WEAK DISORDER LOCALIZATION AND LIFSHITZ TAILS: CONTINUOUS HAMILTONIANS

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Abstract. This paper is devoted to the study of band edge localization for continuous random Schrödinger operators with weak random perturbations. We prove that, in the weak disorder regime, λ small, the spectrum in intervals of size λ at a non-degenerate simple band edge is exponentially and dynamically localized. Upper bounds on the localization length in these energy regions are also obtained. Our results rely on the analysis of Lifshitz tails when the disorder is small; the single site potential need not be of fixed sign.

Résumé. Ce travail est consacré à l'étude des états localisés en bords de spectre pour des opérateurs de Schrödinger aléatoires à faible désordre, λ petit. Nous prouvons que, dans un voisinage de taille λ d'un bord simple et non-dégénéré du spectre, les états sont localisés exponentiellement et dynamiquement. Nous obtenons aussi une majoration de la longueur de localisation. Notre analyse repose sur une étude des asymptotiques de Lifshitz quand le désordre est petit; le potentiel de simple site n'a pas nécessairement un signe constant.

0. Introduction

Consider the following continuous Anderson model

$$H_{\omega,\lambda} = -\Delta + \lambda V_\omega = -\Delta + \lambda \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V_\gamma,$$

where

1. $V_\gamma(\cdot) = V(\cdot - \gamma)$ where $V : \mathbb{R}^d \to \mathbb{R}$ is a bounded, measurable and compactly supported function.
2. the average of V does not vanish i.e.

$$\overline{V} := \int_{\mathbb{R}^d} V(x)dx \neq 0. \tag{0.1}$$

3. the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are i.i.d., bounded, non-trivial and have a bounded density.
4. λ is a positive coupling constant.

It is well known that the spectrum of $H_{\omega,\lambda}$ is almost surely non random ([27]). Denote it by $\Sigma_\lambda = \sigma(H_{\omega,\lambda})$. Let $E_-(\lambda) = \inf \Sigma_\lambda$. Consider the periodic Schrödinger operator

$$\overline{H}_\lambda = -\Delta + \lambda \sum_{\gamma\in\mathbb{Z}^d} V_\gamma$$

and let $\overline{E}(\lambda)$ be the infimum of the spectrum of the periodic Schrödinger operator $\overline{H}_\lambda$. Define $\overline{\omega} = \mathbb{E}(\omega_0)$. Assumption (0.1) ensures that, for some $C > 0$ and λ sufficiently small (see section 5.1),

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1991 *Mathematics Subject Classification.* 82B44, 47B80, 60H25.

Key words and phrases. Random operators, localization, weak disorder, Lifshitz tails.

The author gratefully acknoledges support of the FNS 2000 "Programme Jeunes Chercheurs".

one has

$$\overline{E}(\lambda\overline{\omega}) - E_-(\lambda) \geq \lambda/C. \tag{0.2}$$

Note that $\overline{E}(\lambda\overline{\omega})$ is the infimum of the spectrum of $\mathbb{E}(H_{\omega,\lambda}) = \overline{H}_{\lambda\overline{\omega}}$.
We prove

Theorem 0.1. *Fix $\eta \in (0, d/(4d+4))$. There exists $\lambda_\eta > 0$ such that, for $\lambda \in (0, \lambda_\eta)$, with probability 1, one has*

1. *exponential localization in $I_{\eta,\lambda} := [E_-(\lambda), \overline{E}(\lambda\overline{\omega}) - \lambda^{1+\eta}]$ i.e.*
   - *$\sigma(H_{\omega,\lambda}) \cap I_{\eta,\lambda} = \sigma_{pp}(H_{\omega,\lambda}) \cap I_{\eta,\lambda}$;*
   - *$\sigma_{ac}(H_{\omega,\lambda}) \cap I_{\eta,\lambda} = \sigma_{sc}(H_{\omega,\lambda}) \cap I_{\eta,\lambda} = \emptyset$;*
   - *there exists $a > 0$ such that, an eigenfunction ψ corresponding to an eigenvalue $E \in I_{\eta,\lambda}$ satisfies*

$$\liminf_{|x|\to+\infty} -\frac{\log|\psi(x)|}{|x|} \geq a\sqrt{|E - \overline{E}(\lambda\overline{\omega})|}. \tag{0.3}$$

2. *strong Hilbert-Schmidt dynamical localization in $I_{\eta,\lambda}$ i.e. if $\Pi_\eta(H_{\omega,\lambda})$ denotes the spectral projector of $H_{\omega,\lambda}$ on the interval $I_{\eta,\lambda}$ and X denotes the position operator, then, for $B \subset \mathbb{R}^d$ bounded, one has*

$$\forall q \geq 0, \quad \mathbb{E}\left(\sup_{\|f\|_\infty \leq 1} \left\| |X|^{q/2} f(H_{\omega,\lambda}) \Pi_\eta(H_{\omega,\lambda}) \mathbf{1}_B \right\|_2^2 \right) < +\infty. \tag{0.4}$$

Here, $\sigma_{pp,ac,sc}$ denote respectively the pure point, absolutely continuous and singular continuous part of the spectrum. The function $\mathbf{1}_B$ is the characteristic function of B; $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. In (0.4), the supremum is taken over bounded, Borel measurable functions on $\mathbb{R}$.

Theorem 0.1 is a consequence of the behavior of the integrated density of states near the infimum of the spectrum. More precisely, let Λ_L be the cube of center 0 and side length L in $\mathbb{R}^d$, and let $H_{\omega,\lambda|\Lambda_L}$ be the Hamiltonian $H_{\omega,\lambda}$ restricted to the cube Λ_L with Dirichlet boundary conditions. Define the integrated density of states of $H_{\omega,\lambda}$ by

$$\mathcal{N}_\lambda(E) = \lim_{\#\Lambda\to+\infty} \frac{\#\{\text{eigenvalues of } H_{\omega,\lambda|\Lambda} \leq E\}}{|\Lambda|}. \tag{0.5}$$

The limit in (0.5) exists ω-a.e.; it is non-random and non-decreasing ([3, 27]). We prove

Theorem 0.2. *Fix $\eta \in (0, d/(4d+4))$. Then, there exists $\varepsilon > 0$ and $\lambda_\eta > 0$ such that, for $\lambda \in (0, \lambda_\eta)$, one has*

$$\mathcal{N}_\lambda(\overline{E}(\lambda\overline{\omega}) - \lambda^{1+\eta}) \leq e^{-\lambda^{-\varepsilon}}. \tag{0.6}$$

So, the density of states in the interval $I_{\eta,\lambda}$ is small. This interval is in the fluctuational region of the spectrum (see [26]). That in fluctuation regions, the spectrum is scarce and thus the states are localized is a basic mechanism for localization and has been known for a long time (see e.g. [25, 26]). It also is the only mechanism that has been understood mathematically in dimensions larger than 1; this mechanism is also basic in the study of large disorder localization.

The asymptotics (0.6) is directly related to the celebrated "Lifshitz tail" behavior (see e.g. [18] for a recent review). The main difference with Lifshitz tails is that Lifshitz tails are asymptotics in the limit when one approaches the band edge, and (0.6) holds on an interval. This is a consequence of the weak disorder limit.

Technically speaking the passage from Theorem 0.2 to Theorem 0.1 makes use of multiscale analysis and Wegner estimates (see [30, 8, 11]). These are well known tools in the proof of localization for random Schrödinger operators. In section 3, we explain how we use these tools.

As the main result of this paper, we show that the behavior (0.6), found at the bottom of the spectrum also happens at simple non-degenerate spectral edges. Hence, at such spectral edges, one also has localization. This is the content of Theorems 1.1 and 1.3.

To complete this section, let us note that results similar to Theorems 0.1 and 0.2 have been obtained for discrete random operators in [22].

1. The results

Let W be a bounded $\mathbb{Z}^d$-periodic potential and consider the periodic Schrödinger operator $H = -\Delta + W$ acting on $L^2(\mathbb{R}^d)$. It is self-adjoint on $H^2(\mathbb{R}^d)$; let Σ_0 be its spectrum.
Consider the continuous Anderson model i.e. the random Schrödinger operator defined by

$$H_{\omega,\lambda} = H + \lambda V_\omega = H + \lambda \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V_\gamma, \tag{1.1}$$

where $V_\gamma(\cdot) = V(\cdot - \gamma)$, $V : \mathbb{R}^d \to \mathbb{R}$ is a potential and $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are independent identically distributed random variables. We assume that.

H0.1: $V : \mathbb{R}^d \to \mathbb{R}$ is bounded and decaying faster than $|x|^{-d-1-\epsilon}$ for some $\epsilon > 0$;
H0.2: The random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ are i.i.d., bounded and non-trivial.

Notice that we do not assume that V keeps a fixed sign.

Assumption (H0) ensures that the potential V_ω stays uniformly bounded. Hence, $H_{\omega,\lambda}$ is self-adjoint on $H^2(\mathbb{R}^d)$. By assumption (H0.2), $H_{\omega,\lambda}$ is ergodic; hence, its spectrum is almost surely independent of ω. We denote it by Σ_λ.

Consider $(E_+(0), E_-(0))$ a gap in the spectrum of H (if $E_-(0)$ is the infimum of Σ_0, we set $E_+(0) = -\infty$). For λ sufficiently small, let $E_-(\lambda)$ (resp. $E_+(\lambda)$) be the infimum (resp. supremum) of Σ_λ in $[E_0, +\infty)$ (resp. $(-\infty, E_0]$) where $E_0 = (E_-(0) + E_+(0))/2$,see Fig. 1. So, for λ sufficiently small, one has

$$E_+(\lambda) < E_-(\lambda) \text{ and } \Sigma_\lambda \cap [E_+(0) - \delta, E_-(0) + \delta] \subset [E_+(0) - \delta, E_+(\lambda)] \cup [E_-(\lambda), E_-(0) + \delta].$$

We study the spectrum of $H_{\omega,\lambda}$ near $E_-(\lambda)$ for λ sufficiently small. An analogous study can be done near $E_+(\lambda)$.

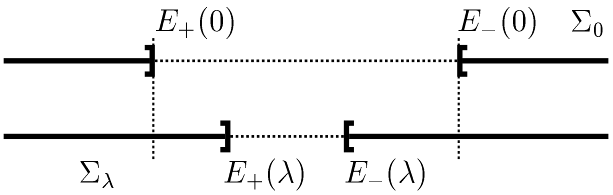

Figure 1. The band edges for Σ and Σ_p

To describe our main assumption, we need to recall some facts from the Floquet theory of periodic Schrödinger operators.

