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LINEARIZATION OF ANALYTIC AND NON–ANALYTIC GERMS OF DIFFEOMORPHISMS OF $(\mathbb{C}, 0)$

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Abstract. We study Siegel's center problem on the linearization of germs of diffeomorphisms in one variable. In addition of the classical problems of formal and analytic linearization, we give sufficient conditions for the linearization to belong to some algebras of ultradifferentiable germs closed under composition and derivation, including Gevrey classes.

In the analytic case we give a positive answer to a question of J.-C. Yoccoz on the optimality of the estimates obtained by the classical majorant series method.

In the ultradifferentiable case we prove that the Brjuno condition is sufficient for the linearization to belong to the same class of the germ. If one allows the linearization to be less regular than the germ one finds new arithmetical conditions, weaker than the Brjuno condition. We briefly discuss the optimality of our results.

1. introduction

In this paper we study the Siegel center problem [He]. Consider two subalgebras $A_1 \subset A_2$ of $z\mathbb{C}[[z]]$ closed with respect to the composition of formal series. For example $z\mathbb{C}[[z]]$, $z\mathbb{C}\{z\}$ (the usual analytic case) or Gevrey–s classes, $s > 0$ (i.e. series $F(z) = \sum_{n\geq 0} f_n z^n$ such that there exist $c_1, c_2 > 0$ such that $|f_n| \leq c_1 c_2^n (n!)^s$ for all $n \geq 0$). Let $F \in A_1$ being such that $F'(0) = \lambda \in \mathbb{C}^*$. We say that F is linearizable in A_2 if there exists $H \in A_2$ tangent to the identity and such that

$$F \circ H = H \circ R_\lambda \tag{1.1}$$

where $R_\lambda(z) = \lambda z$. When $|\lambda| \neq 1$, the Poincaré-Konigs linearization theorem assures that F is linearizable in A_2. When $|\lambda| = 1$, $\lambda = e^{2\pi i \omega}$, the problem is much more difficult, especially if one looks for *necessary and sufficient* conditions on λ which assure that *all* $F \in A_1$ *with the same* λ *are linearizable in* A_2. The only trivial case is $A_2 = z\mathbb{C}[[z]]$ (formal linearization) for which one only needs to assume that λ is not a root of unity, i.e. $\omega \in \mathbb{R} \setminus \mathbb{Q}$.

In the analytic case $A_1 = A_2 = z\mathbb{C}\{z\}$ let S_λ denote the space of analytic germs $F \in z\mathbb{C}\{z\}$ analytic and injective in the unit disk $\mathbb{D}$ and such that $DF(0) = \lambda$ (note that any $F \in z\mathbb{C}\{z\}$ tangent to R_λ may be assumed to belong to S_λ provided that the variable z is suitably rescaled). Let $R(F)$ denote the radius of convergence of the unique tangent to the identity linearization H associated to F. J.-C. Yoccoz [Yo] proved that the *Brjuno condition* (see Appendix A) is necessary and sufficient for having $R(F) > 0$ for all $F \in S_\lambda$. More precisely Yoccoz proved the following

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estimate: assume that $\lambda = e^{2\pi i\omega}$ is a Brjuno number. There exists a universal constant $C > 0$ (independent of λ) such that

$$|\log R(\omega) + B(\omega)| \leq C$$

where $R(\omega) = \inf_{F\in S_\lambda} R(F)$ and B is the Brjuno function (A.3). Thus $\log R(\omega) \geq -B(\omega) - C$.

Brjuno's proof [Br] gives an estimate of the form

$$\log r(\omega) \geq -C' B(\omega) - C''$$

where one can choose $C' = 2$ [He]. Yoccoz's proof is based on a geometric renormalization argument and Yoccoz himself asked whether or not was possible to obtain $C' = 1$ by direct manipulation of the power series expansion of the linearization H as in Brjuno's proof ([Yo], Remarque 2.7.1, p. 21). Using an arithmetical lemma due to Davie [Da] (Appendix B) we give a positive answer (Theorem 2.1) to Yoccoz's question.

We then consider the more general ultradifferentiable case $A_1 \subset A_2 \neq z\mathbb{C}\{z\}$. If one requires $A_2 = A_1$, i.e. the linearization H to be as regular as the given germ F, once again the Brjuno condition is sufficient. Our methods do not allow us to conclude that the Brjuno condition is also necessary, a statement which is in general false as we show in section 2.3 where we exhibit a Gevrey–like class for which the sufficient condition coincides with the optimal arithmetical condition for the associated linear problem. Nevertheless it is quite interesting to notice that given any algebra of formal power series which is closed under composition (as it should if one whishes to study conjugacy problems) and derivation a germ in the algebra is linearizable *in the same algebra* if the Brjuno condition is satisfied.

