Variational principles in the analysis of traffic flows. (Why it is worth to go against the flow.)

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Abstract

By means of a novel variational approach and using dual maps techniques and general ideas of dynamical system theory we derive exact results about several models of transport flows, for which we also obtain a complete description of their limit (in time) behavior in the space of configurations. Using these results we study the motion of a speedy passive particle (tracer) moving along/against the flow of slow particles and demonstrate that the latter case might be more efficient.

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AMS Subject Classification: Primary 37E15; Secondary 37D35, 37B15, 60K.

1 Introduction

During the last decade problems related to transport in complex systems attracted a huge amount of interest in particular due to their evident practical importance. In this paper we deal with theoretical aspects of phenomena arising in the modeling of highway traffic flow. Previously much of the effort in the construction and analysis of such models was concentrated on discrete (on time and on space) stochastic models introduced in [6] and later studied by many authors (see [2] for review and further references). All these models were based on the idea to describe the dynamics in terms of cellular automata and to a large extent were studied by means of numerical simulation (especially because of low computational cost of the numerical realization of cellular automata rules, which made it possible to realize large-scale real-time simulations of urban traffic [8]).

My own interest to this type of problems is mainly due to the following practical observation. Going by foot in a slowly moving crowd it is faster to go against the "flow" than in the same direction as other people go. This effect is especially pronounced if one is moving near a boundary between two "flows" of people going in opposite directions. A standard probabilistic model of a diffusion of a particle against/along the flow clearly contradicts to this observation, which very likely indicates a very special (nonrandom) intrinsic structure of the flow in this case. The main aim of the present paper is to study how this structure emerges from arbitrary (random) initial configurations in some simple models of the traffic flow.

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The main quantity of interest in traffic models is the average velocity of cars V and its dependence on the density of cars ρ (called a fundamental diagram) is typically studied in the steady state. Various approaches starting from the mean-field approximation [4] to combinatorial techniques and statistical mechanics methods [3] were used in the analysis of this type of models. In what follows we shall restrict the consideration to deterministic discrete time and space traffic models. The simplest model (among that we consider) describes the dynamics of cars moving along a one-row motor road and is defined as follows.

The road is associated to a one-dimensional ordered lattice of size N with periodic boundary conditions and each position on the lattice is either occupied by a particle, or empty. Denote the number of particles in the configuration by m. Then the density of particles ρ is equal to $\frac{m}{N}$. On the next time step each particle remains on its place if the next position is occupied, and moves forward by one place otherwise. We shall call this model the traffic model with slow particles to distinguish it from other ones.

Quite recently in [2, 3] this model (and some its generalizations) was studied by means of cellular automata techniques. One of the most intriguing phenomenon related to this model is a drastic change of the shape of the curve describing the dependence $\mathrm{V}(\rho)$ of the average velocity of particles on their density in the steady state when the density passes the $1/2$ value (see Fig. 1). It was found numerically and confirmed analytically in the limit of large N (and for a typical initial configuration) [2, 3] that $\mathrm{V}(\rho)$ is equal to 1 while $\rho < 1/2$ and then goes down to zero as $\frac{1}{\rho} - 1$ for $\rho \geq 1/2$.

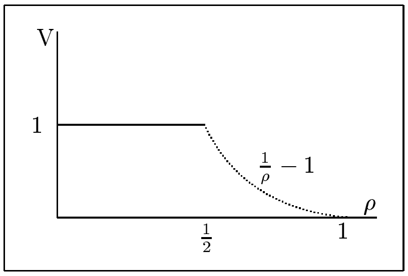

Figure 1: Dependence of the average velocity of particles on their density.

Despite an apparent simplicity of the model with slow particles its dynamics (especially during the transient period and/or in the high density case) is rather nontrivial. The description of the dynamics in terms of cellular automata makes it possible (using a rather complicated combinatorial techniques) to derive an asymptotic description in the limit of large N [3]. Another approach based on the ultradiscrete limit of the Burgers equation was proposed in [7], where the dependence $\mathrm{V}(\rho)$ was proven for a lattice of size N but with the estimate of the transient period $N/2$, which rules out the generalization for the case of the infinite lattice and nonperiodic initial configurations. In this paper we consider this model from a bit more general point of view as a discrete time dynamical system (map T) acting in the space of all possible configurations $\mathbf{X}$ – collections of zeros and ones (describing the positions of particles). We derive the variational approach based on the observation that the average velocity of any configuration does not decrease in time (Proposition 2.4). Simultaneously to the dynamics of particles one can study the dynamics of empty places. Observe that each of these dynamics determines the other one in the unique way. To make use of this observation we introduce a dual map T^* corresponding to the dynamics of empty places. By means of these two basic ideas (variational approach and dual maps techniques) we first prove the formula for the dependence of the average velocity on the density for any (may be small) finite lattice lengths N and any (not necessary typical) initial configurations. We find also that steady state configurations demonstrate certain periodic in time patterns, whose features are described in Theorem 2.1 as well as the convergence to the steady state and the duration of the transient period. Qualitatively our main result about this model is that the following alternative takes place: either the flow consists of only free particles (i.e. there are no clusters of particles), or there are no clusters of empty places.

