Generic Jumps of Fredholm Indices and the Quantum Hall Effect

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Abstract

We describe the generic behavior of Fredholm indices in the space of Toeplitz operators. We relate this behavior to certain conjectures and open problems that arise in the context of the Quantum Hall Effect.

1 Introduction and Motivation

Suppose one interpolates between Fredholm operators with different indices. What can one say about the way the indices change? The answer to this question depends on the choice of the embedding space for the Fredholm operators in question. In the space of bounded operators, little can be said. But, in the space of Toeplitz operators, (and then also for Toeplitz modulu compacts), as we shall explain, the indices change by abrupt discontinuous jumps that tend to be small. We relate this behavior to certain conjectures and open problems that arise in the context of the Quantum Hall Effect (QHE) [Sto].

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1.1 Physical background

In the theory of the integer quantum Hall effect (of non-interacting electrons) [BvES, ASS] one identifies the Hall conductance with the Fredholm index of a rather special operator, namely PUP, thought of as an operator on the range of P. Here $P = P(E)$ is an (infinite dimensional) projection in the Hilbert space $L^2(C)$, namely the projection on the spectrum of the one electron Hamiltonian below the Fermi energy E. U is the multiplication operator $\frac{z}{|z|}$ associated with a singular gauge transformation that introduces an Aharonov-Bohm flux tube at the origin of the Euclidean plane. PUP is Fredholm provided the integral kernel of the projection, $p(z, z'; E)$ has good decay properties as $|z - z'|$ gets large [ASS].

Recent progress in the rigorous theory of random Schrödinger operators relevant to the QHE [Aiz] guarantees good decay properties for $p(z, z'; E)$ provided E lies in certain energy intervals. Percolation arguments [Tru] and scaling theories of localization [Khm] give theoretical evidence that these decay properties persist for all but a discrete set of energies. This implies that the graph of the Hall conductace as a function of E should be a step function. Indeed, experimentally, the Hall conductance in the integer Hall effect, is close to a monotonic step function with ± 1 and ± 2 jumps [Lau]. (Jumps by 2 occur when the Hall conductance is larger than 6 and is attributed to the smallness of the magnetic moment of the electron in these systems.)

The smallness of the jumps of the Fredholm indices in the QHE might, of course, be a special property of a special system. Here, instead, we want to explore the opposite point of view, namely the possibility that the existence of steps and the smallness of the jumps reflects a generic property of Fredholm indices and has little to do with the specific properties of the system in question.

Some support to this point of view comes from the relation of Chern numbers and Fredholm indices. In non-commutative geometry [Con] Chern numbers and Fredholm indices are intimately related. This is also the case in the index theory of elliptic operators [Ati]. For Chern numbers that arise from studies of spectral bundles (of Hamiltonians with discrete spectra), a generic deformation of the Hamiltonian leads to a step function with ± 1 jumps in the first Chern number [Sim]. This follows from the Wigner von Neumann codimension 3 rule for eigenvalue crossing [vNW] and the fact that a generic crossing is a conic crossing and is not system specific.

As far as the QHE goes one might argue that since the Hall conductance can be directly related to a Chern number [Sto, TKNN], the genericity of small jumps follows immediately. The difficulty with this argument has to do with the thermodynamic limit. Normally, the QHE is associated with large systems. The genericity result quoted above for Chern numbers is for operators with discrete spectrum. This is the case for finite sytems, but is in general not the case for extended systems, and in particular does not apply to models of the quantum Hall effect. The main attractive feature of the Fredholm approach to the Hall effect is that it is phrased directly in the thermodynamic limit.

Another way of phrasing the main theme of this paper is: What, if any, is the analog for Fredholm operators of the genericity of small jumps in Chern numbers?

1.2 The mathematical problem

We wish to interpolate between two (or more) Fredholm operators. If the indices of these operators are different this cannot be done within the space of Fredholm operators. At some points in the interpolation the Fredholm property will be lost and the index will be ill defined. For "generic" interpolations, what is the nature of this bad set? Near such a bad point, how big a range of indices can be found?

Working in the space of bounded operators, little can be said. The space is simply too large, and when the Fredholm property is lost we lose all analytic control. However, in the space of sufficiently smooth Toeplitz operators interesting results can be obtained. In systems without symmetry, we find the following behavior: Almost every operator is Fredholm, and sets of codimension n appear as boundaries between regions of Fredholm operators whose indices differ by n. We speak simply of the index "jumping by n" on a set of codimension n.

