Centre de Physique Théorique*, **CNRS Luminy, Case 907**
F-13288 Marseille – Cedex 9

ACCRETIVE PERTURBATIONS AND ERROR ESTIMATES FOR THE TROTTER PRODUCT FORMULA[†]

V. CACHIA[1], **H. NEIDHARDT**[2] **and V.A. ZAGREBNOV**[3]

Abstract

We study the error bound estimate in the operator-norm topology for the exponential Trotter product formula in the case of accretive perturbations. Let A be a semibounded from below self-adjoint operator on a separable Hilbert space. Let B be a closed maximal accretive operator which is, together with B^*, Kato-small with respect to A with relative bounds less than one. We show that in this case the operator-norm error bound estimate for the exponential Trotter product formula is the same as for the self-adjoint B [12]:

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)} \right\| \leq L\frac{\ln n}{n}, \; n = 2, 3, \ldots .$$

We verify that the operator $-(A+B)$ generates a holomorphic contraction semigroup. One gets a similar result when B is substituted by B^*.

Keywords : Semigroups, Trotter product formula, accretive operators, operator-norm estimates.
Mathematics Subject Classification : 47D03, 47B25, 35K22, 41A80

anonymous ftp : ftp.cpt.univ-mrs.fr
web : www.cpt.univ-mrs.fr
February 2000
CPT-2000/P.3961

---

*Unité Propre de Recherche 7061

[†] **To the memory of Tosio Kato**

[1] Allocataire-moniteur at Université de la Méditerranée (Aix-Marseille II)
Email : cachia@cpt.univ-mrs.fr

[2] Weierstraß Institut für angewandte Analysis und Stochastik, Mohrenstr. 39, D-10117 Berlin, Germany - Email : neidhard@wias-berlin.de

[3] Université de la Méditerranée (Aix-Marseille II) - Email : zagrebnov@cpt.univ-mrs.fr

1 Introduction and Setup of the Problem

Let $-A$ and $-B$ be generators of strong continuous semigroups on some separable Hilbert space $\mathfrak{H}$. For those operators the convergence of the Trotter product formula

$$\tau - \lim_{n\to+\infty} (e^{-tA/n}e^{-tB/n})^n = e^{-tH}, \tag{1.1}$$

where H is the sum of the operators A and B defined in a suitable sense, was of interest for many years only with respect to the strong topology $\tau = s$, see [3], [4], [18]. A final result in this direction was obtained by Kato [9], [10] who has shown that the convergence in the strong topology always takes place if the operators A and B are self-adjoint and non-negative, or even more generally they may be m-sectorial. Here and below we follow the terminology of the Kato's book [8]. Moreover, he identified H as the form-sum of A and B and generalized the product formula (1.1) to the form in which the exponential functions are replaced by functions of a certain class which is often called the Kato-class. However, certain applications of the Trotter or the more general Trotter-Kato product formulæ require that

(a) the convergence in (1.1) is in the operator-norm topology or in stronger norms, for instance, in the trace- or the Hilbert-Schmidt-norm,

(b) error estimates in these norms, in particular, in operator-norm, are available,

(c) the convergence and the error estimates are valid not only for self-adjoint operators A and B.

Until now the problems (a) and (b) were solved in a satisfactory manner only for non-negative self-adjoint operators A and B. In papers [11]-[15] the conditions (partially necessary and sufficient) are found for convergence of the Trotter-Kato product formula in the operator-norm or in norms of symmetrically-normed ideals of compact operators, see also [5]. Error bound estimates in the operator-norm for non-negative self-adjoint operators were given in [6], [12], [13] and [17] and for the trace-norm in [7] and [15]. On the other hand there are only few results concerning the problem (c). The first result in this direction was that of Lie who has shown the convergence of the exponential product formula (1.1) for arbitrary matrices A and B. In fact, his proof provides the operator-norm error bound estimate for a couple of bounded operators A and B in a complete normed space:

$$\left\|(e^{-tA/n}e^{-tB/n})^n - e^{-tH}\right\| \le \frac{C}{n}, \quad n = 1, 2, \ldots, \tag{1.2}$$

where $H = A + B$, see [16]. So the Lie's result solves problems (a) - (c) for bounded operators in a Banach space. In [1] the problem (a) in a Hilbert space is solved in the operator-norm topology $\tau = \|\cdot\|$ for a class of unbounded m-sectorial operators A and B with semi-angles $\theta_A, \theta_B \in (0, \pi/2)$. A solution of the problem (b) in a Banach space was presented in [2] for particular domain conditions. Namely, let $-A$ be generator of a holomorphic contraction semigroup and let $-B$ be generator of a contraction semigroup satisfying the conditions: $\mathrm{dom}(B) \supseteq \mathrm{dom}(A^\alpha)$ for some $\alpha \in [0, 1)$ and $\mathrm{dom}(B^*) \supseteq \mathrm{dom}(A^*)$. then there is a constant C_α such that

$$\left\|\left(e^{-tB/n}e^{-tA/n}\right)^n - e^{-tH}\right\| \le C_\alpha \frac{(\ln(n))^{s(\alpha)}}{n^{1-\alpha}}, \quad n = 2, 3, \ldots, \tag{1.3}$$

uniformly in $t \in [0, T]$, $0 < T < +\infty$, where $H = A + B$ on $\mathrm{dom}(H) = \mathrm{dom}(A)$, and $s(\alpha = 0) = 2$, $s(\alpha \neq 0) = 1$. The condition $\mathrm{dom}(B) \supseteq \mathrm{dom}(A^{\alpha})$ implies that B is relatively bounded with respect to A with the relative bound $a = 0$. Thus, comparing with the corresponding results in the self-adjoint case on a Hilbert space, see [12] and [13], the result (1.3) is not optimal. For instance in [12] it is proven that if $A \geq I$, $B \geq 0$ and B is relatively bounded with respect to A with $a < 1$, then there is a constant C such that

$$\left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-tH} \right\| \leq C \frac{\ln(n)}{n}, \quad n = 2, 3, \ldots, \tag{1.4}$$

uniformly in $t \geq 0$, where $H = A + B$ on $\mathrm{dom}(H) = \mathrm{dom}(A)$.

In what follows we suppose that $A = A^*$ be a semibounded from below densely defined self-adjoint operator on the Hilbert space $\mathfrak{H}$ which generates a semigroup $\{U_A(t) = e^{-tA}\}_{t \geq 0}$. Without loss of generality one can suppose that $A \geq I$. Then it is known (see, e.g. the Kato's book [8]) that $U_A(t)$ is a contraction holomorphic semigroup of angle $\omega = \pi/2$, i.e. $\|U_A(t \geq 0)\| \leq 1$ and $U_A(t)$ has a holomorphic extension from $t \geq 0$ into the open sector of the complex plane: $S_{\omega=\pi/2} = \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \pi/2\}$.