The paper is organized as follows. In Section 2 we study in detail the above formulated simplest model, which we call 1D periodic model with slow particles. In Section 3 we consider the same model but on the unbounded lattice, which significantly changes the dynamics and cannot be obtained in the limit of large N from the previous one. To demonstrate the power of our dual maps techniques we introduce in Section 4 the model of the one-row motor road with speedy cars. The latter means that instead of the moving by at most 1 position, a particle moves forward until the next occupied position. We study both periodic and unbounded 1D cases. From the mathematical point of view the main difference of this model from the previous one is that the dynamics is not local, which makes it impossible to describe this model in terms of cellular automata. In Section 5 we generalize the traffic models with slow particles for a more practically interesting case of a multi-row motor road and study its properties. Finally in Section 6 we discuss a model of a passive tracer in the flow generated by the traffic model with slow particles confirming our practical observation above.

It is worth mention that to the best of our knowledge only the simplest 1D periodic traffic model with slow particles (among the models considered in the paper) was discussed in the literature previously. For some missing definitions related to dynamical systems theory (especially for systems acting on discrete phase spaces) we refer the reader to books on ergodic theory of dynamical systems (see, for example, [1]).

2 1D traffic model with slow particles on the finite lattice

Let $\mathbf{X} = \{0,1\}^N$ be the set of all possible configurations – collections X of N elements from the alphabet of 2 letters 0 and 1. We consider a map $T : \mathbf{X} \to \mathbf{X}$ defined as follows:

$$TX(x) := \begin{cases} 1 & \text{if } X(x) = 0 \text{ and } X(x-1) = 1 \\ 1 & \text{if } X(x) = 1 \text{ and } X(x+1) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We assume here periodic boundary conditions, i.e. $N + 1 \equiv 1$ and $0 \equiv N$. Observe that this map is not one-to-one and thus the backward (in time) dynamics cannot be defined in a unique way. See examples of the dynamics under the action of the map T on Fig. 2.

(a)		(b)
0011011100010	t=0	1011011100110
0010111010001	t=1	0110111010101
1001110101000	t=2	1101110101010
0101101010100	t=3	1011101010101
0011010101010	t=4	0111010101011
0010101010101	t=5	1110101010110
1001010101010	t=6	1101010101101
0100101010101	t=7	1010101011011

Figure 2: Two examples of the dynamics of the model with slow particles: (a) $m = 6,\ N = 13,\ \rho = 6/13 < 1/2$, (b) $m = 8,\ N = 13,\ \rho = 8/13 > 1/2$.

We shall say that there is a particle at a position x on the lattice if $X(x) = 1$ and that this position is empty otherwise. A particle at a position x is called free if $X(x+1) = 0$. A group (more than 1) of consecutive particles (empty places) we call a cluster of particles (empty places). Observe that the number of particles is preserved under dynamics. Under the action of the map T on the next time step each particle will either go forward by 1 place if this place is empty (occupied by 0) or will remain on the same place otherwise. Introducing the notion of a local velocity:

$$v(X,x) := \begin{cases} 1 & \text{if } X(x) = 1 \text{ and } X(x+1) = 0 \\ 0 & \text{otherwise,} \end{cases}$$

we define the average (in space) velocity of (particles in) the configuration X as

$$\mathrm{V}(X) := \frac{1}{m(X)} \sum_x v(X,x),$$

where $m(X) \equiv \sum_x X(x)$ is the total number of particles in the configuration. Observe that $\mathrm{V}(X)$ is equal to the number of free particles divided by the total number of particles in the configuration.

Theorem 2.1 *For any N and any initial configuration with $m \le N$ particles after at most $\min(m, N-m)$ iterations the configuration will become periodic (in time) with the period N and the average velocity $\mathrm{V} = \min(1, \frac{N}{m} - 1)$.*

Remark 2.1 The smallest period (in time) of the periodic configuration above might be smaller than N, indeed, for $N = 2m$ the smallest period is equal to 2, i.e. there exists $X \in \mathbf{X}$ such that $T^2 X = X$.

The proof of the theorem consists of the following lemmas.

Lemma 2.2 *The length of any cluster of particles cannot increase, and the number of free particles cannot decrease.*

Proof. Fix a cluster of particles. On the next time step its first particle (with the largest x-coordinate) goes out of the cluster and at most one particle can join the cluster from behind. If the number of particles is 2 and no particle will join the cluster from behind on the next time step, only one particle will remain in it, and according to our definition the cluster disappears.

Consider now the number of free particles. On the next time step each cluster of particles loses the first its particle which becomes a free one and at most the same number of free particles can join clusters coming from behind. Therefore the number of free particles cannot decrease. ∎

Corollary 2.3 *Traffic jams cannot appear from nothing.*

On the other hand, since the average velocity of a configuration is equal to the number of free particles in it and by Lemma 2.2 this number grows in time we come the the following variational principle.

