PERTURBATIONS OF VECTOR FIELDS ON TORI: RESONANT NORMAL FORMS AND DIOPHANTINE PHENOMENA

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Abstract. This paper is concerned with the study of perturbations of smooth vector fields on $\mathbb{T}^n$ (constant if $n \geq 3$) with zero'th order C^∞ and Gevrey G^σ, $\sigma \geq 1$ pseudodifferential operators. A notion of simultaneous resonance is introduced and simultaneous resonant normal forms are shown (via conjugation with an elliptic pseudodifferential operator) under optimal simultaneous Diophantine conditions outside the resonances. In the C^∞ category the results are complete. In the Gevrey category the effect of the loss of the Gevrey regularity of the conjugating operators due to Diophantine conditions is investigated by means of techniques similar to methods for obtaining Nekhoroshev type (effective stability) estimates in hamiltonian dynamics. The normal forms are used to study the global hypoellipticity in C^∞ and Gevrey G^σ. Finally, the exceptional sets associated with the simultaneous Diophantine conditions are studied. Generalized Hausdorff dimension is used to give precise estimates of the "size" of different exceptional sets, including some inhomogenous exceptional sets.

1. Introduction

The purpose of this paper is to study resonant normal forms and the Diophantine phenomena of perturbations of commuting resonant vector fields X_k on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$,

$$X_k = < \omega^k, \partial_x > = \sum_{j=1}^{n} \omega_j^k \partial_{x_j}, \qquad 1 \leq k \leq d, \tag{1.1}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{T}^n$, $\omega^k = (\omega_1^k, \ldots, \omega_n^k) \in \mathbb{R}^n$ and $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$, $\partial_{x_j} = \partial/\partial x_j$. If $n = 2$ we also allow the X_k to be nonconstant vector fields.

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We perturb the X_k with zero'th order classical pseudodifferential operators so that the operators

$$b^k(x,D) := -iX_k + a^k(x,D), \quad 1 \le k \le d, \tag{1.2}$$

are commuting, where $D = (D_{x_1}, \dots , D_{x_n})$, $D_{x_j} = -i\partial_{x_j}$, $j = 1, \dots , n$ and $a^k(x,D)$ is a zero'th order pseudodifferential operator defined by $a^k(x,D)f = \sum_{\xi\in\mathbb{Z}^n} e^{ix\xi}a^k(x,\xi)\hat{f}(\xi)$, where $\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{T}^n} e^{-ix\xi}f(x)dx$ is the discrete Fourier transformation and $a^k(x,\xi) \in S^0(\mathbb{T}^n \times \mathbb{R}^n)$ is a classical symbol of pseudodifferential operators. As a special case of (1.2), we also consider the following first order commuting differential operators

$$\mathcal{L}_j =< \omega^j, D_x > +a_j(x), \qquad a_j \in C^\infty(\mathbb{T}^n),\ j = 1, \dots , d. \tag{1.3}$$

We will actually consider perturbations with functions in the Gevrey class $G^\sigma(\mathbb{T}^n)$. Recall that $G^\sigma(\mathbb{T}^n)$ is the set of all $f \in C^\infty(\mathbb{T}^n)$ such that there exists $C > 0$ satisfying $\max_{x\in\mathbb{T}^n} |\partial^\beta f(x)| \le C^{|\beta|+1}(\beta!)^\sigma$, $\beta \in \mathbb{Z}^n$. Clearly $G^1(\mathbb{T}^n)$ coincides with the set of all analytic functions on $\mathbb{T}^n$.

The study of normal forms is a classical subject, originating with Poincaré's celebrated works on the reduction of differential equations in the neighbourhood of an equilibrium point. It is still one of the most useful tools in Dynamical Systems and PDE. Recall that for a single vector field (respectively, map) with an isolated singular (respectively, fixed) point the formal reduction to its linear part requires a nonresonance condition on the eigenvalues of the linear part, while in order to show the convergence of formal transformations, arithmetic conditions are imposed (see the surveys [25], [37] and the references therein). If commuting vector fields or maps are considered it is not necessary for each vector field or map to satisfy the aforementioned conditions. More precisely, simultaneous arithmetic conditions are required (see [10], [13], [22], [32], [36], [37]). Certain exceptional sets will arise in connection with some of the parameters in these simultaneous arithmetic conditions (cf. [18]) and these will be discussed later.

The present paper deals with four closely related problems. First, after introducing the concept of simultaneous (non)resonance for $\omega^1, \dots \omega^d$, we are interested in the simultaneous resonant normal forms of $\{b^k(x,D)\}_1^d$ and/or $\{\mathcal{L}_k\}_1^d$ via conjugation with an elliptic p.d.o. (which in the case of differential operators reduces to a change of variables on $\mathbb{T}^n$) in the framework of the C^∞ and the Gevrey spaces. Roughly speaking, resonant normal forms mean that lower order terms depend only on resonant variables. For this purpose the geometrical properties of a simultaneously resonant set associated with $\omega := \{\omega^k\}_1^d$ and defined by

$$\Gamma_{\mathbb{Z}}^\omega \quad := \quad \{z \in \mathbb{Z}^n; < \omega^1, z >= \cdots < \omega^d, z >= 0\}. \tag{1.4}$$

will first be considered. We say that $\omega^1, \dots , \omega^d$ are simultaneously nonresonant (respectively resonant) if $\Gamma_{\mathbb{Z}}^\omega = \{0\}$ (respectively, $\Gamma_{\mathbb{Z}}^\omega \ne \{0\}$). Using theorems on finitely generated free abelian groups and exact sequences the existence of suitable bases for $\Gamma_{\mathbb{Z}}^\omega$ will be obtained and this leads naturally to the introduction of resonant variables on the torus. These constructions are invariant under the action of the group of linear automorphisms of the lattice $\mathbb{Z}^n$. A simultaneous resonant normal form for every perturbed system under simultaneous Diophantine conditions for $\omega^1, \dots , \omega^k$ on the set $\mathbb{Z}^n \setminus \Gamma_{\mathbb{Z}}^\omega$

in the class C^∞ or Gevrey G^σ will be exhibited. In particular, if $\Gamma^\omega_{\mathbb{Z}} = \{0\}$ it is possible to transform $\{b^k\}_1^d$ (respectively, $\{\mathcal{L}_k\}_1^d$) simultaneously into a constant p.d.o. (respectively, differential operator). It will also be shown that in general the Diophantine condition is necessary to transform the perturbed systems to their normal forms. We then consider the simultaneous reduction of nonconstant vector fields to constant ones for the case $n = 2$. There are very few results regarding the normal forms of vector fields and differential operators on $\mathbb{T}^n$ except for those vector fields on $\mathbb{T}^2$ of the form $\partial_{x_1} + \lambda(x)\partial_{x_2}$, where the rotation number of the Poincaré map plays an essential role (cf. [2]). For the case $n \geq 3$, the only results known to us are for vector fields close to constant Diophantine vector fields, where however the conjugation is not applied to the original vector field, but to a small perturbation of it cf. [1], [31] (see also the discussion in [37]). There is also a recent result of Chen Wenyi and M.Y. Chi [11] for the reduction of a smooth vector field $\mathcal{L}$ on $\mathbb{T}^n$ to a constant one provided that the adjoint operator $\mathcal{L}^*$ is globally C^∞ hypoelliptic. When the rotation numbers satisfy simultaneous Diophantine conditions a simultaneous reduction to constant vector fields will be shown.

The second part of this paper is related to the regularity of conjugating p.d.o's. if the lower order perturbations are Gevrey G^σ p.d.o's . In fact, a loss of Gevrey regularity due to the simultaneous Diophantine conditions is encountered. Although we are not able to prove that our Gevrey estimates are best possible there is some evidence to suggest this from the sharp loss of Sobolev regularity in each inductive step. For other problems related to the optimal Gevrey index in some KAM type results see [12]. This phenomenon resembles a similar one in the effective stability (Nekhoroshev estimates) of Birkhoff normal forms (cf. [23], [3]; see also [21] for Nekhoroshev estimates for billiard ball maps in $\mathbb{R}^n$, $n \geq 3$, by means of Gevrey techniques).

The third part of this work regards the global hypoellipticity of commuting systems of pseudodifferential operators. Taking advantage of the resonant normal forms, we can make a complete analysis in various cases. Indeed, if $\Gamma^\omega_{\mathbb{Z}} = \{0\}$ we will show that some inhomogeneous Diophantine conditions completely characterize the global properties in C^∞ and in G^θ for large values of θ by virtue of Nekhoroshev type estimates. The case $\Gamma^\omega_{\mathbb{Z}} \neq \{0\}$ is more difficult. A natural extension of completely resonant systems is introduced and it will be shown that the discrete condition is necessary and sufficient for the global hypoellipticity of the perturbations of such systems. We will also consider systems where no discrete phenomena appear.

The final question considered in this paper is that of the "size" of the exceptional sets of ω associated with G^σ Siegel conditions, for which perturbations of the vector fields X_k are not reducible to normal forms. The inhomogeneous Diophantine conditions also play an essential role when studying the global properties of the reduced operators on the torus. Here the problem is twofold: first, in order to obtain information about the exceptional sets in the Gevrey category, the standard Hausdorff dimension is insufficient, hence we need logarithmic Hausdorff dimension; secondly, the inhomogeneity of the Diophantine conditions causes some difficulties.

The paper is organized as follows: section 2 deals with the geometry of the resonant set. Section 3 is devoted to the simultaneous normal forms and the Nekhoroshev type estimates are shown in section 4. The global hypoellipticty and solvability is discussed in section 5. Finally, estimates for the exceptional sets using a generalized concept of Hausdorff dimension are proved in section 6.

2. Simultaneous resonance

We will now investigate the resonant set $\Gamma^{\omega}_{\mathbb{Z}}$. Denote by $SL^{\pm}(n;\mathbb{Z})$ the set of all $n\times n$ matrices P with integral entries and determinant ± 1. In other words, $SL(n;\mathbb{Z}) = SL^{+}(n;\mathbb{Z})\bigcup SL^{-}(n;\mathbb{Z})$ is the group of linear automorphisms of the lattice $\mathbb{Z}^n$. Observe that if $P\in SL(n;\mathbb{Z})$ and $y = {}^tPx$ then $<\omega^j,\partial_x>\to<P\omega^j,\partial_y>$, which implies the invariance property

$$P(\Gamma^{\omega}_{\mathbb{Z}}) = \Gamma^{P\omega}_{\mathbb{Z}}, \qquad P\omega = \{P\omega^1,\dots,P\omega^k\}. \tag{2.1}$$

Also note that if the $\mathbb{R}$–span of $\tilde{\omega}^1,\dots,\tilde{\omega}^\ell\in\mathbb{R}^n$ coincides with the $\mathbb{R}$–span of $\omega^1,\dots,\omega^k\in\ R^n$, then $\Gamma^{\omega}_{\mathbb{Z}} = \Gamma^{\tilde{\omega}}_{\mathbb{Z}}$.

Theorem 2.1. *Let $\Gamma^{\omega}_{\mathbb{Z}}\neq\{0\}$. Then*

i) there exist $\kappa^1,\dots,\kappa^r\in\mathbb{Z}^n\setminus\{0\}$ linearly independent over $\mathbb{R}$ such that $\{\kappa^\nu\}_{\nu=1}^r$ form a basis in $\Gamma^{\omega}_{\mathbb{Z}}$ i.e.

$$\Gamma^{\omega}_{\mathbb{Z}} = \{z\in\mathbb{Z}^n;\, z = \sum_{\nu=1}^{r} t_j\kappa^j,\ \textit{with } t_j\in\mathbb{Z}\textit{ uniquely determined}\}.$$

The number $r = r(\omega) = \dim_{\mathbb{Z}}\Gamma^{\omega}_{\mathbb{Z}}$ is invariant under the action of $SL(n;\mathbb{Z})$ i.e. $r(\omega) = r(P\omega)$ for every $P\in SL(n;\mathbb{Z})$. We can always find κ^j (after changes of indices $x_\mu\to x_\nu$) to satisfy

$$\kappa^j = (\kappa^j_1,\kappa^j_2,\dots,\kappa^j_n), \qquad \kappa^j_j\in\mathbb{N},\ \kappa^j_\nu = 0, 1\le\nu\le j-1. \tag{2.2}$$

Every other basis $\{\tilde{\kappa}^\nu\}_{\nu=1}^r$ is represented in a unique way in the form $(\tilde{\kappa}^1,\cdots,\tilde{\kappa}^r) = (\kappa^1,\cdots,\kappa^r)Q$ for some $Q\in SL^{\pm}(r;\mathbb{Z})$.

ii) Let $\kappa^1,\dots,\kappa^r$ be a basis of $\Gamma^{\omega}_{\mathbb{Z}}$. Then there exists $P\in SL^{+}(n;\mathbb{Z})$ such that its first r columns coincide with $\kappa^1,\dots,\kappa^r$, i.e. we extend the basis of $\Gamma^{\omega}_{\mathbb{Z}}$ into a basis of $\mathbb{Z}^n$. In particular, if $y = {}^{tr}Px$, $y = (y', y'')$, $y' = (y_1,\dots,y_r)$, $y'' = (y_{r+1},\dots,y_n)$, then

$$P\omega^k = (0',\theta^k), \qquad \theta^k\in\mathbb{R}^{n-r}, 1\le k\le d \tag{2.3}$$

with $\Gamma^{\theta}_{\mathbb{Z}} = \{0\}$, i.e. $\theta^1,\dots,\theta^k$ are nonresonant in $\mathbb{R}^{n-r}$. In the y variables the resonant set $\Gamma^{P\omega}_{\mathbb{Z}}$ coincides with $\mathbb{Z}^r_{y'}$. The set of all matrices P from $SL(n;\mathbb{Z})$ with such a property is isomorphic to $\mathcal{M}_{n-r\times r}(\mathbb{Z})\oplus SL(n-r;\mathbb{Z})$, where $\mathcal{M}_{n-r\times r}(\mathbb{Z})$ stands for the group of $n-r\times r$ matrices with integral entries.