1.1. **Floquet theory of periodic Schrödinger operators.** The Floquet spectrum of H is the spectrum of the differential operator H acting on $L^d_{\text{loc}}(\mathbb{R}^d)$ with quasi-periodic boundary conditions. For $\theta \in \mathbb{T}^* = \mathbb{R}^d/\mathbb{Z}^d$, consider the following eigenvalue problem on $L^d_{\text{loc}}(\mathbb{R}^d)$

$$\begin{cases} H\varphi = E\varphi \\ \varphi(x+\gamma) = e^{i2\pi\gamma\theta}\varphi(x), \ \forall x \in \mathbb{R}^d, \ \forall \gamma \in \mathbb{Z}^d. \end{cases} \tag{1.2}$$

As H is elliptic, one knows that the eigenvalues of (1.2) are discrete; when repeated according to multiplicity, we denote them by $E_0(\theta) \leq E_1(\theta) \leq \cdots \leq E_n(\theta) \leq \ldots$. They are called the *Floquet eigenvalues* of H. These functions are Lipschitz continuous in the variable θ; when of multiplicity 1, the Floquet eigenvalues are even analytic in θ. Moreover, Weyl's law tells us that $E_n(\theta) \to +\infty$ as $n \to +\infty$ (uniformly in θ). In regard of (1.2), the Floquet eigenvalues are $\mathbb{Z}^d$-periodic functions of θ. The spectrum of H is given by $\Sigma_0 = \cup_{n\geq 0} E_n(\mathbb{T}^*)$. So the spectrum of H is the union of closed intervals called *the bands* of the spectrum; the connected components of $\mathbb{R} \setminus \Sigma_0$ are called *the gaps* of the spectrum of H.
One says that an energy $E \in \Sigma_0$ is *simple* if there exists exactly one index $n \geq 1$ such that, for some $\theta \in \mathbb{T}^*$, one has $E_n(\theta) = E$. It is proved in [19] that, generically, the band edges are simple. One says that E, a simple band edge, is *non-degenerate* if the Floquet eigenvalue $E_n(\cdot)$ reaching this band edge has only non-degenerate quadratic extrema at that edge, i.e. if θ is such that $E_n(\theta) = E$, then θ is a non-degenerate quadratic extremum of E_n.
One can define $\mathcal{N}_0(E)$, the integrated density of states of H in the same way as for H_ω i.e. by means of (0.5). One proves that

$$\mathcal{N}_0(E) = \sum_{j\geq 1} \int_{\mathbb{T}^*} \mathbf{1}_{E_j(\theta)\leq E} d\theta.$$

The set of points of increase of $\mathcal{N}_0$ coincides with the spectrum. The non-degeneracy of band edges can also be characterized in terms of the density of states. Namely, E_0, a simple band edge of H is non-degenerate if and only if $|\mathcal{N}_0(E) - \mathcal{N}_0(E_0)| \leq C|E - E_0|^{d/2}$ as $E \to E_0$, $E \in \Sigma_0$ (see [20]). Details on the material presented here may be found in [28, 24, 29].

1.2. **The main assumptions.** Let us now state our main assumptions. On the underlying periodic Schrödinger operator, we assume

H1: $E_-(0)$ is a simple non-degenerate band edge.

This condition is known to hold at the bottom of the spectrum of H in any dimension, see [1, 28]. It also holds at any band edge in dimension 1, see [31, 7]. As assumption (H1) is stable under small perturbations, it holds for sufficiently small perturbations of periodic operators with separate variables. For a more general discussion on the validity of assumption (H1), we refer to [4].
Define the periodic operator

$$\overline{H}_\lambda = H + \lambda \sum_{\gamma\in\mathbb{Z}^d} V_\gamma = H + \lambda\overline{V}. \tag{1.3}$$

Let $\overline{E}(\lambda)$ be the spectral band edge of $\overline{H}_\lambda$ closest to $E(0)$. For λ sufficiently small, it is well defined (see section 5.1). We assume that

H2: there exists $C > 0$ such that, for λ sufficiently small

$$|\overline{E}(\lambda) - \overline{E}(0)| = |\overline{E}(\lambda) - E(0)| \geq C|\lambda|. \tag{1.4}$$

It is proved in section 5.1 that, for a given background periodic operator H, assumption (H2) is satisfied for a generic single-site perturbation V. Moreover, (1.4) implies (0.2).

1.3. **The results.** We prove

Theorem 1.1. *Assume (H0), (H1) and (H2). Fix $\eta \in (0, d/(4d+4))$. Then, there exists $\varepsilon > 0$ and $\lambda_\eta > 0$ such that, for $\lambda \in (0, \lambda_\eta)$, one has*

$$\mathcal{N}_\lambda(\overline{E}(\lambda\overline{\omega}) - \lambda^{1+\eta}) - \mathcal{N}_\lambda(E_-(\lambda)) \leq e^{-\lambda^{-\varepsilon}}.$$

This result has strong consequences on the spectral behavior of H_ω. Indeed, let us assume that

H3: the common probability distribution of the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ is absolutely continuous with respect to Lebesgue measure and its density is locally absolutely continuous.

Under this additional assumption, in [11], it is proved that

Theorem 1.2. *Assume (H0), (H1), (H2) and (H3). Fix $0 < \tau < 1$. Then, there exists $c > 0$ and $\lambda_0 > 0$ such that, for $0 < \lambda < \lambda_0$, the integrated density of states of $H_{\omega,\lambda}$, $\mathcal{N}_\lambda$, is Hölder continuous of order τ in the interval $[E_-(\lambda), \overline{E}(\lambda\overline{\omega}) + c\lambda]$ i.e. there exists $C_\lambda > 0$ such that, for $(E, E') \in [E_-(\lambda), \overline{E}(\lambda\overline{\omega}) + c\lambda]^2$, one has*

$$|\mathcal{N}_\lambda(E) - \mathcal{N}_\lambda(E')| \leq C_\lambda |E - E'|^\tau.$$

Theorem 1.2 is a consequence of the Wegner's estimate proved in [11]; a weaker form of such an estimate is proved in Proposition 3.1.

Using Theorems 1.1 and 1.2 (or better said, the Wegner estimate, Proposition 3.1) together with the multi-scale analysis technique (see e.g. [30, 8]), we prove

Theorem 1.3. *Assume (H0), (H1), (H2) and (H3). Fix $\eta \in (0, d/(4d+4))$. There exists $\lambda_\eta > 0$ such that, for $\lambda \in (0, \lambda_\eta)$, with probability 1, one has*

1. *exponential localization in $I_{\eta,\lambda} := [E_-(\lambda), \overline{E}(\lambda\overline{\omega}) - \lambda^{1+\eta}]$ (see the description given in Theorem 0.1); moreover, the eigenfunctions associated to eigenvalues in this interval satisfy* (0.3).
2. *strong Hilbert-Schmidt dynamical localization in $I_{\eta,\lambda}$ (see the description given in Theorem 0.1).*

To our knowledge, these are the first results on localization for sign-indefinite potentials at internal band edges. Localization for such potentials had been established at the bottom of the spectrum assuming some Lifshitz behavior or quickly decaying tails for the distributions of the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ (see e.g. [17, 2]). To our knowledge, for single site potentials that do not have a definite sign, Lifshitz tails have not been proved to hold, even at the bottom of the spectrum. In the present paper, we prove a kind of Lifshitz behavior at internal non-degenerate band edges without assuming that the single site potential has a definite sign. This assumption has always played a crucial role in the analysis of Lifshitz tails. In the present work, it is replaced with assumption (H2) and the assumption that λ, the disorder, is small. Note also that, due to the unique continuation principle, assumption (H2) is satisfied for potentials having a definite sign.

Theorems 0.2 and 0.1 follow immediately from Theorems 1.1 and 1.3. Indeed, for $H = -\Delta$ and $E_-(0) = 0$, it is well known that assumption (H1) is fulfilled at the bottom of the spectrum, see e.g.[28, 1]. In this case, one proves $\overline{E}(\lambda\overline{\omega}) = \lambda\overline{V} + o(\lambda)$ (see section 5.1); hence, assumption (H2) is fulfilled if and only if $\overline{V} \neq 0$.

Analogues of Theorems 1.1 and 1.3 in the case of discrete random Schrödinger operators have been proved in [22]. Finally, let us say that the methods used in the present paper should also apply to other random models, e.g. to Schrödinger operators with weak random magnetic potentials ([9]).

1.4. **The outline of the paper.** Section 2 is devoted to the proof of Theorem 1.1. This proof is made in four steps. In section 2.1, we introduce the periodic approximations that enable us to estimate $\mathcal{N}_\lambda$. Section 2.2 is devoted to the recollection of some facts from the Floquet theory of periodic operators. In section 2.3, we show that estimating $\mathcal{N}_\lambda$ comes up to estimating the probability that, restricted to a sufficiently but not too large cube, the operator $H_{\omega,\lambda}$ has an eigenvalue in $I_{\eta,\lambda}$. This probability is then estimated in section 2.4 by showing that it reduces to large deviation estimates for sums of independent random variables. In section 3, we prove Theorem 1.3 using Theorem 1.1 and Proposition 3.1 that is proved in section 4. At last, in the appendix, section 5, we gathered some useful results.

2. The proof of Theorem 1.1

The scheme of this proof follows the one of the proof of Theorem 1.1 in [22]. We first state an approximation theorem giving precise finite volume approximations to $\mathcal{N}_\lambda$; such a statement was derived in [20, 23]. Then, we study the finite volume approximations.

2.1. **Periodic approximations.** Let $n \in \mathbb{N} \setminus \{0\}$ and define the following periodic Schrödinger operator

$$H^n_{\omega,\lambda} = H + \lambda \sum_{\gamma \in \mathbb{Z}^d_{2n+1}} \omega_\gamma \sum_{\beta \in (2n+1)\mathbb{Z}^d} V(x-\gamma-\beta) = H + \lambda V^n_\omega \tag{2.1}$$

where $\mathbb{Z}^d_{2n+1} := \mathbb{Z}^d/(2n+1)\mathbb{Z}^d$.
For ω fixed and $n \in \mathbb{N}^*$, $H^n_{\omega,\lambda}$ is a $(2n+1)\mathbb{Z}^d$-periodic self-adjoint Schrödinger operator. Let $N^n_{\omega,\lambda}$ be its integrated density of states; it satisfies

$$\mathcal{N}^n_{\omega,\lambda}(E) = \sum_{k\in\mathbb{N}} \int_{\{\theta\in\mathbb{T}^*_{2n+1};\ E_k(n,\omega,\lambda;\theta)\leq E\}} d\theta. \tag{2.2}$$

where $(E_k(n,\omega,\lambda;\theta))_{k\geq 0}$ are the Floquet eigenvalues of $H^n_{\omega,\lambda}$ and the torus $\mathbb{T}^*_n$ is defined by $\mathbb{T}^*_n = \mathbb{R}^d/\frac{1}{n}\mathbb{Z}^d$.
We recall Theorem 1.2 from [21] namely

Theorem 2.1 ([21]). *Pick $\alpha > 0$ and $I \subset \mathbb{R}$, a compact interval. There exists $\nu_0 > 0$ and $\rho > 0$ such that, for $\lambda \in [0,1]$, $E \in I$, $\nu \in (0,\nu_0)$ and $n \geq \nu^{-\rho}$, one has*

$$\begin{aligned} &\mathbb{E}(\mathcal{N}^n_{\omega,\lambda}(E+\nu/2)) - \mathbb{E}(\mathcal{N}^n_{\omega,\lambda}(E-\nu/2)) - e^{-\nu^{-\alpha}} \\ &\qquad \leq \mathcal{N}_\lambda(E+\nu) - \mathcal{N}_\lambda(E-\nu) \leq \\ &\qquad\qquad \mathbb{E}(\mathcal{N}^n_{\omega,\lambda}(E+2\nu)) - \mathbb{E}(\mathcal{N}^n_{\omega,\lambda}(E-2\nu)) + e^{-\nu^{-\alpha}}. \end{aligned} \tag{2.3}$$

In [21], we did not consider the case when the random operator depends on a parameter λ. One easily checks that the proof of Theorem 1.2 of [21] holds uniformly in the parameter λ.