Proposition 2.4 *(Variational principle) The functional $\mathrm{V}(X)$ (average velocity) increases under the dynamics up to the moment when it takes its maximal possible value.*

Lemma 2.5 *Any configuration after a finite number of time steps gets into a periodic one.*

Proof. The phase space $\mathbf{X}$ of the considered dynamical system $(\mathbf{X}, T)$ is finite and therefore any its trajectory $T^t X$ will begin repeating after a finite number of iterations. Thus any limit set of the map T consists of periodic configurations. ∎

The map T describes the dynamics of particles. It turns out that often it is simpler to study the dynamics of empty places instead. To make use of this observation for a configuration X we introduce a dual one $X^*(x) := 1 - X(x)$ for all x and define a dual map T^* acting on the same space $\mathbf{X}$:

$$T^*X(x) := \begin{cases} 0 & \text{if } X(x) = 1 \text{ and } X(x+1) = 0 \\ 0 & \text{if } X(x) = 0 \text{ and } X(x-1) = 1 \\ 1 & \text{otherwise,} \end{cases} \tag{2.2}$$

assuming again periodic boundary conditions. One can easily see that the map T^* describes the motion of empty places under the action of the map T, which in this case satisfies the same rules as the the motion of particles except it goes in the backward direction in space. A direct computation gives the following representation, which can be considered as a definition of the dual map in the general case (not only for this specific model).

Lemma 2.6 $TX = (T^*X^*)^*$.

Corollary 2.7 *All results for the map T hold true also for the dual map and vice versa.*

Proof of Theorem 2.1. Assume first that the density of particles $\rho(X) := \frac{m}{N}$ in the initial configuration X is less or equal to $1/2$. If all the particles in this configuration are free then each particle moves with the velocity 1 and the trajectory starting from this configuration is periodic (in time) with (may be not minimal) period N. Indeed, after N iterations each particle will return to its initial position.

If there are clusters of particles in X we need to show that after at most m iterations all particles will become free. Indeed, by Lemma 2.2 the number of free particles does not decrease. Denote by $\tilde{m}$ the length of the largest cluster of empty places. Let us prove by induction on m that for any pair of positive integers m, n such that $m \le n/2$ and for any initial configuration after at most $t_c := \min(m, \tilde{m})$ iterations all particles will become free. This statement is trivial for $m = 1$ and $n \ge 2$. Assuming that it holds for some m let us prove it for $m+1$. Let X be a configuration with $m+1$ particle on the lattice of size $n \ge 2(m+1)$ and let x_0 be the position of the first particle after (one of) the largest clusters of empty spaces of length $\tilde{m}$. Fix this particle and consider the dynamics of others. By the induction hypothesis during the first $\min(m, \tilde{m})$ iterations other particles do not collide with the chosen one. Therefore their dynamics is the same as if there are only m particles. Again by the induction hypothesis all these particles will become free after $\min(m, \tilde{m})$ iterations and at that moment either the chosen particle is free also, or it forms a cluster of size 2. Therefore after the next iterations it becomes free as well.

It remains to analyze initial configurations with the density of particles greater than $1/2$. This situation is much more complex, because any configuration of this type contains clusters, which are interchanging particles between themselves and never disappear completely. To overcome this difficulty we consider a dual map T^*. The density of empty places is equal to $\frac{N-m}{N} < 1/2$. Therefore by Corollary 2.7 and the first part of the proof under the action of the map T^* a dual configuration X^* after at most $t_0 = N - m$ iterations will get into a periodic configuration $(T^*)^{t_0}X^*$ consisting of $N - m$ free particles with the period (in time) N for the map T^*. Clearly for the dual to it configuration we have the following identity $((T^*)^{t_0}X^*)^* = T^{t_0}X$ and the latter configuration is T-periodic with the same period. To finish the proof we calculate the average velocity:

$$\mathrm{V}(T^{t_0}X) = \frac{N-m}{m} = \frac{N}{m} - 1.$$

■

It is worth notice that our estimate of the duration of the transient period $\min(m, N-m)$ is exact: if $m < N - m$ and the initial configuration consists of only one cluster of length m the duration of the transient period is equal to m.

Corollary 2.8 *Qualitatively our main result about this model is that the following alternative takes place: either the flow consists of only free particles (i.e. there are no clusters of particles), or there are no clusters of empty places.*

On the other hand, there is no a priori information about the distribution of lengths of clusters of particles because this distribution depends on the initial configuration and can differ between various periodic limiting configurations.

3 1D traffic model with slow particles on the unbounded lattice

Consider now the unbounded one-dimensional case, i.e. $\mathbf{X} := \{0,1\}^{\mathbb{Z}^1}$ and the map T is defined by the formula (2.1) in the same way as in the previous section, except for the periodic boundary conditions. For a configuration $X \in \mathbf{X}$ we define the notion of a subconfiguration $X_k^n := \{X(k), X(k+1), \ldots, X(n)\}$, i.e a collection of entries of X between the pair of given indices $k < n$, and introduce the corresponding density and the average velocity:

$$\rho(X_k^n) := \frac{m(X_k^n)}{n-k+1}, \qquad \mathrm{V}(X_k^n) := \frac{1}{m(X_k^{n-1})} \sum_{x=k}^{n-1} v(X,x),$$

where $m(X_k^n)$ stays for the number of particles in the subconfiguration X_k^n.