About the generator B we suppose that :

(*i*) domain $\mathrm{dom}(B) \supseteq \mathrm{dom}(A)$ and there exists $a > 0$ such that

$$\|Bu\| \leq a \|Au\|, \; u \in \mathrm{dom}(A); \tag{1.5}$$

(*ii*) domain $\mathrm{dom}(B^*) \supseteq \mathrm{dom}(A)$ and there exists $a_* > 0$ such that

$$\|B^* u\| \leq a_* \|Au\|, \; u \in \mathrm{dom}(A); \tag{1.6}$$

(*iii*) B is a m-accretive (i.e. closed and maximal accretive) operator on $\mathfrak{H}$, i.e. for $\lambda > 0$ one has

$$\|(B + \lambda)^{-1}\| \leq \lambda^{-1}. \tag{1.7}$$

The latter implies that $\{U_B(t) = e^{-tB}\}_{t \geq 0}$ is a (non self-adjoint) contraction semigroup, but B is not necessarily sectorial: θ_B may be $\pi/2$.

Notice that under conditions (*i*)-(*iii*) the convergence of the Trotter product formula (1.1) is known, due to [3], [4], [18], only in the strong topology $\tau = s$. In our main theorem (see Section 3) we prove that under conditions (*i*)-(*iii*) and $a, a_* < 1$ there is a constant $L > 0$ such that uniformly in $t \geq 0$ one gets (cf (1.4)

$$\left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-tH} \right\| \leq L \frac{\ln(n)}{n} \;, \; n = 1, 2, 3, \ldots \quad . \tag{1.8}$$

Here $H = A + B$ is defined on $\mathrm{dom}(H) = \mathrm{dom}(A)$. Similar result holds for B^* and H^*.

The paper is organized as follows. In Section 2 we prove some technical results, which have an interest in itself, and which are crucial for the proof of the main theorem, which is presented in Section 3. The proof of this main theorem (while inspired by) needs a considerable deviation from the line of reasoning known for the self-adjoint case. Finally, in Section 4 we make some remarks and comments.

2 Preliminaries

Let $-H$ be the generator of a semigroup $\{U_H(t) = e^{-tH}\}_{t\geq 0}$. The semigroup $U_H(t)$ is called a bounded holomorphic semigroup with angle $\omega \in (0, \pi/2]$ if

- (α) $\{U(t)\}_{t\geq 0}$ is the restriction to the positive real axis of an analytic family of bounded operators $\{U(z)\}_{z\in S_\omega}$, $S_\omega = \{z \in \mathbb{C} : z \neq 0,\ |\arg z| < \omega\}$, obeying $U(z + z') = U(z)U(z')$, $z, z' \in S_\omega$, and
- (β) for every $\theta \in (0, \omega)$ the family $\{U(z)\}_{z\in S_\omega}$ is uniformly bounded in the sector S_θ and $U(z)f \longrightarrow f$ as $z \longrightarrow 0$ in $z \in S_\theta$ for each $f \in \mathfrak{H}$.

It turns out that this is the case if and only if the spectrum $\sigma(H)$ of H is contained in the closed sector $\overline{S_{\pi/2-\omega}}$, i.e. $\sigma(H) \subseteq \overline{S_{\pi/2-\omega}}$, and for each $\theta \in [0, \omega)$ there is a constant $M_\theta > 0$ such that

$$\|(H + z)^{-1}\| \leq \frac{M_\theta}{|z|} \tag{2.1}$$

for $z \in S_{\pi/2+\theta}$, see e.g. [8, Ch.IX].

Theorem 2.1 *Let $A \geq I$ and let B be a closed accretive operator satisfying condition (i) with $a < 1$. Then the operator $-H = -(A + B)$ is the boundedly invertible generator of a contraction holomorphic semigroup of angle $\omega = \arccos a$, in the sense that $\|e^{-tH}\| \leq 1$ for $t \geq 0$.*

Proof. Our goal is to verify (2.1) for $z \in S_{\pi/2+\theta}$, $\theta \in (0, \omega)$, where $\omega = \arccos a < \pi/2$. To this end we use the representation

$$(H + z)f = \{I + BA^{-1}A(A + z)^{-1}\}(A + z)f, \qquad f \in \mathrm{dom}(H) = \mathrm{dom}(A). \tag{2.2}$$

Since

$$\|A(A + z)^{-1}\| \leq \sup_{\lambda\geq 1} \left|\frac{\lambda}{\lambda + z}\right| \leq \sup_{\lambda\geq 0} \left|\frac{\lambda}{\lambda + z}\right| \tag{2.3}$$

a straightforward computation shows that

$$\|A(A + z)^{-1}\| \leq \begin{cases} 1 \text{ for } \Re\mathrm{e}\, z \geq 0 \\ |z|/|\Im\mathrm{m}\, z| \text{ for } \Re\mathrm{e}\, z < 0,\ \Im\mathrm{m}\, z \neq 0 \end{cases} \tag{2.4}$$

For $z \in S_{\pi/2+\theta}$ and $\Re\mathrm{e}\, z < 0$, then

$$\frac{|z|}{|\Im\mathrm{m}\, z|} \leq \frac{1}{\cos\theta}. \tag{2.5}$$

Hence (2.4) leads to the estimate

$$\|A(A + z)^{-1}\| \leq \frac{1}{\cos\theta} \tag{2.6}$$

for $z \in S_{\pi/2+\theta}$ and $\Re\mathrm{e}\, z < 0$. If $\Re\mathrm{e}\, z \geq 0$ then (2.4) yields the estimate

$$\|A(A + z)^{-1}\| \leq 1 \leq \frac{1}{\cos\theta}. \tag{2.7}$$

Therefore the estimate (2.6) holds for any $z \in S_{\pi/2+\theta}$.