2.2. **More on Floquet Theory.** It is convenient to introduce some notations. Fix $\theta \in \mathbb{R}^d$. The set of functions φ that are locally square integrable (resp. locally in H^2) on $\mathbb{R}^d$ and that satisfy $\varphi(x+\gamma) = e^{i\gamma\theta}\varphi(x)$, $\forall\gamma \in \mathbb{Z}^d$ is denoted by L^2_θ (resp. H^2_θ); both spaces are endowed with their natural scalar product (see [28]). The operator H acting on the H^2_θ is denoted by $H(\theta)$.
Pick $n \geq 1$. The operator H is $\mathbb{Z}^d$-periodic; hence, it is also $(2n+1)\mathbb{Z}^d$-periodic. To stress this point of view, we sometimes call it H^n. Let $L^2_{n,\theta}$ (resp. $H^2_{n,\theta}$) be the set of functions φ that are locally square integrable (resp. locally in H^2) on $\mathbb{R}^d$ and that satisfy $\varphi(x+\gamma) = e^{i2\pi\gamma\theta}\varphi(x)$, $\forall\gamma \in (2n+1)\mathbb{Z}^d$;

both spaces are endowed with their natural scalar product. The operator H acting on the space $H^2_{n,\theta}$ is denoted by $H^n(\theta)$. Notice that, in this case, we can restrict ourselves to $\theta \in \mathbb{T}^*_{2n+1}$.

The Floquet eigenvalues and eigenvectors of $H^n(\theta)$ are easily computed in terms of the Floquet eigenvalues and eigenvectors of $H(\theta)$. Indeed, let $(E_k(\theta), \varphi_k(\cdot,\theta))_{k\geq 0}$ be the Floquet eigenvalue and eigenvectors for $H(\theta)$ i.e. the solutions to the eigenvalue problem (1.2). One checks that, for $\gamma \in \mathbb{Z}^d_{2n+1}$,

$$H\varphi_{k,\gamma}(\theta) = E_{k,\gamma}(\theta)\varphi_{k,\gamma}(\theta) \text{ where } \begin{cases} E_{k,\gamma}(\theta) = E_k(\theta + \gamma/(2n+1)), \\ \varphi_{k,\gamma}(\theta) = \dfrac{1}{(2n+1)^{d/2}}\varphi_k(\theta + \gamma/(2n+1)). \end{cases}$$

For $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}^d$ and $l' \geq 0$, define

$$C_{\gamma,l'} = \{x = (x_1, \dots, x_d);\ \forall 1 \leq i \leq d,\ -l' - 1/2 \leq x_i - (2l'+1)\gamma_i < l' + 1/2\} \tag{2.4}$$

The vectors $(\varphi_{k,\gamma}(\cdot,\theta))_{k,\gamma}$ form an orthonormal family in $L^2(C_{0,n})$ as

$$\begin{aligned} \int_{C_{0,n}} \varphi_{k,\gamma}(x,\theta)\overline{\varphi_{k',\gamma'}(x,\theta)}dx &= \sum_{\beta\in\mathbb{Z}^d/(2n+1)\mathbb{Z}^d} \int_{\beta + C_{0,0}} \varphi_{k,\gamma}(x,\theta)\overline{\varphi_{k',\gamma}(x,\theta)}dk \\ &= \frac{1}{(2n+1)^d}\left(\sum_{\beta\in\mathbb{Z}^d_{2n+1}} e^{2\pi\beta(\gamma-\gamma')/(2n+1)}\right)\int_{C_{0,0}} \varphi_k(x,\theta+\gamma/(2n+1))\overline{\varphi_{k'}(x,\theta+\gamma'/(2n+1))}dx \\ &= \delta_{kk'}\delta_{\gamma\gamma'}. \end{aligned} \tag{2.5}$$

The family $(\varphi_{k,\gamma}(\cdot,\theta))_{k,\gamma}$ is complete as the family $(\varphi_k(\cdot,\theta))_{k\geq 0}$ is. Indeed, for $u \in L^2(\mathbb{R}^d)$, one has

$$u = \sum_{k\geq 0}\int_{\mathbb{T}^*} \hat{u}_n(\theta)\varphi_n(\cdot,\theta)d\theta = \sum_{k\geq 0}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}}\int_{\mathbb{T}^*_{2n+1}} \hat{u}_{\gamma,n}(\theta)\varphi_{\gamma,n}(\cdot,\theta)d\theta \text{ where } \hat{u}_{\gamma,n}(\theta) = \hat{u}_n(\theta+\gamma/(2n+1)).$$

So, the pairs $(E_{k,\gamma}(\theta), \varphi_{k,\gamma}(\cdot,\theta))_{k\geq 0, \gamma\in\mathbb{Z}^d_{2n+1}}$ form a family of $(2n+1)\mathbb{Z}^d$-periodic Floquet eigenvalues and eigenvectors of H.

We also make use of spectral projectors. We define $\Pi_{<E}$ (resp. $\Pi^n_{<E}$, $\Pi^n_{<E}(\theta)$, etc) to be the spectral projector associated to the interval $(-\infty, E)$ operator H (resp. H^n, $H^n(\theta)$, etc)

2.3. **The asymptotics of the density of states.** We now estimate the density of states of $H^n_{\omega,\lambda}$ for large n. The size of the cube, n, is chosen of order $\lambda^{-\rho}$ for some large ρ.

Pick $\eta < \eta' < d/(4d+4)$. The density of states $\mathcal{N}_\lambda$ being constant in gaps of Σ_λ, as V_ω is uniformly bounded, for λ sufficiently small, one has

$$\forall E \in [E_0, E_-(\lambda)], \quad \mathcal{N}_\lambda(E_-(\lambda)) = \mathcal{N}_\lambda(E).$$

By Theorem 2.1, to estimate $\mathcal{N}_\lambda(\overline{E}(\lambda\overline{\omega}) - \lambda^{1+\eta}) - \mathcal{N}_\lambda(E_-(\lambda))$ to exponential precision, we only need to estimate $\mathbb{E}[\mathcal{N}^n_{\omega,\lambda}(\overline{E}(\lambda\overline{\omega}) - \lambda^{1+\eta'}) - \mathcal{N}^n_{\omega,\lambda}(E_0)]$ for $n \geq \lambda^{-\rho}$ (ρ sufficiently large). A well known characterization of the spectrum of random Schrödinger operators ([14]) tells us that, with probability one, the spectrum of $H^n_{\omega,\lambda}$ is contained in Σ_λ, the almost sure spectrum of $H_{\omega,\lambda}$. Hence, with probability one, $\mathcal{N}^n_{\omega,\lambda}(E_0)=\mathcal{N}^n_{\omega,\lambda}(E_-(\lambda))$. We write

$$\begin{aligned}&\mathbb{E}[\mathcal{N}^n_{\omega,\lambda}(\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta'})-\mathcal{N}^n_{\omega,\lambda}(E_-(\lambda))]\\&\qquad\qquad=\mathbb{E}\left(\int_{\mathbb{T}^*_{2n+1}}\#\{\text{eigenvalues of } H^n_{\omega,\lambda}(\theta) \text{ in } [E_-(\lambda),\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta'}]\}\right).\end{aligned}$$

So, as $H^n_{\omega,\lambda}$ has rank $(2n+1)^d$ and as $\mathbb{T}^*_{2n+1}$ has volume $(2n+1)^{-d}$, one has

$$\mathbb{E}[\mathcal{N}^n_{\omega,\lambda}(\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta'})-\mathcal{N}^n_{\omega,\lambda}(E_-(\lambda))]\leq\mathbb{P}(\Omega(n,\lambda,\eta'))\tag{2.6}$$

where

$$\Omega(n,\lambda,\eta')=\left\{\begin{array}{c}\exists\theta\in\mathbb{R}^d,\\ \omega;\ \ \exists E\in[E_-(\lambda),\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta'}],\ \text{s.t. } H^n_{\omega,\lambda}(\theta)\varphi=E\varphi\\ \exists\varphi\in H^2_{n,\theta},\ \|\varphi\|=1,\end{array}\right\}.$$

Floquet theory gives us another characterization of $\Omega(n,\lambda,\eta')$, namely,

$$\Omega(n,\lambda,\eta')=\left\{\omega;\ \sigma(H^n_{\omega,\lambda})\cap[E_-(\lambda),\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta'}]\neq\emptyset\right\}.\tag{2.7}$$

By (2.6) and Theorem 2.1, Theorem 1.1 is an immediate consequence of

Proposition 2.1. *Fix $\eta\in(0,d/(4d+4))$ and $\rho>d$. Then, there exists $\lambda_{\eta,\rho}>0$ and $\varepsilon>0$ such that, for $\lambda\in(0,\lambda_{\eta,\rho})$, one has*

$$\mathbb{P}[\Omega(n,\lambda,\eta)]\leq e^{-\lambda^{-\varepsilon}}.\tag{2.8}$$

where

$$2n+1=[\lambda^{-1/2+2\eta'}]_o\cdot[\lambda^{-\eta'}]_o\cdot[\lambda^{-\rho}]_o \text{ and } \eta<\eta'<d/(4d+4).$$

Here, $[\cdot]_o$ denotes the largest odd integer smaller than $\cdot$.

2.4. **The proof of Proposition 2.1.** We first reformulate the definition of $\Omega(n,\lambda,\eta)$. It is convenient to work at extremal points of the spectrum e.g. at the bottom of the spectrum. We first transform our problem so as to be in that case. Therefore, we project out the part of the spectrum below $E_0=(E_-(0)+E_+(0))/2$.