By the density and the average velocity (of particles) of a entire configuration $X \in \mathbf{X}$ we mean the following limits (if they are well defined):

$$\rho(X) := \lim_{n\to\infty} \rho(X_{-n}^n), \qquad \mathrm{V}(X) := \lim_{n\to\infty} \mathrm{V}(X_{-n}^n),$$

otherwise one can consider the corresponding upper and lower limits, which we denote by $\rho_\pm(X)$ and $\mathrm{V}_\pm(X)$.

Notice that in distinction to the finite case these quantities make sense not for all possible configurations. We shall say that a configuration X satisfies the regularity assumption (or simply regular) if there exists a number ρ and a monotonous one-to-one function $\varphi(n) \to 0$ as $n \to \infty$, such that for any $n \in \mathbb{Z}^1$, $N \in \mathbb{Z}_+^1$ and any subconfiguration X_{n+1}^{n+N} of length N the number of particles in this subconfiguration $m(X_{n+1}^{n+N})$ satisfies the inequality

$$\left| \frac{m(X_{n+1}^{n+N})}{N} - \rho \right| \le \varphi(N). \tag{3.1}$$

It is clear that at least for a configuration X satisfying the regularity assumption the density $\rho(X)$ is well defined and is equal to the value ρ in the formulation of the assumption.

Theorem 3.1 *Let the initial configuration X satisfies the regularity assumption with $\rho \neq 1/2$. Then after a finite number of iterations the average velocity of particles becomes equal to* $\min(1, \frac{1}{\rho} - 1)$.

Proof. Notice that Lemma 2.2 holds true in this case also. Moreover it can be applied for the dual map as well and thus the length of any cluster of empty places cannot decrease. This shows that the variational principle (Proposition 2.4) holds in this case also. Moreover it can be reformulated to take care about configurations for which neither the density, nor the average velocity are well defined.

Proposition 3.1 *(Variational principle) For any configuration X any pair of its particle denote by $x'(t) < x''(t)$ their positions at the moment t. Then the average velocity $\mathrm{V}(X_{x'(t)}^{x''(t)})$ of the subconfiguration $X_{x'(t)}^{x''(t)}$ increases monotonically with t up to the moment (may be infinite) when it takes its maximal possible value.*

The basic technical step of the proof of Theorem 3.1 is given by the following statement.

Lemma 3.2 *If X satisfies the regularity assumption, then the same holds true for T^tX for any $t > 0$.*

Proof. Clearly it is enough to prove this statement for $t = 1$. Assume that this is not true and for some $N > \varphi^{-1}(\rho)$ there is a subconfiguration $(TX)_{n+1}^{n+N}$ of length N in the configuration TX such that

$$m((TX)_{n+1}^{n+N}) < (\rho - \varphi(N))N.$$

This can happen only if the following equalities are satisfied simultaneously

$$m(X_{n+1}^{n+N}) = (\rho - \varphi(N))N, \quad X(n) + X(n + N + 1) = 0, \quad X(n + N) = 1,$$

i.e. on the next time step no particle from behind will come to this interval and there is a free particle in the last position of the considered interval, which leaves it. On the other hand, from these equalities we immediately deduce that

$$m(X_n^{n+N-1}) = (\rho - \varphi(N))N - 1,$$

which contradicts the regularity assumption. In the same way one can prove that the inequality

$$m(X_n^{n+N-1}) \le (\rho + \varphi(N))N$$

cannot break as well. ■

Now we can return to the proof of Theorem 3.1. Assume first that $\rho < 1/2$. Denoting by $[\cdot]$ the integer part of a number, we get that for any $N \ge N_c := [\varphi^{-1}(\frac{1}{2} - \rho)]$ the number of particles in any subconfiguration $(T^tX)_{n+1}^{n+N}$ can be estimated as

$$m((T^tX)_{n+1}^{n+N}) \le (\rho + \varphi(N))N \le (\rho + \varphi(N_c))N \le N/2.$$

Applying the same machinery as in the previous section one can show that after at most

$$t_c := (\rho + \varphi(N_c))N_c \le \frac{N_c}{2} \le \frac{1}{2}\varphi^{-1}(\frac{1}{2} - \rho)$$

iterations all particles will become free ones, which implies the unit average velocity.