In order to prove the theorem we recall the following well known assertion (cf. [30] for more general statements on sequences of maps for finitely generated abelian groups)

Lemma 2.2. *Let $A \subset B := \mathbb{Z}^n$ be a subgroup. Set $C = A^{\perp}_{\mathbb{Z}} = \{\xi \in \mathbb{Z}^n : \xi \perp A\}$. Then*

$$B = A \oplus C \tag{2.4}$$

holds if and only if the sequence

$$0 \to A \overset{id}{\to} B \overset{\pi}{\to} C \to 0 \tag{2.5}$$

is exact (i.e. $Ker\pi = id(A)$) where π (respectively, id) stands for the orthogonal projection on C (respectively, the identity map).

Proof. The existence of a basis follows from the well known fact that any subgroup A (in our case $A = \Gamma^{\omega}_{\mathbb{Z}}$) of a finitely generated abelian group B (in our case $B = \mathbb{Z}^n$) is also a finitely generated abelian group cf. [30]. Define $\pi_j : \mathbb{R}^n \to \mathbb{R}^n$ by $\pi_j(\xi) =< \xi, \vec{e_j} > \vec{e_j} = \xi_j \vec{e_j}$, where $\vec{e_j}$ is the j-th unit vector. Without loss of generality we may assume that for each $j \in \{1, \dots, n\}$ we have $\pi_j \omega^k \neq 0$ for some k, otherwise we are reduced to a space of dimension $n-1$. Because $\Gamma^{\omega}_{\mathbb{Z}}$ is a linear $\mathbb{Z}$–submodule of $\mathbb{Z}^n$, $\pi_1 \Gamma^{\omega}_{\mathbb{Z}}$ is closed under summation, and therefore there exists $p_1 \in \mathbb{N}$ such that $\pi_1(\Gamma^{\omega}_{\mathbb{Z}}) = p_1 \mathbb{Z} \vec{e_1}$. Let $\kappa^1 \in \Gamma^{\omega}_{\mathbb{Z}}$ be such that $\pi_1 \kappa^1 = p_1 \vec{e_1}$. Let $\Lambda_2 \subset \Gamma^{\omega}_{\mathbb{Z}}$ be the $\mathbb{Z}$–submodule such that $\pi_1 \Lambda_2 = \{0\}$. In fact, Λ_2 is uniquely determined: $\Lambda_2 = \{(\xi_1, \dots, \xi_n) \in \Gamma^{\omega}_{\mathbb{Z}} : \xi_1 = 0\}$. If $\Lambda_2 = \{0\}$ the set $\Gamma^{\omega}_{\mathbb{Z}}$ is generated by κ^1 and $r = 1$. If $\Lambda_2 \neq \{0\}$, a similar argument as above shows that $\pi_2 \Lambda_2 = p_2 \mathbb{Z} \vec{e_2}$ for some $p_2 \in \mathbb{N}$. Let $\kappa^2 \in \Lambda_2$ be a vector such that $\pi_2 \kappa^2 = p_2 \vec{e_2}$. Clearly $\kappa^2 = (0, p_2, \dots)$. Let $\Lambda_3 \subset \Lambda_2$ be the uniquely determined $\mathbb{Z}$–submodule such that $\pi_2 \Lambda_3 = \{0\}$. If $\Lambda_3 = \{0\}$ we see that Λ_2 is generated by κ^2 and $r = 2$. Therefore $\Gamma^{\omega}_{\mathbb{Z}}$ is generated by κ^1 and κ^2. If $\Lambda_3 \neq \{0\}$ this procedure is continued and sequences $\{\kappa^k\} \subset \Gamma^{\omega}_{\mathbb{Z}}$, $\{p_k\} \subset \mathbb{N}$ are constructed together with a nested family of $\mathbb{Z}$–submodules $\{\Lambda_k\}$, $\Lambda_k \subset \Lambda_{k-1}$ such that $\pi_k \Lambda_k = p_k \mathbb{Z} \vec{e_k}$, $\pi_k \kappa^k = p_k \vec{e_k}$. Clearly, this procedure ends after a finite number of steps. Let $r \leq n-1$ be the smallest integer such that $\Lambda_{r+1} = \{0\}$, then the vectors κ^j form a basis of $\Gamma^{\omega}_{\mathbb{Z}}$ and satisfy (2.2).

Let now $\{\tilde{\kappa}^j\}_1^r \subset \Gamma^{\omega}_{\mathbb{Z}}$ be another basis of $\Gamma^{\omega}_{\mathbb{Z}}$. Then there exist two $r \times r$ matrices Q and $\tilde{Q}$ with integral entries such that $(\kappa^1 \cdots \kappa^r)Q = (\tilde{\kappa}^1 \cdots \tilde{\kappa}^r)$ and $(\tilde{\kappa}^1 \cdots \tilde{\kappa}^r)\tilde{Q} = (\kappa^1 \cdots \kappa^r)$. Hence $\tilde{Q} = Q^{-1}$ and $\det Q \det \tilde{Q} = 1$. Because $\det Q$ and $\det \tilde{Q}$ are integers it follows that $\det Q = \det \tilde{Q} = \pm 1$, i.e. $Q, \tilde{Q} \in SL^{\pm}(r; \mathbb{Z})$. Conversely, for any $Q \in SL^{\pm}(r; \mathbb{Z})$ $(\kappa^1 \cdots \kappa^r)Q$ gives a basis of $\Gamma^{\omega}_{\mathbb{Z}}$. This proves (i).

We will prove (ii). Set $A = \Gamma^{\omega}_{\mathbb{Z}}$ and $C = A^{\perp} = \{\xi \in \mathbb{Z}^n : \xi \perp A\}$. Evidently $A \subset C^{\perp}_{\mathbb{Z}}$. If we show the exactness of (2.5), namely

$$\rho \in \mathbb{Z}^n, \, \rho \perp C \Rightarrow \rho \in A \tag{2.6}$$

then Lemma 2.2 yields the first assertion of ii) since by (2.4) we can always extend a basis in A with a basis in C and obtain a basis in $\mathbb{Z}^n$.

Let $\rho \in \mathbb{Z}^n$, $\rho \perp C$. Set $C^{\perp}_{\mathbb{R}} = \{\xi \in \mathbb{R}^n ; \xi \perp C\}$. Fix a basis as in (2.2). Since $\kappa^j \perp C$ for $j = 1, \dots, r$, and $\dim_{\mathbb{R}} C^{\perp}_{\mathbb{R}} = r$ we obtain that $\kappa^1, \dots, \kappa^r$ is a basis in the vector subspace $C^{\perp}_{\mathbb{R}}$ and ρ is a linear combination of $\kappa^1, \dots, \kappa^r$ implying that $< \omega^k, \rho >= 0$ for all $1 \leq k \leq d$. Since $\rho \in \mathbb{Z}^n$, this implies $\rho \in A$.

Next, note that $\kappa^1, \dots \kappa^r$ become the first r unit vectors in $\mathbb{R}^n_y$. Thus the invariance of the resonant set implies that $\Gamma^{P\omega}_{\mathbb{Z}} = \{\zeta \in \mathbb{Z}^n : \zeta_{r+1} = \dots = \zeta_n = 0\}$ and (2.3) is therefore true. If $\theta^1, \dots \theta^{n-r}$ are resonant in $\mathbb{R}^{n-r}_{y''}$ there will exist a vector $\zeta^0 \in \mathbb{Z}^{n-d} \setminus \{0\}$ orthogonal to $\theta^1, \dots, \theta^k$, so that $(0', \zeta^0) \in \Gamma^{P\omega}_{\mathbb{Z}}$, which is impossible by $\zeta^0 \neq 0''$.

The proof is concluded by observing that $P \in SL(n;\mathbb{Z})$ preserves the basis $\vec{e_1}, \dots \vec{e_r}$ of $\Gamma^{P\omega}_{\mathbb{Z}} = \mathbb{Z}^r \times 0$ iff $P = \begin{pmatrix} I_r & 0_{n-r\times r} \\ M & P'' \end{pmatrix}$ with $P'' \in SL(n-r;\mathbb{Z})$ and $M \in \mathcal{M}_{n-r\times r}(\mathbb{Z})$. $\square$

We call the basis $\kappa^1, \dots, \kappa^r$ a canonical basis of $\Gamma^{\omega}_{\mathbb{Z}}$, and r a ($\mathbb{Z}-$)dimension of $\Gamma^{\omega}_{\mathbb{Z}}$ over $\mathbb{Z}$ and write $\dim_{\mathbb{Z}} \Gamma^{\omega}_{\mathbb{Z}} = r$. We stress that $\mathbb{Z}^n \setminus \Gamma^{\omega}_{\mathbb{Z}}$ is not a linear $\mathbb{Z}$ submodule of $\mathbb{Z}^n$.

Remark 2.3. *Take $A = \{(2t, -2t);\, t \in \mathbb{Z}\}$, $C = A^{\perp}_{\mathbb{Z}} = \{(t,t);\, t \in \mathbb{Z}\}$. Then although A, $B = \mathbb{Z}^2$ and C are finitely generated abelian groups, the decomposition (2.4) is not true (the sequence (2.5) is not exact because $C^{\perp}_{\mathbb{Z}} \neq A$).*

We say that vectors $\omega^k \in \mathbb{R}^n$, $(1 \le k \le d)$ satisfy the simultaneous resonant σ-Siegel condition if, for every $\varepsilon > 0$

$$\liminf_{|\xi|\to\infty, \xi \in \mathbb{Z}^n \setminus \Gamma^{\omega}_{\mathbb{Z}}} \left(e^{\varepsilon |\xi|^{1/\sigma}} \max_{1\le k\le d} | < \omega^k, \xi > | \right) > 0, \qquad (Si)_\sigma$$

for $1 \le \sigma < \infty$, and

$$\liminf_{|\xi|\to\infty, \xi \in \mathbb{Z}^n \setminus \Gamma^{\omega}_{\mathbb{Z}}} \left(|\xi|^{\tau} \max_{1\le k\le d} | < \omega^k, \xi > | \right) > 0, \qquad (Si)_\infty$$

when $\sigma = \infty$ for some $\tau \in \mathbb{R}$.

Remark 2.4. *Observe that $(Si)_\sigma$ and $(Si)_\infty$ are invariant under the action of $SL(n;\mathbb{Z})$. In particular, extending to a basis of $\mathbb{Z}^n$ and in view of (2.3), we get that $(Si)_\sigma$ (respectively $(Si)_\infty$) becomes*

$$\liminf_{|\xi''|\to\infty, \xi'' \in \mathbb{Z}^{n-r}} \left(e^{\varepsilon |\xi''|^{1/\sigma}} \max_{1\le k\le d} | < \theta^k, \xi'' > | \right) > 0, \qquad \forall \varepsilon > 0 \qquad (Si)'_\sigma$$

(respectively,

$$\liminf_{|\xi''|\to\infty, \xi'' \in \mathbb{Z}^{n-r}} \left(|\xi''|^{\tau} \max_{1\le k\le d} | < \theta^k, \xi'' > | \right) > 0, \qquad (Si)'_\infty)$$

for the nonresonant vectors $\theta^1, \dots, \theta^k$ in $\mathbb{R}^{n-k}_{y''}$).

We point out that if $d = 1$ and $\omega^1 \in \mathbb{R}^n$ is nonresonant, i.e. $\Gamma^{\omega}_{\mathbb{Z}} = \{0\}$, then $(Si)_\infty$ (respectively, $(Si)_\sigma$, $1 \le \sigma < \infty$) is the well known small divisor condition for ω^1 and the number τ in $(Si)_\infty$ is necessarily not less than $n-1$ (cf. [35],[31] and [37]) (respectively, coincides with the Gevrey type Siegel condition used in [20]. See also [34] and [5] for $\sigma = 1$). Furthermore, if $n \ge 3$ and $d = n-1$, $(Si)_\infty$ in the nonresonance case is the same as the simultaneous Diophantine condition used in [6].

Let $C^\infty_{2\pi}(\mathbb{R}^n_x)$ be the space of all 2π periodic C^∞ functions (i.e. the lift on $\mathbb{R}^n$ of all functions from $C^\infty(\mathbb{T}^n)$). It is readily verified that this space is invariant under the action of $SL(n;\mathbb{Z})$, namely

(2.7)
$$f(y) \in C^\infty_{2\pi}(\mathbb{R}^n_y) \Leftrightarrow P^*f(x) := f(Px) \in C^\infty_{2\pi}(\mathbb{R}^n_x) \quad \text{for some} \quad P \in SL(n;\mathbb{Z})$$

We will identify $C^\infty(\mathbb{T}^n)$ and $G^\sigma(\mathbb{T}^n)$ with $C^\infty_{2\pi}(\mathbb{R}^n)$ and $G^\sigma_{2\pi}(\mathbb{T}^n)$, respectively.