Using the condition (i) we find the estimate

$$\|BA^{-1}A(A+z)^{-1}\| \leq \|BA^{-1}\|\|A(A+z)^{-1}\| \leq \frac{a}{\cos\theta} < 1 \tag{2.8}$$

for $z \in S_{\pi/2+\theta}$. Hence the operator $I + B(A+z)^{-1}$ is invertible for $z \in S_{\pi/2+\theta}$ and we have the estimate

$$\|(I + B(A+z)^{-1})^{-1}\| \leq \frac{\cos\theta}{\cos\theta - a} \tag{2.9}$$

for $z \in S_{\pi/2+\theta}$. Then from (2.2) we obtain that $z \in S_{\pi/2+\theta} \subseteq \rho(-H)$ the resolvent set of the operator $-H$ and the representation

$$(H+z)^{-1} = (A+z)^{-1}(I + B(A+z)^{-1})^{-1}, \tag{2.10}$$

and the estimate

$$\|(H+z)^{-1}\| \leq \frac{\cos\theta}{\cos\theta - a}\|(A+z)^{-1}\| \tag{2.11}$$

for $z \in S_{\pi/2+\theta}$. Since A is self-adjoint, we have

$$\|(z+A)^{-1}\| \leq \frac{1}{\cos\theta}\frac{1}{|z|} \tag{2.12}$$

for any $z \in S_{\pi/2+\theta}$. Thus we conclude

$$\|(H+z)^{-1}\| \leq \frac{M_\theta}{|z|} \tag{2.13}$$

for all $z \in S_{\pi/2+\theta}$ with

$$M_\theta := \frac{1}{\cos\theta - a} \tag{2.14}$$

which shows that $-H$ is the generator of a holomorphic semigroup with angle $\omega = \arccos a$.

Since B is accretive and A is self-adjoint positive, $H = A + B$ is accretive and since $\lambda + H$ is boundedly invertible for $\lambda > 0$, H is m-accretive, i.e. $-H$ generates a contraction semigroup. Finally by the representation $H = (I + BA^{-1})A$ with $\|BA^{-1}\| \leq a < 1$, the operator H is boundedly invertible.

□

Remark: Since $\sigma(H) \subseteq S_{\pi/2-\omega}$ implies $\sigma(H^*) \subseteq S_{\pi/2-\omega}$, the characterisation (2.1) entails that $-H^*$ is also the generator of a holomorphic semigroup.

Below we denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathrm{D}_{1/2} = \{z \in \mathbb{C} : |z - 1/2| \leq 1/2\}$, two discs in the complex plane $\mathbb{C}$.

Lemma 2.2 *Let $A \geq I$. Then for any $\tau > 0$*

$$\|(I - e^{-\tau A})(z - e^{-\tau A})^{-1}\| \leq \begin{cases} |z|^{-1} & z \in \mathbb{D} \setminus \mathrm{D}_{1/2} \\ |1-z|/|\Im\mathrm{m}\, z| & z \in \mathrm{D}_{1/2} \setminus [0,1] \end{cases} \tag{2.15}$$

Proof: We use the functional calculus for self-adjoint operators. Obviously, we have

$$\|(I - e^{-\tau A})(z - e^{-\tau A})^{-1}\| \leq \sup_{\lambda \geq 1}\left|\frac{1 - e^{-\tau\lambda}}{z - e^{-\tau\lambda}}\right| \leq \sup_{\lambda \geq 0}\left|\frac{1 - e^{-\lambda}}{z - e^{-\lambda}}\right|. \tag{2.16}$$

Let $z = x + iy$ and

$$m(\lambda) := \frac{1 - e^{-\lambda}}{|z - e^{-\lambda}|} = \frac{1 - e^{-\lambda}}{\sqrt{(x - e^{-\lambda})^2 + y^2}}, \quad \lambda \geq 0. \tag{2.17}$$

If $z \in \mathbb{D} \setminus \mathrm{D}_{1/2}$, then $m'(\lambda) \geq 0$ for $\lambda \geq 0$. Therefore $m(\lambda)$ is a non-decreasing function such that

$$\sup_{\lambda \geq 0} m(\lambda) = \lim_{\lambda \to +\infty} m(\lambda) = \frac{1}{|z|}, \tag{2.18}$$

which proves the first estimate (2.15). If $z \in \mathrm{D}_{1/2} \setminus [0,1]$, then $m(\lambda)$ attains its maximum at λ_0 such that

$$e^{-\lambda_0} = \frac{\frac{1}{4} - (x - \frac{1}{2})^2 - y^2}{1 - x}. \tag{2.19}$$

Inserting (2.19) into (2.17) we obtain

$$m(\lambda_0) = \frac{|1 - z|}{|\Im\mathrm{m}\, z|}, \tag{2.20}$$

which yields

$$0 \leq m(\lambda) \leq \frac{|1 - z|}{|\Im\mathrm{m}\, z|}, \tag{2.21}$$

The inequality (2.21) verifies the second estimate in (2.15). □

Since for $A \geq I$ and an m-accretive B the operator

$$T(\tau) = e^{-\tau A} e^{-\tau B}, \qquad \tau > 0, \tag{2.22}$$

is a contraction, its spectrum $\sigma(T(\tau)) \subseteq \mathbb{D}$, $\tau > 0$. However, for B small relative to A we can show more. To this end we introduce the family of closed convex sets in $\mathbb{C}$, for $r \in [0,1]$

$$\Sigma_r := \Sigma_r^0 \cup \Sigma_r^1, \tag{2.23}$$

where

$$\Sigma_r^0 := \{z \in \mathrm{D}_{1/2} : \ |\Im\mathrm{m}\, z| \leq r|1 - z|\} \tag{2.24}$$

and

$$\Sigma_r^1 := \{z \in \mathbb{D} \setminus \mathrm{D}_{1/2} : \ |z| \leq r\}, \tag{2.25}$$

see Figure 1. We remark that the family Σ_r is increasing with r: for $r < r'$ one gets $\Sigma_r \subset \Sigma_{r'}$.

On the other hand, we denote by R_r the decreasing family $\mathbb{D} \setminus \Sigma_r$.

Theorem 2.3 *Let $A \geq I$ and let B be an m-accretive operator satisfying the condition (ii) with $a_* < 1$. Then $\sigma(T(\tau)) \subseteq \Sigma_{a_*}$ for each $\tau > 0$. Moreover, for any $r \in (a_*, 1)$, one obtains the estimate*

$$\|(z - T(\tau))^{-1}\| \leq \frac{r}{r - a_*} \|(z - e^{-\tau A})^{-1}\|, \ z \in R_r = \mathbb{D} \setminus \Sigma_r. \tag{2.26}$$