In the remaining case $\rho > 1/2$ we follow the same idea as in the proof of Theorem 2.1 and pass to the dual map and the dual configuration. Observe that if a configuration X satisfies the regular assumption with the density ρ and the rate function φ, then the dual configuration

X^* also satisfies it with the density $1-\rho$ and the same rate function φ. Indeed $m((X^*)_{n+1}^{n+N}) = N - m(X_{n+1}^{n+N})$ for any n, N and thus

$$\left|\frac{m((X^*)_{n+1}^{n+N})}{N} - (1-\rho)\right| = \left|\frac{m(X_{n+1}^{n+N})}{N} - \rho)\right| \le \varphi(N).$$

On the other hand, the density of the dual configuration $1-\rho \le 1/2$ which, having in mind that the only difference between the maps T and T^* is the direction of motion, gives us the possibility to apply the first part of the proof. ∎

Observe that in the proof of Theorem 3.1 we actually derived an estimate of the length of the transient period as $t_c := (\rho + \varphi(N_c))N_c$ which goes to infinity as $\rho \to 1/2$. This is the reason why Theorem 3.1 does not cover the boundary case $\rho = 1/2$, which we discuss below.

Theorem 3.2 *Let the initial configuration X satisfies the regularity assumption with $\rho = 1/2$ and let $x'(t) < x''(t)$ be positions of two fixed particles at the arbitrary moment t. Then the average velocity of the subconfiguration $X_{x'(t)}^{x''(t)}$ converges to 1 as $t \to \infty$.*

Proof. Choose an integer $\hat{N}$ and consider a configuration $\hat{X}$ obtained from the configuration X by the following operation: for each k we remove from the configuration X the closest from behind particle to the position $k\hat{N}$.

$$\left|\frac{m(\hat{X}_{n+1}^{n+N})}{N} - (\rho - \frac{1}{\hat{N}})\right| \le \left|\frac{m(X_{n+1}^{n+N})}{N} - \rho\right| + \left|\frac{m(\hat{X}_{n+1}^{n+N})}{N} - \frac{m(X_{n+1}^{n+N})}{N} + \frac{1}{\hat{N}}\right| \le \varphi(N) + \frac{1}{\hat{N}}.$$

Thus the configuration $\hat{X}$ is also regular but with the density $\frac{1}{2} - \frac{1}{\hat{N}} < \frac{1}{2}$. Thus according to Theorem 3.1 after a finite number of iterations t_c the average velocity of the configuration $T^{t_c}\hat{X}$ becomes equal to 1.

Making an opposite operation, namely inserting a particle to the configuration X to the closest from behind to $k\hat{N}$ empty position for each k, we obtain a regular configuration with the density $\frac{1}{2} + \frac{1}{\hat{N}} > 1/2$. Again by Theorem 3.1 after a finite number of iterations the average velocity of this configuration becomes equal to

$$\frac{1}{\frac{1}{2} - \frac{1}{\hat{N}}} - 1 = 1 + \frac{4}{\hat{N} - 2} \to 1 \quad \text{as} \quad \hat{N} \to \infty.$$

Thus both (arbitrary close as $\hat{N} \to \infty$) approximations to the configuration X have after a finite number of iterations (depending on $\hat{N}$) the average velocity deviating from 1 by $O(\hat{N})$. It remains to show that the average velocity of a subconfiguration of the configuration X can be estimated from above and from below by those from above approximations. Let X and Y are two configurations such that $X(x) \le Y(x)$ for all x and let $x'(t) < x''(t)$ be positions of two fixed particles in the configuration X at the arbitrary moment t. Denote by $y'(t) < y''(t)$ positions of the same particles in the configuration Y. Then

$$\mathrm{V}(X_{x'(t)}^{x''(t)}) \ge \mathrm{V}(X_{x'(t)}^{x''(t)})$$

for any moment of time t. Indeed, additional particles in the configuration Y present only obstacles to the motion of other particles, thus making the average velocity slower (or at least not faster). ∎

4 1D traffic model with speedy particles

The dual map T^* that we made use of in the previous sections was mirror symmetric with respect to the map T. In the model considered in this section we show that the dual map might have a different structure as well.

Let us start with the one-dimensional finite periodic case, but with speedy particles instead of particles moving with the velocity 1 as above. This means that if in front of a particle there are exactly n empty places it moves by n places forward and remains in the same place if the next place is occupied. By the average velocity of (particles in) the configuration X we mean the total distance covered by the particles from X during the next time step divided by the number of particles. The corresponding map of the space $\mathbf{X} = \{0,1\}^N$ into itself we again denote by T. See examples of the dynamics under the action of the map T on Fig. 3.

(a)		(b)
0011011100010	t=0	1011011100110
0110111000100	t=1	0110111001101
1101110001000	t=2	1101110011010
1011100010001	t=3	1011100110101
0111000100011	t=4	0111001101011
1110001000110	t=5	1110011010110
1100010001101	t=6	1100110101101
1000100011011	t=7	1001101011011

Figure 3: Two examples of the dynamics of the model with speedy particles: (a) $m = 6,\ N = 13,\ \rho = 6/13 < 1/2$, (b) $m = 8,\ N = 13,\ \rho = 8/13 > 1/2$.