In systems with a Z_2 symmetry (e.g. time reversal symmetry or complex conjugation symmetry), sets of codimension n appear as common boundaries of regions of Fredholm operators whose indices differ by as much as $2n$. That is, the index can jump by as much as $2n$ on a set of codimension n.

2 Basic Definitions and Properties

We review here the basic definitions and properties of Fredholm operators on separable Hilbert spaces. For a more complete treatment see [Dou].

Definition 1 *A bounded operator A on a separable Hilbert space if* Fredholm *if there exists another bounded operator B such that $1 - AB$ and $1 - BA$ are compact.*

In particular, the kernel and cokernel of A are finite dimensional, and we define

Definition 2 *The* index *of a Fredholm operator F is*

$$Index(F) = dim\, Ker(F) - dim\, Ker(F^\dagger). \tag{1}$$

Fredholm operators are stable under compact perturbations and under small bounded perturbations. That is, if A is Fredholm, there exists an $\epsilon > 0$ such that, for any bounded operator B with operator norm $\|B\| < \epsilon$ and for any compact operator K, the operator $A+B+K$ is Fredholm with the same index as A.

The simplest example of a Fredholm operator with nonzero index is the shift operator. Let $e_0, e_1, e_2, \ldots$ be an orthonormal basis for a Hilbert space, and let the operator a act by

$$a(e_n) = \begin{cases} e_{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}. \tag{2}$$

The adjoint of a acts by

$$a^\dagger(e_n) = e_{n+1} \tag{3}$$

Since $aa^\dagger = a^\dagger a + |e_0\rangle\langle e_0|$ is the identity, a is Fredholm. The kernel of a is 1-dimensional. The cokernel of a, which is the same as the kernel of $a^\dagger$, is 0 dimensional. Thus the index of a is 1. Similarly, $a^\dagger$ is Fredholm with index -1.

The following theorem is standard:

Theorem 1 *If $A_1, \ldots A_n$ are Fredholm operators, then the product $A_1 A_2 \cdots A_n$ is also Fredholm, and $Index(A_1 \cdots A_n) = \sum_{i=1}^n Index(A_i)$.*

Finally we consider connectedness in the space of Fredholm operators. If A and A' are Fredholm operators on the same Hilbert space, then there is a continuous path of Fredholm operators from A to A' if and only if $Index(A) = Index(A')$. (By continuous, we mean relative to the operator norm). Put another way, the path components of $Fred(H)$, the space of Fredholm operators on H, is indexed (pun intended) by the integers. The n-th path component is precisely the set of Fredholm operators of index n [Dou].

3 Fredholm Operators in the Space of Bounded Operators

The most natural setting for our problem is consider arbitrary bounded operators, with the topology defined by the operator norm. We ask how many parameters must be varied in order to reach the common boundary of two regions, whose indices differ by k. Unfortunately, the answer is independent of k:

Theorem 2 *Let U_n be the set of Fredholm operators of index n. Every point on the boundary of U_n is also on the boundary of U_m, for every integer m.*

Proof: Let A be a (not Fredholm) operator on the boundary of U_n. Given $\epsilon > 0$, we must find an operator in U_m within a distance ϵ of A.

Suppose that the kernel and cokernel of A are infinite dimensional, and that there is a gap in the spectrum of $A^\dagger A$ at zero. (If this is not the case, we may perturb A by an arbitrarily small amount to make it so). Now let B be a unitary map from the kernel of A to the cokernel. Let P, (P') be the orthogonal projection onto $ker(A)$, $(coker(A))$, and let a be a shift operator on $ker(A)$. For each $m \geq 0$, $A(\epsilon) = A + \epsilon B a^m P$ has a bounded right inverse

$$A^\dagger \frac{1}{P' + AA^\dagger} P'_\perp + \frac{1}{\epsilon}(a^\dagger)^m B^\dagger P'. \tag{4}$$

It follows that the cokernel of $A(\epsilon)$ is empty. It is easy to see that the kernel of $A(\epsilon)$ is m dimensional hence $Index(A(\epsilon)) = m$. Similarly, $A + \epsilon B(a^\dagger)^m P$ has index $-m$. ∎

This theorem tells us that, in the space of all bounded operators there is no specific notion of being at a transition point from index n to index m.

As long as an operator stays Fredholm, its index cannot change, and when it fails to be Fredholm it can change into anything.