Proof: One has to prove that each point $z \in \mathbb{D} \setminus \Sigma_{a_*} = R_{a_*}$ belongs to the resolvent set $\rho(T(\tau))$ of $T(\tau)$ for any $\tau > 0$. To this end we use the representation

$$z - T(\tau) = z - e^{-\tau A} + e^{-\tau A}(I - e^{-\tau B}). \tag{2.27}$$

If $\tau > 0$, then the operator $(I - e^{-\tau A})$ is invertible. Let

$$X(\tau, z) := (I - e^{-\tau A})(z - e^{-\tau A})^{-1} \tag{2.28}$$

and

$$Y(\tau) := (I - e^{-\tau A})^{-1} e^{-\tau A}(I - e^{-\tau B}). \tag{2.29}$$

Then one obviously gets the representation

$$z - T(\tau) = (z - e^{-\tau A})\{I + X(\tau, z)Y(\tau)\}. \tag{2.30}$$

For each $f \in \mathrm{dom}(B)$ one has

$$(I - e^{-\tau B})f = \int_0^\tau ds B e^{-sB} f. \tag{2.31}$$

Hence we get

$$A^{-1}(I - e^{-\tau B})f = \int_0^\tau ds A^{-1} B e^{-sB} f. \tag{2.32}$$

This leads to the estimate

$$\|A^{-1}(I - e^{-\tau B})\| \leq \tau a_* \tag{2.33}$$

Using for (2.29) the representation

$$Y(\tau) = (I - e^{-\tau A})^{-1} A e^{-\tau A} A^{-1}(I - e^{-\tau B}) \tag{2.34}$$

we obtain

$$\|Y(\tau)\| \leq \tau a_* \|(I - e^{-\tau A})^{-1} A e^{-\tau A}\|. \tag{2.35}$$

By the functional calculus for self-adjoint operators one verifies that

$$\tau \|(I - e^{-\tau A})^{-1} A e^{-\tau A}\| \leq 1 \tag{2.36}$$

for each $\tau > 0$ and by consequence

$$\|Y(\tau)\| \leq a_* \,. \tag{2.37}$$

Let $r > a_*$. To estimate $X(\tau, z)$ we have to consider two cases:

1. Let $z \in \mathrm{D}_{1/2} \cap R_r$. Then $|\Im\mathrm{m}\, z| > r|1 - z|$. Since $z \in \mathrm{D}_{1/2}$ by Lemma 2.2 we find

$$\|X(\tau, z)\| \leq \frac{|1 - z|}{|\Im\mathrm{m}\, z|}. \tag{2.38}$$

Hence we have

$$\|X(\tau, z)\| < \frac{1}{r} \tag{2.39}$$

for $z \in R_r \cap \mathrm{D}_{1/2}$ and $\tau > 0$.

2. Let $z \in R_r \setminus \mathrm{D}_{1/2}$. Since $|z| > r$ by Lemma 2.2 we find

$$\|X(\tau, z)\| \leq \frac{1}{|z|} < \frac{1}{r}. \tag{2.40}$$

Finally we see that the estimate

$$\|X(\tau, z)\| < \frac{1}{r} \tag{2.41}$$

is valid for any $z \in R_r$ and $\tau > 0$. Moreover, by (2.37) and (2.41) we have

$$\|X(\tau, z)Y(\tau)\| \leq \|X(\tau, z)\|\|Y(\tau)\| < \frac{a_*}{r} \leq 1 \tag{2.42}$$

for $z \in R_r$, $\tau > 0$. This yields that the operator $\{I + Y(\tau)X(\tau, z)\}$ is invertible for $z \in R_{a_*} = \bigcup_{r>a_*} R_r$ and $\tau > 0$, with the following estimate for $z \in R_r$, $r > a_*$:

$$\|(I + X(\tau, z)Y(\tau, z))^{-1}\| \leq \frac{1}{1 - a_*/r}, \ \tau > 0. \tag{2.43}$$

Since the operator $(z - e^{-\tau A})$ is invertible for $z \in R_{a_*}$ and $\tau > 0$, we obtain from the representation (2.30) that the operator $T(\tau)$ is invertible for $z \in R_{a_*}$, $\tau > 0$, and the announced estimate for $z \in R_r$, $r \in (a_*, 1)$. □

Let us define the closed curve $\Gamma_r = \partial\Sigma_r$. Outside the disc $\mathrm{D}_{1/2}$, it coincides with the arc of centre 0 and of radius r, whereas inside $\mathrm{D}_{1/2}$, this curve consists of two segments of tangents to the preceding arc passing the point 1 (see Figure 1).

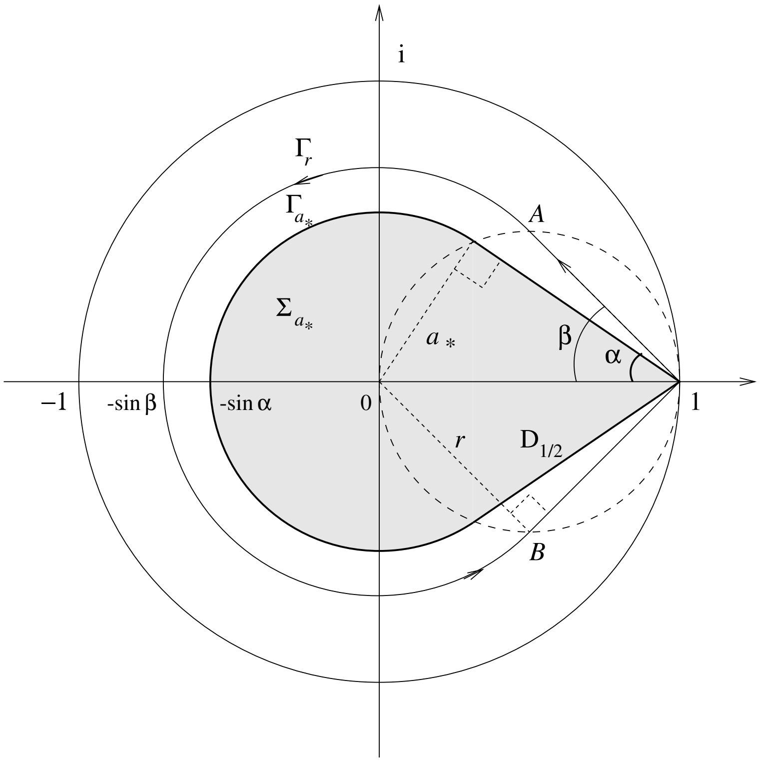

Figure 1: Illustration of the set Σ_{a_*} (shaded domain) with boundary $\Gamma_{a_*} = \partial\Sigma_{a_*}$, where $a_* = \sin\alpha$, as well as of our choice of the contour Γ_r in the resolvent set $\rho(F(\tau))$, where $r = \sin\beta > a_*$. The contour Γ_r consists of two segments of tangent straight lines $(1, A)$ and $(1, B)$ and the arc (A, B) of radius r. The dotted circle $\partial\mathrm{D}_{1/2}$ corresponds to the set of tangent points for the different values of $r \in [0, 1]$.