Theorem 4.1 *The dual map in this case satisfies the relation $T^*X^* = TX^*$, and for any nontrivial (having both zeros and ones) initial configuration $X \in \mathbf{X}$ the average velocity does not depend on time and is equal to $\frac{N}{m(X)} - 1$, where $m(X)$ is the number of particles in the configuration X.*

Proof. We consider this result as an illustration of the dual maps techniques and and a base for the introduction of the passive tracer model considered in Section 6. Therefore we do not give explicit representations for the maps T, T^* and present only a sketch of the proof in this case.

The relation between the map T and the dual one T^* can be checked by a direct computation (which we leave for the reader). Notice that unlike the dual map in Section 2 the dynamics of empty places is exactly the same as the dynamics of particles and occurs in the same direction in space.

According to the definition each particle on the next time step moves forward by the number of empty places in front of it. Therefore the total number of places to which the particles in the configuration X move forward is equal to the number of empty places. Thus the average velocity is equal to $\mathrm{V}(X) = \frac{N-m(X)}{m(X)} = \frac{N}{m(X)} - 1$.

Using the same argument as in the proof of Theorem 2.1 we deduce that any trajectory of the map T gets periodic after a finite number of iterations and a trivial observation that the number of empty places is invariant under dynamics finishes the proof. ∎

Consider now the model of speedy particles on the unbounded 1D lattice $\mathbb{Z}^1$ in analogy to the model in Section 3. The map T is defined as above and acts in the space $\mathbf{X} = \{0,1\}^{\mathbb{Z}^1}$.

We shall say that a configuration $X \in \mathbf{X}$ satisfies the Law of Large Numbers if for any integer n the limit

$$\lim_{N\to\infty} \frac{m(X_{n+1}^{n+N})}{N}$$

is well defined and does not depend on n. Clearly the above limit (if it exists) coincides with the density of particles $\rho(X)$ in the configuration X.

The dual map in the considered case satisfies the same relation $T^*X^* = TX^*$ as in Theorem 4.1 and the dynamics is described by the following statement.

Theorem 4.2 *Let an initial configuration $X \in \mathbf{X}$ satisfy the Law of Large Numbers and let $\rho(X) \cdot (1 - \rho(X)) \neq 0$. Then the average velocity does not depend on time and is equal to $\frac{1}{\rho(X)} - 1$.*

Proof. Let $x' < x''$ be positions of two particles in the configuration X. Then the average velocity of the subconfiguration $X_{x'}^{x''}$ is equal to the number of empty places in this subconfiguration, divided by the number of particles in it, i.e.

$$\mathrm{V}(X_{x'}^{x''}) = \frac{x'' - x' - m(X_{x'}^{x''})}{m(X_{x'}^{x''})} = \frac{x'' - x'}{m(X_{x'}^{x''})}.$$

Now taking into account that this relation holds for arbitrary positions of particles, we immediately get the desired statement. ∎

Remark 4.1 It is of interest that the average velocity in both discussed speedy particles models coincides with the average velocity in the models with slow particles in the case of high density (i.e. when $\rho(X) > 1/2$). The explanation is that in the models with slow particles with the density $\rho(X) > 1/2$ the typical distance between particles (in the steady state) is 0 or 1, which makes no difference between the dynamics of speedy and slow particles.

Observe that in the models with speedy particles the dynamics is richer than in the models with slow particles, for example the statement of Lemma 2.2 does not hold, i.e. clusters of particles can grow and traffic jams become typical even for the case of low density of particles.

5 2D traffic model

From the point of view of real traffic problems a clear shortage of all models considered above is the absence of the possibility to go around particles staying in a traffic jam. Indeed in a one-row motor road (which was the main considered example) this is not possible. To overcome this restriction we consider a model describing the motion of slow particles on a multi-row motor road.

A lattice in our model is a $N \times K$-strip, describing a one-way cyclic road of length N consisting of K parallel rows. We denote the first coordinate corresponding to the spread of the road by $x \in \{1, \ldots, N\}$, and the second one (the row number) by $y \in \{1, \ldots, K\}$, and assume periodic boundary conditions on x. In terms of particles the dynamics is defined as follows. If there is a particle at a position (x, y) (i.e. $X(x, y) = 1$) then this particle moves forward by one place to the position $(x+1, y)$ if this place is not occupied, else moves to the left to the position $(x, y+1)$ if this place and the place before it are empty, else moves to the right to the position $(x, y-1)$ if this place and the place before and the next place to the right are not occupied, and remains on its place otherwise. To be consistent we assume that the 0-th, $(K+1)$-th and

$(K+2)$-th (virtual) rows are completely occupied. Recall that the space is N-periodic on the x-coordinate.

Observe that this model is nothing more than the simplest formulation of the standard traffic rules.

The space of all possible configurations of this model is $\mathbf{X} = \{0,1\}^{NK}$ and the corresponding map describing the dynamics of the configurations in this space can be written as follows:

$$TX(x,y) := \begin{cases} 1 & \text{if } X(x,y) = 0 \text{ and } X(x-1,y) = 1 \\ 1 & \text{if } X(x,y) = 1 \text{ and } X(x+1,y) = 1 \\ 1 & \text{if } X(x,y) + X(x-1,y) = 0 \\ & \quad \text{and } X(x,y+1) \cdot X(x+1,y+1) = 1 \\ 1 & \text{if } X(x,y) + X(x-1,y) + X(x,y+1) = 0 \\ & \quad \text{and } X(x,y-1) \cdot X(x+1,y+1) = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

The dual map for this model can be easily defined and is described by the following statement. Again to be consistent we need to add an additional (-1)-th completely occupied virtual row to the lattice.