To achieve useful results, we must work on a smaller space.

4 Linear Combinations of Shifts

In this section and the next we show that "generic" behavior is indeed achieved in some finite dimensional spaces, and in some infinite-dimensional spaces with sufficiently fine topologies. We see also how control is lost as the space is enlarged and the topology is coarsened.

4.1 Shift by one

We begin by considering linear combinations of the shift operator a and the identity operator 1. That is, we consider the operator

$$A = c_1 a + c_0$$

where c_1 and c_0 are constants.

Theorem 3 *If $|c_1| \neq |c_0|$, then A is Fredholm. The index of A is 1 if $|c_1| > |c_0|$ and zero if $|c_1| < |c_0|$. If $|c_1| = |c_0|$, then A is not Fredholm.*

Proof: First suppose $|c_0| > |c_1|$. Then A is invertible:

$$A^{-1} = c_0^{-1}(1 + (c_1/c_0)a)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n c_1^n}{c_0^{n+1}} a^n,$$

as the sum converges absolutely. Thus A has neither kernel nor cokernel, and has index zero.

If $|c_1| > |c_0|$, then the kernel of A is 1-dimensional, namely all multiples of $|\psi\rangle = \sum_{n=0}^{\infty} z_0^n e_n$, where $z_0 = -c_0/c_1$. Notice how the norm of $|\psi\rangle$ goes to infinity as $|z_0| \to 1$. However, $A^\dagger$ has no kernel, since for any unit vector $|\phi\rangle$, $\|A^\dagger|\phi\rangle\| = \|\bar{c}_1 a^\dagger|\phi\rangle + \bar{c}_0|\phi\rangle\| \geq \|\bar{c}_1 a^\dagger|\phi\rangle\| - \|\bar{c}_0|\phi\rangle\| = |c_1| - |c_0|$. Thus the index of A is 1.

If $|c_1| = |c_0|$, then A is at the boundary between index 1 and index 0, and so cannot be Fredholm. ∎

4.2 Finite linear combinations of shifts

Next we consider linear combinations of $1, a, a^2, \ldots$ up to some fixed a^n. That is, we consider operators of the form

$$A = c_n a^n + c_{n-1} a^{n-1} + \cdots + c_0. \tag{5}$$

This is closely related to the polynomial

$$p(z) = c_n z^n + \cdots + c_0. \tag{6}$$

Theorem 4 *If none of the roots of p lie on the unit circle, then A is Fredholm, and the index of A equals the number of roots of p* inside *the unit circle, counted with multiplicity. If any of the roots of p lie* on *the unit circle, then A is not Fredholm.*

Proof: The polynomial $p(z)$ factorizes as $p(z) = c_k \prod_{i=1}^k (z - \zeta_i)$, where k is the degree of p (typically $k = n$, but it may happen that $c_n = 0$). But then $A = c_k \prod_{i=1}^k (a - \zeta_i)$. If none of the roots ζ_i lie on the unit circle, then each term in the product is Fredholm, so the product is Fredholm, and the index of the product is the sum of the indices of the factors. By Theorem 3, this exactly equals the number of roots ζ_i inside the unit circle.

If any of the roots lie on the unit circle, then a small perturbation can push those roots in or out, yielding Fredholm operators with different indices. This borderline operator therefore cannot be Fredholm. ∎

The last theorem easily generalizes to linear combination of left-shifts and right-shifts. The index of an operator

$$A = c_n a^n + \cdots + c_1 a + c_0 + c_{-1} a^\dagger + \cdots + c_{-m} (a^\dagger)^m \tag{7}$$

equals the number of roots of

$$p(z) = \sum_{i=-m}^{n} c_i z^i \tag{8}$$

inside the unit circle, minus the degree of the pole at $z = 0$ (that is m, unless $c_{-m} = 0$). This follow from the fact that

$$A = (\sum_{i=-m}^{n} c_i a^{i+m})(a^\dagger)^m. \tag{9}$$

Since there is no qualitative difference between combinations of left-shifts and combinations of both left- and right-shifts, we restrict our attention to left-shifts only, and consider families of operators of the form (5).