Theorem 2.4 *Let $A \geq I$ and let B be an m-accretive operator obeying condition (ii) with $a_* < 1$. Then there is a constant C such that*

$$\|T(\tau)^k(I - T(\tau))\| \leq \frac{C}{k+1}, \qquad k = 0, 1, 2, \ldots, \tag{2.44}$$

holds for any $\tau > 0$.

Proof: By Theorem 2.3, one gets that for any $r > a_*$ and $\tau > 0$, $\Gamma_r \in \rho(T(\tau))$, the resolvent set of $T(\tau)$. Using the Dunford-Taylor calculus we get the representation

$$T(\tau)^k(I - T(\tau)) = \frac{1}{2\pi i} \int_{\Gamma_r} z^k(1-z)(z - T(\tau))^{-1} dz. \tag{2.45}$$

We estimate the integral (2.45) with help of the estimate (2.26):

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_r} z^k(1-z)(z - T(\tau))^{-1} dz \right\| \leq \frac{1}{2\pi} \int_{\Gamma_r} |z^k(1-z)| \frac{r}{r - a_*} \|(z - e^{-\tau A})^{-1}\| |dz|. \tag{2.46}$$

Since the integral in the right hand side of (2.46) is invariant with respect to conjugation $z \rightarrow \overline{z}$, it is sufficient to estimate the integral on the path Γ_r^+: the branch of Γ_r with $\Im\mathrm{m}\, z > 0$. This branch consists of segment $(1, A)$ and arc $(\pi/2 - \beta, \pi)$ of radius r, see Figure 1.

Let us parametrize $(1, A)$ by

$$z = 1 - se^{-i\beta}, \ 0 \leq s \leq \cos\beta, \text{ with } r = \sin\beta. \tag{2.47}$$

Since for (2.47) and the self-adjoint $e^{-\tau A}$ one has $\|(z - e^{-\tau A})^{-1}\| \leq (\Im\mathrm{m}\, z)^{-1} \leq (s \sin\beta)^{-1}$, we get

$$\int_{(1,A)} |z^k(1-z)| \|(z - e^{-\tau A})^{-1}\| |dz| \leq \int_0^{\cos\beta} ds |1 - se^{-i\beta}|^k \frac{1}{\sin\beta} \tag{2.48}$$

Notice that for $0 \leq s \leq \cos\beta$ one gets by convexity that

$$|1 - se^{-i\beta}| \leq 1 - s\frac{1 - \sin\beta}{\cos\beta} \tag{2.49}$$

Therefore, we obtain for (2.48) the estimate

$$\int_{(1,A)} |z^k(1-z)| \|(z - e^{-\tau A})^{-1}\| |dz| \leq \frac{\cos\beta}{\sin\beta(1 - \sin\beta)} \frac{1 - (\sin\beta)^k}{k+1} \tag{2.50}$$

For the arc $(\pi/2 - \beta, \pi)$ of radius r, one has the estimate $\|(z - e^{-\tau A})^{-1}\| \leq (r\cos\beta)^{-1}$. Then we find:

$$\int_{\Gamma_r^+ \setminus (1,A)} |z^k(1-z)| \|(z - e^{-\tau A})^{-1}\| |dz| \leq (\pi/2 + \beta) r \frac{2r^k}{r\cos\beta} \leq \frac{2\pi}{\cos\beta} \frac{1/r}{e\ln(1/r)} \frac{1}{k+1}. \tag{2.51}$$

In the last inequality we use that: $\sup_{k \geq 0}(k+1)e^{-k\ln x} = x/(e\ln x)$.

Finally, combining (2.46), (2.50) and (2.51) we obtain:

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_r} z^k (1-z)(z - T(\tau))^{-1} dz \right\| \leq \tag{2.52}$$
$$2\left(\frac{\cos\beta}{2\pi \sin\beta (1-\sin\beta)} + \frac{1/r}{e\cos\beta \ln(1/r)} \right) \frac{r}{r-a_*} \frac{1}{k+1}$$

which gives the announced estimate (2.44) with

$$C = \left(\frac{r\cos\beta}{2\pi \sin\beta (1-\sin\beta)} + \frac{1}{e\cos\beta \ln(1/r)} \right) \frac{2}{r-a_*}. \tag{2.53}$$

□

Remark: If $T(\tau) = e^{-\tau A} e^{-\tau B}$ is replaced by $F(\tau) = e^{-\tau B} e^{-\tau A}$, then the above Theorems 2.3 and 2.4 remain valid if B satisfies, instead of (ii), the condition (i) with $a < 1$. For the proof one has to consider the adjoint family $F(\tau)^* = e^{-\tau A} e^{-\tau B^*}$. In this case we have simply to replace a_* by a in the course of proofs.

3 Proof of the main Theorem

First we recall the following property of holomorphic semigroups (see e.g. [8, Ch.IX]):

Proposition 3.1 *Let $-H$ be the generator of a bounded holomorphic semigroup. Then* $\operatorname{ran}(e^{-tH}) \subseteq \operatorname{dom}(H)$*, and we have the estimate:*

$$\| He^{-tH} \| \leq \frac{C_H}{t}, \qquad t > 0. \tag{3.1}$$

Lemma 3.2 *Let A be a self-adjoint operator, $A \geq I$. Let operator B be m-accretive, satisfying (ii) with $a_* < 1$. Then operator $I - T(\tau)$ is boundedly invertible for each $\tau > 0$ and*

$$\| (I - T(\tau))^{-1} u \| \leq \frac{1}{1-a_*} \left(\|u\| + \frac{1}{\tau} \|A^{-1}u\| \right), \qquad u \in \mathfrak{H}, \tag{3.2}$$

Proof: We use representation

$$I - T(\tau) = I - e^{-\tau A} + e^{-\tau A}(I - e^{-\tau B}). \tag{3.3}$$

Since $\|e^{-\tau A}\| \leq e^{-\tau} < 1$, the operator $I - e^{-\tau A}$ is boundedly invertible provided $\tau > 0$. Hence, one gets representation

$$I - T(\tau) = (I - e^{-\tau A})(I + Y(\tau)). \tag{3.4}$$

where we have used the notation (2.29). By (2.37) and $a_* < 1$ we obtain that $I - T(\tau)$ is invertible. Moreover, we have the estimate :

$$\|(I - T(\tau))^{-1} u\| \leq \frac{1}{1-a_*} \left\| \left(I - e^{-\tau A} \right)^{-1} u \right\|, \quad u \in \mathfrak{H} \tag{3.5}$$

By Lemma 2.3 of [12], we have for any $u \in \mathfrak{H}$ and $\tau > 0$ that

$$\left\| \left(I - e^{-\tau A} \right)^{-1} u \right\| \leq \|u\| + \frac{1}{\tau} \|A^{-1} u\|. \tag{3.6}$$

