end{aligned} \tag{B.6}$$

We choose now $M = M(l) = m(\varepsilon)^{-1} \beta \log l$ in (B.6). From (B.5) we see that this implies (B.4) for any $0 < \alpha < \beta$ and l sufficiently large. $\square$

Let $f_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, be a finite collection of local functions. Given a number $l > 0$, we say that $f_1, f_2, \ldots, f_n$ are *l-separated* if their supports $S_1, S_2, \ldots, S_n$ are such that

$$d(S_i, S_j) > l\,, \quad \text{for all } i \neq j\,.$$

Lemma B.2. *There exists a constant $l_0 > 0$ and a non-increasing function*

$$\rho : [0, \infty) \to [1, \infty)\,,$$

with $\rho(l) \to 1$, as $l \to \infty$, such that the following holds for arbitrary integers $n \in \mathbb{Z}_+$, $l > l_0$ and $L > l$. Suppose $f_1, f_2, \ldots, f_n$ are l-separated local functions in $L^1(P^\varepsilon)$. Suppose there exist sites $x_i \in (L + l)\mathbb{Z}^d$, $i = 1, \ldots, n$ with

$$S_i \subset B_i \equiv x_i + [0, L + l]^d \cap \mathbb{Z}^d\,, \quad i = 1, \ldots, n\,.$$

Then

$$\Big\| \prod_{i=1}^{n} f_i \Big\|_1 \leqslant \prod_{i=1}^{n} \|f_i\|_{\rho(l)} \tag{B.7}$$

Proof. Let us consider the case $d = 2$. The general case requires an obvious generalization of the argument. Let $l > 0$, $L > l$ and let $f_1, f_2, \ldots, f_n$ and $B_1, \ldots, B_n$ be as above. We group the cubes B_i, $i = 1, \ldots, n$ into four families A,B,C and D, depending on whether the coordinates of the corresponding x_i are (even,even),(odd,even),(odd,odd) or (even,odd) respectively. Define

$$f_A \equiv \prod_{i:\, S_i \in A} f_i\,,$$

We define f_B, f_C and f_D in the same way. In this way f_A, f_B, f_C and f_D are l-separated functions, and Lemma B.1 yields

$$\begin{aligned}\Big\| \prod_{i=1}^{n} f_i \Big\|_1 &\leqslant \|f_A\|_{1+\tau(l)} \, \|f_B f_C f_D\|_{1+\tau(l)} \\ &\leqslant \|f_A\|_{1+\tau(l)} \, \|f_B\|_{(1+\tau(l))^2} \, \|f_C\|_{(1+\tau(l))^3} \, \|f_D\|_{(1+\tau(l))^3}\,.\end{aligned}$$

We can use the fact that $\|f\|_p \leqslant \|f\|_{p'}$, for $1 < p \leqslant p'$, to see that, for some constant $\gamma \in (0, \infty)$ (independent of n, L and l)

$$\Big\| \prod_{i=1}^{n} f_i \Big\|_1 \leqslant \|f_A\|_{1+\gamma\tau(l)} \, \|f_B\|_{1+\gamma\tau(l)} \, \|f_C\|_{1+\gamma\tau(l)} \, \|f_D\|_{1+\gamma\tau(l)}\,. \tag{B.8}$$

Since $L > l$, we see that, within each family A,B,C,D, the functions are now at least $2l$-separated. Therefore we can repeat the argument by dividing the family A into four subfamilies AA,AB,AC and AD. Within each subfamily the functions are now $4l$-separated. Implementing this idea, and iterating (B.8), we have

$$\Big\| \prod_{i=1}^{n} f_i \Big\|_1 \leqslant \prod_{i=1}^{n} \|f_i\|_{\rho(l)}$$

where we have defined

$$\rho(l) \equiv \prod_{k=1}^{\infty} (1 + \gamma\tau(2^k l))\,.$$

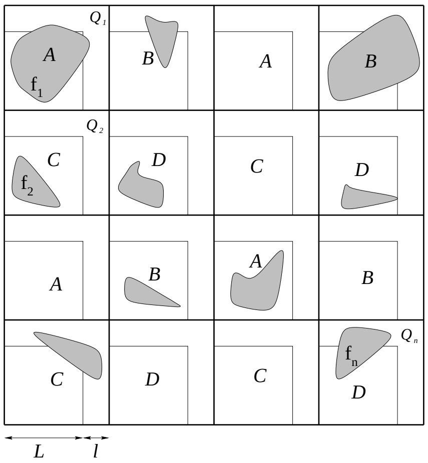

FIGURE 1. Geometric construction in the proof of Lemma B.2 .