We decompose $L^2=L^2(\mathbb{R}^d)$ into $L^2=\Pi_{\leq E_0}L^2+\Pi_{>E_0}L^2$ and write

$$H^n_{\omega,\lambda}-E=\begin{pmatrix}\Pi_{\leq E_0}(H^n_{\omega,\lambda}-E)\Pi_{\leq E_0} & \lambda\Pi_{\leq E_0}V^n_\omega\Pi_{>E_0}\\ \lambda\Pi_{>E_0}V^n_\omega\Pi_{\leq E_0} & \Pi_{>E_0}(H^n_{\omega,\lambda}-E)\Pi_{>E_0}\end{pmatrix}.\tag{2.9}$$

For $E\in I_{\eta,\lambda}$, the operator $\Pi_{\leq E_0}(H^n_{\omega,\lambda}-E)\Pi_{\leq E_0}$ is invertible and its inverse bounded by a constant independent of λ and ω. This follows from the fact that V_ω is uniformly bounded and that $\Pi_{\leq E_0}H^n\Pi_{\leq E_0}\leq E_+(0)\Pi_{\leq E_0}$.
Hence, $E\in I_{\eta,\lambda}$ is in the spectrum of $H^n_{\omega,\lambda}$ if and only if it is in the spectrum of

$$\Pi_{>E_0}H^n_{\omega,\lambda}\Pi_{>E_0}-\lambda^2\Pi_{>E_0}V^n_\omega\Pi_{\leq E_0}\left[\Pi_{\leq E_0}(H^n_{\omega,\lambda}-E)\Pi_{\leq E_0}\right]^{-1}\Pi_{\leq E_0}V^n_\omega\Pi_{>E_0}.\tag{2.10}$$

As the second term in (2.10) is bounded by $C\lambda^2$, we see that $E\in I_{\eta,\lambda}$ is in the spectrum of $H^n_{\omega,\lambda}$ implies that the operator $\Pi_{>E_0}H^n_{\omega,\lambda}\Pi_{>E_0}$ has spectrum below the energy $\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta}+C\lambda^2$. Thus, we have proved

$$\Omega(n,\lambda,\eta)\subset\left\{\omega;\ \exists\varphi\in\Pi_{>E_0}H^2,\ \|\varphi\|=1,\ \text{s.t. } \langle H^n_{\omega,\lambda}\varphi,\varphi\rangle\leq\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta}+C\lambda^2\right\}.\tag{2.11}$$

Let us estimate the probability of this last event. As V_ω^n is uniformly bounded, perturbation theory tells us that, for some $C>0$ and λ sufficiently small,

$$|E_-(\lambda)-E_-(0)|+|\overline{E}(\lambda\overline{\omega})-E_-(0)|\leq C|\lambda|.$$

Hence, for φ as in (2.11), one has

$$\langle H-E_-(0)\varphi,\varphi\rangle\leq|E-E_-(0)|+\lambda\|V_\omega^n\|_\infty\leq C\lambda. \tag{2.12}$$

Rewrite the condition in (2.11) as

$$\langle(\overline{H}^n_{\lambda\overline{\omega}}-\overline{E}(\lambda\overline{\omega}))\varphi,\varphi\rangle\leq(-\lambda^{1+\eta}+C\lambda^2)\|\varphi\|^2+\lambda\langle(\mathbb{E}(V_\omega^n)-V_\omega^n)\varphi,\varphi\rangle. \tag{2.13}$$

Note that $\mathbb{E}(V_\omega)=\mathbb{E}(V_\omega^n)=\overline{\omega}\overline{V}$. By the definition of $\overline{E}(\lambda\overline{\omega})$, for λ sufficiently small, $\overline{H}^n_{\lambda\overline{\omega}}$, hence, $\Pi_{>E_0}\overline{H}^n_{\lambda\overline{\omega}}\Pi_{>E_0}$ has no spectrum in the interval $[E_0,\overline{E}(\lambda\overline{\omega}))$; hence, equation (2.13) implies that, for some $\varphi\in\Pi_{>E_0}H^2$,

$$\langle(\mathbb{E}(V_\omega^n)-V_\omega^n)\varphi,\varphi\rangle\geq\lambda^\eta/2\|\varphi\|^2. \tag{2.14}$$

Indeed, if this is not the case, then (2.13) implies that, for some $\varphi\in\Pi_{>E_0}H^2$,

$$\langle(\overline{H}^n_{\lambda\overline{\omega}}-\overline{E}(\lambda\overline{\omega}))\varphi,\varphi\rangle\leq(-\lambda^{1+\eta}/2+C\lambda^2)\|\varphi\|^2.$$

which means that, for λ sufficiently small, $\Pi_{>E_0}\overline{H}^n_{\lambda\overline{\omega}}\Pi_{>E_0}$ has spectrum in the interval $[E_0,\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta}/4)$. So, taking (2.11) into account, we proved that

$$\Omega(n,\lambda,\eta)\subset\Omega'(n,\lambda,\eta)$$

where

$$\Omega'(n,\lambda,\eta)=\left\{\omega;\ \exists\varphi\in\Pi_{>E_0}H^2,\ \|\varphi\|=1,\ \text{s.t. equations (2.12) and (2.14) hold}\right\}.$$

Let us now estimate the probability of $\Omega'(n,\lambda,\eta)$. Write $\varphi\in\Pi_{>E_0}H^2$ using the Floquet decomposition introduced in section 2.2

$$\varphi=\sum_{p\geq p_0}\int_{\mathbb{T}^*}\chi_p(\theta)\varphi_p(\cdot,\theta)d\theta. \tag{2.15}$$

The index p_0 is the largest index p so that, for all $\theta\in\mathbb{T}^*$, one has $E_{p-1}(\theta)\leq E_0$.
By assumption (H.2), $E_-(0)$ is a simple degenerate Floquet eigenvalue of H i.e., hence, there exists $C>0$ such that

- for $p\neq p_0$, $\forall\theta\in\mathbb{T}^*$,

$$|E_p-E_-(0)|\geq 1/C; \tag{2.16}$$

- there exists $Z=\{\theta_j;\ 1\leq j\leq n_z\}$ such that $E_{p_0}(\theta_j)=E_-(0)$ and for $\theta\in\mathbb{R}^d$, one has

$$|E_{p_0}-E_-(0)|\geq 1/C\inf_{1\leq j\leq n_z}|\theta-\theta_j|^2. \tag{2.17}$$

We refer to section 5.1 for more details on these properties.
Equation (2.12) implies that

$$\sum_{p\geq p_0}\int_{\mathbb{T}^*}|E_p(\theta)-E_-(0)|^2|\chi_p(\theta)|^2d\theta\leq C\lambda^2. \tag{2.18}$$

Fix $2l+1=[\lambda^{-1/2+2\eta'}]_o\cdot[\lambda^{-\eta'}]_o$ and $2k+1=[\lambda^{-\rho}]_o$ where $\eta<\eta'<d/(4d+4)$ is fixed as in Proposition 2.1. Note that $2n+1=(2l+1)(2k+1)$. Equations (2.18), (2.16) and (2.17) imply that

$$\sum_{p>p_0}\int_{\mathbb{T}^*}|\chi_p(\theta)|^2d\theta+\sum_{1\leq j\leq n_z}\int_{|\theta-\theta_j|>1/l}|\chi_{p_0}(\theta)|^2d\theta\leq C\lambda^2l^2\leq C\lambda^{2\eta'}.$$

Hence, we write

$$\varphi=\sum_{1\leq j\leq n_z}\varphi_j+\varphi_e \text{ where } \varphi_j=\int_{|\theta-\theta_j|\leq 1/l}\chi_{p_0}(\theta)\varphi_{p_0}(\cdot,\theta)d\theta \text{ and } \|\varphi_e\|\leq C\lambda^{\eta'}. \tag{2.19}$$

We note that

$$\sum_{1\leq j\leq n_z}\|\varphi_j\|^2=\|\varphi\|^2-C\lambda^{2\eta'}=1-C\lambda^{2\eta'}. \tag{2.20}$$

Pluging (2.19) into (2.14), for λ sufficiently small, we get

$$\sum_{1\leq j,j'\leq n_z}\langle(\mathbb{E}(V^n_\omega)-V^n_\omega)\varphi_j,\varphi_{j'}\rangle\geq\lambda^\eta/4.$$

By (2.20), this implies that, for some $1\leq j,j'\leq n_z$, one has

$$\langle(\mathbb{E}(V^n_\omega)-V^n_\omega)\varphi_j,\varphi_{j'}\rangle\geq\lambda^\eta/(2n_z)^2. \tag{2.21}$$

Let us study the functions $(\varphi_j)_{1\leq j\leq n_z}$ in more detail. We prove

Lemma 2.1. *Fix $1\leq j\leq n_z$. For $1\leq l'\leq l$, there exists $\tilde\varphi_j\in L^2(\mathbb{R}^d)$ such that*

1. *$\tilde\varphi_j$ is constant on each cube*
$$C_{\gamma,l'}=\{x=(x_1,\dots,x_d);\ \forall 1\leq i\leq d,\ -l'-1/2\leq x_i-(2l'+1)\gamma_i<l'+1/2\}$$
where $\gamma=(\gamma_1,\dots,\gamma_d)\in\mathbb{Z}^d$;
2. *there exists $C>0$ (depending only on φ_{p_0})*

$$\|\varphi_j(\cdot)-\tilde\varphi_j(\cdot)\cdot\psi_{p_0}(\cdot,\theta_j)\|_{L^2}\leq Cl'/l \tag{2.22}$$

where $\psi_{p_0}(\cdot,\theta)$ is the periodic component of $\varphi_{p_0}(\cdot,\theta)$ i.e. $\varphi_{p_0}(\cdot,\theta)=e^{ix\theta}\psi_{p_0}(\cdot,\theta)$.