If the linearization is allowed to be less regular than the given germ (i.e. A_1 is a proper subset of A_2) one finds a new arithmetical condition, weaker than the Brjuno condition. This condition is also optimal if the small divisors are replaced with their absolute values as we show in section 2.4. We discuss two examples, including Gevrey–s classes.[1]

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2. the Siegel center problem

Our first step will be the formal solution of equation (1.1) assuming only that $F \in z\mathbb{C}[[z]]$. Since $F \in z\mathbb{C}[[z]]$ is assumed to be tangent to R_λ then $F(z) = \sum_{n\geq 1} f_n z^n$ with $f_1 = \lambda$. Analogously since $H \in z\mathbb{C}[[z]]$ is tangent to the identity $H(z) = \sum_{n=1}^{\infty} h_n z^n$ with $h_1 = 1$. If λ is not a root of unity equation (1.1) has a unique solution $H \in z\mathbb{C}[[z]]$ tangent to the identity: the power series coefficients satisfy the recurrence relation

$$h_1 = 1 \; , \; h_n = \frac{1}{\lambda^n - \lambda} \sum_{m=2}^{n} f_m \sum_{n_1+\ldots+n_m = n \, , \, n_i \geq 1} h_{n_1} \ldots h_{n_m} \; . \tag{2.1}$$

In [Ca] it is shown how to generalize the classical Lagrange inversion formula to non–analytic inversion problems on the field of formal power series so as to obtain an explicit non–recursive formula for the power series coefficients of H.

[1] We refer the reader interested in small divisors and Gevrey–s classes to [Lo, GY1, GY2].

2.1. **The analytic case: a direct proof of Yoccoz's lower bound.** Let S_λ denote the space of germs $F \in z\mathbb{C}\{z\}$ analytic and injective in the unit disk $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$ such that $DF(0) = \lambda$ and assume that $\lambda = e^{2\pi i\omega}$ with $\omega \in \mathbb{R}\setminus\mathbb{Q}$. With the topology of uniform convergence on compact subsets of $\mathbb{D}$, S_λ is a compact space. Let $H_F \in z\mathbb{C}[[z]]$ denote the unique tangent to the identity formal linearization associated to F, i.e. the unique formal solution of (1.1). Its power series coefficients are given by (2.1). Let $R(F)$ denote the radius of convergence of H_F. Following Yoccoz ([Yo], p. 20) we define

$$R(\omega) = \inf_{F\in S_\lambda} R(F)\ .$$

We will prove the following

Theorem 2.1. Yoccoz's lower bound.

$$\log R(\omega) \geq -B(\omega) - C \tag{2.2}$$

where C is a universal constant (independent of ω) and B is the Brjuno function (A.3).

Our method of proof of Theorem 2.1 will be to apply an arithmetical lemma due to Davie (see Appendix B) to estimate the small divisors contribution to (2.1). This is actually a variation of the classical majorant series method as used in [Si, Br].

Proof. Let $s(z) = \sum_{n\geq 1} s_n z^n$ be the unique solution analytic at $z = 0$ of the equation $s(z) = z + \sigma(s(z))$, where $\sigma(z) = \frac{z^2(2-z)}{(1-z)^2} = \sum_{n\geq 2} n z^n$. The coefficients satisfy

$$s_1 = 1\ ,\ s_n = \sum_{m=2}^{n} m \sum_{n_1+\ldots+n_m=n\,,\, n_i\geq 1} s_{n_1}\ldots s_{n_m}\ . \tag{2.3}$$

Clearly there exist two positive constants γ_1, γ_2 such that

$$|s_n| \leq \gamma_1\gamma_2^n\ . \tag{2.4}$$

From the recurrence relation (2.1) and Bieberbach–De Branges's bound $|f_n| \leq n$ for all $n \geq 2$ we obtain

$$|h_n| \leq \frac{1}{|\lambda^n - \lambda|}\sum_{m=2}^{n} m \sum_{n_1+\ldots+n_m=n\,,\, n_i\geq 1} |h_{n_1}|\ldots|h_{n_m}|\ . \tag{2.5}$$

We now deduce by induction on n that $|h_n| \leq s_n e^{K(n-1)}$ for $n \geq 1$, where K is defined in Appendix B. If we assume this holds for all $n' < n$ then the above inequality gives

$$|h_n| \leq \frac{1}{|\lambda^n - \lambda|}\sum_{m=2}^{n} m \sum_{n_1+\ldots+n_m=n\,,\, n_i\geq 1} s_{n_1}\ldots s_{n_m} e^{K(n_1-1)+\ldots K(n_m-1)}\ . \tag{2.6}$$

But $K(n_1-1)+\ldots K(n_m-1) \leq K(n-2) \leq K(n-1) + \log|\lambda^n - \lambda|$ and we deduce that

$$|h_n| \leq e^{K(n-1)}\sum_{m=2}^{n} m \sum_{n_1+\ldots+n_m=n\,,\, n_i\geq 1} s_{n_1}\ldots s_{n_m} = s_n e^{K(n-1)}\ , \tag{2.7}$$

as required. Theorem 2.1 then follows from the fact that $n^{-1}K(n) \leq B(\omega) + \gamma_3$ for some universal constant $\gamma_3 > 0$ (Davie's lemma, Appendix B).