Theorem 5 *In the space of complex linear combinations of 1, a, ..., a^n, almost every operator is Fredholm. For every $k \leq n$, the points where the index can jump by k (by which we mean the common boundaries of regions of Fredholm operators whose indices differ by k) is a set of real codimension k.*

In the space of real linear combinations of 1, a, ..., a^n, almost every operator is Fredholm. For every $k \leq n$, the points where the index jumps by k is a stratified space, the largest stratum of which has real codimension $\lfloor (k+1)/2 \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x.

Proof: Our parameter space is the space of coefficients c_i, or equivalently the space of polynomials of degree $\leq n$. This is either $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$, depending on whether we allow real or complex coefficients. In either case, the set U_k of Fredholm operators of index k is identical to the set of polynomials with k roots inside the unit circle and the remaining $n-k$ roots outside (if $c_n = 0$, we say there is a root at infinity; if $c_n = c_{n-1} = 0$, there is a double root at infinity, and so on. Counting these roots at infinity, there are always exactly n roots in all.) The boundary of U_k is the set of polynomials with at most k roots inside the unit circle, at most $n-k$ outside the unit circle, and at least one root on the unit circle. (Strictly speaking, the zero polynomial is also on this boundary. This is of such high codimension that it has no effect on the phase portrait we are developing.). We consider the common boundary of U_k and $U_{k'}$. If $k < k'$, a nonvanishing polynomial is on the boundary of both U_k and $U_{k'}$ if it has at most k roots inside the unit circle and at most $n - k'$ roots outside. It must therefore have at least $k' - k$ roots on the unit circle.

If we are working with complex coefficients, this is a set of codimension $k' - k$. The roots themselves, together with an overall scale c_n, can be used to parametrize the space of polynomials. For each root, being on the unit circle is codimension 1, while being inside or outside are open conditions. Since the roots are independent, placing $k' - k$ roots on the unit circle is codimension $k' - k$.

If we are working with real coefficients, the roots are not independent, as non-real roots come in complex conjugate pairs. Thus, the common boundary of U_k and $U_{k'}$ breaks into several strata, depending on how many real roots and how many complex conjugate pairs lie on the unit circle. If $k' - k$ is even, the biggest stratum consists of having $(k'-k)/2$ pairs, and has codimension $(k'-k)/2$. If $k'-k$ is odd, the biggest stratum consists of having $(k'-k-1)/2$

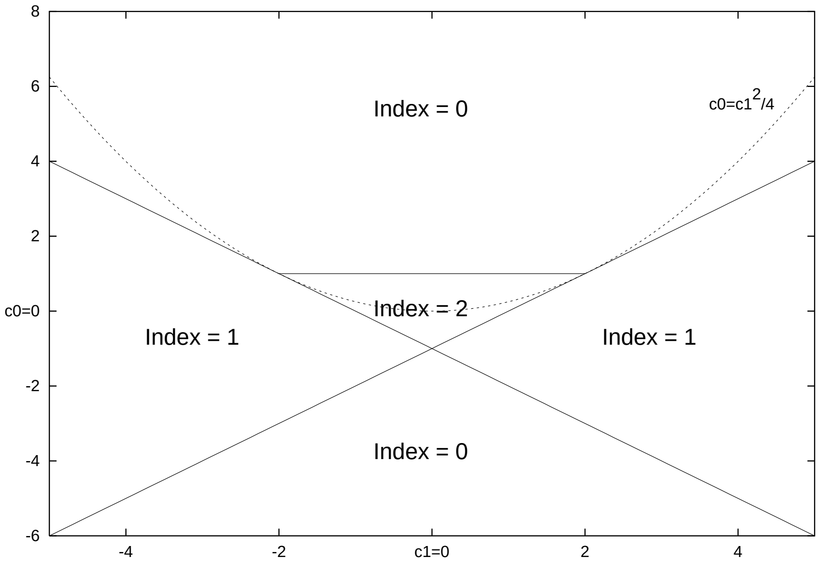

Figure 1: A phase plot for $A = a^2 + c_1 a + c_0$.

pairs and one real root on the unit circle, and has codimension $(k' + 1 - k)/2$. ∎

Theorem 5 is illustrated in Figure 1, where the phase portrait is shown for $n = 2$ with real coefficients, with c_2 fixed to equal 1. The points above the parabola $c_0 = c_1^2/4$ have complex conjugate roots, while points below have real roots. Notice that the transition from index 2 to index 0 occurs at an isolated point when the roots are real, but on an interval when the roots come in complex-conjugate pairs.