Using the bound $\tau(l) \leqslant l^{-\alpha}$, cf. Lemma B.1, it is straightforward to check that $\rho(l)$ satisfies the desired properties. $\square$

The hypothesis of Lemma B.2 are tailored in such a way that they apply to the following situation (see [11], Lemma 5.4.13, for the one-dimensional case). Suppose a local bounded function f is given, whose support S_f is contained in the cube of side $q \in \mathbb{Z}_+$

$$Q = [0, q-1]^d \cap \mathbb{Z}^d \,.$$

We show that the limit

$$\Lambda_Q(f) \;=\; \lim_{N\to\infty} \frac{1}{N^d} \, \log P^\varepsilon\big(\exp \sum_{x\in\Lambda_N} f\circ\theta_x\big) \tag{B.9}$$

exists. To this end we may assume $f \geqslant 0$, otherwise replace f with $f + \|f\|_\infty$. By shift invariance of P^ε, we may replace Λ_N in (4.45) by the cube

$$B_N = [0, N-1]^d \cap \mathbb{Z}^d \,.$$

We also let $B_N(x)$ denote the translate $B_N + x$, $x \in \mathbb{Z}^d$. Let $M \in \mathbb{Z}_+$, $M < N$ and divide B_N into translates of the cube B_M. If $m = [N/M]$, the integer part, then $B_N = B_{mM} \cup R_{N,M}$, with $|R_{N,M}| \leqslant c_M N^{d-1}$, for some M-dependent constant c_M. We now enlarge the cubes B_M contained in B_{mM} by changing their side from M to $M+q+l$, for some $l > 0$. We choose m^d sites $x_i \in (M+q+l)\mathbb{Z}^d$, in such a way that,

$$B_{mM} \,\subset\, \bigcap_{i=1}^{m^d} B_{M+q+l}(x_i) \,.$$

We may estimate

$$\sum_{x\in B_N} f\circ\theta_x \;\leqslant\; \sum_{i=1}^{m^d} \; \sum_{x\in B_{M+q+l}(x_i)} f\circ\theta_x \;+\; c_M \, N^{d-1} \|f\|_\infty \,.$$

Moreover, there is a finite constant $c_{q,l}$, depending on q and l, such that for any $i = 1, \dots, m^d$ and for any $M \in \mathbb{Z}_+$

$$|\, B_{M+q+l}(x_i) \setminus B_M(x_i)\,| \;\leqslant\; c_{q,l}\, M^{d-1}\,,$$

and therefore

$$\sum_{x\in B_N} f\circ\theta_x \;\leqslant\; (c_{q,l}\, m^d\, M^{d-1} \;+\; c_M\, N^{d-1})\,\|f\|_\infty + \sum_{i=1}^{m^d}\sum_{x\in B_M(x_i)} f\circ\theta_x \tag{B.10}$$

By construction, the supports S_i of the functions

$$f_i \;\equiv\; \sum_{x\in B_M(x_i)} f\circ\theta_x$$

are contained in $B_{M+q}(x_i)$, so that $d(S_i, S_j) > l$, for all $i \neq j$ and, setting $L = M + q$, we see that Lemma B.2 applies. If l is large enough we can then use the estimate (B.7) to obtain

$$P^\varepsilon\Big(\prod_{i=1}^{m^d}\exp f_i\Big) \;\leqslant\; \prod_{i=1}^{m^d}\Big[P^\varepsilon(\exp\rho(l)\, f_i)\Big]^{1/\rho(l)} \tag{B.11}$$

where $\rho(l) \to 1$, as $l \to \infty$. Also, notice that for each $i = 1, \dots, m^d$,

$$\frac{1}{\rho(l)}\,\log\, P^\varepsilon\big(\exp\rho(l) f_i\big) \;\leqslant\; \log\, P^\varepsilon\big(\exp f_i\big) \;+\; (\rho(l)-1)\, M^d\, \|f\|_\infty\,.$$

Let us define

$$\Gamma_N(f) \;=\; \log P^\varepsilon\Big(\exp\sum_{x\in B_N} f\circ\theta_x\Big)\,.$$

Using shift invariance of P^ε, from (B.11) and (B.10) we have

$$\Gamma_N(f) \;\leqslant\; m^d\Gamma_M(f) \;+\; \big(m^d\, M^d\,(\rho(l)-1) \;+\; c_{q,l}\, m^d\, M^{d-1} \;+\; c_M\, N^{d-1}\big)\|f\|_\infty.$$

For fixed M and l, this implies

$$\limsup_{N\to\infty}\frac{\Gamma_N(f)}{N^d} \;\leqslant\; \frac{\Gamma_M(f)}{M^d} \;+\; \big[(\rho(l)-1) \;+\; \frac{c_{q,l}}{M}\big]\,\|f\|_\infty$$

By letting first $M \to \infty$, and then $l \to \infty$, we have

$$\limsup_{N\to\infty}\frac{\Gamma_N(f)}{N^d} \;\leqslant\; \liminf_{M\to\infty}\frac{\Gamma_M(f)}{M^d}.$$

This proves the existence of the limit in (4.45).