Lemma 5.1 *The dual map T^* satisfies the general statement of Lemma 2.6 and the explicite formula for $T^*X(x,y)$ can be obtained from the relation (5.1) by changing everywhere $x+1$ by $x-1$, $x-1$ by $x+1$, and $y+1$ by $y-1$.*

This statement can be checked by a direct computation.

Corollary 5.2 *The dynamics of empty places under this model is the same as the dynamics of particles except it occurs in the opposite direction in space along the x-coordinate, i.e. the correspondence between the dynamics is the mirror symmetry.*

We are interested in the average velocity along the x-coordinate, which we define in the same way as in the previous sections. From the first sight it seems that the presence of additional possibilities to go to the right or to the left (if the way forward is blocked) should improve the traffic. The following statement shows that this is not the case and moreover in the 2D case the average velocity might be much slower than the velocity predicted by the 1D model with slow particles.

Theorem 5.1 *For any initial configuration X in the case of the periodic bounded lattice and any regular initial configuration X with $\rho \neq 1/2$ in the case of the unbounded (on the x-coordinate) lattice after a sufficiently large number of iterations the upper limit for the average velocity satisfies the inequality*

$$\mathrm{V}_+(T^tX) \le \min(1, \frac{1}{\rho(X)} - 1),$$

while the lower limit can be arbitrary close to 0 even in the case of low density of particles ($\rho < 1/2$).

Notice that in the bounded case exactly as in the 1D model on the bounded lattice any initial configuration becomes periodic after a finite number of iterations. Moreover in this case both limits $\mathrm{V}_\pm$ coincide.

Proof. We start with the case $\rho \le 1/2$ and the periodic (on the x-coordinate) bounded lattice of size NK. Our first aim is to show that for any number $\rho \in [0, 1/KN, \dots, 1]$ a configuration of the density ρ cannot have the average velocity larger than the one predicted by the $1D$ model with slow particles. If $\rho N \le [N/2]$, then one can construct a configuration X_+ consisting of exactly ρN free particles on each of K rows. Clearly the configuration X_+ is N-periodic and has the maximal possible average velocity 1. Otherwise at least in one of the rows there is a cluster of particles and hence the average velocity is strictly smaller than 1.

To get the lower bound we construct the following example. Assuming that N is even, we choose a number $k \in \{K/3, \dots, K/2\}$. Consider the configuration X such that $X(x,l) = 1$ for all x and $l = \{1, \dots, k\}$, while in the $(k+1)$-th row we assume that $X(l, k+1) = 0$ for odd l and is equal to 1 otherwise. Then independently on the positions of other particles, neither particle from the first k rows can move to the rows with numbers larger than k, no particles from other rows can get into the first k rows. On the other hand, this configuration X_k is time periodic with the period 2 and its average velocity is equal to

$$\mathrm{V}(X_k) = \frac{N/2}{kN + N/2} = \frac{1}{2k+1} \le \frac{1}{2K/3+1}.$$

Therefore $\mathrm{V}(X_k) \to 0$ as $K \to \infty$.

Moreover choosing $k = 1$ in the above example we get the average velocity $\frac{N/2}{N+N/2} = 1/3$ for the configuration of density $\rho = \frac{N+N/2}{N \cdot K} = \frac{3}{2K} \to 0$ as $K \to \infty$. Thus for an arbitrary low density we may have the average velocity less than 1.

If the density $\rho > 1/2$ we use the same trick as usual and pass to the dual map and dual configuration, which permits to use the first part of the proof to get the needed estimates.

The proof in the unbounded case is basically the same and we leave it to the reader. ∎

There are other possible ways to model multi-row traffic flows, for example one might consider different rules for the row changing, the presence of high velocity particles, periodic boundary conditions between the boundary rows, etc. However if the model respects the law that on the next step a particle can follow its way along the same row (assuming that the next place is not occupied) independently on particles on neighboring rows, the example in the above proof demonstrating the phenomenon of the low average velocity under a low density of particles remains generic.

6 Passive tracer in the traffic flow

In the Introduction we have mentioned the practical observation that it is beneficial in some cases to go against the flow than in the same direction as it goes. To study this phenomenon we consider in this section a simple model of a passive tracer imitating the behavior of a person moving in a hurry in the traffic flow. As usual we assume that the tracer have its own (forward or backward) chosen direction of motion and does not make any impact to the flow.

Let $X(t)$ describes the 1D flow of particles and let at time t the passive tracer occupies the position x. Then before the next time step of the model of the flow the tracer moves in its chosen direction to the closest (in this direction) position of a particle of the configuration $X(t)$. For example, if the going forward tracer occupies the position 2 and the closest particle in this direction occupies the position 5, then the tracer moves to the position 5. Then the next iteration of the flow occurs, the tracer moves to its new position, etc.