It is clear that an almost identical theorem applies to linear combinations of left-shifts up to a^n and right-shifts up to $(a^\dagger)^m$. The results are essentially independent of n and m (their only effect being to limit the size of possible jumps to $n + m$). We can therefore extend the results to the space of all (finite) linear combinations of left- and right-shifts, which is topologized as the union over all n and m of the spaces considered above. Our result, restated for that space, is

Theorem 6 *In the space of finite complex linear combinations of left- and right-shifts of arbitrary degree, almost every operator is Fredholm. For every* *integer $k \geq 1$, the points where the index can jump by k (by which we mean the common boundaries of regions of Fredholm operators whose indices differ by k) is a set of real codimension k.*

If we restrict the coefficients to be real, then, for every $k \leq n$, the points where the index jumps by k is a stratified space, the largest stratum of which has real codimension $\lfloor (k+1)/2 \rfloor$.

5 Toeplitz operators

Although Theorem 6 refers to an infinite-dimensional space, this space is still extremely small – each point is a *finite* linear combination of shifts. In this section we consider *infinite* linear combinations of shifts. This is equivalent to studying Toeplitz operators.

Definition 3 *The* Hardy space *H is the subspace of $L^2(S^1)$ consisting of functions whose Fourier transforms have no negative frequency terms. Equivalently, if we give $L^2(S^1)$ a basis of Fourier modes $e_n = e^{in\theta}$, where the integer n ranges from $-\infty$ to ∞, then H is the closed linear span of $e_0, e_1, e_2, \ldots$.*

We think of S^1 as sitting in the complex plane, with $z = e^{i\theta}$. Now let $f(z)$ be a bounded, measurable function on S^1, and let P be the orthogonal projection from $L^2(S^1)$ to H. If $|\psi\rangle \in H$, then $|f\psi\rangle$ (pointwise product) is in $L^2(S^1)$, and $P|f\psi\rangle \in H$. We define the operator T_f by

$$T_f|\psi\rangle = P|f\psi\rangle. \tag{10}$$

Definition 4 *An operator of the form (10) is called a* Toeplitz operator. *We call a Toeplitz operator T_f continuous if the underlying function f is continuous, and apply the terms "differentiable", "smooth" and "analytic" similarly.*

Remark: Toeplitz operators can be represented by semi-infinite matrices that have constant entries on diagonals, and the various classes we have defined correspond to the decay away from the main diagonal.

Notice that

$$T_{e_m} e_n = \begin{cases} e_{n+m} & \text{if } n+m \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

so T_{e_m} is simply a shift by m, a right shift if $m > 0$ and a left-shift if $m < 0$. All our results about shifts can therefore be understood in the context of Toeplitz operators. Theorem 5 refers to operators T_f, where f is a polynomial in z^{-1} of limited degree. Theorem 6 considers polynomials or arbitrary degree in z and z^{-1}. We will see that the results carry over to analytic functions on an annulus around S^1, and to a lesser extent to C^k Toeplitz operators, but with results that weaken as k is decreased.

Here are some standard results about Toeplitz operators. For details, see [Dou].

Theorem 7 *A C^1 Toeplitz operator T_f is Fredholm if and only if f is everywhere nonzero on the unit circle. In that case the index of T_f is minus the winding number of f around the origin, namely*

$$Index(T_f) = -Winding(f) = \frac{-1}{2\pi i}\int_{S^1} \frac{df}{f}, \tag{12}$$

Given the first half of the theorem, the equality of index and winding number is easy to understand. We simply deform f to a function of the form $f(z) = z^n$, while keeping f nonzero on all of S^1 throughout the deformation (this is always possible, see e.g. [GuP]). In the process of deformation, neither the index of T_f nor the winding number of f can change, as they are topological invariants. Since the winding number of z^n is n, and since $T_{z^n} = (a^\dagger)^n$ (if $n \geq 0$, a^{-n} otherwise), which has index $-n$, the result follows.

We now consider functions f on S^1 that can be analytically continued (without singularities) an annulus $r_0 \leq |z| \leq r_1$, where the radii $r_0 < 1$ and $r_1 > 1$ are fixed. This is equivalent to requiring that the Fourier coefficients $\hat{f}_n$ decay exponentially fast, i.e. that the sum

$$\sum_{n=-\infty}^{\infty} |\hat{f}_n|(r_0^n + r_1^n) \tag{13}$$

converges. For now we do not impose any reality constraints or other symmetries on the coefficients $\hat{f}_n$. This space of functions is a Banach space, with norm given by the sup norm on the annulus. This norm is stronger than any Sobolev norm on the circle itself.