We postpone the proof of Lemma 2.1 to complete the proof of Proposition 2.1. We set $2l'+1=[\lambda^{-1/2+2\eta'}]_o$ and $2k'+1=[\lambda^{-\rho}]_o\cdot[\lambda^{-\eta'}]_o$ where $\eta<\eta'<d/(4d+4)$ is fixed as in Proposition 2.1. Note that $2n+1=(2l'+1)(2k'+1)$. Using point (1) of Lemma 2.1, we define

$$\Psi_j(x)=\psi_{p_0}(x,\theta_j)\tilde\varphi_j(x)=\psi_{p_0}(x,\theta_j)\sum_{\beta\in\mathbb{Z}^d}(2l'+1)^{-d/2}a_j(\beta)\mathbf{1}_{(2l'+1)\beta+C_{0,l'}}.$$

As $\psi_{p_0}(\cdot,\theta_j)\tilde\varphi_j(\cdot)\in L^2(\mathbb{R}^d)$, using the periodicity of $\psi_{p_0}(\cdot,\theta_j)$, we compute

$$\begin{aligned}\|\psi_{p_0}(\cdot,\theta_j)\tilde\varphi_j\|^2_{L^2(\mathbb{R}^d)}&=\sum_{\beta\in\mathbb{Z}^d}(2l'+1)^{-d}|a_j(\beta)|^2\int_{(2l'+1)\beta+C_{0,l'}}|\psi_{p_0}(x,\theta_j)|^2dx\\&=\sum_{\beta\in\mathbb{Z}^d}|a_j(\beta)|^2\int_{C_{0,0}}|\psi_{p_0}(x,\theta_j)|^2dx.\end{aligned}$$

Hence, as $\int_{C_{0,0}}|\varphi_{p_0}(x,\theta_j)|^2dx=\int_{C_{0,0}}|\psi_{p_0}(x,\theta_j)|^2dx$ is positive, by (2.22), we get

$$\sum_{\beta\in\mathbb{Z}^d}|a_j(\beta)|^2\leq C\|\varphi_j\|^2_{L^2(\mathbb{R}^d)}<+\infty. \tag{2.23}$$

By (2.22) and the definitions of l and l', the lower bound (2.21) implies that

$$\langle(\mathbb{E}(V_\omega^n) - V_\omega^n)\Psi_j, \Psi_{j'}\rangle \geq \lambda^\eta/(2n_z)^2 - C\lambda^{\eta'}. \tag{2.24}$$

Let us compute $\langle(\mathbb{E}(V_\omega^n) - V_\omega^n)\Psi_j, \Psi_{j'}\rangle$. One has

$$\begin{aligned}
\langle(\mathbb{E}(V_\omega^n) - V_\omega^n)\Psi_j, \Psi_{j'}\rangle &= \sum_{\beta\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}} (\overline{\omega} - \omega_\gamma)(2l'+1)^{-d} a_j(\beta)\overline{a_{j'}(\beta)} \\
&\qquad \int_{(2l'+1)\beta + C_{0,l'}} V^n(x-\gamma)\psi_{p_0}(x,\theta_j)\overline{\psi_{p_0}(x,\theta_{j'})}dx \\
&= \sum_{\beta''\in\mathbb{Z}^d}\sum_{\beta'\in\mathbb{Z}^d_{2k'+1}}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}} (\overline{\omega} - \omega_\gamma)(2l'+1)^{-d} a_j(\beta' + (2k'+1)\beta'')\overline{a_{j'}(\beta' + (2k'+1)\beta'')} \\
&\qquad \int_{C_{0,l'}} V^n(x-\gamma+(2l'+1)\beta')\psi_{p_0}(x,\theta_j)\overline{\psi_{p_0}(x,\theta_{j'})}dx.
\end{aligned}$$

as V^n is $(2n+1)\mathbb{Z}^d$-periodic and $(2l'+1)(2k'+1) = (2n+1)$.
Then, setting

$$c_{jj'}(\beta') = \sum_{\beta''\in\mathbb{Z}^d} a_j(\beta' + (2k'+1)\beta'')\overline{a_{j'}(\beta' + (2k'+1)\beta'')}$$

we obtain

$$\begin{aligned}
\langle(\mathbb{E}(V_\omega^n) - V_\omega^n)\Psi_j, \Psi_{j'}\rangle &= (2l'+1)^{-d}\sum_{\gamma\in\mathbb{Z}^d_{2n+1}} (\overline{\omega} - \omega_\gamma)\left(\sum_{\beta'\in\mathbb{Z}^d_{2k'+1}} c_{jj'}(\beta')\right. \\
&\qquad \left.\int_{C_{0,l'}} V^n(x-\gamma+(2l'+1)\beta')\psi_{p_0}(x,\theta_j)\overline{\psi_{p_0}(x,\theta_{j'})}dx\right) \\
&= (2l'+1)^{-d}\sum_{\gamma\in\mathbb{Z}^d_{2l'+1}}\left(\sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} (\overline{\omega} - \omega_{\gamma+(2l'+1)\gamma'})\left[\sum_{\beta'\in\mathbb{Z}^d_{2k'+1}} c_{jj'}(\beta')\right.\right. \\
&\qquad \left.\left.\int_{C_{0,l'}} V^n(x-\gamma+(2l'+1)(\beta'-\gamma'))\psi_{p_0}(x,\theta_j)\overline{\psi_{p_0}(x,\theta_{j'})}dx\right]\right) \\
&= (2l'+1)^{-d}\sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} X_{jj'}(\gamma)
\end{aligned} \tag{2.25}$$

where

$$X_{jj'}(\gamma) = \sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} (\overline{\omega} - \omega_{\gamma+(2l'+1)\gamma'})A_{jj'}(\gamma,\gamma'),$$

$$A_{jj'}(\gamma,\gamma') = \sum_{\beta'\in\mathbb{Z}^d_{2k'+1}} c_{jj'}(\beta')\int_{C_{0,l'}} V^n(x-\gamma+(2l'+1)(\beta'-\gamma'))\psi_{p_0}(x,\theta_j)\overline{\psi_{p_0}(x,\theta_{j'})}dx.$$

The random variables $(X_{jj'}(\gamma))_{\gamma\in\mathbb{Z}^d_{2l'+1}}$ are independent and centered (i.e. $\mathbb{E}(X_{jj'}(\gamma))=0$). They are bounded; indeed, one computes

$$\begin{aligned}
|X_{jj'}(\gamma)| &\leq C \sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} |A_{jj'}(\gamma,\gamma')| \\
&\leq C \sum_{\beta'\in\mathbb{Z}^d_{2k'+1}} \sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} |c_{jj'}(\beta')| \\
&\qquad\qquad \int_{C_{0,l'}} |V^n(x-\gamma+(2l'+1)(\beta'-\gamma'))||\psi_{p_0}(x,\theta_j)||\psi_{p_0}(x,\theta_{j'})|dx \\
&\leq C \sum_{\beta'\in\mathbb{Z}^d_{2k'+1}} |c_{jj'}(\beta')| \sum_{\gamma'\in\mathbb{Z}^d_{2k'+1}} \int_{C_{0,l'}} |V^n(x-\gamma+(2l'+1)\gamma')||\psi_{p_0}(x,\theta_j)||\psi_{p_0}(x,\theta_{j'})|dx.
\end{aligned}$$

Using the fact that V^n is $(2n+1)\mathbb{Z}^d$-periodic, the relation $(2n+1)=(2l'+1)(2k'+1)$ and estimate (2.23), we obtain

$$\begin{aligned}
|X_{jj'}(\gamma)| &\leq C \sum_{\beta\in\mathbb{Z}^d} |a_j(\beta)|\cdot|a_{j'}(\beta)| \sum_{\gamma'\in\mathbb{Z}^d_{2n+1}} \int_{C_{0,0}} |V^n(x-\gamma')||\psi_{p_0}(x,\theta_j)||\psi_{p_0}(x,\theta_{j'})|dx \\
&\leq C\|\varphi_j\|\|\varphi_{j'}\| \sum_{\gamma'\in\mathbb{Z}^d} \int_{C_{0,0}} |V(x-\gamma')||\psi_{p_0}(x,\theta_j)||\psi_{p_0}(x,\theta_{j'})|dx \\
&\leq C\|\varphi_j\|\|\varphi_{j'}\| < +\infty
\end{aligned}$$

By (2.24) and (2.25), to estimate the probability of $\Omega(n,\lambda,\eta)$, it suffices to estimate the probability that

$$\lambda^\eta/(4n_z)^2 \leq \lambda^\eta/(2n_z)^2 - C\lambda^{\eta'} \leq (2l'+1)^{-d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} X_{jj'}(\gamma).$$

This is a large deviation estimate; it is bounded above in the usual way using exponential Markov inequalities (see e.g. [6, 5]) and yields

$$\mathbb{P}\left[\lambda^\eta/(4n_z)^2 \leq (2l'+1)^{-d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} X_{jj'}(\gamma)\right] \leq e^{-c(l')^d\lambda^{2\eta}} \leq e^{-c\lambda^{-d/2+2d\eta'+2\eta}}$$

recalling the definition of l'. As $\eta<\eta'<d/(4d+4)$, one has $-d/2+2d\eta'+2\eta<0$. Hence, for some $\varepsilon>0$ and λ sufficiently small, one has

$$\mathbb{P}\left[\lambda^\eta/(4n_z)^2 \leq (2l'+1)^{-d} \sum_{\gamma\in\mathbb{Z}^d_{2l'+1}} X_{jj'}(\gamma)\right] \leq e^{-\lambda^{-\varepsilon}}.$$

On the other hand, the probability of $\Omega(n,\lambda,\eta)$ is bounded by the sum over j and j' of the probability estimate above. This yields (2.8) and completes the proof of Proposition 2.1. □

Proof of Lemma 2.1. Recall that

$$\varphi_j = \int_{\mathbb{T}^*} \tilde{\chi}_{p_0}(\theta)\varphi_{p_0}(\cdot,\theta)d\theta.$$

where $\tilde\chi_{p_0} = \mathbf{1}_{|\theta-\theta_j|\leq 1/l}\cdot\chi_{p_0}$ (see (2.19)). Using the periodic components, we write

$$\varphi_j = \left(\int_{\mathbb{T}^*}\tilde\chi_{p_0}(\theta)e^{i\cdot\cdot\theta}d\theta\right)\psi_{p_0}(\cdot,\theta_j) + \int_{\mathbb{T}^*}\tilde\chi_{p_0}(\theta)e^{i\cdot\cdot\theta}(\psi_{p_0}(\cdot,\theta)-\psi_{p_0}(\cdot,\theta_j))d\theta.$$

As discussed in section 5.1, under our assumptions, for θ close to θ_j, the function $(x,\theta)\mapsto\psi_{p_0}(x,\theta)$ is analytic in θ valued in the $\mathbb{Z}^d$-periodic, locally square integrable functions in x; hence, we compute

$$\begin{aligned}
&\left\|\int_{\mathbb{T}^*}\tilde\chi_{p_0}(\theta)e^{i\cdot\cdot\theta}(\psi_{p_0}(\cdot,\theta)-\psi_{p_0}(\cdot,\theta_j))d\theta\right\|^2_{L^2(\mathbb{R}^d)}\\
&=\sum_{\gamma\in\mathbb{Z}^d}\int_{C_{\gamma,0}}\left|\int_{\mathbb{T}^*}\tilde\chi_{p_0}(\theta)e^{ix\cdot\theta}(\psi_{p_0}(x,\theta)-\psi_{p_0}(x,\theta_j))d\theta\right|^2dx\\
&=\sum_{\gamma\in\mathbb{Z}^d}\int_{C_{0,0}}\left|\int_{\mathbb{T}^*}\tilde\chi_{p_0}(\theta)e^{ix\theta}e^{i\gamma\theta}(\psi_{p_0}(x+\gamma,\theta)-\psi_{p_0}(x+\gamma,\theta_j))d\theta\right|^2dx
\end{aligned}$$

In the first step, we rewrote the x-integral over $\mathbb{R}^d$ as the sum of the integrals over cubes covering $\mathbb{R}^d$; then, we shifted the argument x so as to center the cube at 0.
In the next step, we use the periodicity of the periodic components, and Parseval's formula to compute the θ-integral.