□

2.2. **The ultradifferentiable case.** A classical result of Borel says that the map $J_{\mathbb{R}} : \mathcal{C}^\infty([-1,1],\mathbb{R}) \to \mathbb{R}[[x]]$ which associates to f its Taylor series at 0 is surjective. On the other hand, $\mathbb{C}\{z\} = \varinjlim_{r>0} \mathcal{O}(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z \in \mathbb{C},\, |z| < r\}$ and $\mathcal{O}(\mathbb{D}_r)$ is the $\mathbb{C}$–vector space of $\mathbb{C}$–valued functions analytic in $\mathbb{D}_r$. Between $\mathbb{C}[[z]]$ and $\mathbb{C}\{z\}$ one has many important algebras of "ultradifferentiable" power series (i.e. asymptotic expansions at $z=0$ of functions which are "between" $\mathcal{C}^\infty$ and $\mathbb{C}\{z\}$).

In this part we will study the case A_1 or A_2 (or both) is neither $z\mathbb{C}\{z\}$ nor $z\mathbb{C}[[z]]$ but a general ultradifferentiable algebra $z\mathbb{C}[[z]]_{(M_n)}$ defined as follows.

Let $(M_n)_{n\geq 1}$ be a sequence of positive real numbers such that:

0. $\inf_{n\geq 1} M_n^{1/n} > 0$;
1. There exists $C_1 > 0$ such that $M_{n+1} \leq C_1^{n+1} M_n$ for all $n \geq 1$;
2. The sequence $(M_n)_{n\geq 1}$ is logarithmically convex;
3. $M_n M_m \leq M_{m+n-1}$ for all $m, n \geq 1$.

Definition 2.2. *Let $f = \sum_{n\geq 1} f_n z^n \in z\mathbb{C}[[z]]$; f belongs to the algebra $z\mathbb{C}[[z]]_{(M_n)}$ if there exist two positive constants c_1, c_2 such that*

$$|f_n| \leq c_1 c_2^n M_n \quad \textit{for all}\, n \geq 1\,. \tag{2.8}$$

The role of the above assumptions on the sequence $(M_n)_{n\geq 1}$ is the following: 0. assures that $z\mathbb{C}\{z\} \subset z\mathbb{C}[[z]]_{(M_n)}$; 1. implies that $z\mathbb{C}[[z]]_{(M_n)}$ is stable for derivation. Condition 2. means that $\log M_n$ is convex, i.e. that the sequence (M_{n+1}/M_n) is increasing; it implies that $z\mathbb{C}[[z]]_{(M_n)_{n\geq 1}}$ is an algebra, i.e. stable by multiplication. Condition 3. implies that this algebra is *closed for composition*: if $f, g \in z\mathbb{C}[[z]]_{(M_n)_{n\geq 1}}$ then $f \circ g \in z\mathbb{C}[[z]]_{(M_n)_{n\geq 1}}$. This is a very natural assumption since we will study a *conjugacy* problem.

Let $s > 0$. A very important example of ultradifferentiable algebra is given by the algebra of *Gevrey–s* series which is obtained chosing $M_n = (n!)^s$. It is easy to check that the assumptions 0.–3. are verified. But also more rapidly growing sequences may be considered such as $M_n = n^{an^b}$ with $a > 0$ and $1 < b < 2$.

We then have the following

Theorem 2.3.
1. If $F \in z\mathbb{C}[[z]]_{(M_n)}$ and ω is a Brjuno number then also the linearization H belongs to the same algebra $z\mathbb{C}[[z]]_{(M_n)}$.
2. If $F \in z\mathbb{C}\{z\}$ and ω verifies

$$\limsup_{n\to+\infty} \left(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} - \frac{1}{n}\log M_n \right) < +\infty \tag{2.9}$$

where $k(n)$ is defined by the condition $q_{k(n)} \leq n < q_{k(n)+1}$, then the linearization $H \in z\mathbb{C}[[z]]_{(M_n)}$.
3. Let $F \in z\mathbb{C}[[z]]_{(N_n)}$, where the sequence (N_n) verifies 0,1,2,3 and is asymptotically bounded by the sequence (M_n) (i.e. $M_n \geq N_n$ for all sufficiently large n). If ω verifies

$$\limsup_{n\to+\infty} \left(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} - \frac{1}{n}\log \frac{M_n}{N_n} \right) < +\infty \tag{2.10}$$

where $k(n)$ is defined by the condition $q_{k(n)} \leq n < q_{k(n)+1}$, then the linearization $H \in z\mathbb{C}[[z]]_{(M_n)}$.