The analysis of the corresponding Toeplitz operators is straightforward and similar to the proof of Theorem 5. Since f has no poles in the annulus, we just have to keep track of the zeroes of f. For the index of T_f to change, a zero of f must cross the unit circle. For the index to jump from k to k', $|k - k'|$ zeroes must cross simultaneously. In the absence of symmetry, the locations of the zeroes are independent and can be freely varied, so this is a codimension-$|k-k'|$ event.

If we impose a reality condition: $f(\bar{z})=\overline{f(z)}$, then zeroes appear only on the real axis or in complex conjugate pairs. In that case, changing the index by 2 is merely a codimension-1 event. Combining these observations we obtain

Theorem 8 *In the space of Toeplitz operators that are analytic in a (fixed) annulus containing S^1, almost every operator is Fredholm. For every integer $k\geq 1$, the points where the index can jump by k is a set of real codimension k.*

If we impose a reality condition $f(\bar{z})=\overline{f(z)}$ then, for every $k\leq n$, the points where the index jumps by k is a stratified space, the largest stratum of which has real codimension $\lfloor (k+1)/2 \rfloor$.

Finally we consider Toeplitz operators that are not necessarily analytic, but are merely ℓ times differentiable, and we use the C^ℓ norm. Our result is

Theorem 9 *In the space of Toeplitz C^ℓ operators, almost every operator is Fredholm. For every integer k with $1\leq k\leq 2\ell+1$, the points where the index can jump by k is a set of real codimension k. For every integer $k\geq 2\ell+1$, the points where the index can jump by k is a set of real codimension $2\ell+1$.*

In other words, our familiar results hold up to codimension $2\ell+1$, at which point we lose all control of the change in index.

Proof: As long as f is everywhere nonzero, T_f is Fredholm. To get a change in index, therefore, we need one or more points where f, and possibly some derivatives of f with respect to θ, vanish. Suppose then that for some angle θ_0, $f(\theta_0)=f'(\theta_0)=\cdots=f^{(n-1)}(\theta_0)=0$ for some $n\leq\ell$, but that the n-th derivative $f^{(n)}(\theta_0)\neq 0$. This is a codimension $2n-1$ event, since we are setting the real and imaginary parts of n variables to zero, but have a 1-parameter choice of points where this can occur. Without loss of generality, we suppose that this n-th derivative is real and positive. By making a C^ℓ-small perturbation of f, we can make the value of f highly oscillatory near θ_0, thereby wrapping around the origin a number of times. However, since a C^ℓ-small perturbation does not change the n-th derivative by much, the sign of the real part of f can change at most n times near θ_0, so the argument of f can only increase or decrease by $n\pi$ or less. The difference between these two extremes is $2n\pi$, or a change in winding number of n.

To change the index by an integer m, therefore, we must have the function vanish to various orders at several points, with the sum of the orders of vanishing adding to m. The generic event is for f (but not f') to vanish at m different points – this is a codimension m event, analagous to having m zeroes of a polynomial cross the unit circle simultaneously at m different points. All other scenarios have higher codimension and are analogous to having 2 or more zeroes of the m zeroes crossing the unit circle at the same point.

The situation is different, however, when the function f and the first ℓ derivatives all vanish at a point θ_0. Then the higher-order derivatives are not protected from C^ℓ-small perturbations and, by making such a perturbation, we can change f into a function that is identically zero on a small neighborhood of $\theta = \theta_0$. By making a further small perturbation, we can make f wrap around the origin as many times as we like near $\theta = \theta_0$. More specifically, if f is zero on an interval of size δ, then, for small ϵ, $\tilde{f}(\theta) = f(\theta) + \epsilon e^{iN\theta}$ will wrap around the origin approximately $N\delta/2\pi$ times near θ_0. By picking N as large (positive or negative) as we wish, we can obtain arbitrarily positive or negative indices. As long as we take $\epsilon \ll N^{-\ell}$, this perturbation will remain small in the C^ℓ norm. ∎

6 The Quantum Hall Effect

We have seen in the previous section that the Fredholm index of a generic one dimensional family of Toeplitz operators is a step function with small jumps. This is reminiscent of what one observes for the Hall conductance for random Schrödinger operators.

In this section we want to discuss some of the difficulties, and what one would still need to know, for the strategy in this paper to yield useful results for the QHE.