3. Global canonical form of a resonant system

First we recall the basic properties of G^σ pseudodifferential operators. (cf. [7], [33], [15]). In the C^∞ case, the constants in the estimates below may depend on the multiindices $\alpha, \beta \in \mathbb{Z}^n_+$ and the index of homogeneity j. (cf. [26]). Let $FS^m_\sigma = FS^m_\sigma(\mathbb{T}^n \times \mathbb{R}^n)$ be the set of all formal sums $\sum_{j=0}^\infty a_{m-j}(x,\xi)$ such that $a_{m-j} \in G^\sigma(\mathbb{T}^n \times (\mathbb{R}^n \setminus \{0\}))$, $ord_\xi a_{m-j} = m-j$ (i.e a_{m-j} is (positively) homogeneous of order $m-j$) and that there exists $A > 0$ satisfying

$$\sup_{x\in\mathbb{T}^n} |\partial_x^\beta \partial_\xi^\gamma a_{m-j}(x,\xi)| \le A^{j+|\gamma|+|\beta|+1}(\beta! j!)^\sigma \gamma! |\xi|^{m-j-|\gamma|} \tag{3.1}$$

for all $\beta \in \mathbb{Z}^n_+$, $\gamma \in \mathbb{Z}^n_+$, $j \in \mathbb{Z}_+$, $\xi \in \mathbb{R}^n$. By Cauchy's integral formula and the homogeneity of a_{m-j} (3.1) is equivalent to the existence of a neighbourhood $\tilde{S}$ in $\mathbb{C}^n$ of S^{n-1} such that

$$\sup_{x\in\mathbb{T}^n, \xi\in\tilde{S}} |\partial_x^\beta a_{m-j}(x,\xi)| \le A^{j+|\beta|+1}(\beta! j!)^\sigma, \quad \forall \beta \in \mathbb{Z}^n_+, j \in \mathbb{Z}_+. \tag{3.2}$$

A function $a(x,\xi) \in G^\sigma(\mathbb{T}^n \times \mathbb{R}^n)$ is said to be a G^σ realization of $\sum_{j=0}^\infty a_{m-j}(x,\xi) \in FS^m_\sigma$; we write $a(x,\xi) \sim \sum_{j=0}^\infty a_{m-j}(x,\xi)$ if for every $\gamma \in \mathbb{Z}^n_+$ one can find $C = C_\gamma > 0$ such that

$$|\partial_x^\beta \partial_\xi^\gamma (a(x,\xi) - \sum_{j=0}^N a_{m-j}(x,\xi))| \le C^{N+|\gamma|+|\beta|+1}(\beta! N!)^\sigma |\xi|^{m-N-|\gamma|-1} \tag{3.3}$$

for all $\beta \in \mathbb{Z}^n_+$, $N \in \mathbb{Z}_+$, $x \in \mathbb{T}^n$, $\xi \in \mathbb{R}^n$, $|\xi| \ge 1$. We denote the set of G^σ realizations of order m by $S^m_\sigma(\mathbb{T}^n \times \mathbb{R}^n)$ and the set of pseudofifferential operators with symbol in $S^m_\sigma(\mathbb{T}^n \times \mathbb{R}^n)$ by $OS^m_\sigma(\mathbb{T}^n)$. We also recall that for each $\tilde{a} \in FS^m_\sigma$ there exists a realization $a \in S^m_\sigma$ such that $a \sim \tilde{a}$. If $b(x,\xi)$ is another G^σ realization of $\tilde{a}$, then $R(x,\xi) = a(x,\xi) - b(x,\xi) \in S^{-\infty}_\sigma(\mathbb{T}^n \times \mathbb{R}^n)$ i.e. R is a G^σ smoothing symbol, namely for some $c > 0$ the next estimate holds

$$\sup_{x\in\mathbb{T}^n} |\partial_x^\beta R(x,\xi))| \le c^{-|\beta|-1}(\beta!)^\sigma e^{-c|\xi|^{1/\sigma}}, \qquad \xi \in \mathbb{R}^n, \beta \in \mathbb{Z}^n_+. \tag{3.4}$$

We note that the composition rule for p.d.o. on $\mathbb{T}^n$ is valid as in the local case, namely if $a(x,\xi) \sim \sum_{j=0}^\infty a_{\mu-j} \in S^\mu_\sigma(\mathbb{T}^n \times \mathbb{R}^n)$ and $b(x,\xi) \sim \sum_{j=0}^\infty b_{\nu-j}(x,\xi) \in S^\nu_\sigma(\mathbb{T}^n \times \mathbb{R}^n)$ then $c(x,D) := a(x,D) \circ b(x,D) \in OS^{\nu+\mu}_\sigma(\mathbb{T}^n)$ with symbol $c(x,\xi) = a(x,\xi) \circ b(x,\xi) \sim \sum_{j=0}^\infty c_{\mu+\nu-j}(x,\xi)$ given by (cf. [15])

$$c_{\mu+\nu-j}(x,\xi) = \sum_{p+q+|\beta|=j} \frac{1}{\beta!} D_\xi^\beta a_{\mu-p}(x,\xi) \partial_x^\beta b_{\nu-q}(x,\xi), \qquad j \in \mathbb{Z}_+. \tag{3.5}$$

Furthermore, if $a(x,\xi)$ is elliptic, i.e. $a_\mu(x,\xi) \neq 0$ for $(x,\xi) \in \mathbb{T}^n \times (\mathbb{R}^n \setminus 0)$, then $a(x,D)$ admits a (formal) inverse $a^{-1}(x,D) \in OS^{-\mu}$ satisfying $a \circ a^{-1} - id \in OS_\sigma^{-\infty}$ and $a^{-1} \circ a - id \in OS_\sigma^{-\infty}$. Recall that the theory of symplectic maps (cf. [26]) shows that for a given $P \in SL(n;\mathbb{Z})$, the change of variables $y = Px$ induces the linear symplectic change $(x,\xi) \to (y,\eta)$, $x = P^{-1}y$ and $\xi = {}^{tr}P\eta$. We write

$$(P^*a)(y,\eta) = a(P^{-1}y, {}^{tr}P\eta), \qquad y \in \mathbb{T}^n,\, \eta \in \mathbb{R}^n. \tag{3.6}$$

Thus the composition is invariant, i.e. for two p.d.o.-s $a = a(x,D)$ and $b = b(x,D)$ we have

$$P^*(a \circ b) = P^*a \circ P^*b. \tag{3.7}$$

Moreover, if a is a p.d.o. having a symbol depending in x only on resonant variables, the same is true for P^*a.

Now we state the first main result on simultaneous normal forms for $\{b^k(x,D)\}_1^d$.

Theorem 3.1. *Let $1 \le \sigma \le \infty$ and suppose that $(Si)_\sigma$ holds for ω. Fix a resonant (canonical) basis $\{\kappa^j\}_1^r$ if $r = dim_{\mathbb{Z}}\Gamma_{\mathbb{Z}}^\omega \ge 1$. If $\sigma < \infty$ we also require that $a_{-j}^k(x,\xi)$ is $C^\infty(\mathbb{R}^n \setminus \{0\}; G^\sigma(\mathbb{T}_x^n))$, $k = 1,\dots,d$, $j \in \mathbb{Z}_+$. Then there exists an elliptic symbol $q(x,\xi) \sim \sum_{j=0}^\infty q_{-j}(x,\xi) \in FS^0(\mathbb{T}^n \times \mathbb{R}^n)$ and zero order symbols $\alpha^k(y,\xi) \sim \sum_{j=0}^\infty \alpha_{-j}^k(y,\xi) \in FS^0(\mathbb{T}^r \times \mathbb{R}^n)$ such that for $1 \le k \le d$*

$$(-iX_k + a^k(x;D)) \circ q(x;D) = q(x;D) \circ (-iX_k + \alpha^k(\kappa x; D) + R_k(x,D)), \tag{3.8}$$

where $R_k(x,\xi) \in S^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n)$ and $\kappa x := (<\kappa^1, x>, \dots, <\kappa^r, x>)$. If $\Gamma_{\mathbb{Z}}^\omega = \{0\}$, the symbols $q_{-j}(x,\xi)$, $j \in \mathbb{Z}_+$ are uniquely determined by the requirements $[q_0(\cdot,\xi)]_n = 1$, $[q_{-j}(\cdot,\xi)]_n = 0$, for $j \ge 1$, where $[a]_n := (2\pi)^{-n}\int_{\mathbb{T}^n} a(t)dt$ is the mean value of a.

Proof. Let $P \in SL(n;\mathbb{Z})$ be as in Theorem 2.1, with the first r rows being $\kappa^1,\dots,\kappa^n$, and define $y = Px$. In view of (3.7), we will consider in the new variables (y,η), with $\kappa^j = \vec{e_j}$. Indeed, in the original variables $x = P^{-1}y$ we observe that $y_j =< \kappa^j, x >$, $1 \le j \le r$ if $r \ge 1$. The resonance set is the lattice $\mathbb{Z}_{y'}^r$ in $\mathbb{R}_{y'}^r$, where $y = (y', y''), y' = (y_1,\dots,y_r)$. For brevity $a(y,\eta)$ is used instead of $P^*a(y,\eta) = a(P^{-1}y, {}^{tr}P\eta)$ for a symbol $a(x,\xi)$. We write terms of the same order of homogeneity in both sides of (3.8) in the new variables y. The first system of d equations involves both $q_0(y,\eta)$ and $\alpha_0^k(y',\eta)$ $(1 \le k \le d)$

$$< \theta^k, D_{y''} > q_0 + a_0^k(y,\eta)q_0 = \alpha_0^k(y',\eta)q_0, \quad \eta \in \mathbb{R}^n \setminus 0. \tag{3.9}$$

The advantage of the expression (3.9) is that we can regard the resonant variables y' as parameters if $r \ge 1$. We can find a nonzero solution of (3.9) if we choose α_0^k so as to satisfy

$$\int_{\mathbb{T}^{n-r}} a_0^k(y', y'', \eta)dy'' = \alpha_0^k(y',\eta), \qquad 1 \le k \le d,\, y' \in \mathbb{T}^r,\, \eta \in \mathbb{R}^n \setminus \{0\} \tag{3.10}$$

i.e. the Fourier coefficients $\mathcal{F}_{y\to\zeta}(a^k(y,\eta)-\alpha_0^k(y',\eta))$ vanish on the resonant set, which in the canonical variables is defined by $\zeta_{r+1}=\cdots\zeta_n=0$. By the commutativity we have $<\theta^j,D_{y''}>a_0^k(y,\eta)=<\theta^k,D_{y''}>a_0^j$ for $1\le j,k\le d$. Because $<\theta^j,D_{y''}>\alpha_0^k\equiv 0$, it follows from the Frobenius theorem and simple calculations that the general solution of (3.9) is given by

(3.11)
$$q_0(y,\eta)=\ell(y',\eta)e^{i\psi(y,\eta)},\quad \ell(y',\eta)\in G^\sigma(\mathbb{T}^r\times(\mathbb{R}^n\setminus 0)),\qquad ord_\eta\ell=0,$$

where $\psi(y,\eta)$ is the unique solution of the following system of equations with Fourier coefficients vanishing on the resonant set $\Gamma_{\mathbb{Z}}^{P\omega}$

$$<\theta^k,\partial_{y''}>\psi=f_0^k(y,\eta):=-a_0^k(y,\eta)+\alpha_0^k(y',\eta),\qquad 1\le k\le d. \tag{3.12}$$

In fact, ψ is defined by

$$\mathcal{F}_{y''\to\zeta''}(\psi(y',y'',\eta))=\frac{\mathcal{F}_{y''\to\zeta''}(f_0^k(y',y'',\eta))}{i<\theta^k,\zeta''>} \tag{3.13}$$

if $<\theta^k,\zeta''>\neq 0$ for some $1\le k\le d$, and $=0$, if otherwise. In view of $(Si)_\sigma$ and results in [20], $\psi(y,\eta)$ (and therefore $q_0(y,\eta)$) is G^σ in x. The composition rule for p.d.o. yields

(3.14)
$$q_0^{-1}(y,\eta)\circ b^k(y,\eta)\circ q_0(y,\eta)=<\theta^k,\eta''>+\alpha_0^k(y',\eta)+(O(|\eta|^{-1})),\ |\eta|\to\infty$$

for all $1\le k\le d$, where b^k stands for $P^*(iX_k+a^k)$. We proceed by induction; namely we suppose that there exist $\alpha_{-j}^k(y',\eta)\in C^\infty(\mathbb{R}^n\setminus\{0\};G^\sigma(\mathbb{T}_{y'}^r))$, $ord_\eta\alpha_{-j}^k(y',\eta)=-j$, $0\le j\le N-1$, $1\le k\le d$, such that

$$a^k(y,\eta)\sim\sum_{j=0}^{N-1}\alpha_{-j}^k(y,\eta)+\sum_{j=N}^{\infty}a_{-j}^k(y',\eta),\qquad 1\le k\le d. \tag{3.15}$$

Now we look for an elliptic symbol $q(y,\eta)\sim 1+q_{-N}(y,\eta)$, $ord_\eta q_{-N}=-N$, and $\alpha_{-N}^k(y',\eta)\in C^\infty(\mathbb{R}^n\setminus\{0\};G^\sigma(\mathbb{T}^r))$ such that

(3.16)
$$q^{-1}(y,\eta)\circ b^k(y,\eta)\circ q(y,\eta)=<\theta^k,\eta''>+\sum_{j=0}^{N}\alpha_{-j}^k(y',\eta)+O(|\eta|^{-N-1}),$$

when $|\eta|\to\infty$ for all $1\le k\le d$. We obtain the system of equations, $k=1,\dots,d$ for $q_{-N}(y,\eta)$ and $\alpha_{-N}^k(y',\eta)$

$$-i<\theta^k,\partial_{y''}>q_{-N}(y,\eta)=f_{-N}^k(y,\eta):=\alpha_{-N}^k(y',\eta)-a_{-N}^k(y,\eta). \tag{3.17}$$

In view of (3.10) and (3.12), we define α_{-N}^k by

$$\mathcal{F}_{y'\to\zeta'}(\alpha_{-N}^k(y',\eta))=\mathcal{F}_{y\to\zeta}(a_{-N}^k(y',y'',\eta))|_{\zeta=(\zeta',0)}. \tag{3.18}$$

Bt the Frobenius theorem, (3.18) and $(Si)_\sigma$, (3.17) admits a unique solution $q_{-N}(y,\xi)$, if $\mathcal{F}_{y'\to\zeta'}(q_{-N}(y',y'',\eta))=0$ for $\zeta'\in\mathbb{Z}^r$. Moreover, in view of $(Si)_\sigma$, $q_{-N}(y,\xi)$ is $C^\infty(\mathbb{R}^n\setminus\{0\};G^\sigma(\mathbb{T}^n))$.