The estimates (3.5), (3.6) lead to the desired inequality (3.2) □

Lemma 3.3 *Let $-A$, boundedly invertible, and $-B$ be generators of bounded semigroups, with B satisfying the conditions (i), (ii) for some (not necessarily bounded by 1) a and a_*. Assume that operator $-H = -(A+B)$ is the boundedly invertible generator of a bounded semigroup. Then there exists a constant L_1 such that for all $\tau > 0$:*

$$\begin{aligned} \left\|A^{-1}\left(e^{-\tau A}e^{-\tau B} - e^{-\tau(A+B)}\right)\right\| &\leq L_1\tau && (3.7) \\ \left\|\left(e^{-\tau A}e^{-\tau B} - e^{-\tau(A+B)}\right)A^{-1}\right\| &\leq L_1\tau && (3.8) \end{aligned}$$

Proof. See Lemma 5 in [2]. □

Lemma 3.4 *Let A, B and $H = A+B$ be the same as in Lemma 3.3. Then there exists a constant L_2 such that for all $\tau \geq 0$:*

$$\left\|A^{-1}\left(e^{-\tau A}e^{-\tau B} - e^{-\tau(A+B)}\right)A^{-1}\right\| \leq L_2\tau^2 \tag{3.9}$$

Proof. See Lemma 6 in [2]. □

Now we are in position to prove our main Theorem.

Theorem 3.5 *Let A be a self-adjoint operator and $A \geq I$. Let B satisfy conditions (i), (ii), (iii), and $a, a_* < 1$. Then there is a constant $L > 0$ such that the estimate*

$$\left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH}\right\| \leq L\,\frac{\ln(n)}{n}, \qquad n = 3, 4, \ldots, \tag{3.10}$$

holds uniformly in $t \geq 0$, where $H = A+B$ with $\mathrm{dom}(H) = \mathrm{dom}(A)$.

Proof: Let $\tau = t/n$ and $U(\tau) = e^{-\tau H}$, which is a holomorphic contraction semigroup by Theorem 2.1. We start with the representation

$$\begin{aligned} & T(\tau)^n - U(\tau)^n && (3.11) \\ = \ & T(\tau)^{n-1}(T(\tau) - U(\tau)) + \sum_{m=1}^{n-1} T(\tau)^{n-m-1}(T(\tau) - U(\tau))U(\tau)^m. \end{aligned}$$

Since by Lemma 3.2 the operator $(I - T(\tau))^{-1}$ exists for any $\tau > 0$, we find

$$\begin{aligned} T(\tau)^n - U(\tau)^n = \ & T(\tau)^{n-1}(T(\tau) - U(\tau)) && (3.12) \\ & + \sum_{m=1}^{n-1} T(\tau)^{n-m-1}(I - T(\tau))(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m. \end{aligned}$$

This leads to the estimate

$$\begin{aligned} \|T(\tau)^n - U(\tau)^n\| \leq \ & \|T(\tau)^{n-1}(T(\tau) - U(\tau))\| && (3.13) \\ & + \sum_{m=1}^{n-1} \|T(\tau)^{n-m-1}(I - T(\tau))\|\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m\|. \end{aligned}$$

Then by (3.2) we obtain

$$\begin{aligned} & \|(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m\| && (3.14) \\ \leq \ & \frac{1}{1 - a_*}\left\{\|(T(\tau) - U(\tau))U(\tau)^m\| + \frac{1}{\tau}\left\|A^{-1}(T(\tau) - U(\tau))U(\tau)^m\right\|\right\}. \end{aligned}$$

Since

$$(T(\tau) - U(\tau))U(\tau)^m = (T(\tau) - U(\tau))A^{-1}AH^{-1}HU(\tau)^m, \tag{3.15}$$

one gets

$$\|(T(\tau) - U(\tau))U(\tau)^m\| \le \left\|(T(\tau) - U(\tau))A^{-1}\right\| \|AH^{-1}\|\|HU(\tau)^m\|. \tag{3.16}$$

Applying now Proposition 3.1 we obtain

$$\|(T(\tau) - U(\tau))U(\tau)^m\| \le \left\|A^{-1}(T(\tau) - U(\tau))\right\| \|AH^{-1}\|\frac{C_H}{\tau}\frac{1}{m}. \tag{3.17}$$

By $AH^{-1} = (I + BA^{-1})^{-1}$ and $\|BA^{-1}\| \le a < 1$ one gets

$$\|AH^{-1}\| \le \frac{1}{1-a}. \tag{3.18}$$

Hence

$$\|(T(\tau) - U(\tau))U(\tau)^m\| \le \left\|A^{-1}(T(\tau) - U(\tau))\right\| \frac{C_H}{(1-a)\tau}\frac{1}{m}. \tag{3.19}$$

Similarly we get

$$\|A^{-1}(T(\tau) - U(\tau))U(\tau)^m\| \le \left\|A^{-1}(T(\tau) - U(\tau))A^{-1}\right\| \frac{C_H}{(1-a)\tau}\frac{1}{m}. \tag{3.20}$$

Inserting (3.19) and (3.20) into (3.14) we find

$$\begin{aligned} &\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m\| \qquad (3.21)\\ \le\ &\left\{\frac{1}{\tau}\|(A^{-1}(T(\tau) - U(\tau))\| + \frac{1}{\tau^2}\left\|A^{-1}(T(\tau) - U(\tau))A^{-1}\right\|\right\}\frac{C_H}{(1-a_*)(1-a)}\frac{1}{m}. \end{aligned}$$

Applying to (3.21) Lemma 3.3 and Lemma 3.4 we find

$$\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m\| \le C_H\frac{L_1 + L_2}{(1-a_*)(1-a)}\frac{1}{m} = \frac{L_3}{m}, \tag{3.22}$$

where we put $L_3 := C_H(L_1 + L_2)/(1 - a_*)(1 - a)$. Further, by Theorem 2.4 we have

$$\|T(\tau)^{n-m-1}(I - T(\tau))\| \le \frac{C}{n-m}, \qquad n = 2, 3, \ldots, \qquad m = 0, 1, 2, \ldots, n-1. \tag{3.23}$$

Inserting (3.22) and (3.23) into (3.13) we get

$$\|T(\tau)^n - U(\tau)^n\| \le \|T(\tau)^{n-1}(T(\tau) - U(\tau))\| + CL_3\sum_{m=1}^{n-1}\frac{1}{(n-m)m}. \tag{3.24}$$