$$\begin{aligned}
&\left\|\int_{\mathbb{T}^*}\tilde\chi_{p_0}(\theta)e^{i\cdot\cdot\theta}(\psi_{p_0}(\cdot,\theta)-\psi_{p_0}(\cdot,\theta_j))d\theta\right\|^2_{L^2(\mathbb{R}^d)}\\
&=\int_{C_{0,0}}\left(\sum_{\gamma\in\mathbb{Z}^d}\left|\int_{\mathbb{T}^*}\tilde\chi_{p_0}(\theta)e^{ix\theta}e^{i\gamma\theta}(\psi_{p_0}(x,\theta)-\psi_{p_0}(x,\theta_j))d\theta\right|^2\right)dx\\
&=\int_{C_{0,0}}\int_{\mathbb{T}^*}\left|\tilde\chi_{p_0}(\theta)(\psi_{p_0}(x,\theta)-\psi_{p_0}(x,\theta_j))\right|^2d\theta dx\\
&=\int_{\mathbb{T}^*}|\tilde\chi_{p_0}(\theta)|^2\left(\int_{C_{0,0}}|\psi_{p_0}(x,\theta)-\psi_{p_0}(x,\theta_j)|^2dx\right)d\theta\leq\frac{C}{l^2}.
\end{aligned}$$

To conclude, we used the analyticity of $\theta\mapsto\psi_{p_0}(\cdot,\theta)$.
So we now are left with proving Lemma 2.1 when φ_j is replaced with the function

$$x\mapsto\left(\int_{\mathbb{T}^*}\tilde\chi_{p_0}(\theta)e^{ix\cdot\theta}d\theta\right)\psi_{p_0}(x,\theta_j).$$

This is an immediate consequence of Lemma 5.1 when one picks $\varepsilon = 1/l$ and $\eta = (2l'+1)/(\pi l)$. This completes the proof of Lemma 2.1. ☐

3. The proof of Theorem 1.3

Theorem 1.3 is derived from Theorem 1.1 using the multiscale analysis done in [30, 8]. In order to apply the results of [30, 8], we have to check two assumptions:

- the Wegner estimate;
- the initial length scale estimate.

We first state the Wegner estimate. This estimate is an estimate of the probability of presence of eigenvalues in a given interval. More precisely, we prove

Proposition 3.1. *Fix $\tau\in(0,1)$. Then, there exists $\lambda_0>0$ and $c_0>0$ such that, for $\lambda\in(0,\lambda_0)$, one has, for $n\geq 1$ and $\theta\in\mathbb{T}^*_{2n+1}$, for I, an interval in $[E_-(\lambda),\overline{E}(\lambda\overline{\omega})+c_0\lambda]$, one has*

$$\mathbb{P}\left(\{H^n_{\omega,\lambda}(\theta) \text{ has an eigenvalue in } I\}\right)\leq\mathbb{E}\left(\#\{\text{eigenvalues of } H^n_{\omega,\lambda}(\theta) \text{ in } I\}\right)\leq C\lambda^{-1}n^{2d}|I|^{\tau}. \tag{3.1}$$

Proposition 3.1 is proved in section 4. It does not imply Theorem 1.2. The volume dependence in (3.1) is not optimal. But it is sufficient for our purpose. In [11], the optimal volume dependence was obtained but for an approximation scheme different from the periodic one used in the present paper.

We now state the initial length scale estimate; it is a consequence of Theorem 1.1.

Proposition 3.2. *Fix $\eta\in(0,d/(4d+4))$ and $\rho>d$ sufficiently large. Then, there exists $a>0$, $\varepsilon_0>0$ and $\lambda_0>0$ such that, for $0<\lambda<\lambda_0$ and $0<\varepsilon<\varepsilon_0$, for $\lambda^{-\rho}<n<e^{\lambda^{\varepsilon}}$, with probability $1-e^{-\gamma^{-\varepsilon}}$, for $E\in I_{\eta,\lambda}$ and $\theta\in\mathbb{T}^*_{2n+1}$, one has*

$$\|\mathbf{1}_{C_{0,2n+1}\setminus C_{0,2n-1}}(E-H^n_{\omega,\lambda}(\theta))^{-1}\mathbf{1}_{C_{0,n}}\|\leq e^{a\sqrt{|E-\lambda\overline{\omega}|}n}. \tag{3.2}$$

Using Propositions 3.1 and 3.2 for $\theta=0$, Theorem 1.2 is an immediate consequence of Theorem 12.2 and Proposition 13.1 of [30], and Theorem 3.8 of [8].

3.1. **The proof of the initial length scale estimate.** We now prove Proposition 3.2. Fix η,η' as in Proposition 3.2 i.e. $\eta<\eta'<d/(4d+4)$ and fix $\rho>d$. Define the event $\Omega_{\gamma,\eta',n}=\{$there exists an eigenvalue of $H^n_{\omega,\lambda}$ in $[E_-(\lambda),\overline{E}(\lambda\overline{\omega})-\lambda^{1+\eta'}]\}$. Using Theorem 2.1 in conjunction with Theorem 1.1, we obtain that, there exists $\gamma_\eta>0$ and $\varepsilon>0$ such that, for $0<\lambda<\lambda_\eta$ and $\lambda^{-\rho}\leq n\leq e^{\lambda^{-\varepsilon/2}}$, one has

$$\mathbb{P}(\Omega_{\lambda,\eta',n})\leq Ce^{d\lambda^{-\varepsilon/2}}e^{-\lambda^{-\varepsilon}}\leq e^{-\frac{1}{2}\lambda^{-\varepsilon}}.$$

By a Combes-Thomas estimate (see e.g. [10] and references therein), for $E\in I_{\eta,\lambda}$ and $\omega\notin\Omega_{\lambda,\eta',n}$, one has

$$\left\|\mathbf{1}_{C_{\gamma,0}}(E-H^n_{\omega,\lambda}(\theta))^{-1}\mathbf{1}_{C_{\gamma',0}}\right\|\leq C\lambda^{-1-\eta}e^{-\sqrt{|E-\overline{E}(\lambda\overline{\omega})+\lambda^{\eta'+1}|}|\gamma-\gamma'|/C}, \tag{3.3}$$

for some constant C independent of λ and η. Define $\delta(E):=\sqrt{|E-\overline{E}(\lambda\overline{\omega})+\lambda^{1+\eta'}|}$. We notice that, for $E\in I_{\eta,\lambda}$, $E-\overline{E}(\lambda\overline{\omega})\geq\lambda^{1+\eta}$; as $\eta<\eta'$, for λ sufficiently small, one has

$$\delta(E)\geq\frac{1}{2}\sqrt{|E-\overline{E}(\lambda\overline{\omega})|}.$$

Pick $0<\varepsilon_0<\varepsilon$. Summing (3.3) over $\gamma\in C_{0,2n+1}\setminus C_{0,2n-1}$ and $\gamma'\in C_{0,n}$ and using the fact that $\lambda^{-\rho}<n<e^{\lambda^{-\varepsilon_0}}$, we get that, for $\omega\notin\Omega_{\gamma,\eta',n}$ and $E\in I_{\eta,\lambda}$, estimate (3.2) holds. This completes the proof of Theorem 3.2.

4. Proof of the Wegner estimate

The proof of Proposition 3.1 follows the philosophy introduced in [17] and is quite similar to the proof of Theorem 1.2 in [11]. For the reader's convenience, we nevertheless reproduce the details here.

As above, we fix a band edge $E_-(0)$ and $\theta\in\mathbb{T}^*_{2n+1}$. All the operators we now discuss are considered with θ-quasi-periodic boundary condition in the sense explained in section 2.2; in our notations, we forget about the parameter θ i.e. we write E_k instead of $E_k(\theta)$, H instead of $H(\theta)$, etc. Our statement are uniform in the parameter θ. Recall that the Floquet eigenvalues of H are denoted by $(E_k)_{k\geq 1}$ and ordered increasingly. Define $\Pi_0 = \Pi^n_{<E_-(0)+1/2}$ and $\Pi_0^\perp = \mathbf{1} - \Pi_0$. One has

$$\Pi_0^\perp H^n_{\omega,\lambda}\Pi_0^\perp \geq (E_-(0)+1/2)\Pi_0^\perp.$$

Moreover, by the discussion of section 2.2, the projector $\Pi^n_{<E_-(0)+1/2}$ is of rank at most Cn^d.
Using the decomposition $\mathbf{1} = \Pi_0 + \Pi_0^\perp$, we see that $E \leq E_-(0)+1/2$ is an eigenvalue of $H^n_{\omega,\lambda}$ if and only if it is an eigenvalue of the operator

$$\Pi_0 H^n_{\omega,\lambda}\Pi_0 - \lambda^2 \Pi_0 V^n_\omega \Pi_0^\perp (E - \Pi_0^\perp H^n_{\omega,\lambda}\Pi_0^\perp)^{-1}\Pi_0^\perp V^n_\omega \Pi_0. \tag{4.1}$$

Define

$$\Gamma^0(\omega,E) = \Pi_0 V^n_\omega \Pi_0^\perp (E - \Pi_0^\perp H^n_{\omega,\lambda}\Pi_0^\perp)^{-1}\Pi_0^\perp V^n_\omega \Pi_0.$$

The operator $\Gamma^0(\omega,E)$ and all its derivatives in ω and E are bounded for $E \leq E_-(0)+1/2$ and ω bounded.

By assumption (H2), there exists $\omega^+ \in \mathrm{supp}(\omega_0)$ and $C>0$ such that, for λ sufficiently small,

$$\overline{E}(\lambda\varpi) - \overline{E}(\lambda\omega^+) \geq \lambda/C. \tag{4.2}$$

For more details, we refer to section 5.1. We rewrite $H_{\omega,\lambda} = \overline{H}_{\lambda\omega^+} + \lambda V_{\tilde\omega}$ where $(\tilde\omega_\gamma)_\gamma = (\omega_\gamma - \omega^+)_\gamma$. So, for λ sufficiently small, the operator $\overline{H}_{\lambda\omega^+}$ has no spectrum in the interval $[(E_-(0)+2E_+(0))/3, \overline{E}(\lambda\varpi)+C\lambda]$ (for some $C>0$). Hence, for any $n\geq 1$, the operator $\overline{H}^n_{\lambda\omega^+}$ has no spectrum in $[(E_-(0)+2E_+(0))/3, \overline{E}(\lambda\varpi)+C\lambda]$. We decompose $\Pi_0 = \Pi_0^+ + \Pi_0^-$ where $\Pi_0^- = \Pi^n_{<E_0}$ (where we recall $E_0 = (E_-(0)+E_+(0))/2$).
For $E \in [E_0, \overline{E}(\lambda\varpi)+C\lambda/2]$, $E\Pi_0^- - \Pi_0^- H_{\omega,\lambda}\Pi_0^- = E\Pi_0^- - \Pi_0^-\Pi_0 H_{\omega,\lambda}\Pi_0\Pi_0^-$ is invertible and its inverse is bounded independently of ω and λ sufficiently small. Hence, $E \in [E_0, \overline{E}(\lambda\varpi)+C\lambda/2]$ is an eigenvalue of the operator defined in (4.1) if and only if E is an eigenvalue of

$$\Pi_0^+ H^n_{\omega,\lambda}\Pi_0^+ - \lambda^2\left[\Pi_0^+\Gamma^0(\omega,E)\Pi_0^+ + \Gamma^{0+}(\omega,E)\right] \tag{4.3}$$

where

$$\Gamma^{0+}(\omega,E) = \Pi_0^+ V^n_\omega \Pi_0^- (E - \Pi_0^- H^n_{\omega,\lambda}\Pi_0^-)^{-1}\Pi_0^- V^n_\omega \Pi_0^+.$$