Finally we construct an ellpitic p.d.o. q by

$$q(y,\eta)=\ldots\circ(1+q_{-N}(y,\eta))\circ\ldots\circ(1+q_{-1}(y,\eta))\circ q_0(y,\eta). \tag{3.19}$$

The right-hand side in (3.19) defines a formal symbol in $FS^0(\mathbb{T}^n \times (\mathbb{R}^n \setminus 0))$. Taking any realization of this formal symbol, the above calculations yield (3.8) and the estimates on the reminders. $\square$

The next assertion shows that $(Si)_\sigma$ is optimal.

Proposition 3.2. *Let $\omega \in \mathbb{R}^n$ be nonresonant and assume that ω does not satisfy $(Si)_\sigma$ for some $1 \leq \sigma \leq \infty$. Then we can find $f(x) \in G^\sigma(\mathbb{T}^n)$ such that the pseudo–differential operator with symbol $< \omega, \xi > + f(x)\xi|\xi|^{-1}$ cannot be transformed via conjugation (3.8) into a constant p.d.o.*

Proof. In view of the assumption on ω there exists a positive constant $\delta > 0$ and a sequence $\rho^k \in \mathbb{Z}^n \setminus 0$ $(k = 1, 2, \dots)$ such that

$$\lim_{k\to\infty} e^{-2\delta|\rho^k|^{1/\sigma}} | < \omega, \rho^k > |^{-1} = \infty. \tag{3.20}$$

We choose a function $f(x) = \sum_\eta \hat{f}(\eta)e^{i\eta x} \in G^\sigma(\mathbb{T}^n)$ such that $\hat{f}(\eta) = e^{-\delta|\rho^k|^{1/\sigma}}$ if $\eta = \rho^k$, $\hat{f}(\eta) = 0$ for $\eta \neq \rho^k$. The solution $q_0(x, \xi)$ of the equation $< \omega, D_x > q_0 = f(x)\xi/|\xi|$ satisfies $|\mathcal{F}_{x\to\eta}(q_0(x, \xi))| \geq Ce^{\delta|\eta|^{1/\sigma}}$ for $\eta = \rho^k$, $k \in \mathbb{N}$, which contradicts the fact that $q_0 \in G^\sigma$ in x. $\square$

Remark 3.3. *In Theorem 3.1 we consider perturbations of constant vector fields X_k. If X_k is a variable coefficient vector field, the problem is essentially a global nonlinear one. There follows two examples of a system of overdetermined vector fields being simultaneously transformed into constant vector fields. First, any nonsingular hypoelliptic real vector field on $\mathbb{T}^2$ is diffeomorphic to a nonzero multiple of $\partial_t + \rho\partial_x$ with ρ being an irrational non Liouville number (cf.* [27]*).*

The second example is an overdetermined system of vector fields, $L = d_t + \omega(t) \wedge \partial_x$, $x \in \mathbb{T}^1$, $t = (t_1, \dots, t_n) \in \mathbb{T}^n$, $\omega(t) = \sum_{j=1}^n \omega_j(t)dt_j$ being a real-valued smooth closed one form on $M \equiv \mathbb{T}^n$. The corresponding family of n commuting vector fields associated with L (cf. [6]*) is given by $L_j = \partial_{t_j} + \omega_j(t)\partial_x$, $1 \leq j \leq n$. Straightforward calculations show the existence of a real-valued function $h(t)$ on $\mathbb{T}^n$ such that $\partial_{t_j}h(t) = \omega_j(t) - c_j$, with c_j being the mean value $[\omega_j(\cdot)]_n$ of ω_j, $1 \leq j \leq n$. Then the family $\{L_j\}_1^n$ is transformed into the one $\{\partial_{s_j} + c_j\partial_y\}_1^n$ via the diffeomorphism of $\mathbb{T}^{n+1}$: $y = x - h(t)$, $s = t$.*

We now study the simultaneous reduction of a family of commuting vector fields with variable coefficients on $\mathbb{T}^2$

$$X_j = \partial_t + h_j(t, x)\partial_x, \qquad j = 1, \dots, d. \tag{3.21}$$

The commutativity relations are given by

$$\partial_t(h_k - h_j) = h_k\partial_x h_j - h_j\partial_x h_k, \qquad j, k = 1, \dots, d. \tag{3.22}$$

Let $\psi_j^t : \mathbb{T}^1 \to \mathbb{T}^1$ be the time map of X^j, defined by $\psi_j^t(\xi) = x_j(t; \xi)$, where $x_j(t; \xi)$ is a solution of

$$\dot{x_j} = h_j(t, x_j), \quad x_j(0) = \xi. \tag{3.23}$$

Clearly (3.22) is equivalent to $\psi_j^t \circ \psi_k^s = \psi_k^s \circ \psi_j^t$, $s, t \in \mathbb{R}$, $j, k = 1, \dots, d$.

We recall that the Poincaré map of X_j, given by $P_j := \psi_j^{2\pi} : \mathbb{T}^1 \to \mathbb{T}^1$, is a diffeomorphism of $\mathbb{T}^1$ and we assume the orientation preserving property of P_j. The rotation number of P_j is denoted by $\rho_j \in \mathbb{R}$ (ρ_j is also called the rotation number of the vector field X_j). Moreover, we suppose that there exists a smooth diffeomorphism u simultaneously conjugating the maps $P_1, \dots, P_d$ to the rotations $R_{\rho_1}, \dots, R_{\rho_d}$ where $R_{\rho_j}(z) = z + \rho_j$, $z \in \mathbb{T}$ ($j = 1, \dots, d$), so that

$$u^{-1} \circ P_j \circ u = R_{\rho_j}, \qquad j = 1, \dots, d. \tag{3.24}$$

Theorem 3.4. *Let $1 \le \sigma \le +\infty$. If $d \ge 2$ we assume that the Poincaré maps $P_1, \dots, P_d$ are orientation preserving and that there exists an index $j \in \{1, \dots, d\}$ such that $(2\pi)^{-1}\rho_j$ is irrational. Then if we can find a G^σ diffeomorphism u on $\mathbb{T}^1$ satisfying (3.24) the map Φ: $t = s$, $x = \phi(s, y)$, where $\phi(s,y) = x_1(s, u(y - s(2\pi)^{-1}\rho_1))$ if $d = 1$ and $\phi(s,y) = x_j(s, u(y - s(2\pi)^{-1}\rho_j)$ if $d \ge 2$, is a G^σ diffeomorphism of $\mathbb{T}^2$ transforming $\partial_t + h_k(t,x)\partial_x$ to $\partial_s + (2\pi)^{-1}\rho_k\partial_y$ for all $1 \le k \le d$.*

Proof. We want to reduce the vector fields X_j to $\partial_t + \rho_j\partial_x$ by the change of variables on $\mathbb{T}^2$, $(t,x) = \Phi(s,y)$ with $t = s$ and $x = \phi(s,y)$ simultaneously. By simple computations it can be seen that $\partial_s + (2\pi)^{-1}\rho_j\partial_y = \partial_t + (\phi_s + \rho_j\phi_y)\partial_x$. Hence it is necessary to solve the equation

$$\phi_s + \rho_j\phi_y = h_j(s, \phi(s,y)), \qquad j = 1, \dots, d. \tag{3.25}$$

By the definition of $x_j(t;\xi)$ in (3.23), we obtain for $\xi = u(y - s(2\pi)^{-1}\rho_j)$, $t = s$

$$\begin{aligned} \partial_s\phi + (2\pi)^{-1}\rho_j\partial_y\phi &= \dot{x_j}(t,\xi) - (2\pi)^{-1}\rho_j\partial_\xi x_j(t;\xi)u'(y - s(2\pi)^{-1}\rho_j) \\ &+ (2\pi)^{-1}\rho_j\partial_\xi x_j(t;\xi)u'(y - s(2\pi)^{-1}\rho_j) = \dot{x_j}(t;\xi) \\ &= h_j(t, x_j(t;\xi)) = h_j(s, \phi(s,y)). \end{aligned} \tag{3.26}$$

Hence ϕ transforms $\partial_t + h_j(t,x)\partial_x$ into $\partial_s + (2\pi)^{-1}\rho_j\partial_y$. If $d \ge 2$ for $i \ne j$, $\partial_t + h_i(t,x)\partial_x$ is transformed to $\partial_s + g_i(s,y)\partial_y$ for some $g_i \in G^\sigma(\mathbb{T}^2;\mathbb{R})$. The commutativity implies that

$$(\partial_s + (2\pi)^{-1}\rho_j\partial_y)g_i(s,y) = 0, \qquad i = 1, \dots, d, \ i \ne j. \tag{3.27}$$

Since $(2\pi)^{-1}\rho_j$ is irrational it is readily verified from the Fourier expansion that the g_i are constants. Because the rotation number is invariantly defined for vector fields on $\mathbb{T}^2$, we have $g_i(s,y) \equiv (2\pi)^{-1}\rho_i$.

In order to prove that the map Φ is a G^σ diffeomorphism on $\mathbb{T}^2$ we note that the Jacobian of Φ does not vanish, because it is equal to $\partial_y\phi$. For $\xi = u(y - s(2\pi)^{-1}\rho_j)$ we have $\partial_y\phi(s,y) = \partial_\xi x_j(t;\xi)u'(y - s(2\pi)^{-1}\rho_j)$. Since u is a G^σ diffeomorphism we have $u'(y) \ne 0$ for every $y \in \mathbb{R}$. On the other hand, $\theta(t;z) := \partial_z x_j(t;z)$ satisfies

$$\dot{\theta} = \partial_x h_j(t, x_j(t;z))\theta, \qquad \theta|_{t=0} = 1 \tag{3.28}$$

which implies that $\theta(t;\xi) = \partial_\xi x_j(t;\xi) \ne 0$ for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}$. Thus $\partial_y\phi(s,y) \ne 0$ for all $s \in \mathbb{R}$, $y \in \mathbb{R}$ and the map Φ is a G^σ diffeomorphism. In order to complete the proof, we will show that $\phi(s + 2\pi, y) = \phi(s,y)$ and $\phi(s, y + 2\pi) = \phi(s,y)$. The latter relation is easy. To see the former one, we recall that $x_j(2\pi, u(y)) = u(y + \rho_j)$ by the definition of u. The function $x_j(t + 2\pi, u(y - \rho_j - s\rho_j/2\pi))$ is a solution of (3.23) with initial value $\xi = x_j(2\pi, u(y - \rho_j - s\rho_j/2\pi)) = u(y - s\rho_j/2\pi)$. By uniqueness it is therefore equal to $x_j(t, u(y - s\rho_j/2\pi))$. ∎

Remark 3.5. *a) Let $d = 1$ and suppose either that $\sigma = 1$ and $\rho/(2\pi) = \rho_1/(2\pi)$ satisfies the Bruno condition* [9] *or that $\sigma = \infty$ and $\rho/(2\pi)$ satisfies the usual Siegel condition. Then by the global reduction theorems due to M. Herman and A. Bruno, the circle map $P = P_1$ is conjugated to a rotation by a G^1 or C^∞ diffeomorphism.*
b) Let $d = 1$ and $1 < \sigma < +\infty$. We are not aware of any result on global reduction in G^σ Gevrey classes.
c) Let $d \geq 2$. If $\sigma = +\infty$ or $\sigma = 1$, there is a local simultaneous reduction theorem due to J. Moser [32]*, under a simultaneous Siegel condition. A global C^∞ simultaneous reduction theorem is shown by Y. Katznelson, B. Kra and D. Ornstein* [28].
d) If $d \geq 1$ and $1 \leq \sigma < +\infty$, T. Gramchev and M. Yoshino [22] *proved local simultaneous reductions of commuting G^σ circle maps under weak exponential arithmetic conditions.*

4. Nekhoroshev type estimates for G^σ p.d.o.

The next result shows that perturbations with G^σ p.d.o. are conjugated with G^θ elliptic p.d.o. for some $\theta > \sigma$ if simultaneous small divisor conditions in the nonresonant case are satisfied. This shows sharp contrast to the case of commuting diffeomorphisms, where even weaker arithmetic conditions are sufficient.

Theorem 4.1. *Let $\Gamma^\omega_{\mathbb{Z}} = \{0\}$ and let $(Si)_\infty$ in §2 hold for some $\tau \leq -1$. Let $\theta = \max\{1, 1+\tau\}$. Asssume that $a^k(x,D)$ $(1 \leq k \leq d)$ is a G^σ p.d.o. Then there exist symbols $q_{-j}(x,\xi)$ of the conjugating p.d.o. with the following properties: there exists a neighbourhood $\tilde{S} \subset \mathbb{C}^n$ of the unit sphere S^{n-1} such that for every $s > n/2$, $\varepsilon > 0$ we can find $M > 0$, $N > 0$ satisfying*

$$\sup_{\xi \in \tilde{S}} \|\partial_x^\gamma q_{-j}(.,\xi)\|_{H^s} \leq \varepsilon M^j N^{|\gamma|}((\theta j + |\gamma|)!)^\sigma, \quad \gamma \in \mathbb{Z}^n_+,\ j \in \mathbb{N}, \tag{4.1}$$

$$\sup_{\xi \in \tilde{S}} |\alpha^k_{-j}(\xi)| \leq \varepsilon M^j((\theta j)!)^\sigma, \qquad k = 1, \dots, d,\ j \in \mathbb{N}, \tag{4.2}$$

where $\|\cdot\|_{H^s}$ denotes the usual Sobolev norm. It follows that $\sum_{j=0}^\infty q_{-j}(x,\xi) \in FS^0_{\theta\sigma}$ and $\sum_{j=0}^\infty \alpha_{-j}(\xi) \in FS^0(\mathbb{R}^n \setminus \{0\})$. Set $\theta' = \theta$ if $\sigma > 1$ or $\tau > 0$, and $\theta' = 1 + \rho$, $0 < \rho \ll 1$ if $\theta = 1$ (i.e. $\sigma = 1$ and $\tau \leq 0$). Then there exist $G^{\theta\sigma}$ realizations $q(x,\xi)$, $\alpha^k(\xi)$, $1 \leq k \leq d$ and positive constants C, δ such that

$$\sup_{x \in \mathbb{T}^n} |\partial_x^\gamma \partial_\xi^\beta R_k(x,\xi)| \leq C_2^{|\beta|+|\gamma|}(\gamma!)^\sigma(\beta!)^{\sigma\theta'}|\xi|^{-|\gamma|}\exp(-\delta|\xi|^{1/(\sigma\theta)}), \tag{4.3}$$

for all $\beta, \gamma \in \mathbb{Z}^n_+$, $\xi \in \mathbb{R}^n$ and $1 \leq k \leq d$. Thus $R_k(x,D) \in OS^{-\infty}_{\sigma\theta}(\mathbb{T}^n)$, $1 \leq k \leq d$. Finally, if the symbols $a_j(x,\xi)$ depend on one base variable, the system can be transformed into a simultaneous normal form without any restrictions on τ in the Diophantine condition for $\omega^1, \dots, \omega^d$. In this case there is no loss of regularity in the remainders; more precisely if $a_k(x,D)$, $1 \leq k \leq d$ are G^σ p.d.o. then $R_k(x,D)$ are G^σ smoothing p.d.o.