Since

$$\begin{aligned} &\sum_{m=1}^{n-1}\frac{1}{(n-m)m} \qquad (3.25)\\ =\ &\frac{2}{n}\sum_{m=1}^{n-1}\frac{1}{m} \le \frac{2}{n}\left(1 + \ln(n-1)\right) \le 4\frac{\ln(n)}{n}, \qquad n = 2, 3, \ldots, \end{aligned}$$

one has

$$\|T(\tau)^n - U(\tau)^n\| \leq \|T(\tau)^{n-1}(F(\tau) - U(\tau))\| + 4CL_3\frac{\ln(n)}{n}. \tag{3.26}$$

It remains to estimate $\|T(\tau)^{n-1}(T(\tau) - U(\tau))\|$. Again by invertibility of $I - T(\tau)$ (see Lemma 3.2) we find

$$\|T(\tau)^{n-1}(T(\tau) - U(\tau))\| \leq \|T(\tau)^{n-1}(I - T(\tau)\|\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))\|. \tag{3.27}$$

By (3.2) we get

$$\begin{aligned} &\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))\| \\ \leq\ & \frac{1}{1-a_*}\left\{\|T(\tau) - U(\tau)\| + \frac{1}{\tau}\left\|A^{-1}(T(\tau) - U(\tau))\right\|\right\} \\ \leq\ & \frac{1}{1-a_*}\left\{2 + \frac{1}{\tau}\left\|A^{-1}(T(\tau) - U(\tau))\right\|\right\}. \end{aligned} \tag{3.28}$$

Applying to the right-hand side of (3.28) Lemma 3.3 we find

$$\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))\| \leq \frac{2 + L_1}{1 - a_*}. \tag{3.29}$$

Inserting (3.29) into (3.27) one gets

$$\|T(\tau)^{n-1}(T(\tau) - U(\tau))\| \leq \frac{2 + L_1}{1 - a_*}\|T(\tau)^{n-1}(I - F(\tau)\|. \tag{3.30}$$

Then applying Theorem 2.4 to the right-hand side of (3.30) we obtain

$$\|T(\tau)^{n-1}(T(\tau) - U(\tau))\| \leq \frac{2 + L_1}{1 - a_*}\,\frac{C}{n}, \ n = 1, 2, \ldots\ . \tag{3.31}$$

Inserting (3.31) into (3.26) we find the estimate (3.10):

$$\|T(\tau)^n - U(\tau)^n\| \leq L\,\frac{\ln(n)}{n}, \qquad n = 3, 4, \ldots\ . \tag{3.32}$$

Here $L := C(2 + L_1)(1 - a_*)^{-1} + 4CL_3$ and we used $1/n < \ln(n)/n$ to estimate (3.31), for $n = 3, 4, \ldots$ □

Corollary 3.6 *Let A and B satisfy the conditions of Theorem 3.5. Then there are constants L', L'' such that*

$$\left\|\left(e^{-tB/n}e^{-tA/n}\right)^n - e^{-tH}\right\| \leq L'\,\frac{\ln(n)}{n}, \qquad n = 3, 4, \ldots, \tag{3.33}$$

$$\left\|\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n - e^{-tH}\right\| \leq L''\,\frac{\ln(n)}{n}, \qquad n = 3, 4, \ldots\ . \tag{3.34}$$

holds uniformly in $t \geq 0$.

Proof. Since B^* satisfies conditions of Theorem 3.5, we have

$$\left\|\left(e^{-tA/n}e^{-tB^*/n}\right)^n - e^{-tH^*}\right\| \leq L'\,\frac{\ln(n)}{n}. \tag{3.35}$$

Notice that $H^* = A + B^*$ is well defined and m-accretive. By taking the adjoint sequence of operators, we find (3.33). In order to obtain (3.34), one has to observe that the line of reasoning in the proof of Theorem 3.5 follows thought verbatim for the symmetrized formula. □

4 Conclusion

The aim of the present paper was to show that the operator-norm error bound estimates for the exponential Trotter product formula persist if for a couple $A = A^*, B$ we pass from a self-adjoint A-small perturbation B to an m-accretive A-small operator B. This generalizes the previous result [12] to the case of a non-self-adjoint B under conditions that $\|Bu\| \leq a\|Au\|$, $u \in \mathrm{dom}(A)$ with $a \in (0,1)$ and $\|B^*u\| \leq a_*\|Au\|$, $u \in \mathrm{dom}(A)$ with $a_* \in (0,1)$, see Theorem 3.5.

Let us consider the following special case: $B(z) = zV$ where V is a self-adjoint operator which is relatively small with respect to $A \geq I$, i.e. $\mathrm{dom}(A) \subseteq \mathrm{dom}(V)$ and

$$\|Vf\| \leq a\|Af\|, \quad f \in \mathrm{dom}(A), \quad \text{for some} \quad a \in (0,1). \tag{4.1}$$

Obviously, the operators $B(z) = zV$ and $B(z)^* = \overline{z}V$ are m-accretive and satisfy the conditions (i)-(ii) for $\Re\mathrm{e}\, z \geq 0$. If, in addition, $|z| \leq 1$, then the constants a, a_* for $B(z)$ remain smaller than 1. Applying Theorem 3.5 we find, for instance, that the error estimate

$$\|(e^{-tA/n}e^{-tzV/n})^n - e^{-tH(z)}\| \leq L\frac{\ln(n)}{n}, \quad n = 2, 3\ldots, \tag{4.2}$$

holds uniformly in $t \geq 0$ and in z such that $\Re\mathrm{e}\, z \geq 0$, $|z| \leq 1$, where $H(z) = A + zV$ on $\mathrm{dom}(H(z)) = \mathrm{dom}(A)$. In the case $z = i$, we consider the perturbation of a self-adjoint contraction semigroup by a unitary group.

Since a, a^* may be strictly positive, the present paper is also a generalisation of [2], where the estimate (1.3) was found for B which is *infinitesimally* Kato-small with respect to A. Notice that in [2] we did not suppose that the operator A is self-adjoint but only that it generates a contraction holomorphic semigroup. It is the case in particular when A is an m-sectorial operator with vertex in 0, see [8, Ch.IX].

Therefore, it is relevant to ask: why do we need $A = A^*$ in the main Theorem of the present paper ? First, this condition is important for estimate $\|(z - e^{-\tau A})^{-1}\|$, see (2.48) and (2.51) in the proof of Theorem 2.4. Second, it makes possible to use the spectral theorem to obtain the estimates (2.15), see Lemma 2.2, and (3.2), see Lemma 3.2.