To be precise for a fixed configuration $X \in \mathbf{X}$ with $m(X) > 1$ we introduce two maps τ_X^+ and τ_X^- of the ordered periodic lattice $\mathbf{L}$ into itself defined as follows:

$$\tau_X^+ y := \min(x \in \mathbf{L} : \ \mathbf{y} < \mathbf{x}, \ \mathbf{X}(\mathbf{x}) = \mathbf{1}),$$

$$\tau_X^- y := \max(x \in \mathbf{L} : \ \mathbf{y} > \mathbf{x}, \ \mathbf{X}(\mathbf{x}) = \mathbf{1})$$

for any $y \in \mathbf{L}$, where the order relation $y < x$ is induced by the order on the lattice $\mathbf{L}$. Then the simultaneous dynamics of the configuration of particles (describing the flow) and the tracer is defined by the skew product of two maps – the map T and one of the maps $\tau_{\cdot}^{\pm}$, i.e.

$$(X, y) \to \mathcal{T}_{\pm}(X, y) := (TX, \tau_X^{\pm} y),$$

acting on the extended phase space $\bar{\mathbf{X}} := \mathbf{X} \times \mathbf{L}$.

Let $S(t)$ denotes the total distance covered by the tracer up to the moment t with the positive sign if the tracer moves forward, and the negative sign otherwise. Then we define the average (in time) velocity of the tracer $v(t)$ as $S(t)/t$.

Theorem 6.1 *For an arbitrary configuration X with $m(X) > 1$ of the 1D model with slow particles on the finite periodic lattice and for an arbitrary initial configuration X satisfying the regularity assumption (3.1) with the density $\rho \notin \{0, \frac{1}{2}, 1\}$ in the 1D unbounded case the average velocity of the passive tracer converges to 1 if the tracer moves forward (along the flow)) and $\rho \le 1/2$, and to $-\max(1, \frac{1}{\rho} - 1)$ if it moves backward (against the flow).*

Proof. We start with the finite periodic lattice and the tracer moving in the forward direction. According to our definition after the first iteration of the map $\mathcal{T}$ the tracer will occupy the same position as some particle. Therefore it is enough to consider only extended configurations (X, y), such that $X(y) = 1$. Since $\rho \le 1/2$, then after a finite number of iterations the flow will consist of only free particles. Therefore the tracer will run down one of them and will follow it, but cannot outstrip. Indeed after each iteration of the flow this free particle occurs just one position behind the tracer.

We do not discuss the case $\rho > 1/2$ because the average velocity of the tracer in this case sensitively depends on the choice of the initial configuration. For example in the case $n = 8$ and $m = 6$ (i.e. $\rho = 3/4$) consider two initial configurations 01011111 and 01111011. In the first case the tracer after several iterations obtains the constant velocity 1, while in the second case the velocity of the tracer is periodic with the period 3 and consists of the repeating groups of $(2 + 2 + 1)$, i.e. the average speed is equal to $5/3$.

Consider now the case when the tracer is moving backward. Then each time the tracer encounters a particle, on the next time step this particle moves in the opposite direction and does not disturb the movement of the tracer until the collision with the next particle. Assume that the density of particles is less than $1/2$. Then by Theorem 2.1 after a finite number of iterations only free particles are present in the flow, which results in the convergence of the average velocity of the passive tracer to $-(\frac{1}{\rho} - 1)$. Indeed on the spread of length N there are m particles, i.e. m obstacles for the tracer, which gives the average velocity $\frac{N-m}{m}$. On the other hand, if the density of particles is greater or equal to $1/2$ again by Theorem 2.1 there are no clusters of empty places in the flow and thus after each iteration the tracer can move only by one position, which finishes the proof for the model of the flow with periodic boundary conditions.

The proof in the unbounded case is practically the same with the only difference that one should use Theorem 3.1 instead of Theorem 2.1. Notice that additionally to the regularity assumption we need to assume that $\rho \notin \{0, 1\}$ to be consistent with the definition of the dynamics of the passive tracer. ∎

Observe that the motion against the flow is efficient only in the case of low density of particles. Certainly, this model is oversimplified and probably its predictions are unrealistic for the case of very high density of particles. However we believe that while the density is not low and not very high our description of the passive tracer is reasonable at least on the qualitative level.

7 Conclusion

This paper represents one of the first steps in the mathematical foundation of the analysis of traffic flows and we restrict ourselves here to the pure deterministic settings. The next step should describe ergodic (statistical) properties of the considered models with initial conditions chosen at random and random versions of these models as well. This circle of questions is especially interesting in the case of models on infinite lattices, where the dynamics of typical configurations cannot be obtained in the limit of infinitively large lattice sizes from our results about regular configurations even in the deterministic setting. Indeed, for a reasonable choice of the class of random initial conditions their realizations do not satisfy our regular assumption with the probability 1.

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