First we need a technical assertion, where $C_0^\infty(\mathbb{T}^n)$ denotes the set of smooth functions with mean value zero.

Lemma 4.2. *Suppose that $(Si)_\infty$ holds. Then for every $f_k(x) \in C_0^\infty(\mathbb{T}^n)$ such that $X_k f_r = X_r f_k$, $1 \le r, k \le d$ there exists a unique solution $q(x) \in C_0^\infty(\mathbb{T}^n)$ of the system $< \omega^k, \partial_x > q(x) = f_k(x)$, $1 \le k \le d$, satisfying*

$$\hat{u}(\xi) = 0, \qquad \xi \in \Gamma_{\mathbb{Z}}^\omega. \tag{4.4}$$

Futhermore, for some $C_\tau > 0$ this unique solution satisfies the estimate

$$\|q\|_{H^s} \le C_\tau \max_{1\le k\le d} \|f_k\|_{H^{s+\tau}}, \qquad s \ge 0. \tag{4.5}$$

Proof. By Fourier transform we obtain $< \omega^k, \eta > \hat{q}(\eta) = \widehat{f}_k(\eta)$. It follows that $\widehat{f}_k(\eta) = 0$ when $< \omega^k, \eta >= 0$. The compatibility condition and the definition of $\Gamma_{\mathbb{Z}}^\omega$ implies that we can decompose

$$\mathbb{Z}^n \setminus \Gamma_{\mathbb{Z}}^\omega = \bigcup_{m=1}^d Z'(m), \qquad Z'(m) \bigcap Z'(j) = \emptyset,\ 1 \le j < m \le d \tag{4.6}$$

with the property $< \omega^k, \eta > \neq 0$ for $\eta \in Z'(k)$, $1 \le k \le d$. Hence for $1 \le k \le d$ the inverse Fourier transform gives,

$$\|q\|_{H^s} \le (2\pi)^{-n/2} \sum_{k=1}^d \left(\sum_{\eta \in Z'(k)} \frac{|\eta|^{2s}}{| < \omega^k, \eta > |^2} |\widehat{f}_k(\eta)|^2 \right)^{1/2}$$

$$\le \ C_\tau \max_{1\le k\le d} \left(\sum_{\eta \in \mathbb{Z}^n \setminus 0} < \eta >^{2(s+\tau)} |\widehat{f}_k(\eta)|^2 \right)^{1/2} = C_\tau \max_{1\le k\le d} \|f_k\|_{H^{s+\tau}},$$

where $< \eta >= (1 + |\eta|^2)^{1/2}$. ☐

Proof of Theorem 4.1. We will suppose that τ is an integer (the proof is the same for $\tau \notin \mathbb{Z}$, but there are some technical complications). Clearly, after conjugation with the symbol $e^{i\psi(x,\xi)}$, we may suppose that $a_0^k(x, \xi) = \alpha_0(\xi)$ for $1 \le k \le d$ and hence that $\psi = 0$, $q_0 = 1$ and the conjugating p.d.o. is of the form $q(x, \xi) \sim 1 + \sum_{j=1}^\infty q_{-j}(x, \xi)$. Then, by the composition rule, q_{-j} satisfies the following equations

$$< \omega^k, D_x > q_{-j} = F_{-j}^k(x, \xi) + G_{-j}^k(x, \xi) - a_{-j}^k(x, \xi) + \alpha_{-j}^k(\xi), \quad 1 \le k \le d, \tag{4.7}$$

where

$$F_{-j}^k(x, \xi) = -\sum_{r=1}^{j-1} \sum_{|\beta| \le j-r} \frac{1}{\beta!} D_\xi^\beta a_{-(j-r-|\beta|)}^k(x, \xi) \partial_x^\beta q_{-r}(x, \xi) \tag{4.8}$$

and

$$G_{-j}^k(x, \xi) = \sum_{r=1}^{j-1} q_{-r}(x, \xi) \alpha_{-(j-r)}^k(\xi). \tag{4.9}$$

Clearly in order to solve (4.7) we must choose

$$\alpha_{-j}^k(\xi) = -[F_{-j}^k(\cdot, \xi)]_n - [G_{-j}^k(\cdot, \xi)]_n + [a_{-j}^k(\cdot, \xi)]_n, \qquad 1 \le k \le d. \tag{4.10}$$

Because $a^k(x,\xi) \in S^0_\sigma(\mathbb{T}^n)$, $1 \le k \le d$ there exists a complex neighbourhood $\tilde{S}$ in $\mathbb{C}^n$ of the unit sphere S^{n-1} in $\mathbb{R}^n$ so that for fixed $s > n/2$ and τ there exists a positive constant A such that

$$\sup_{\xi\in\tilde{S}} \|\partial_x^\beta a^k_{-j}(\cdot,\xi)\|_{H^{s+\tau}} \le A^{j+|\beta|+1}(j!\beta!)^\sigma, \qquad j \in \mathbb{Z}_+,\ \beta \in \mathbb{Z}^n_+. \tag{4.11}$$

By the Schauder lemma for multiplication in Sobolev spaces, if $s+\tau > n/2$ there exists $\omega > 0$ such that

$$\|fg\|_{H^{s+\tau}} \le \omega \|f\|_{H^{s+\tau}} \|g\|_{H^{s+\tau}}, \qquad f,g \in H_0^{s+\tau}(\mathbb{T}^n). \tag{4.12}$$

Estimates (4.1) and (4.2) will be proved by induction. The estimates (4.1) and (4.2) hold for $j = 0$. Suppose that (4.1) and (4.2) are true for $0 \le j \le i-1$. By (4.12) we obtain for $\xi \in \tilde{S}$ that

$$\begin{aligned} \|\partial_x^\gamma F^k_{-i}(\cdot,\xi)\|_{H^{s+\tau}} &\le \omega \sum_{r=1}^{i-1} \sum_{|\beta|\le i-r} \sum_{\gamma'\le\gamma} \binom{\gamma}{\gamma'} \frac{1}{\beta!} \\ \times\ & \|\partial_x^{\gamma-\gamma'} D_\xi^\beta a^k_{-(i-r-|\beta|)}(\cdot,\xi)\|_{H^{s+\tau}} \|\partial_x^{\beta+\gamma'} q_{-r}(\cdot,\xi)\|_{H^{s+\tau}} \\ \le\ & \varepsilon\omega A \sum_{r=1}^{i-1} \sum_{|\beta|\le i-r} \sum_{\gamma'\le\gamma} \binom{\gamma}{\gamma'} A^{i-r+|\gamma|-|\gamma'|}((i-r-|\beta|)!(\gamma-\gamma')!)^\sigma M^r \\ \times\ & N^{|\beta|+\tau+|\gamma'|}((\theta r+|\beta|+|\gamma'|+\tau)!)^\sigma \le \varepsilon M^i N^{|\gamma|}((\theta i+|\gamma|)!)^\sigma \omega A N^\tau K \end{aligned} \tag{4.13}$$

with

$$\begin{aligned} K &= \sum_{r=1}^{i-1} (AN/M)^{i-r} \sum_{|\beta|\le i-r} \sum_{\gamma'\le\gamma} \binom{\gamma}{\gamma'} \\ \times\ & (A/N)^{|\gamma|-|\gamma'|} \left(\frac{(i-r-|\beta|)!(\gamma-\gamma')!(\theta r+|\beta|+\tau+|\gamma'|)!}{(\theta i+|\gamma|)!} \right)^\sigma \\ \le\ & \sum_{r=1}^{i-1} (AN/M)^{i-r} \sum_{|\beta|\le i-r} \sum_{\gamma'\le\gamma} \frac{(|\gamma'|+1)\dots(|\gamma'|+|\gamma|-|\gamma'|)}{(\gamma-\gamma')!} \\ \times\ & (A/N)^{|\gamma|-|\gamma'|} \left(\frac{(\gamma-\gamma')!}{(|\gamma'|+1)\dots(|\gamma'|+|\gamma|-|\gamma'|)} \right)^\sigma \\ \times\ & \left(\frac{(i-r-|\beta|)!(|\gamma'|+1)\dots(|\gamma'|+|\gamma|-|\gamma'|)(\theta r+|\beta|+|\gamma'|+\tau)!}{(\theta i+|\gamma|)!} \right)^\sigma \\ \le\ & \sum_{r=1}^{i-1} (AN/M)^{i-r} \sum_{|\beta|\le i-r} \sum_{\gamma'\le\gamma} (A/N)^{|\gamma|-|\gamma'|} C_1^{\sigma-1} C_2^\sigma \end{aligned} \tag{4.14}$$

where

$$C_1 = \frac{(\gamma-\gamma')!}{(|\gamma'|+1)\dots(|\gamma'|+|\gamma|-|\gamma'|)} \le 1 \tag{4.15}$$

(4.16)

$$\begin{aligned} C_2 &= \frac{(i-r-|\beta|)!(|\gamma'|+1)\dots(|\gamma'|+|\gamma|-|\gamma'|)(\theta r+|\beta|+|\gamma'|+\tau)!}{(\theta i+|\gamma|)!} \\ &\le \frac{(i-r-|\beta|)!(\theta r+|\beta|+|\gamma|+\tau)!}{(\theta i+|\gamma|)!} \le \frac{(\theta i+|\gamma|-(\theta-1)(i-r)+\tau)!}{(\theta i+|\gamma|)!} \le 1 \end{aligned}$$

for all $\gamma,\gamma',\beta\in\mathbb{Z}^n_+$, $\gamma'\le\gamma$, $r,i\in\mathbb{N}$, $i-r\ge 1$. In (4.16) the inequality $(\theta-1)(i-r)\ge\theta-1\ge\tau$ if $i-r\ge 1$ was used. Putting (4.15) and (4.16) into (4.14), and choosing first $N>A$ and then $M>2AN$ large enough so that $\overline{\delta}:=AN/M$ satisfies the inequality $\overline{\delta}(1-A/N)^{-n}\sum_{\nu=0}^\infty 2^{-\nu}(\nu+2)^n\le(6C_\tau\omega AN^\tau)^{-1}$, we get

$$
\begin{aligned}
(4.17)\qquad K &\le \sum_{r=1}^{i-1}(AN/M)^{i-r}\sum_{|\beta|\le i-r}\sum_{\gamma'\le\gamma}(A/N)^{|\gamma|-|\gamma'|}\\
&\le \sum_{r=1}^{i-1}(AN/M)^{i-r}\sum_{|\beta|\le i-r}(1-A/N)^{-n}\\
&\le \frac{AN}{M}(1-A/N)^{-n}\sum_{r=1}^{i-1}(AN/M)^{i-r-1}(i-r+1)^n\\
&\le \overline{\delta}(1-A/N)^{-n}\sum_{\nu=0}^{\infty}2^{-\nu}(\nu+2)^n\le(6C_\tau\omega AN^\tau)^{-1}.
\end{aligned}
$$

Hence, by the choice of M and N, we obtain

$$(4.18)\qquad \sup_{\xi\in\tilde S}\|\partial_x^\gamma F^k_{-i}(\cdot,\xi)\|_{H^{s+\tau}}\le(3C_\tau)^{-1}\varepsilon M^iN^{|\gamma|}((\theta i+|\gamma|)!)^\sigma,\quad \gamma\in\mathbb{Z}^n_+.$$

Next, for any $\xi\in\tilde S$

$$
\begin{aligned}
(4.19)\qquad \|\partial_x^\gamma G^k_{-i}(\cdot,\xi)\|_{H^{s+\tau}} &\le \sum_{r=1}^{i-1}\|\partial^\gamma q_{-r}(\cdot,\xi)\|_{H^{s+\tau}}|\alpha^k_{-(i-r)}(\xi)|\\
(4.20)\qquad &\le \varepsilon^2M^iN^{|\gamma|+\tau}((\theta i+|\gamma|)!)^\sigma L
\end{aligned}
$$

where

$$
\begin{aligned}
(4.21)\qquad L&=\sum_{r=1}^{i-1}\left(\frac{(\theta r+|\gamma|+\tau)!(\theta(i-r))!}{(\theta i+|\gamma|)!}\right)^\sigma\\
&=\left(\frac{(\tau+1)!}{\theta i+|\gamma|}\right)^\sigma\sum_{r=1}^{i-1}\left(\frac{(\tau+2)\dots(\tau+1+\theta r+|\gamma|-1)}{(\theta(i-r)+1)\dots(\theta(i-r)+\theta r+|\gamma|-1)}\right)^\sigma.
\end{aligned}
$$

Since $i-r\ge 1$, $r\ge 1$ and $\theta\ge(1+\tau)$ we have

$$
\begin{aligned}
\tilde c &:= \frac{(\tau+2)\dots(\tau+1+\theta r+|\gamma|-1)}{(\theta(i-r)+1)\dots(\theta(i-r)+\theta r+|\gamma|-1)}\\
(4.22)\qquad &\le \frac{\theta\dots(\theta+\theta r+|\gamma|-1)}{(\theta(i-r)+1)\dots(\theta(i-r)+\theta r+|\gamma|-1)}\le 1.
\end{aligned}
$$