Note that conditions (2.9) and (2.10) are generally weaker than the Brjuno condition. For example if given F analytic one only requires the linearization H to be Gevrey–s then one can allow the denominators q_k of the continued fraction expansion of ω to verify $q_{k+1} = \mathcal{O}(e^{\sigma q_k})$ for all $0 < \sigma \leq s$ whereas an exponential growth rate of the denominators of the convergents is clearly forbidden from the Brjuno condition. If the linearization is required only to belong to the class $z\mathbb{C}[[z]]_{(M_n)}$ with $M_n = n^{an^b}$, with $a > 0$ and $1 < b < 2$, one can even have $q_{k+1} = \mathcal{O}(e^{\alpha q_k^\beta})$ for all $\alpha > 0$ and $1 < \beta < b$ and the series $\sum_{k\geq 0} \frac{\log q_{k+1}}{q_k^b}$ converges. This kind of series have been studied in detail in [MMY].

Proof. We only prove (2.10) which clearly implies (2.9) (choosing $N_n \equiv 1$) and also assertion 1. (choosing $M_n \equiv N_n$).

Since it is not restrictive to assume $c_1 \geq 1$ and $c_2 \geq 1$ in $|f_n| \leq c_1 c_2^n N_n$ one can immediately check by induction on n that $|h_n| \leq c_1^{n-1} c_2^{2n-2} s_n N_n e^{K(n-1)}$, where s_n is defined in (2.3). Thus by (2.4) and Davie's lemma one has

$$\frac{1}{n} \log \frac{|h_n|}{M_n} \leq c_3 + \frac{1}{n} \log \frac{N_n}{M_n} + \sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k}$$

for some suitable constant $c_3 > 0$. □

Problem. Are the arithmetical conditions stated in Theorem 2.3 optimal? In particular is it true that given any algebra $A = z\mathbb{C}[[z]]_{(M_n)}$ and $F \in A$ then $H \in A$ *if and only if* ω is a Brjuno number?

We believe that this problem deserves further investigations and that some surprising results may be found. In the next two sections we will give some preliminary results.

2.3. A Gevrey–like class where the linear and non linear problem have the same sufficient arithemtical condition.

Let $\mathbb{C}[[z]]_s$ denote the algebra of Gevrey–s complex formal power series, $s > 0$. If $s' > s > 0$ then $z\mathbb{C}[[z]]_s \subset z\mathbb{C}[[z]]_{s'}$; let

$$A_s = \bigcap_{s'>s} z\mathbb{C}[[z]]_{s'} \ .$$

Clearly A_s is an algebra stable w.r.t. derivative and composition. This algebra can be equivalently characterized requiring that given $f(z) = \sum_{n\geq 1} f_n z^n \in z\mathbb{C}[[z]]$ one has

$$\limsup_{n\to\infty} \frac{\log|f_n|}{n \log n} \leq s \tag{2.11}$$

Consider Euler's derivative (see [Du], section 4)

$$(\delta_\lambda f)(z) = \sum_{n=2}^{\infty} (\lambda^n - \lambda) f_n z^n \ , \tag{2.12}$$

with $\lambda = e^{2\pi i\omega}$. It acts linearly on zA_s and it is a linear automorphism of zA_s if and only if

$$\lim_{k\to\infty} \frac{\log q_{k+1}}{q_k \log q_k} = 0 \tag{2.13}$$

where, as usual, $(q_k)_{k\in\mathbb{N}}$ is the sequence of the denominators of the convergents of ω. This fact can be easily checked by applying the law of the best approximation (Lemma A.3, Appendix A) and the charaterization (2.11) to

$$h(z) = (\delta_\lambda^{-1} f)(z) = \sum_{n\geq 2} \frac{f_n}{\lambda^n - \lambda} z^n \, .$$

Note that the arithmetical condition $\log q_{k+1} = o\left(q_k \log q_k\right)$ is much weaker than Brjuno's condition.