6.1 Landau levels

The Hall conductance is related to the Index of PUP (on *Range* P) with P a spectral projection in $L^2(\mathbb{C})$ and U a multiplication by $\frac{z}{|z|}$. This operator is closely related to a Toeplitz operator in the case of a basic paradigm for the Hall effect:

Theorem 10 *Let P be a projection on the lowest Landau level in $\mathbb{R}^2$, then PUP differs from a Toeplitz operator by a compact operator.*

Proof: A basis for the lowest Landau level is

$$|n\rangle = \frac{1}{\sqrt{\pi\, n!}}\, z^n\, e^{-|z|^2/2}, \quad n \geq 0. \tag{14}$$

As a consequence

$$\langle n|U|m\rangle = \delta_{n,m+1}\, \frac{(m+1/2)!}{m!\sqrt{m+1}} \approx \delta_{n,m+1} \left(1 - \frac{1}{8m}\right). \ \blacksquare \tag{15}$$

The same result also holds if P is a projection on a higher Landau level, but the calculation is more involved. If P is a projection onto multiple Landau levels, then PUP is a compact perturbation of a direct sum of Toeplitz operators, one for each Landau level. This suggests that the class of Toeplitz operators is indeed related to the QHE.

For (spinless) electrons/holes on the Euclidean and hyperbolic planes, with homogeneous magnetic field, and without disorder, $Index(PUP)(E)$ has been explicitly computed as a function of the "Fermi energy" E. In the Euclidean plane one finds a monotonic step function with jumps ± 1 [APn]. (One needs both signs for electrons and holes.) The same results apply in the hyperbolic plane for all energies below the continuous spectrum [APn]. This implies that also for (relatively) compact perturbations of these Hamiltonians the Fredholm index in the QHE behaves as does the Fredholm index of Toeplitz operators. The situation is, however, quite different for Schrödinger operators with periodic potentials where $PUP(E)$ failes to be Fredholm on intervals of "energy bands" and where the Fredholm index in adjacent gaps can jump by large integers [TKNN].

6.2 An open problem

For applications to the Hall effect one considers PUP (on the range of P) where the projection P depends on a parameter such as the Fermi energy or the external magnetic field. The family PUP is therefore defined on different spaces, since the range of P is not fixed. Our strategy, so far, has been to study a family of operators on a *fixed* Hilbert space. To adapt the QHE to this strategy one must replace PUP by something like

$$C = PUP + 1 - P, \tag{16}$$

acting on the full Hilbert space, as $Index(C)$ on the full space coincides with $Index(PUP)$ on $Range(P)$. Now, a deformation of P leads to a deformation of C and gives a family of bounded operators on a fixed space, say, $L^2(\mathbb{C})$. However, this modification is not without a price since now, even for the simple case of a full Landau level, C is not strictly a Toeplitz operator. It is a rather silly generalization of a Toeplitz operator to a direct sum of a Toeplitz operator and the identity.

A more serious problem has to do with what should one pick as a good family P. In particular, when one considers a variation of the Fermi energy E the corresponding projection $P(E)$ is not continuous in the operator norm. Hence, a smooth variation of E is not even a smooth variation of C in the operator norm (much less in the sharper norms considered above).

Using the fact that the Fredholm index does not change under small changes in the norm of the operator, there is no harm done if one replaces the spectral projection $P(E)$ by the Fermi function

$$P_\beta(E) = \frac{1}{\exp\beta(H-E)+1}, \tag{17}$$

for β large. Unlike $P(E)$, $P_\beta(E)$ is a smooth function of E, and so the family $C_\beta(E)$ is smooth. The price one pays is that $P_\beta(E)$ is not a projection, which leads to ambiguities as to what one might want to choose for $C_\beta(E)$. For example, instead of (16) one might choose

$$C_\beta(E) = P_\beta(E)UP_\beta(E) + (1 - P_\beta^2(E)). \tag{18}$$

The trouble is that it is not clear what, if anything, the results about families of Toeplitz operators imply for the family $C_\beta(E)$.

We therefore pose the following questions:

For random Scrödinegr operators on the plane, with β sufficiently large, what are the properties of the family of operators $C_\beta(E)$? Is it Fredholm away from a discrete set of energies E, or does it fail to be Fredholm on bigger sets? If it fails to be Fredholm at isolated points, are the jumps generically small?

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