It is readily verified from (4.21) that, for all $\gamma\in\mathbb{Z}^n_+$

$$
\begin{aligned}
(4.23)\qquad L &\le \left(\frac{(\tau+1)!}{\theta i+|\gamma|}\right)^\sigma\sum_{r=1}^{i-1}1\\
&= \left(\frac{(\tau+1)!}{\theta i+|\gamma|}\right)^\sigma(i-1)\le\frac{((\tau+1)!)^\sigma}{(\theta i+|\gamma|)^{\sigma-1}}\le((\tau+1)!)^\sigma.
\end{aligned}
$$

Hence, by choosing $\varepsilon > 0$ and $N > 0$ so that $\varepsilon N^{\tau}((\tau+1)!)^{\sigma} \leq 1/3$ we obtain, from (4.1) and (4.2) with $1 \leq j \leq i-1$

$$\sup_{\xi\in\tilde{S}} \|\partial_x^{\gamma} G^k_{-i}(\cdot,\xi)\|_{H^{s+\tau}} \leq 3^{-1}\varepsilon M^i N^{|\gamma|}((\theta i + |\gamma|)!)^{\sigma}, \qquad \gamma \in \mathbb{Z}^n_+. \tag{4.24}$$

Finally, observe that (4.11) leads to

$$\sup_{\xi\in\tilde{S}} \|\partial_x^{\gamma} a^k_{-j}(\cdot,\xi)\|_{H^{s+\tau}} \leq 3^{-1}\varepsilon M^j N^{|\gamma|}((\theta j + |\gamma|)!)^{\sigma}, \qquad j \in \mathbb{N},\ \gamma \in \mathbb{Z}^n_+. \tag{4.25}$$

provided that $M \geq 3A/\varepsilon$ and $N \geq A$. By Lemma 4.2, (4.10), (4.24) and (4.25) we have, for any $\xi \in \tilde{S}$

$$\begin{aligned}
|\alpha^k_{-i}(\xi)| &\leq |[F^k_{-i}(\cdot,\xi)]_n| + |[G^k_{-i}(\cdot,\xi)]_n| + |[a^k_{-i}(\cdot,\xi)]_n| \\
&\leq \|F^k_{-i}(\cdot,\xi)\|_{L^2} + \|G^k_{-i}(\cdot,\xi)\|_{L^2} + \|a^k_{-i}(\cdot,\xi)\|_{L^2} \\
&\leq \|F^k_{-i}(\cdot,\xi)\|_{H^{s+\tau}} + \|G^k_{-i}(\cdot,\xi)\|_{H^{s+\tau}} + \|a^k_{-i}(\cdot,\xi)\|_{H^{s+\tau}} \\
&\leq \varepsilon(C_\tau)^{-1}M^i((\theta i)!)^{\sigma} \leq \varepsilon M^i((\theta i)!)^{\sigma}.
\end{aligned} \tag{4.26}$$

Thus (4.1) and (4.2) for $1 \leq j \leq i-1$ imply (4.2) for $j = i$, while (4.1) for $j = i$ follows from Lemma 4.2 applied to

$$f_k = F^k_{-i}(x,\xi) + G^k_{-i}(x,\xi) - a^k_{-i}(x,\xi) + \alpha^k_{-i}(\xi), \qquad 1 \leq k \leq d, \tag{4.27}$$

and the fact that the $H^{s+\tau}(\mathbb{T}^n)$ norm of f_k given by (4.27), does not depend on α^k_{-i}. □

5. Global properties of perturbations of resonant vector fields

Let $\mathcal{L}_j$ be defined by (1.3) with $a_j \in G^{\sigma}(\mathbb{T}^n)$. We say that $\mathcal{L}_j$ is G^{σ} globally hypoelliptic ($1 \leq \sigma \leq \infty$) if every distribution u such that $\mathcal{L}_j u \in G^{\sigma}$ ($j = 1,\dots,d$) is in G^{σ}. Set $r = \dim_{\mathbb{Z}} \Gamma^{\omega}_{\mathbb{Z}}$ and for $r \geq 1$, take a canonical basis $\eta^1,\dots,\eta^r$ of $\Gamma^{\omega}_{\mathbb{Z}}$. We suppose that it is possible to find $\alpha_j(y) \in G^{\sigma}(\mathbb{T}^r)$ ($j = 1,\dots,d$) for which the system

$$< \omega^j, \partial_x > \psi = a_j(x) - \alpha_j(< \eta^1, x >,\dots,< \eta^r, x >), \qquad 1 \leq j \leq d \tag{5.1}$$

admits a unique solution ψ with $\hat{\psi}(\xi) = 0$ for $\xi \in \Gamma^{\omega}_{\mathbb{Z}}$. Note that (5.1) is weaker than the Siegel type conditions in $\mathbb{Z}^n \setminus \Gamma^{\omega}_{\mathbb{Z}}$.

If the multiplication operator q is defined by $qv(x) = e^{i\psi(x)}v(x)$, the proof of Theorem 3.1 and (5.1) imply that

$$q^{-1} \circ \mathcal{L}_j \circ q = L_j := < \omega^j, D_x > + \alpha_j(< \eta^1, x >,\dots,< \eta^r, x >) \quad 1 \leq j \leq d. \tag{5.2}$$

Therefore the system $\mathcal{L}_j u = f_j$, ($1 \leq j \leq d$) is equivalent to the system

$$L_j v = g_j, \qquad 1 \leq j \leq d \tag{5.3}$$

where $g_j = e^{i\psi(x)}f_j(x)$, $1 \leq j \leq d$.

We define the change of variables $x = T\tilde{x}$, $\tilde{x} = (y,z)$ with $y = (y_1,\dots,y_r)$ and $z = (z_1,\dots,z_{n-r})$ by $y_j =< \eta^j, x >$, ($1 \leq j \leq r$), and $z_j =< \kappa^j, x >$, ($1 \leq j \leq n-r$). Then the system (5.3) becomes

$$< \theta^j, D_z > w + \alpha_j(y)w = h_j(y,z), \qquad j = 1,\dots,d, \tag{5.4}$$

where $\theta^j \in \mathbb{R}^{n-r}$ are given by $(0, \theta^j) = {}^{tr}T\omega^j$, $1 \le j \le d$. The partial Fourier transform $z \to \zeta$ reduces (5.4) to

$$(<\theta^j, \zeta> + \alpha_j(y))\hat{w}(y, \zeta) = \hat{h_j}(y, \zeta), \qquad \zeta \in \mathbb{Z}^{n-r},\ j = 1, \dots, d. \tag{5.5}$$

Define $\Phi(y, \zeta) = \sum_{j=1}^d | <\theta^j, \zeta> + \alpha_j(y)|$ and let $d_0 \le n - r$ be the rank of $\theta^1, \dots, \theta^d$. Then we have

Theorem 5.1. *Suppose $\Gamma^\omega_{\mathbb{Z}} \ne \{0\}$ and let $\eta^1, \dots, \eta^r$ be a canonical basis of $\Gamma^\omega_{\mathbb{Z}}$. Assume that (5.1) has a unique solution and that $a^k_{-j}(x, \xi) \in C^\infty(\mathbb{R}^n_\xi \setminus \{0\}; G^\sigma(\mathbb{T}^n_x))$ if $\sigma < \infty$. Then*
(i) The system (5.4) is G^σ globally hypoelliptic if there exists $c > 0$ such that $\Phi(y, \zeta) \ge c$ for all $\zeta \in \mathbb{Z}^{n-r}$ and $y \in \mathbb{T}^r$. If $d_0 = n - r$ the preceeding condition is replaced by a discrete condition $\Phi(y, \zeta) \ne 0$ for all $\zeta \in \mathbb{Z}^{n-r}$ and all $y \in \mathbb{T}^n$. We note that the completely resonant case $r = n - 1$ satisfies $d_0 = n - r$.
(ii) Suppose $d_0 = n - r$. Then the system (5.4) is G^σ globally hypoelliptic for $1 < \sigma \le \infty$ if and only if the discrete condition is satisfied.

Proof. By the representaion (5.5), every solution w satisfies $\hat{w}(y; \zeta) = \hat{h_j}(y; \zeta)(<\theta^j, \zeta> + \alpha_j(y))^{-1}$ for some $1 \le j \le d$. Hence w is in G^σ. If $d_0 = n - r$, we note that $\sum_{j=1}^d | <\theta^j, \zeta> | \to \infty$ as $|\zeta| \to \infty$. Hence the condition $\Phi \ge c$ can be replaced by the discrete condition. Thus (i) is proved.

As to (ii) we define the distribution $w(y, z)$ by $\hat{w}(y, \zeta) \equiv 0$ if $\zeta \ne \zeta^0$, and $\hat{w}(y, \zeta^0) = \delta(y - y^0)$, with $y^0 \in \mathbb{T}^r$ and $\zeta^0 \in \mathbb{Z}^{n-r}$ satisfying $\zeta^0 + \alpha_k(y^0) = 0$ for $1 \le k \le d$. Here $\delta(y - y^0)$ stands for the Dirac measure massed at $y = y^0$. The distribution $w(y, z)$ gives a nonsmooth zero solution of (5.4). ☐

Remark 5.2. *(a) If $d_0 < n - r$ the condition $\Phi(y, \zeta) \ge c$ is not a discrete condition because the vectors $(<\theta^1, \zeta>, \dots, <\theta^d, \zeta>)$ $(\zeta \in \mathbb{Z}^{n-r})$ fill a dense subset of a d_0-dimensional subspace of $\mathbb{R}^d$. We note that if $d_0 < d$ this actually occurs.*

(b) If the simultaneous G^σ Siegel condition holds, the solvability of (5.1) is superfluous; that is instead of the Diophantine conditions on ω^j, the solvability of (5.1) for special $a_j(x)$ is enough. This allows us to characterize larger classes of first order operators on $\mathbb{T}^2$ with nonresonant $\omega = \omega^1$ than those in Proposition 3.2 of [4]. *Note that the Gevrey spaces $G^\sigma(\mathbb{T}^n)$ are not metrisable if $1 \le \sigma < \infty$ and therefore Baire's theorem is not applicable.*

Now we study the global properties of $\{b^k(x, D)\}$ in the nonresonant case $\Gamma^\omega_{\mathbb{Z}} = \{0\}$. Let a corresponding normal form $\{\omega^k \cdot \xi + \alpha^k(\xi)\}_1^d$ be as in Theorem 3.1 and $1 \le \theta \le \infty$. We say that $\{\omega^k \xi + \alpha^k(\xi)\}_1^d$ satisfies the simultaneous inhomogeneous θ-Siegel condition if

$$\liminf_{|\xi| \to \infty, \xi \in \mathbb{Z}^n_+ \setminus 0} \exp(\varepsilon |\xi|^{1/\theta}) \max_{1 \le k \le d} |\omega^k \cdot \xi + \alpha^k(\xi)| > 0, \quad \text{for every } \varepsilon > 0 \quad (TSi)_\theta$$

for $1 \le \theta < \infty$, and

$$\liminf_{|\xi| \to \infty, \xi \in \mathbb{Z}^n_+ \setminus 0} |\xi|^N \max_{1 \le k \le d} |\omega^k \cdot \xi + \alpha^k(\xi)| > 0, \qquad (TSi)_\infty$$

for some $N \in \mathbb{R}$ when $\theta = \infty$.

If $\theta = \infty$, $(\mathrm{TSi})_\infty$ is well defined in the sense that it does not depend on realizations of $\sum_{r=0}^{\infty} \alpha_{-r}(\xi)$. However, if $1 \le \theta < \infty$, the condition $(TSi)_\theta$ is independent of realizations if q is a G^θ p.d.o. In this case, $\alpha^j(D)$ and the remainders $R^k(x, D)$ are G^θ p.d.o. as well.

Theorem 5.3. *Assume that ω^j, $j = 1, \dots, d$, are simultaneously nonresonant and satisfy $(Si)_\sigma$ for some $1 \le \sigma \le \infty$. Let the hypotheses of Theorem 3.1 be true. Then $(TSi)_\infty$ holds for one simultaneous canonical form of $\{b_j\}_1^d$ if and only if it holds for every simultaneous canonical form of $\{b^j\}_1^d$. Moreover, $\{b^j\}_1^d$ is globally hypoelliptic on $\mathbb{T}^n$ if and only if a normal form of $\{b^j\}_1^d$ satisfies $(TSi)_\infty$. If $(Si)_\infty$ holds and $a^j(x, D)$, $j = 1, \dots, d$, are G^σ p.d.o. for some $1 \le \sigma < \infty$ then the operator $\{b^j\}_1^d$ is G^θ globally hypoelliptic if and only if a normal form of $\{b^j\}_1^d$ satisfies $(TSi)_\theta$ for $\theta \ge \rho\sigma$, where $\rho = \max\{1, 1 + \tau\}$.*

Proof. The proof follows from Theorem 3.1 (respectively, the Nekhoroshev type estimates) if $\theta = \infty$ (respectively, $\theta \ge \rho\sigma$), and the following properties: Since $q(x, D)$ is elliptic in $G^{\rho\sigma}$ either both systems $\{\omega^j D_x + a^j(x, D)\}$ and $\{\omega^j D_x + \alpha^j(D) + R^j(x, D)\}$ are globally hypoelliptic or neither are globally hypoelliptic. The estimates for $q(x, D)$ and the remainders $R^j(x, \xi)$ imply the global G^θ hypoellipticity for $\theta \ge \rho\sigma$ if $(TSi)_\theta$ is true. $\square$

6. Logarithmic Hausdorff dimension

For $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d) \in R^d$ and $1 \le \sigma \le \infty$ we define

$$NG^\sigma(\boldsymbol{\beta}; d) = \{\omega = (\omega^1, \dots, \omega^d) \in \mathbb{R}^{nd} : \{< \omega^j, \xi > + \beta_j\}_1^d \tag{6.1}$$

$$\text{does not satisfy } (TSi)_\sigma\}, \tag{6.2}$$

$$NG^\sigma(d) = NG^\sigma(0; d). \tag{6.3}$$

Clearly $NG^\sigma(d) \backslash \Gamma_{\mathbb{Z}}^\omega$ is the exceptional set for which $(Si)_\sigma$ fails, and therefore no reduction to normal forms is possible. Similarly, if $\boldsymbol{\beta} \ne 0$, $NG^\sigma(\boldsymbol{\beta}; d)$ is related to the exceptional sets of $\{\omega^j\}_1^d$ for which the system $\{L_k + \beta_k\}_1^d$ is not globally G^σ hypoelliptic. The sizes of $NG^\sigma(d)$ and $NG^\sigma(\beta; 1)$ will be estimated in terms of the Hausdorff dimension and logarithmic Hausdorff dimension associated with the Gevrey index σ.