We now consider the Siegel problem associated to a germ $F \in A_s$. Applying the third statement of Theorem 2.3 with $N_n = (n!)^{s+\eta}$ and $M_n = (n!)^{s+\epsilon}$ for any positive fixed $\epsilon > \eta > 0$ one finds that if the following arithmetical condition is satisfied

$$\lim_{k\to\infty} \frac{1}{\log q_k} \sum_{i=0}^{k} \frac{\log q_{i+1}}{q_i} = 0 \tag{2.14}$$

then the linearization H_F also belongs to A_s.[2]

The equivalence of (2.14) and (2.13) is the object of the following

Lemma 2.4. *Let $(q_l)_{l\geq 0}$ be the sequence of denominators of the convergents of $\omega \in \mathbb{R}\setminus\mathbb{Q}$. The following statements are all equivalent:*

1. $\lim_{n\to\infty} \frac{1}{\log n} \sum_{l=0}^{k(n)} \frac{\log q_{l+1}}{q_l} = 0$
2. $\sum_{l=0}^{k(n)} \frac{\log q_{l+1}}{q_l} = o\left(\log q_k\right)$
3. $\log q_{k+1} = o\left(q_k \log q_k\right)$

Proof. 1. $\Longrightarrow$ 2. is trivial (choose $n = q_{k(n)}$).

2. $\Longrightarrow$ 3. Writing for short k istead of $k\,(n)$

$$\begin{aligned}\frac{1}{\log q_k} \sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} &= \frac{\log q_{k+1}}{q_k \log q_k} + \frac{1}{\log q_k} \sum_{l=0}^{k-1} \frac{\log q_{l+1}}{q_l} \\ &= \frac{\log q_{k+1}}{q_k \log q_k} + \frac{o\left(\log q_{k-1}\right)}{\log q_k}\end{aligned}$$

Since $\lim_{k\to\infty} \frac{o(\log q_{k-1})}{\log q_k} = 0$ we get 3.

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[2]In Theorem 2.3 we proved that a sufficient condition with this choice of M_n and N_n is

$$\limsup_{n\to+\infty} \left(\sum_{i=0}^{k(n)} \frac{\log q_{i+1}}{q_i} - \frac{\epsilon-\eta}{n} \log\left(n!\right) \right) \leq C < +\infty$$

which can be rewritten as

$$\limsup_{n\to+\infty} \left(\sum_{i=0}^{k(n)} \frac{\log q_{i+1}}{q_i} - (\epsilon-\eta) \log q_{k(n)} - C \right) = 0$$

from which (2.14) is just obtained dividing by $\log q_{k(n)}$.

3. $\Longrightarrow$ 1. First of all note that since $q_{k(n)} \leq n$ 2. trivially implies 1. Thus it is enough to show that 3. $\Longrightarrow$2.

$\log q_{k+1} = o\left(q_k \log q_k\right)$ means:

$$\forall \epsilon > 0 \ \ \exists \hat{n}\left(\epsilon\right) \ \text{ such that } \forall l > \hat{n}\left(\epsilon\right) \quad \frac{\log q_{l+1}}{q_l \log q_l} < \epsilon$$

If $\log q_{l+1} < a q_l^{\alpha}$ for some positive constants a and $\alpha < 1$ then:

$$\frac{1}{\log q_k} \sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} \leq \frac{a}{\log q_k} \sum_{l=0}^{\infty} \frac{1}{q_l^{1-\alpha}} \leq \frac{aC}{\log q_k}$$

for some universal constant C thanks to (A.2).

If $\log q_{l+1} \geq a q_l^{\alpha}$ and $\frac{1}{2} < \alpha < 1$, consider the decomposition:

(2.15)

$$\frac{1}{\log q_k} \sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} = \underbrace{\frac{\log q_{k+1}}{q_k \log q_k}}_{1} + \underbrace{\frac{1}{\log q_k} \sum_{l=0}^{\hat{n}(\epsilon)} \frac{\log q_{l+1}}{q_l}}_{2} + \underbrace{\frac{1}{\log q_k} \sum_{l=\hat{n}(\epsilon)+1}^{k-1} \frac{\log q_{l+1}}{q_l}}_{3}$$

if $k-1 \geq \hat{n}\left(\epsilon\right)+1$ otherwise the second and the third terms are replaced by $\frac{1}{\log q_k} \sum_{l=0}^{k-1} \frac{\log q_{l+1}}{q_l}$. The third term can be bounded from above by:

$$\frac{1}{\log q_k} \sum_{l=\hat{n}(\epsilon)+1}^{k-1} \frac{\log q_{l+1}}{q_l} \leq \frac{\epsilon}{\log q_k} \sum_{l=\hat{n}(\epsilon)} +1^{k-1} \log q_l \leq \epsilon\left(k-1-\hat{n}\left(\epsilon\right)\right) \frac{\log q_{k-1}}{\log q_k} .$$

Since $\log q_j \leq \frac{2}{e} q_j^{\frac{1}{2}}$, from (A.1) and the hypothesis $\log q_{l+1} \geq a q_l^{\alpha}$ we obtain:

$$\begin{aligned} \frac{1}{\log q_k} \sum_{l=\hat{n}(\epsilon)+1}^{k-1} \frac{\log q_{l+1}}{q_l} &\leq \left(k-1\right) \frac{\epsilon}{a q_{k-1}^{\alpha}} \frac{2}{e} q_{k-1}^{\frac{1}{2}} \leq \\ &\leq \frac{\epsilon 2}{ea} \left(k-1\right) e^{-(k-2)\left(\alpha-\frac{1}{2}\right)\log G} \leq \epsilon C_1 \end{aligned}$$

with $C_1 = \frac{2}{ea} \frac{e^{-1+\left(\alpha-\frac{1}{2}\right)\log G}}{\left(\alpha-\frac{1}{2}\right)\log G}$, $G = \frac{\sqrt{5}+1}{2}$.