Suppose, for example, that instead to be self-adjoint the operator A is m-sectorial. Then operator-norm continuity of the holomorphic semigroup $U_A(t) = e^{-tA}$ for $t > 0$, allows to localize the *spectrum* of this semigroup $\sigma(e^{-tA})$ but *not* the numerical range $\Theta(e^{-tA})$! However to estimate the norm of the resolvent $(z - e^{-\tau A})^{-1}$, the spectrum is not sufficient in general, except for normal (in particular self-adjoint) A, for which $\overline{\Theta(e^{-tA})}$ is the convex hull of $\sigma(e^{-tA})$. Thus we solve the problem here only for the couple of self-adjoint A and m-accretive B, i.e. the limiting case of angles $\theta_A = 0$ and $\theta_B = \pi/2$.

Although the above mentioned result of [2] allows to pass to an m-sectorial operator A with $0 < \theta_A < \pi/2$, the price is that B and B^* should be infinitesimally $A-$small if one insists on the *error bound estimate*. If *not*, then [1] for two m-sectorial operators A and B ($\theta_A, \theta_B \in (0, \pi/2)$) such that:

– either $(I + A)^{-1}$ or $(I + \Re\mathrm{e}\, A)^{-1/2}(I + \Re\mathrm{e}\, B)^{1/2}$ is compact,

– or A and B satisfy conditions

(i) $\Re\mathrm{e}\ A, \Re\mathrm{e}\ B \geq \gamma > 0$;

(*ii*) dom($\Re\text{e } A$) $\subseteq$ dom($\Im\text{m } A$) ;

(*iii*) dom($\Re\text{e } B$) $\subseteq$ dom($\Im\text{m } B$) ;

(*iv*) dom($(\Re\text{e } A)^\alpha$) $\subseteq$ dom($(\Im\text{m } B)^\alpha$) for some $\alpha \in (1/2, 1]$;

(*v*) $\|(\Re\text{e } B)^\alpha u\| \leq c\|(\Re\text{e } A)^\alpha u\|$, $u \in \text{dom}((\Re\text{e } A)^\alpha)$, $0 < c < 1$ and some $\alpha \in (1/2, 1]$,

one gets

$$\|.\| - \lim_{n\to+\infty}(e^{-tA/n}e^{-tB/n})^n = e^{-tH}. \tag{4.3}$$

The operator-norm convergence in (4.3) is uniform in t for any compact set of the sector $S_\omega = \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \omega\}$, where $\omega = \pi/2 - \max\{\theta_A, \theta_B\}$ and H is the form-sum of A and B.

We conclude by remark that our main Theorem 3.5 is valid if, instead of $e^{-tA/n}$, one considers operator $f(tA/n)$. Here $f : [0, +\infty)\to[0, 1]$ is a Borel function such that, cf.[12] :

$$C_0 := \sup_{x>0}\left\{\frac{x\sqrt{f(x)}}{1 - f(x)}\right\} < +\infty \tag{4.4}$$

$$C_1 := \sup_{x>0}\left\{\frac{1 - f(x)}{x}\right\} < +\infty \tag{4.5}$$

$$C_2 := \sup_{x>0}\left|\left(f(x) - \frac{1}{1+x}\right)\frac{1}{x^2}\right| < +\infty \tag{4.6}$$

$$\tag{4.7}$$

as well as

$$0 < C_0 a < 1. \tag{4.8}$$

Then we get

$$\left\|\left(f(tA/n)e^{-tB/n}\right)^n - e^{-t(A+B)}\right\| \leq L_0\frac{\ln n}{n},\ n \geq 3 \tag{4.9}$$

uniformly in $t \geq 0$. The proof needs only minor modifications of Lemmata 3.2-3.4 as it is outlined in [12].

Acknowledgements

H.N. would like to thank Deutsche Forschungsgemeinschaft for financial support and Centre de Physique Théorique for hospitality.

References

[1] Cachia, V.; Zagrebnov, V.A.: Operator-norm convergence of the Trotter product formula for sectorial generators, to appear in *Lett. Math. Phys.* (2000).

[2] Cachia, V.; Zagrebnov, V. A.: Operator-norm convergence of the Trotter product formula for holomorphic semigroups, to appear in *J. Operator Theory* (2000).

[3] Chernoff, P.R.: Note on product formulas for operator semigroups. *J. Funct. Anal.* **2** (1968), 238-242.

[4] Chernoff, P.R.: Product formulas, nonlinear semigroups and addition of unbounded operators. *Mem. Am. Math. Soc.* **140** (1974), 1-121.

[5] Hiai, F.: Trace norm convergence of exponential product formula *Lett. Math. Phys.* **33** (1995), 147-158.

[6] Ichinose, T.; Tamura, H.: Error estimate in operator norm for Trotter-Kato product formula. *Integr. Equ. Oper. Theory* **27** (1997), 195-207.

[7] Ichinose, T.; Tamura, H.: Error bound in Trace norm for Trotter-Kato product formula of Gibbs semigroups. *Asymptotic Analysis* **17** (1998), 239-266.

[8] Kato T.: Perturbation theory for linear operators, Springer Verlag, Berlin 1966.

[9] Kato, T: On the Trotter-Lie product formula. *Proc. Japan Acad.* **50** (1974), 694-698.

[10] Kato, T.: Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups. *Topics in Funct. Anal., Ad. Math. Suppl. Studies* Vol. **3**, 185-195 (I.Gohberg and M.Kac eds.). Acad. Press, New York 1978.

[11] Neidhardt, H.; Zagrebnov V.A.: The Trotter product formula for Gibbs semigroups. *Comm. Math. Phys.* **131** (1990), 333-346.

[12] Neidhardt, H.; Zagrebnov V.A.: On error estimates for the Trotter-Kato product formula. *Lett. Math. Phys.* **44** (1998), 169-186.

[13] Neidhardt, H.; Zagrebnov V.A.: Fractional powers of self-adjoint operators and Trotter-Kato product formula. *Integr. Equ. Oper. Theory,* **35** (1999), 209-231.

[14] Neidhardt, H.; Zagrebnov V.A.: Trotter-Kato product formula and operator-norm convergence. *Comm. Math. Phys.* **205** (1999), 129-159.

[15] Neidhardt, H.; Zagrebnov V.A.: Trotter-Kato product formula and symmetrically normed ideals. *J. Funct. Anal.* **167** (1999), 113-167.

[16] Reed, M.; Simon, B.: Methods of Modern Mathematical Physics, Vol.I: Functional Analysis, Acad. Press, New York 1972.

[17] Rogava, D.L.: Error bounds for Trotter-type formulas for self-adjoint operators.*Funct. Anal. Application* **27**, No. 3 (1993), 217-219.

[18] Trotter, H.F.: On the products of semigroups of operators. *Proc. Am. Math. Soc.* **10** (1959), 545-551.