Define the following set for $n > d$

$$\begin{aligned} E_{\boldsymbol{\beta}}(\sigma) &= E_{\boldsymbol{\beta}}(\sigma; n, d) \\ &= \{X \in \mathbb{R}^{nd} : |\mathbf{q}X - \boldsymbol{\beta}| < e^{-|\mathbf{q}|^{1/\sigma}} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\}. \end{aligned} \tag{6.4}$$

Note that if X is identified with $\omega = (\omega^1, \dots, \omega^d)$, we have

$$E_0(\sigma) \subset NG^\sigma(d) \tag{6.5}$$

$$E_{\boldsymbol{\beta}}(\sigma) \subset NG^\sigma(\boldsymbol{\beta}; d). \tag{6.6}$$

Definition 6.1. *Let J be a set in $\mathbb{R}^{nd}$ and define the function f as $f(r) = r^{(n-1)d}(\log 1/r)^{-s}$. The logarithmic Hausdorff measure of J is*

$$\mathfrak{L}^s(J) := \lim_{\delta \to 0} \inf_{\mathfrak{C}_\delta} \sum_{C \in \mathfrak{C}_\delta : \cup C \supset J} f(L(C)),$$

where the infimum is over all δ covers, $\mathfrak{C}_\delta$ of J and $L(C)$ is the diameter of C. The logarithmic Hausdorff dimension is

$$\mathrm{Ldim}\,(J) := \sup\{s : \mathfrak{L}^s(J) = \infty\} = \inf\{s : \mathfrak{L}^s(J) = 0\}.$$

The definition of Hausdorff dimension and its properties can be found in [19]. To denote the information given by the two dimensions, ordinary and logarithmic the dimension of J will be written as the ordered pair $\vec{\dim} J = (\dim J, \mathrm{Ldim}\ J)$.

For the homogeneous case the following theorem will be proved.

Theorem 6.2. *Let $\sigma \in \mathbb{R}^+$. Then $\vec{\dim} E_0(\sigma; n, d) = ((n-1)d, n\sigma)$.*

By modifying the proof of Theorem 6.2 an inhomogeneous version will also be proved for the case $d = 1$.

Theorem 6.3. *Let $\sigma \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$. Then $\vec{\dim} E_\beta(\sigma; n, 1) = (n-1, n\sigma)$.*

Remark 6.4. *Let us identify $\mathbb{R}^{nd}$ with $(\mathbb{R}^n)^d$. Then we note that: a) the sets $NG^\sigma(\boldsymbol{\beta}; d)$ are invariant under the action of $SL(n;\mathbb{Z})$ ($P(NG^\sigma(\boldsymbol{\beta}; d)) = NG^\sigma(\boldsymbol{\beta}; d)$ for all $P \in SL(n;\mathbb{Z})$). Next, if we introduce an equivalence relation in $NG^\sigma(\boldsymbol{\beta}; d)$ as follows: $\omega \sim \omega'$ if and only if $\omega = P\omega'$ for some $P \in SL(n;\mathbb{Z})$ (i.e., they lie in the same orbit), then the space of $SL(n;\mathbb{Z})$ orbits $NG^\sigma(\boldsymbol{\beta}; d)/$ has the same $\vec{\dim}$.*

For both theorems it is readily verified that, without loss of generality, we can restrict ourselves to the set $E_{\boldsymbol{\beta}}(\sigma; n, d) \cap I^{nd}$ where $I = (-\frac{1}{2}, \frac{1}{2}]$. The proofs are done in three parts, firstly the Hausdorff dimensions are obtained. Following this an upper bound for the logarithmic Hausdorff dimension of $E_{\boldsymbol{\beta}}(\sigma; n, d)$ is determined for all $\boldsymbol{\beta}$, n and d and finally lower bounds for the logarithmic Hausdorff dimensions of the two sets are found. We recall that in [14] the Hausdorff dimension of the set

$$\begin{aligned} W_0(\tau) &= W_0(n, d; \tau) \\ &= \{X \in \mathbb{R}^{nd} : |\mathbf{q}X| < |\mathbf{q}|^{-\tau} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n\} \end{aligned}$$

was shown to be $(n-1)d + n/(\tau+1)$. It is readily verified that the sets $R_{\mathbf{q}} = \{X \in I^{nd} : \mathbf{q}X = 0\}$ are $(n-1)d$–dimensional hyperplanes contained in $E_0(\sigma)$ for all $\mathbf{q} \in \mathbb{Z}^n$. Also $E_0(\sigma) \subset W_0(\tau)$ for all τ and all σ. Thus $\dim E_0(\sigma) = (n-1)d$.

For the inhomogeneous case the sets $R_{\beta,\mathbf{q}} = \{\mathbf{x} \in I^n : \mathbf{q}.\mathbf{x} + \beta = 0\}$ are $(n-1)$–dimensional hyperplanes contained in $E_\beta(\sigma; n, 1)$ implying that $\dim E_\beta(\sigma; n, 1) \geq n-1$. Using the upper bound argument in the next section with $f(L(C))$ replaced by $L(C)^s$ it is readily verified that for any $s > n-1$ the Hausdorff s–measure of $E_\beta(\sigma; n, 1)$ is zero which implies that $\dim E_\beta(\sigma; n, 1) \leq n-1$ completing the result.

Now we begin to deal with the logarithmic component of $\vec{\dim}$. In what follow the Vinogradov notation will be used; that is $a \ll b$ (respectively $a \gg b$) means that there exists a constant $c > 0$ such that $a \leq cb$ ($a \geq cb$). If $a \ll b$ and $a \gg b$ then $a \asymp b$.

First the upper bounds are obtained for both theorems.

Lemma 6.5. *Let $n, d \in \mathbb{N}$, $n \geq 2$, $\sigma \in \mathbb{R}^+$ and $\boldsymbol{\beta} \in \mathbb{R}$. Then we have $\mathrm{Ldim}\ E_{\boldsymbol{\beta}}(\sigma; n, d) \leq n\sigma$.*

Proof. For future reference note that

$$f(e^{-r^{1/\sigma}}/r) = \frac{\left(e^{-r^{1/\sigma}}\right)^{(n-1)d}}{r^{(n-1)d}(r^{s/\sigma}+\log r)} \asymp \frac{\left(e^{-r^{1/\sigma}}\right)^{(n-1)d}}{r^{(n-1)d+s/\sigma}}. \tag{6.7}$$

The 'resonant sets' $R_{\boldsymbol{\beta},\mathbf{q}}$ are defined as $R_{\boldsymbol{\beta},\mathbf{q}} = \{X \in I^{nd} : \mathbf{q}X + \boldsymbol{\beta} = 0\}$. Let $C(\mathbf{q})$ denote a collection of hypercubes of sidelength $L(C)$ $= 2n^{1/2}e^{-|\mathbf{q}|^{1/\sigma}}/|\mathbf{q}|$ centred on $R_{\boldsymbol{\beta},\mathbf{q}}$ with centres at a distance of $e^{-|\mathbf{q}|^{1/\sigma}}/|\mathbf{q}|$ apart. There are $\ll \left(|\mathbf{q}|e^{(|\mathbf{q}|^{1/\sigma})}\right)^{(n-1)d}$ such hypercubes. Denote by C_N the set $C_N = \{C(\mathbf{q}) : |\mathbf{q}| > N\}$. For each N this set is a cover of $E_{\boldsymbol{\beta}}(\sigma)$. Therefore

$$\begin{aligned}\mathfrak{L}^s(E_{\boldsymbol{\beta}}(\sigma)) &\leq \sum_{r>N}\sum_{|\mathbf{q}|=r}\sum_{C\in C(\mathbf{q})} f(L(C)) \ll \sum_{r>N}\sum_{|\mathbf{q}|=r}\left(|\mathbf{q}|e^{|\mathbf{q}|^{1/\sigma}}\right)^{(n-1)d} \\ &\times \frac{\left(e^{-|\mathbf{q}|^{1/\sigma}}\right)^{(n-1)d}}{|\mathbf{q}|^{(n-1)d+s/\sigma}} \ll \sum_{r>N} r^{(n-1)-s/\sigma} < \varepsilon\end{aligned}$$

for $s > n\sigma$ and N sufficiently large. Therefore $\mathfrak{L}^s(E_{\boldsymbol{\beta}}(\sigma)) = 0$ for $s > n\sigma$ which implies that Ldim $E_{\boldsymbol{\beta}}(\sigma) \leq n\sigma$. □

Proof of Theorem 6.2. In view of the lemma in the above we obtain the lower bound for Theorem 6.2. To sketch the ideas: firstly, a Cantor subset $\mathbb{K}$ of $E_0(\sigma; n, d)$ and a probability measure supported on $\mathbb{K}$ are constructed. Then the Mass Distribution Principle is used to show that the logarithmic Hausdorff measure of $\mathbb{K}$ is infinity for $s < n\sigma$. This implies that the logarithmic Hausdorff dimension of $\mathbb{K}$ and hence $E_0(\sigma; n, d)$ is greater than or equal to $n\sigma$.

For the rest of the proof it is assumed that $s < n\sigma$. Also note that the following inequality will continually be used:

$$\frac{N^{n/d}}{16n\log N} \leq \left[\frac{N^{n/d}}{8n\log N}\right] \leq \frac{N^{n/d}}{8n\log N}, \tag{6.8}$$

where $[\ .\ ]$ denotes integer part.

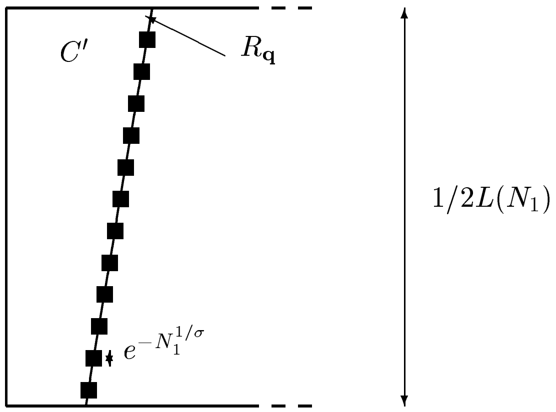
First we recall the following assertion (Lemma 2 in [14]) (Almost all means for all but a set of measure zero).

Lemma 6.6. *For every $N > 2$ and almost every $X \in I^{nd}$ there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \mathbf{0}$ with $N/\log N < |\mathbf{q}| \leq N$ such that* dist $(X, R_{\mathbf{q}}) < nN^{-n/d}\log N$.

Similar Cantor constructions can be found in [8, p 347] and [16] (for general dimension functions). Choose N_1 large enough so that

$$e^{N_1^{1/\sigma}} > \frac{5N_1^{n/d-1}}{2\log N_1}, \tag{6.9}$$

$$\frac{15N_1^{n-s/\sigma}}{5n^d 2^{4nd+4}\log^d N_1} > 1. \tag{6.10}$$



Divide I^{nd} into $T(N_1) = [N_1^{n/d}/8n\log N_1]^{nd}$ hypercubes C with

$$L_{N_1} = L(C) = [N_1^{n/d}/8n\log(N_1)]^{-1}.$$

Let C' be a hypercube with the same centre as C and $L(C') = \frac{1}{2}L(C)$. Let $R_N = \{R_{\mathbf{q}} : N \leq |\mathbf{q}| \leq 2N\}$ and let $G(N_1)$ denote the set of cubes C for which there exists a resonant set $R_{\mathbf{q}} \in R_{N_1}$ such that $C' \cap \mathbb{R}_q \geq \left(\frac{1}{10}L(C)\right)^{(n-1)d}$. From Lemma 2 in [14] it can be verified that for N_1 sufficiently large

$$\#\mathrm{G}(N_1) \geq \left[\frac{N_1^{n/d}}{8n\log N_1}\right]^{nd}(1-\varepsilon) \tag{6.11}$$

for any $\varepsilon > 0$.