The second term of (2.15) is bounded by

$$\frac{1}{\log q_k} \sum_{l=0}^{\hat{n}(\epsilon)} \frac{\log q_{l+1}}{q_l} \leq \frac{C_2}{\left(k-1\right)\log G - \log 2} \leq \epsilon C_2$$

if $k > k\left(\epsilon\right) > \hat{n}(\epsilon)$, for some positive constant C_2.

Putting these estimates together we can bound (2.15) with:

$$\frac{1}{\log q_k} \sum_{l=0}^{k} \frac{\log q_{l+1}}{q_l} \leq \epsilon + \epsilon C_1 + \epsilon C_2$$

for all $\epsilon > 0$ and for all $k > k\left(\epsilon\right)$, thus $\sum_{l=0}^{k} \frac{\log_{l+1}}{q_l} = o\left(\log q_k\right)$ $\square$

2.4. **Divergence of the modified linearization power series when the artihmetical conditions of Theorem 2.3 are not satisfied.** In Theorem 2.3 we proved that if $F \in z\mathbb{C}\{z\}$ and ω verifies condition (2.9) then the linearization $H \in z\mathbb{C}[[z]]_{(M_n)}$. The power series coefficients h_n of H are given by (2.1).

Let us define the sequence of strictly positive real numbers $(\tilde{h}_n)_{n\geq 0}$ as follows:

$$\tilde{h}_0 = 1 \ , \ \ \tilde{h}_n = \frac{1}{|\lambda^n - 1|} \sum_{m=2}^{n+1} |f_m| \sum_{n_1+\ldots+n_m = n+1-m \ , \ n_i \geq 0} \tilde{h}_{n_1} \ldots \tilde{h}_{n_m} \ . \tag{2.16}$$

Clearly $|h_n| \leq \tilde{h}_{n+1}$. Let $\tilde{H}$ denote the formal power series associated to the sequence $(\tilde{h}_n)_{n\geq 0}$

$$\tilde{H}(z) = \sum_{m=1}^{\infty} \tilde{h}_{n-1} z^n \tag{2.17}$$

Following closely [Yo], Appendice 2, in this section we will prove that if condition (2.9) is violated then $\tilde{H}$ doesn't belong to $z\mathbb{C}[[z]]_{(M_n)}$.

Note that since it is not restrictive to assume that $|f_2| \geq 1$ one has

$$\tilde{h}_n > \sum_{k=0}^{n-1} \tilde{h}_k \tilde{h}_{n-1-k} \geq \tilde{h}_{n-1} \ , \tag{2.18}$$

thus the sequence $(\tilde{h}_n)_{n\geq 0}$ is strictly increasing.

Let ω be an irrational number which violates (2.9) and let $U = \{q_j : q_{j+1} \geq (q_j+1)^2\}$ where $(q_j)_{j\geq 1}$ are the denominators of the convergents of x. Since $\inf_n \frac{1}{n} \log M_n = c > -\infty$ we have:

$$\sum_{q_j \notin U, j=0}^{k(n)} \frac{\log q_{j+1}}{q_j} - \frac{\log M_n}{n} \leq \sum_{q_j \notin U, j=0}^{k(n)} \frac{2\log(q_j+1)}{q_j} - c = \tilde{c} < +\infty$$

where $k(n)$ is defined by: $q_{k(n)} \leq n < q_{k(n)+1}$.

On the other hand $\limsup_{n\to\infty} \left(\sum_{j=0}^{k(n)} \frac{\log q_{j+1}}{q_j} - \frac{\log M_n}{n} \right) = \infty$ thus

$$\limsup_{n\to\infty} \left(\sum_{q_j \in U : j=0}^{k(n)} \frac{\log q_{j+1}}{q_j} - \frac{\log M_n}{n} \right) = \infty \tag{2.19}$$

this implies that U is not empty. From now on the elements of U will be denoted by: $q'_0 < q'_1 < \ldots$.

Let $n_i = \lfloor \frac{q'_{i+1}}{q'_i + 1} \rfloor$.