References

[1] E. Bolthausen, J. D. Deuschel, Critical large deviations for Gaussian fields in the phase transition regime, *Ann. Probab.* **21**,1876-1920, 1993

[2] E. Bolthausen, J. D. Deuschel, O. Zeitouni, Entropic repulsion of the lattice free field, *Commun. Math. Phys.* **170**, 417-443, 1995

[3] W. Bryc, A. Dembo, On large deviations of empirical measures for stationary Gaussian processes, *Stochastic Processes Appl.* **58**, 23-34, 1995

[4] D. Brydges, J. Fröhlich, T. Spencer, The random walk representation of classical spin systems and correlation inequalities, *Commun. Math. Phys.* **83**, 123-150, 1982

[5] P. Caputo, J.D. Deuschel, Critical large deviations for massless Gaussian fields with long range interaction, *in preparation.*

[6] T. Chyonobu, S. Kusuoka, The large deviations principle for hypermixing processes, *Probab. Th. Rel. Fields* **78**, 627-649, 1988

[7] F. Comets, Grandes deviations pour des champs de Gibbs sur $\mathbb{Z}^d$, *C. R. Acad. Sci. Paris I* **303**, 511-513, 1986.

[8] J.T. Cox, A. Greven, T. Shiga, Finite and infinite systems of interacting diffusions, *Probab. Th. Rel. Fields* **102**, 165-197, 1995.

[9] A. Dembo, O. Zeitouni, Large deviations, Jones and Burtlett publishers, Boston 1993

[10] J.D. Deuschel, G. Giacomin, D. Ioffe, Large deviations and concentration properties for $\nabla\varphi$ interfaces, *preprint* 1998

[11] J.D. Deuschel, D.W. Stroock, Large deviations, Pure and Applied Math. **137**, Academic Press 1989

[12] R.L. Dobrushin, Gaussian random fields: Gibbsian point of view, *in*: Multicomponent random systems, R.L. Dobrushin and Ya.G. Sinai eds., Marcel Dekker, New York, 1980

[13] M.D. Donsker, S.R.S. Varadhan, Asymptotic evaluation of certain Markov process expectations for large time IV, *Commun. Pure Appl. Math.* **28**, 1975, 1-47.

[14] M.D. Donsker, S.R.S. Varadhan, Large deviations for stationary Gaussian processes, *Commun. Math. Phys.* **97**, 187-210, 1985

[15] R.M. Dudley, Real analysis and probability, Wadsworth and Brooks/Cole, Monterey, 1989

[16] A.C.D. van Enter, R. Fernandez, A.D. Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: scope and limitations of Gibbsian theory, *J. Stat. Phys.* **72**, 879-1167, 1993.

[17] H. Föllmer, On entropy and information gain in random fields, *Z. Wahrsch. verw. Gebiete*, **26**, 207-217, 1973

[18] H. Föllmer, S. Orey, Large deviations for the empirical field of a Gibbs measure, *Ann. Probab.* **16**, 1988, 961-977

[19] J. Fröhlich, C. Pfister, On the absence of spontaneous symmetry breaking and of crystalline ordering in two-dimensional systems, *Commun. Math. Phys.* **81**, 277-298, 1981.

[20] H.O. Georgii, Gibbs measures and phase transitions, de Gruyter Studies in Math. 9, 1988.

[21] H.O. Georgii, Large deviations and maximum entropy principle for interacting random fields on $\mathbb{Z}^d$, *Ann. Probab.* **21**, 1845-1875, 1993

[22] H. Künsch, Thermodynamics and statistical analysis of Gaussian random fields, *Z. Wahrsch. verw. Gebiete* **58**, 407-421, 1981

[23] H. Künsch, Almost sure entropy and variational principle for random fields with unbounded state space, *Z. Wahrsch. verw. Gebiete 58*, 69-85, 1981

[24] O.E. Lanford III, Entropy and equilibrium states in classical statistical mechanics, *in*: Statistical mechanics and mathematical problems, Lecture Notes in Physics **20**, 1-113, Springer Berlin 1973.

[25] G.F. Lawler, Intersections of random walks, Birkhäuser Boston, 1991.

[26] J.L. Lebowitz, E. Presutti, Statistical mechanics of unbounded spins systems, *Commun. Math. Phys.* **50**, 195-218, 1976

[27] E.W. Montroll, B.J. West, On an enriched collection of stochastic processes, *in* Fluctuation phenomena, Studies in statistical mechanics vol. VII, eds. E.W. Montroll and J.L. Lebowitz, North-Holland, 1979

[28] E. Nelson, The free Markov field, J. Funct. Anal. **12**, 211-227, 1973

[29] S. Olla, Large deviations for Gibbs random fields, *Probab. Th. Rel. Fields* **77**, 1988, 343-357

[30] C. Preston, Random fields, Lecture Notes in Mathematics **534**, Springer 1976

[31] Y. A. Rozanov, On Gaussian fields with given conditional distributions. *Th. Prob. Appl.* **12**, 381-391, 1967

[32] Spitzer F. Principles of random walks, 2nd ed. Berlin, Heidelberg, New York: Springer 1976.

Fachbereich Mathematik, Sekr. MA 7-4, TU-Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: `caputo@math.tu-berlin.de`

Fachbereich Mathematik, Sekr. MA 7-4, TU-Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany

E-mail address: `deuschel@math.tu-berlin.de`