For each $C \in \mathrm{G}(N_1)$ pick one such intersection and call it $I(C)$. Now draw further, smaller hypercubes centred on $I(C)$ and contained in C' of sidelength $e^{-N_1^{1/\sigma}}/N_1$ with centres at a distance of $2e^{-N_1^{1/\sigma}}/N_1$ apart, so that they are disjoint, (see figure for $n = 2$). Using (6.9) it can be shown that

$$\begin{aligned}\left(\frac{1}{40}N_1 L(C)e^{N_1^{1/\sigma}}\right)^{(n-1)d} &\leq \left(\frac{1}{20}L(C)N_1 e^{N_1^{1/\sigma}} - 1\right)^{(n-1)d} \leq \#I(C)\\ &\leq \left(N_1 e^{N_1^{1/\sigma}}L(C)\right)^{(n-1)d},\end{aligned}$$

where $\#I(C)$ is the number of cubes lying on a segment $I(C)$. This is the first level K_1 of the Cantor set. From (6.8), (6.11) (with $\varepsilon = 1/16$) and above

$$\begin{aligned}\#K_1 &\geq \frac{15}{16}\left[\frac{N_1^{n/d}}{8n\log N_1}\right]^{nd}\left(\frac{1}{40}N_1 L(C)e^{N_1^{1/\sigma}}\right)^{(n-1)d}\\ &\geq \frac{15N_1^{nd-d+n}\left(e^{N_1^{1/\sigma}}\right)^{(n-1)d}}{2^{nd+4}(8n)^d 40^{(n-1)d}(\log N_1)^d}\end{aligned}$$

Let E_1 denote a hypercube in K_1, so that $L(E_1) = e^{-N_1^{1/\sigma}}/N_1$, then (6.10) yields

$$\#K_1 f(L(E_1)) \geq \frac{15N_1^{n-s/\sigma}}{5n^d 2^{4nd+4}\log^d N_1} > 1.$$

To get the second level K_2 precisely the same construction is made but instead of I^{nd} the cubes from the previous level K_1 are used. Choose N_2 large enough so that

$$\frac{N_2^{n/d}}{16n\log N_2} > N_1 e^{N_1^{1/\sigma}}, \tag{6.12}$$

$$\frac{3N_2^{n-s/\sigma}}{5n^d \cdot 2^{5nd+2}\log^d N_2} > \frac{e^{ndN_1^{1/\sigma}} N_1^{nd}}{\#K_1} \tag{6.13}$$

and $\varepsilon > 0$ such that

$$\varepsilon \leq 2^{-1} e^{-2N_1^{1/\sigma}} N_1^{-nd}. \tag{6.14}$$

As before divide I^{nd} into $T(N_2)$ hypercubes of sidelength L_{N_2} $= [N_2^{n/d}/8n\log N_2]^{-1}$. Since $N_2 > N_1$ from (6.8)

$$\#\mathrm{G}(N_2) \geq \left[\frac{N_2^{n/d}}{8n\log N_2}\right]^{nd}(1-\varepsilon) > \frac{N_2^{n^2}}{(16n)^{nd}\log^{nd} N_2}(1-\varepsilon).$$

Let $T_{E_1}(N_2)$ denote the number of hypercubes contained in E_1 and $\#\mathrm{G}_{E_1}(N_2)$ the number of those with large intersection. It is readily verified that

$$\left(\frac{e^{-N_1^{1/\sigma}}}{N_1 L_{N_2}} - 1\right)^{nd} \leq T_{E_1}(N_2) \leq \left(\frac{e^{-N_1^{1/\sigma}}}{N_1 L_{N_2}} + 2\right)^{nd},$$

and (6.12) implies that

$$\left(\frac{e^{-N_1^{1/\sigma}}}{2N_1 L_{N_2}}\right)^{nd} \leq T_{E_1}(N_2) \leq \left(\frac{2e^{-N_1^{1/\sigma}}}{N_1 L_{N_2}}\right)^{nd}.$$

Equations (6.14) and (6.11) can be used to show that $\#\mathrm{G}(N_2) \geq T(N_2) - T_{E_1}(N_2)/4$ whence

$$\mathrm{G}_{E_1}(N_2) > \frac{3T_{E_1}(N_2)}{4} > \frac{3e^{-ndN_1^{1/\sigma}}}{4\cdot 2^{nd}(16n)^{nd} N_1^{nd} N_2^{-d^2}\log^{nd} N_2}.$$

Do exactly the same as before with the segments $I(C)$ to construct K_2, then we have $\#K_2 = \#K_1\#H_2$, where $\#H_2$ is the number of hypercubes of K_2 in one E_1 with

$$\begin{aligned} \#\mathrm{G}_{E_1}(N_2)(L_{N_2}N_2 e^{N_2^{1/\sigma}}/40)^{(n-1)d} &\leq \#H_2 \\ &\leq \#\mathrm{G}_{E_1}(N_2)\left(L_{N_2}N_2 e^{N_2^{1/\sigma}}\right)^{(n-1)d}. \end{aligned}$$

Thus

$$\#K_2 \geq \#K_1 \frac{3N_2^{nd-d+n} e^{(n-1)dN_2^{1/\sigma}}}{4\cdot 2^{2nd}(8n)^n \cdot 40^{(n-1)d}\log^d N_2}\ \frac{e^{-ndN_1^{1/\sigma}}}{N_1^{nd}}$$

and

$$\#K_2 f(L(E_2)) = \frac{3N_2^{n-s/\sigma}}{5n^d \cdot 2^{5nd+2}\log^d N_2}\,\frac{e^{-ndN_1^{1/\sigma}}}{N_1^{nd}}\,\#K_1 > 1,$$

from (6.13).

Similarly for the r'th level. Assume that K_{r-1} has been constructed with cubes of sidelength $e^{-N_{r-1}^{1/\sigma}}/N_{r-1}$. Choose N_r sufficiently large so that

$$\frac{N_r^{n/d}}{16n\log N_r} > N_r e^{N_{r-1}^{1/\sigma}}, \tag{6.15}$$

$$\frac{3N_r^{n-s/\sigma}}{5n^d \cdot 2^{5nd+2}\log^d N_r} > \frac{e^{ndN_{r-1}^{1/\sigma}}N_{r-1}^{nd}}{\#K_{r-1}} \tag{6.16}$$

and $\varepsilon > 0$ so that

$$\varepsilon \leq 2^{-1}e^{-2N_{r-1}^{1/\sigma}}N_{r-1}^{-nd} \tag{6.17}$$

Let $T_{E_{r-1}}(N_r)$ denote the number of hypercubes contained in E_{r-1} and $\#\mathrm{G}_{E_{r-1}}(N_r)$ the number which have large intersection. Following the argument exactly as for the second level but using equations (6.15) and (6.17) rather than equations (6.12) and (6.14) we get

$$\#K_r \geq \#K_{r-1}\frac{3N_r^{nd-d+n}e^{(n-1)dN_r^{1/\sigma}}}{4\cdot 2^{2nd}(8n)^d 40^{(n-1)d}\log^d N_r}\,\frac{e^{-ndN_{r-1}^{1/\sigma}}}{N_{r-1}^{nd}}, \tag{6.18}$$

and

$$\#K_r f(L(E_r)) = \frac{3N_2^{n-s/\sigma}}{5n^d \cdot 2^{5nd+2}\log^d N_r}\,\frac{e^{-ndN_{r-1}^{1/\sigma}}}{N_{r-1}^{nd}}\,\#K_{r-1} > 1,$$

from (6.16).

Let $\mathbb{K} = \cap_{r=1}^{\infty}K_r$. It is easy to show that $\mathbb{K} \subset E_0(\sigma; n, d)$. In order to construct a measure on $\mathbb{K}$ we need the following lemma which follows from [19, page 55].

Lemma 6.7 (The Mass distribution principle)**.** *Let μ be a probability measure supported on a subset F of $\mathbb{R}^k$. Suppose there are positive constants λ and δ such that $\mu(C) \leq \lambda f(L(C))$, for any cube C with sidelength $L(C) \leq \delta$. Then we have $\mathfrak{L}^s(F) \geq \lambda^{-1}$.*

Recall that E_i represents a cube in K_i, that is in the i'th level of $\mathbb{K}$ and define a probability measure on $\mathbb{K}$ in the following way: $\mu(E_i) = (\#K_i)^{-1} < f(L(E_i))$, $i \in \mathbb{N}$. Thus the sum of the measures of all hypercubes in the r'th level under one particular box on the $r-1$'th level adds up to $\mu(E_{r-1})$, i.e. $\mu(E_r) = \mu(E_{r-1})/(\#H_r)$. To use Lemma 6.7 we need to show that the measure of an arbitrary cube B is $\ll f(L(B))$ where $L(B)$ is the sidelength of the cube. This obviously holds for any cube in $\mathbb{K}$. Without loss of generality we can choose an arbitrary cube B contained in some hypercube E_{r-1} and containing at least two cubes of the r'th level so that $L(B) \leq e^{-N_{r-1}^{1/\sigma}}/N_{r-1}$. There are two cases to consider. The first is when B intersects hypercubes lying on at least two different segments $I(C)$ of resonant sets. In this case $L(B) \geq L_{N_r} = [N_r^{n/d}/8n\log N_r]^{-1}$. Let S be the number of segments $I(C)$ that B intersects. Then we have $S \leq (L(B)/L_{N_r})^{nd}$. The maximum number of hypercubes of K_r that B can intersect on each segment is $\left(L_{N_r} N_r e^{(N_r^{1/\sigma})}\right)^{(n-1)d}$. Evidently

$$\mu(B) \leq \frac{S\left(L_{N_r} N_r e^{N_r^{1/\sigma}}\right)^{(n-1)d}}{\# K_r} \leq \frac{L(B)^{nd} N_r^{nd-d+n} e^{(n-1)dN_r^{1/\sigma}}}{\log^d N_r \# K_r}.$$

¿From (6.18) we obtain

$$\mu(B) \ll \frac{L(B)^{nd} e^{ndN_{r-1}^{1/\sigma}} N_{r-1}^{nd}}{\# K_{r-1}}.$$

However, $\# K_{r-1} \geq 1/f(L(E_{r-1}))$ which implies that

$$\mu(B) \ll L(B)^{nd} e^{ndN_{r-1}^{1/\sigma}} N_{r-1}^{nd} f(L(E_{r-1})).$$

For $r < 1$ the function $f(r)/r^{nd}$ decreases, hence, as $L(B) \leq \exp(-N_{r-1}^{1/\sigma})/N_{r-1} < 1$ we obtain $\mu(B) \leq f(L(B))$.

For the second case assume that B intersects cubes in K_r from only one segment $I(C)$. As before $L(B) \leq e^{-N_{r-1}^{1/\sigma}}/N_{r-1}$, and $L(B) \geq L(E_r) = e^{-N_r^{1/\sigma}}/N_r$. The number of cubes that B can intersect is $\ll (L(B)/L(E_r))^{(n-1)d}$. Therefore

$$\mu(B) \ll \frac{L(B)^{(n-1)d}}{L(E_r)^{(n-1)d} \# K_r} \leq \frac{L(B)^{(n-1)d} f(L(E_r))}{L(E_r)^{(n-1)d}}.$$

As $f(r)/r^{(n-1)d} = 1/\log^s(1/r)$ is an increasing function and $L(B) \geq L(E_r)$ it is easy to verify that $\mu(B) \leq f(L(B))$.

Thus for $s < n\sigma$ from Lemma 6.7 we have $\mathfrak{L}^s(F) \geq 1$ which implies that Ldim $(F) \geq n\sigma$ completing the prove of Theorem 6.2. $\square$

In order to prove Theorem 6.3 we prepare a lemma. Let $I_\delta^m = \{\mathbf{x} \in I^n : |\mathbf{x}| > \delta\}$ and let $||\ .\ ||$ denote the distance from the nearest integer, i.e. $||\mathbf{x}|| = \min_{\mathbf{k} \in \mathbb{Z}^2} |\mathbf{x} - \mathbf{k}|$.

Lemma 6.8. *For every $N > 2$ and almost all $\mathbf{x} \in I_\delta^n$ there exists $\mathbf{q} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ with $N/\log^4 N \leq |\mathbf{q}| \leq N$ such that* dist $(\mathbf{x}, R_{\beta,\mathbf{q}}) < N^{-2} \log^7 N$.

Proof. ¿From [29], pp. 74-75 it can be verifed that for every $N > 2$ and almost all $\mathbf{x} \in I^n$ there exists $\mathbf{q} \in \mathbb{Z}^n \backslash \{\mathbf{0}\}$ with $N/\log^4 N \leq |\mathbf{q}| \leq N$ such that

$$||\mathbf{q} \cdot \mathbf{x} - \beta|| \leq N^{-n} \log^3 N \tag{6.19}$$

The following argument is done in detail for $n = 2$ and any decreasing function $\phi(N)$ and is not difficult to extend to $n \geq 3$.

Consider the line $x_2 = 1$. Then $|\mathbf{q} \cdot \mathbf{x} + \beta| < \phi(N)$ implies that $|q_1 x_1 + q_2 + \beta| < \phi(N)$. Thus we can consider the inequality $||q_1 x + \beta|| < \phi(N)$. Similarly consider the line $x_2 = \alpha$ for any $1 \geq \alpha > 0$. Then the inequality $|q_1 x_1 + q_2 \alpha + \beta| < \phi(N)$ is the same as the inequality $||qx + \beta'|| < \alpha^{-1}\phi(N)$ where $x = x_1/\alpha$ and $\beta' = \beta/\alpha$. Thus, if there exists a function ϕ for which almost all points on the lines $x_2 = \alpha$ are covered by intervals centered at points $(p + \beta)/q$ of length $\alpha^{-1}\phi(N)/q$ with $N/\log^4 N < q < N$ then for some fixed $\delta > 0$ with $\alpha > \delta$ almost all points in I_δ^2 will be covered by the set

$$\bigcup_{\mathbf{q}:N>|\mathbf{q}|>N/\log^4 N} \{\mathbf{x} : |\mathbf{q}\cdot\mathbf{x}+\beta| < \phi(|\mathbf{q}|)\}$$

for N sufficiently large. On letting $\phi(N) = N^{-1}\log^3 N$ and using (6.19) and the fact that $N > |\mathbf{q}| > N/\log^4 N$ the lemma is proved. Exactly the same argument can be used for $n \geq 3$. □

Proof of Theorem 6.3. Only the lower bound is left to prove. Using the ubiquity argument from [29] it can be shown that as in the homogeneous case with $L_N = [N^2/8n\log^7 N]^{-1}$ we have

$$\#G(N) \geq \left[\frac{N^2}{8m\log^7 N}\right]^n (1-\varepsilon).$$

Theorem 6.3 can now be proved by using the above and following the proof of Theorem 6.2 exactly. □

Theorems 6.2 and 6.3 show that the sets $E_0(\sigma; m, n)$ and $E_\beta(\sigma; m, 1)$ consist of more than just the resonant hyperplanes $R_{\mathbf{q},\beta}$.

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