For $E \in [E_0, \overline{E}(\lambda\varpi)+C\lambda/2]$ and for ω bounded, the operator $\Gamma^{0+}(\omega,E)$ and all its derivatives in ω and E are bounded.
By the Birman-Schwinger principle (see e.g. [16] and references therein), an energy E in the interval $[E_0, \overline{E}(\lambda\varpi)+C\lambda/2]$ is an eigenvalue (with multiplicity m) of the operator defined in (4.3) if and only if -1 is an eigenvalue (with multiplicity m) of the operator

$$\begin{aligned} G(E,\omega,\lambda) = \lambda(\Pi_0^+\overline{H}^n_{\lambda\omega^+}\Pi_0^+ - E)^{-1/2}\big[\Pi_0^+ V^n_{\tilde\omega,\lambda}\Pi_0^+ - \\ \lambda\left(\Pi_0^+\Gamma^0(\omega,E)\Pi_0^+ + \Gamma^{0+}(\omega,E)\right)\big](\Pi_0^+\overline{H}^n_{\lambda\omega^+}\Pi_0^+ - E)^{-1/2}. \end{aligned} \tag{4.4}$$

Notice that, for $E, E' \in [E_0, \overline{E}(\lambda\varpi)+C\lambda/2]$, the operator $G(E,\omega,\lambda)$ satisfies

$$\|G(E,\omega,\lambda) - G(E',\omega,\lambda)\| \leq C|E-E'|\lambda^{-1}. \tag{4.5}$$

For $E_0 \in [E_0, \overline{E}(\lambda\varpi)+C\lambda/2]$ and $\varepsilon>0$ such that $[E_0-\varepsilon, E_0+\varepsilon] \subset [E_0, \overline{E}(\lambda\varpi)+C\lambda/2]$, if E is an eigenvalue in $[E_0-\varepsilon, E_0+\varepsilon]$, then, by (4.5), $G(E_0,\omega,\lambda)$ has an eigenvalue in $[-1-C\varepsilon\lambda^{-1}, -1+C\varepsilon\lambda^{-1}]$.

This yields

$$\begin{aligned}&\#\{\text{eigenvalues of } H^n_{\omega,\lambda} \text{ in } [E_0-\varepsilon, E_0+\varepsilon]\}\\ &\qquad\qquad\le \#\{\text{eigenvalues of } G(E_0,\omega,\lambda) \text{ in } [-1-C\varepsilon\lambda^{-1}, -1+C\varepsilon\lambda^{-1}]\},\end{aligned}$$

the eigenvalues being counted with multiplicity.
Hence, one has

$$\begin{aligned}\text{(4.6)}\quad &\mathbb{E}(\#\{\text{eigenvalues of } H^n_{\omega,\lambda} \text{ in } [E_0-\varepsilon, E_0+\varepsilon]\})\\ &\qquad\qquad\le \mathbb{E}(\#\{\text{eigenvalues of } G(E_0,\omega,\lambda) \text{ in } [-1-C\varepsilon\lambda^{-1}, -1+C\varepsilon\lambda^{-1}]\}).\end{aligned}$$

To estimate the last expectation, we use the computations done in [11]. Define $I_\varepsilon = [-1-\kappa, -1+\kappa]$ where $\kappa = C\varepsilon\lambda^{-1}$. Let ρ be a nonnegative, smooth, monotone decreasing function such that $\rho(x) = 1$, for $x < -\kappa/2$, and $\rho(x) = 0$, for $x \ge \kappa/2$. We can assume that ρ has compact support since $G(E_0,\omega,\lambda)$ is lower semibounded independently of n. Then, one has

$$\text{(4.7)}\quad \begin{aligned}&\mathbb{E}(\#\{\text{eigenvalues of } G(E_0,\omega,\lambda) \text{ in } [-1-\kappa,-1+\kappa]\})\\ &\le \mathbb{E}\{\mathrm{tr}[\rho(\lambda G(E_0,\omega,\lambda)+1-3\kappa/2) - \rho(G(E_0,\omega,\lambda)+1+3\kappa/2)]\}\\ &\le \mathbb{E}\left\{\mathrm{tr}\left[\int_{-3\kappa/2}^{3\kappa/2} \frac{d}{dt}\rho(G(E_0,\omega,\lambda)+1-t)\,dt\right]\right\}.\end{aligned}$$

In order to evaluate the ρ' term in (4.7), define the vector field

$$\mathcal{V} = \sum_{\gamma\in\mathbb{Z}^d_{2n+1}} \tilde{\omega}_\gamma \partial_{\omega_\gamma}$$

and compute

$$\mathcal{V}G(E_0,\omega,\lambda) = G(E_0,\omega,\lambda) + \lambda G_1(E_0,\omega,\lambda)$$

where $G_1(E_0,\omega,\lambda)$ is bounded uniformly in ω, λ and $E_0 \in [E_0, \overline{E}(\lambda\overline{\omega}) + C\lambda/2]$. This follows from the fact that $\Pi_0^+ \overline{H}^n_{\lambda\omega^+} \Pi_0^+$ does not depend on ω_γ, that $\mathcal{V}V^n_{\tilde{\omega},\lambda} = V^n_{\tilde{\omega},\lambda}$, that $\mathcal{V}\Gamma^0(\omega,E_0)$ and $\mathcal{V}\Gamma^{0+}(\omega,E_0)$ stay bounded, and that for $E_0 \in [E_0, \overline{E}(\lambda\overline{\omega}) + C\lambda/2]$, $\mathrm{dist}(E, \sigma(\overline{H}_{\lambda\overline{\omega}})) \ge \lambda/C$ for some $C > 1$.
We now write the ρ' term in (4.7) as

$$\begin{aligned}\mathcal{V}\rho(G(E_0,\omega,\lambda)+1-t) &= \rho'(G(E_0,\omega,\lambda)+1-t)\mathcal{V}G(E_0,\omega,\lambda)\\ &= \rho'(G(E_0,\omega,\lambda)+1-t)(G(E_0,\omega,\lambda)+O(\lambda))\end{aligned}$$

We now note that $\rho' \le 0$, and that on $\mathrm{supp}\rho'$, one has $G(E_0,\omega,\lambda) \le (-1+2\kappa)$. So, for λ sufficiently small, we obtain

$$\text{(4.8)}\qquad -\rho'(G(E_0,\omega,\lambda)+1-t) \le -\frac{1}{2(1-2\kappa)} \sum_{\omega\in\mathbb{Z}^d_{2n+1}} \tilde{\omega}_\gamma \frac{\partial\rho}{\partial\omega_\gamma}(G(E_0,\omega,\lambda)+1-t).$$

With this estimate, and the fact that $d\rho(x+1-t)/dt = -\rho'(x+1-t)$, the right side of (4.7) is bounded above by

$$\text{(4.9)}\qquad -\frac{1}{2(1-2\kappa)} \sum_{\gamma\in\mathbb{Z}^d_{2n+1}} \int_{-3\kappa/2}^{3\kappa/2} \mathbb{E}\{\tilde{\omega}_\gamma \frac{\partial}{\partial\omega_\gamma}\mathrm{tr}[\rho(G(E_0,\omega,\lambda)+1-t)]\}dt.$$

In order to evaluate the expectation, we select one random variable, say ω_γ ($\gamma \in \mathbb{Z}^d_{2n+1}$), and first integrate with respect to this variable using hypothesis (H3). Let h_0 be the common density of the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$; by (H3), there is a decomposition $\mathrm{supp}(\omega_0) = \cup_{l=0}^{N-1}(M_l, M_{l+1})$ so that h_0 is absolutely continuous on each subinterval. Let $\tilde{h}_0$ be the function $\tilde{h}_0(x) := xh_0(x)$. As $\tilde{h}_0$ is locally absolutely continuous, we can integrate by parts and obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}} d\omega_\gamma \tilde{h}_0(\omega_\gamma)\frac{\partial}{\partial\omega_\gamma}\mathrm{tr}\{\rho(G(E_0,\omega,\lambda)+1-t) - \rho(G(E_0,\omega,\lambda)^{M_0,\gamma}+1-t)\}\right| \\
&= \left|\sum_{l=0}^{N-1}\int_{M_l}^{M_{l+1}} d\omega_\gamma \tilde{h}_0(\omega_\gamma)\frac{\partial}{\partial\omega_\gamma}\mathrm{tr}\{\rho(G(E_0,\omega,\lambda)+1-t) - \rho(G(E_0,\omega,\lambda)^{M_0,\gamma}+1-t)\}\right| \\
&\leq \|\tilde{h}_0\|_\infty |\mathrm{tr}\{\rho(G(E_0,\omega,\lambda)^{M_N,\gamma}+1-t) - \rho(G(E_0,\omega,\lambda)^{M_0,\gamma}+1-t)\}| \\
&+ \|\tilde{h}_0'\|_\infty \sup_{x\in\mathrm{supp}\tilde{\omega}} |\mathrm{tr}\{\rho(G(E_0,\omega,\lambda)^{x,\gamma}+1-t) - \rho(G(E_0,\omega,\lambda)^{M_0,\gamma}+1-t)\}|,
\end{aligned}$$

where $G(E_0,\omega,\lambda)^{x,\gamma}$ is the operator $G(E_0,\omega,\lambda)$ with the coupling constant ω_γ at the γ^{th}-site fixed at the value $\omega_\gamma = x$. Similarly, the value 0 or M denotes the coupling constant ω_γ fixed at those values. Now, as $G(E_0,\omega,\lambda)$ is of rank at most Cn^d, we get that, for some $C>0$, one has

$$\left|\int_{\mathbb{R}} d\omega_\gamma \tilde{h}_0(\omega_\gamma)\frac{\partial}{\partial\omega_\gamma}\mathrm{tr}\{\rho(G(E_0,\omega,\lambda)+1-t) - \rho(G(E_0,\omega,\lambda)^{M_0,\gamma}+1-t)\}\right| \leq Cn^d.$$

Pluging this in turn into (4.9), (4.8), (4.7), and finally (4.6), for λ sufficiently small, we obtain

$$\mathbb{E}(\#\{\text{eigenvalues of } H^n_{\omega,\lambda} \text{ in } [E_0-\varepsilon, E_0+\varepsilon]\}) \leq Cn^{2d}\kappa = \tilde{C}n^{2d}\varepsilon\lambda^{-1}.$$

This completes the proof of Proposition 3.1. □

5. Appendix

5.1. **The dependence of Σ_λ on λ.** In this section, we study the dependence on Σ_λ (i.e. of $E_-(\lambda)$) in λ in a neighborhood of a gap of Σ_0. In the case of a nonnegative single site potential V, such a study was done in [15].