Lemma 2.5. *The subsequence* $\left(\tilde{h}_{q'_i}\right)_{i\geq 0}$ *verifies:*

$$\tilde{h}_{q'_{i+1}} \geq \frac{1}{|\lambda^{q'_{i+1}} - 1|} \tilde{h}_{q'_i}^{n_i} \ . \tag{2.20}$$

Proof. From the definition (2.16) and the assumption $|f_2| \geq 1$ it follows that

$$\tilde{h}_{2s-1} \geq \frac{|f_2|}{|\lambda^{2s-1} - 1|} \tilde{h}_{s-1}^2 \geq \frac{\tilde{h}_{s-1}^2}{2}$$

thus for all $i \geq 2$ and $s \geq 1$ one has

$$\tilde{h}_{2s-1} \geq \frac{\tilde{h}^i_{s-1}}{2} . \tag{2.21}$$

Choosing $s = q'_i + 1$, $i = n_i$ this leads to the desired estimate:

$$\tilde{h}_{q'_{i+1}} \geq \frac{2|f_2|}{|\lambda^{q'_{i+1}} - 1|}\tilde{h}_{q'_{i+1}-1} \geq \frac{2|f_2|}{|\lambda^{q'_{i+1}} - 1|}\tilde{h}_{n_i(q'_i+1)-1} \geq \frac{\tilde{h}^{n_i}_{q'_i}}{|\lambda^{q'_{i+1}} - 1|} .$$

□

By means of the previous lemma we can now prove that $\limsup_{n\to\infty} \frac{1}{n}\log\frac{\tilde{h}_n}{M_n} = +\infty$.

Let $\alpha_i = n_i \frac{q'_i}{q'_{i+1}}$. Then $1 \geq \alpha_i \geq \left(1 - \frac{1}{q'_i+1}\right)^2$, which assures that $\prod_{i\geq 0}\alpha_i = c$ for some finite constant c (depending on ω). Then from (2.20) we get:

$$\frac{1}{q'_{i+1}}\log\frac{\tilde{h}_{q'_{i+1}}}{M_{q'_{i+1}}} \geq c\left[\sum_{j=1}^{i+1} -\frac{\log|\lambda^{q'_j} - 1|}{q'_j} - \frac{1}{q'_{i+1}}\log M_{q'_{i+1}}\right] + c_4$$

which diverges as $i \to \infty$.

Appendix A. continued fractions and Brjuno's numbers

Here we summarize briefly some basic notions on continued fraction development and we define the Brjuno numbers.

For a real number ω, we note $\lfloor\omega\rfloor$ its integer part and $\{\omega\} = \omega - \lfloor\omega\rfloor$ its fractional part. We define the Gauss' continued fraction algorithm:

- $a_0 = \lfloor\omega\rfloor$ and $\omega_0 = \{\omega\}$
- for all $n \geq 1$: $a_n = \lfloor\frac{1}{\omega_{n-1}}\rfloor$ and $\omega_n = \{\frac{1}{\omega_{n-1}}\}$

namely the following representation of ω:

$$\omega = a_0 + \omega_0 = a_0 + \frac{1}{a_1 + \omega_1} = \dots$$

For short we use the notation $\omega = [a_0, a_1, \dots, a_n, \dots]$.

It is well known that to every expression $[a_0, a_1, \dots, a_n, \dots]$ there corresponds a unique irrational number. Let us define the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows:

$$q_{-2} = 1,\ q_{-1} = 0,\ q_n = a_n q_{n-1} + q_{n-2}$$
$$p_{-2} = 0,\ p_{-1} = 1,\ p_n = a_n p_{n-1} + p_{n-2}$$

It is easy to show that: $\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n]$.

For any given $\omega \in \mathbb{R}\setminus\mathbb{Q}$ the sequence $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ satisfies

$$q_n \geq \left(\frac{\sqrt{5}+1}{2}\right)^{n-1} ,\quad n \geq 1 \tag{A.1}$$

thus

$$\sum_{k\geq 0}\frac{1}{q_k} \leq \frac{\sqrt{5}+5}{2} \quad\text{and}\quad \sum_{k\geq 0}\frac{\log q_k}{q_k} \leq \frac{1}{e}\frac{2^{\frac{5}{4}}}{2^{\frac{3}{4}} - 1} , \tag{A.2}$$

and it has the following important properties:

Lemma A.1. *for all $n \geq 1$ then:* $\frac{1}{q_n+q_{n+1}} \leq |q_n\omega - p_n| < \frac{1}{q_{n+1}}$.