As $E(0)$ is simple, there exists a unique Floquet eigenvalue of H, say $E_n(\cdot)$, such that, for some $\theta \in \mathbb{T}^*$, $E_-(0) = E_n(\theta)$. Let $Z \subset \mathbb{T}^*$ be the set of points θ such that $E_n(\theta) = E_-(0)$. This set is discrete by assumption (H1), thus, finite, as $\mathbb{T}^*$ is compact. To fix ideas, set $Z = \{\theta_j;\ 1 \leq j \leq n_z\}$. As $E_-(0)$ is simple, for $1 \leq j \leq n_z$, the Floquet eigenspace associated to $E_-(0)$ and θ_j is one-dimensional. Hence, by standard analytic perturbation theory (cf [13, 28]), for θ sufficiently close to θ_j, the Floquet eigenvalue $E_n(\theta)$ is analytic in θ, and one can find a Floquet eigenvector associated to $E_n(\theta)$, say $\varphi_n(x,\theta)$, that is normalized and analytic in θ.

Applying now analytic perturbation theory to $\overline{H}_\lambda(\theta)$ (i.e. $\overline{H}_\lambda$ with θ-periodic boundary conditions) for small λ, we obtain that, there exists $\delta > 0$ and $\lambda_0 > 0$ such that, for $|\lambda| < \lambda_0$ and $1 \leq j \leq n_z$,

- in the interval $]E_-(0)-\delta, E_-(0)+\delta[$, $\overline{H}_\lambda(\theta)$ has a unique Floquet eigenvalue for the Floquet parameter $|\theta-\theta_j| < \delta$; let $E_n(\theta,\lambda)$ be this eigenvalue;
- $E_n(\theta,\lambda)$ is real analytic in $(\theta,\lambda) \in \{|\theta-\theta_j| < \delta\}\times]E_-(0)-\delta, E_-(0)+\delta[$, and, there exists a Floquet eigenvector, say $\varphi_n(x,\theta,\lambda)$, that is normalized and analytic in (θ,λ);
- the rest of the spectrum of $\overline{H}_\lambda(\theta)$ lies outside of $]E_0, E_-(0)+\delta[$.

This proves that $\overline{E}(\lambda)$ is equal to one of the numbers $(E_n(\tilde{\theta}_j(\lambda),\lambda))_{1\leq j\leq n_z}$ where $\tilde{\theta}_j(\lambda)$ is unique minimum of $E_n(\theta,\lambda)$ in $\{|\theta-\theta_j| < \delta\}$. The points $\tilde{\theta}_j(\lambda)$ are analytic in λ; they satisfy the equation $\nabla E_n(\tilde{\theta}_j(\lambda),\lambda) = 0$ and the Hessian matrix of $\theta \mapsto E_n(\theta,\lambda)$ is non degenerate in a neighborhood of θ_j

for λ small. This is a consequence of perturbation theory as it holds for $\lambda = 0$ by assumption. One computes

$$
\begin{aligned}
\partial_\lambda E_n(\tilde\theta_j,\lambda)_{\restriction\lambda=0} &= \left(\langle V\varphi_n(x,\tilde\theta_j(\lambda),\lambda),\varphi_n(x,\tilde\theta_j(\lambda),\lambda)\rangle + \nabla_\theta E_n(\tilde\theta(\lambda),\lambda)\partial_\lambda\tilde\theta_j(\lambda)\right)_{\restriction\lambda=0} \\
&= \int_{\mathbb{R}^d} V(x)|\varphi_n(x,\theta_j)|^2 dx.
\end{aligned} \tag{5.1}
$$

Hence, assumption (H2) is satisfied if and only if

$$
\exists 1\le j\le n_z,\quad \int_{\mathbb{R}^d} V(x)|\varphi_n(x,\theta_j)|^2 \neq 0. \tag{5.2}
$$

For a given operator H, by the Unique Continuation Principle (see e.g. [12]), this condition is satisfied for a generic V.

The computation (5.1) and the analyticity of the $(E_n(\tilde\theta_j(\lambda),\lambda))_{1\le j\le n_z}$ implies that, for some $1\le j\le n_z$,

$$
\overline{E}(\lambda) = E_-(0) + \lambda\int_{\mathbb{R}^d} V(x)|\varphi_n(x,\theta_j)|^2 + O(\lambda^2) \tag{5.3}
$$

On the other hand, using the characterization of Σ_λ, the almost sure spectrum of $H_{\omega,\lambda}$ in terms of admissible periodic spectra (see [14, 27]), we know that, for λ sufficiently small and $t\in$ess-supp(ω_0), we have

$$
\frac{1}{2}(E_-(0)+E_+(0)) \le E_-(\lambda) \le \overline{E}(\lambda t).
$$

Let ω_+ and ω_- respectively be the essential supremum and infimum of the random variables $(\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$; as the random variables are not trivial, using (5.2) and the computation (5.1), one sees that, for some $C>0$ and for λ sufficiently small

$$
|\overline{E}(\lambda\overline{\omega}) - \overline{E}(\lambda\omega_+)| \ge |\lambda|/C \text{ and } |\overline{E}(\lambda\overline{\omega}) - \overline{E}(\lambda\omega_-)| \ge |\lambda|/C.
$$

This implies that (0.2) holds for some $C>0$ and λ sufficiently small.

5.2. **A useful lemma.** We prove

Lemma 5.1. *Pick $\varepsilon>0$ and $u\in L^2(\mathbb{R}^d)$ such that supp$(u)\subset\{|x|\le\varepsilon\}$. Then, for any $\eta\in(0,1)$, there exists $\tilde u\in L^2(\mathbb{R}^d)$ such that*

- $\|\tilde u\|_{L^2} = \|u\|_{L^2}$,
- $\|u-\tilde u\|_{L^2}\le C\eta\|u\|_{L^2}$,
- *$\hat u_\eta$ is constant on each cube $C_{\gamma,\frac{\eta}{2\varepsilon}}$, $\gamma\in\mathbb{Z}^d$ where*

$$
C_{\gamma,r} = \{x=(x_1,\dots,x_d);\ -\pi r\le x_i - 2\pi r\gamma_i < \pi r\}.
$$

Lemma 5.1 is a very simple quantitative version of the Uncertainty Principle. It is the analogue of Lemma 6.2 in [22] developed for the discrete setting.

Proof. Pick ε, η and u as in Lemma 5.1. Consider u as a $(2\varepsilon/\eta)\mathbb{Z}^d$-periodic function (continue it by 0 on the fundamental cell of $(2\varepsilon/\eta)\mathbb{Z}^d$ and periodically to the rest of $\mathbb{R}^d$). Expand it in a Fourier series to get

$$
u(x) = \sum_{\gamma\in\mathbb{Z}^d}\hat u_\gamma e^{i\frac{\pi\eta}{\varepsilon}\gamma x} \text{ where } \hat u_\gamma = \left(\frac{\eta}{2\varepsilon}\right)^d\int_{\mathbb{R}^d} u(x)e^{-i\frac{\pi\eta}{\varepsilon}\gamma x}dx.
$$

Parseval's identity then reads

$$\|u\|_{L^2}^2 = \sum_{\gamma\in\mathbb{Z}^d} \left(\frac{2\varepsilon}{\eta}\right)^d |\hat{u}_\gamma|^2. \tag{5.4}$$

Define the function $v:\ \mathbb{R}^d \to \mathbb{R}^d$ by

$$v(\xi) = \left(\frac{2\varepsilon}{\eta}\right)^d \hat{u}_\gamma \text{ for } \xi \in C_{\gamma,\frac{\eta}{2\varepsilon}},\ \gamma \in \mathbb{Z}^d.$$

By (5.4), $v \in L^2(\mathbb{R}^d)$. Let $\tilde{u}$ be the inverse Fourier transform of v. Check that it satisfies the the properties stated in Lemma 5.1. First, (5.4) yields

$$\|\tilde{u}\|_{L^2}^2 = \frac{1}{(2\pi)^d}\|v\|_{L^2}^2 = \left(\frac{2\varepsilon^2}{\pi\eta^2}\right)^d \sum_{\gamma\in\mathbb{Z}^d} \int_{C_{0,\frac{\eta}{2\varepsilon}}} |\hat{u}_\gamma|^2 d\xi = \|u\|_{L^2}^2.$$

Let $\hat{u}$ be the Fourier transform of u; we compute

$$\int_{\mathbb{R}^d} |\hat{u}(\xi) - v(\xi)|^2 d\xi = \sum_{\gamma\in\mathbb{Z}^d} \int_{C_{0,\frac{\eta}{2\varepsilon}}} \left|\hat{u}(\xi + \gamma\frac{\pi\eta}{\varepsilon}) - \left(\frac{2\varepsilon}{\eta}\right)^d \hat{u}_\gamma\right|^2 d\xi. \tag{5.5}$$

On the other hand, we note

$$\hat{u}(\xi + \gamma\frac{\pi\eta}{\varepsilon}) - \left(\frac{2\varepsilon}{\eta}\right)^d \hat{u}_\gamma = \int_{\mathbb{R}^d} u(x)(e^{ix\xi} - 1)e^{-i\frac{\pi\eta}{\varepsilon}\gamma x} dx.$$

As we did for u, we consider $x \mapsto u(x)(e^{ix\xi} - 1)$ as a $(2\varepsilon/\eta)\mathbb{Z}^d$-periodic function; then, Parseval's identity yields

$$\sum_{\gamma\in\mathbb{Z}^d} \left|\hat{u}(\xi + \gamma\frac{\pi\eta}{\varepsilon}) - \left(\frac{2\varepsilon}{\eta}\right)^d \hat{u}_\gamma\right|^2 = \left(\frac{2\varepsilon}{\eta}\right)^d \int_{|x|\le\varepsilon} |u(x)(e^{ix\xi} - 1)|^2 dx$$

Substituting in (5.5), we obtain

$$\int_{\mathbb{R}^d} |\hat{u}(\xi) - v(\xi)|^2 d\xi = \int_{|x|\le\varepsilon} |u(x)|^2 \left(\left(\frac{2\varepsilon}{\eta}\right)^d \int_{C_{0,\frac{\eta}{2\varepsilon}}} |e^{ix\xi} - 1|^2 d\xi\right) dx$$

that is

$$\int_{\mathbb{R}^d} |\hat{u}(\xi) - v(\xi)|^2 d\xi \le C\eta^2 \int_{|x|\le\varepsilon} |u(x)|^2 dx = C\eta^2\|u\|_{L^2}^2.$$

This completes the proof of Lemma 5.1.

□

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