Lemma A.2. *If for some integer r and s, $|\ \omega - \frac{r}{s}\ | \leq \frac{1}{2s^2}$ then $\frac{r}{s} = \frac{p_k}{q_k}$ for some integer k.*

Lemma A.3. *The law of best approximation: if $1 \leq q \leq q_k$, $(p,q) \neq (p_n,q_n)$ and $n \geq 1$ then $|qx - p| > |q_n x - p_n|$. Moreover if $(p,q) \neq (p_{n-1},q_{n-1})$ then $|qx-p| > |q_{n-1}x - p_{n-1}|$.*

For a proof of these standard lemmas we refer to [HW].

The growth rate of $(q_n)_{n\in\mathbb{N}}$ describes how rapidly ω can be approximated by rational numbers. For example ω is a diophantine number [Si] if and only if there exist two constants $c > 0$ and $\tau \geq 1$ such that $q_{n+1} \leq c q_n^\tau$ for all $n \geq 0$.

To every $\omega \in \mathbb{R}\setminus\mathbb{Q}$ we associate, using its convergents, an arithmetical function:

$$B(\omega) = \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n} \tag{A.3}$$

We say that ω is a *Brjuno number* or that it satisfies the *Brjuno condition* if $B(\omega) < +\infty$. The Brjuno condition gives a limitation to the growth rate of $(q_n)_{n\in\mathbb{N}}$. It was originally introduced by A.D.Brjuno [Br]. The Brjuno condition is weaker than the Diophantine condition: for example if $a_{n+1} \leq c e^{a_n}$ for some positive constant c and for all $n \geq 0$ then $\omega = [a_0, a_1, \dots, a_n, \dots]$ is a Brjuno number but is not a diophantine number.

Appendix B. Davie's lemma

In this appendix we summarize the result of [Da] that we use, in particular Lemma B.4. Let $\omega \in \mathbb{R}\setminus\mathbb{Q}$ and $\{q_n\}_{n\in\mathbb{N}}$ the partial denominators of the continued fraction for ω in the Gauss' development.

Definition B.1. *Let $A_k = \left\{n \geq 0 \mid \|n\omega\| \leq \frac{1}{8q_k}\right\}$, $E_k = \max(q_k, q_{k+1}/4)$ and $\eta_k = q_k/E_k$. Let A_k^* be the set of non negative integers j such that either $j \in A_k$ or for some j_1 and j_2 in A_k, with $j_2 - j_1 < E_k$, one has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any non negative integer n define:*

$$l(n) = \max\left\{(1+\eta_k)\frac{n}{q_k} - 2, (m_n\eta_k + n)\frac{1}{q_k} - 1\right\}$$

where $m_n = \max\{j \mid 0 \leq j \leq n, j \in A_k^*\}$. We then define the function $h_k(n)$

$$h_k(n) = \begin{cases} \frac{m_n + \eta_k n}{q_k} - 1 & \text{if } m_n + q_k \in A_k^* \\ l(n) & \text{if } m_n + q_k \notin A_k^* \end{cases}$$

The function $h_k(n)$ has some properties collected in the following proposition

Proposition B.2. *The function $h_k(n)$ verifies;*

1. $\frac{(1+\eta_k)n}{q_k} - 2 \leq h_k(n) \leq \frac{(1+\eta_k)n}{q_k} - 1$ *for all n.*
2. *If $n > 0$ and $n \in A_k^*$ then $h_k(n) \geq h_k(n-1) + 1$.*
3. *$h_k(n) \geq h_k(n-1)$ for all $n > 0$.*
4. *$h_k(n + q_k) \geq h_k(n) + 1$ for all n.*

Now we set $g_k(n) = \max\left(h_k(n), \lfloor\frac{n}{q_k}\rfloor\right)$ and we state the following proposition

Proposition B.3. *The function g_k is non negative and verifies:*

1. $g_k(0) = 0$
2. $g_k(n) \leq \frac{(1+\eta_k)n}{q_k}$ *for all* n
3. $g_k(n_1) + g_k(n_2) \leq g_k(n_1 + n_2)$ *for all* n_1 *and* n_2
4. *if* $n \in A_k$ *and* $n > 0$ *then* $g_k(n) \geq g_k(n-1) + 1$

The proof of these propositions can be found in [Da].

Let $k(n)$ be defined by the condition $q_{k(n)} \leq n < q_{k(n)+1}$. Note that k is non–decreasing.

Lemma B.4. Davie's lemma *Let*

$$K(n) = n \log 2 + \sum_{k=0}^{k(n)} g_k(n) \log(2q_{k+1}) \ .$$

The function $K(n)$ *verifies:*

1. *There exists a universal constant* $\gamma_3 > 0$ *such that*
$$K(n) \leq n \left(\sum_{k=0}^{k(n)} \frac{\log q_{k+1}}{q_k} + \gamma_3 \right) \ ;$$
2. $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ *for all* n_1 *and* n_2*;*
3. $-\log|\lambda^n - 1| \leq K(n) - K(n-1)$.

The proof is a straightforward application of Proposition B.3.

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