Preprint-KUL-TF-2000/09

BEC for a Coupled Two-type Hard Core Bosons Model

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March, 2000

Abstract

We study a solvable model of two hard core Bose particles. A complete analysis is given of its equilibrium states including the proof of existence of Bose-Einstein condensation. The plasmon frequencies and the quantum normal modes corresponding to these frequencies are rigorously constructed. In particular we show a two-fold degeneracy of these frequencies. We show that all this results from spontaneous gauge symmetry breakdown.

PACS - Keywords

03.75.F - 11.15.E - 05.40
Bose-Einstein condensation, spontaneous symmetry breakdown, fluctuation operators, normal coordinates, plasmon frequencies.

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1 Introduction

The study of lattice Bose gas models was proposed [1, 2] in order to understand the effect of interaction on Bose-Einstein Condensation (BEC). It was argumented that, if the interaction has a hard core, i.e. each site can be occupied by at most one particle, then the model can be mapped onto a quantum spin-$\frac{1}{2}$ XY-model on a $\mathbb{Z}^d$-lattice.

The exact solution, exhibiting condensation, for the hard core Bose gas on a complete graph has been obtained [3, 4]. In spin language this work corresponds to the mean field spin-$\frac{1}{2}$ XY-model. Other exact results on the full model are an upper bound on the condensate [5] and an existence proof of condensation for the system in a half filled $\mathbb{Z}^d$-lattice for dimensions $d \geq 3$ [6]. In this context we mention also the relevant result [7] about the occurence of long-range order in the XY-model with spins and dimensions greater than one.

In this paper we turn our attention to a system of two types of interacting hard core Bose particles on a complete graph and interacting through a repulsive interaction if the particles are of different type. (Compare this with the Hubbard model for electron systems.) The model is exactly soluble. We study in detail the equilibrium states in the case of a half filled lattice and equal density for the two particle types. If the system is weakly interacting we obtain an explicit and complete analysis of the equilibrium states. We analyse the BEC phenomenon and in particular we are interested in the appearence of spontaneous symmetry breaking (SSB) linked with it. The interesting aspect of this model is that the broken symmetry is two-dimensional. While, as is well known for a single type Bose system, the broken symmetry is the one-dimensional gauge symmetry. We compute rigorously the plasmon frequencies related to these modes, and find the quantum normal modes of these plasmon motions. We find out that there are two different frequencies, but that both frequencies are two-fold degenerate. The four quantum degrees of freedom have normal modes which we build up as the quantum fluctuation operators of the components of the infinitesimal generators of the spontaneously broken gauge symmetries, with adjoints the quantum fluctuation operators of corresponding order parameter operators. The degeneracy of the plasmon spectrum is the new phenomenon in this model study of Bose-Einstein condensation. Another revealing fact is that the adjoint fluctuation operators are the total and the relative current fluctuations between the two types of bosons.

2 The Model and its Equilibrium States

2.1 The Model

The model is defined on the lattice $\mathbb{Z}$, by the local Hamiltonians H_Λ, with Λ a finite interval in $\mathbb{Z}$:

$$H_\Lambda = \frac{t}{|\Lambda|} \sum_{\alpha\in\{1,2\}} \sum_{i\neq j\in\Lambda} \sigma^+_{i\alpha}\sigma^-_{j\alpha} + \mu \sum_{\alpha\in\{1,2\}} \sum_{i\in\Lambda} \sigma^+_{i\alpha}\sigma^-_{i\alpha} + U \sum_{i\in\Lambda} \sigma^+_{i1}\sigma^-_{i1}\sigma^+_{i2}\sigma^-_{i2}. \tag{1}$$

The first term is the hopping contribution, the particles jump freely from one site to another with a hopping rate $t/|\Lambda|$, $|\Lambda|$ is the number of points in Λ; the index α specifies the type of particles; μ is the chemical potential; U is taken to be positive and is the coupling constant for the interaction between the types of particles. If two particles of different type are on the same lattice point the potential energy is raised with an amount U. Clearly the $\sigma^\pm, \sigma^z$ are the usual Pauli matrices and are here the creation an annihilation operators for the Bose particles with hard core, satisfying:

$$\begin{aligned}
&[\sigma^+_{i\alpha}, \sigma^-_{j\beta}] = \delta_{ij}\delta_{\alpha\beta};\\
&\left(\sigma^+_{i\alpha}\right)^2 = 0; \quad \left(\sigma^+_{i\alpha}\right)^* = \sigma^-_{i\alpha};\\
&\sigma^+_{i\alpha}\sigma^-_{i\alpha} = \frac{1}{2}\left(\sigma^z_{i\alpha} + \mathbf{1}\right).
\end{aligned}$$

We look first for the equilibrium states of this model in the thermodynamic limit. The Hamiltonians (1) are permutation invariant with respect to the lattice index. This means [8] that the equilibrium states at inverse temperatures β are convex combinations of product states on the infinite tensor product algebra of observables $\mathcal{M} = \bigotimes_{i\in\mathbb{Z}} (M_2 \otimes M_2)_i$ of the system; $(M_2 \otimes M_2)_i$ represents the tensor product of the 2×2 complex matrices M_2 with itself. Each set M_2 is referring to a different type of hard core bosons, both at site $i \in \mathbb{Z}$. In particular, if one looks only at the product states which are invariant for the interchange of the two types of particles, then the equilibrium states are given by the states $\omega_{\beta\lambda}(\,.\,)$ of the form [8]:

$$\omega_{\beta\lambda}(A_1A_2\ldots A_n) = \prod_{i=1}^n \operatorname{tr}\rho_{\beta\lambda}A_i; \quad \text{for any } A_i \in (M_2\otimes M_2)_i\,, i = 1,\ldots n$$

where $\rho_{\beta\lambda}$ is a density matrix in $M_2 \otimes M_2$, satisfying the self-consistency equation:

$$\rho_{\beta\lambda} = \frac{\exp[-\beta h_\lambda]}{\operatorname{tr}\exp[-\beta h_\lambda]}, \tag{2}$$

where

$$h_\lambda = t\lambda\left(\sigma^+_1 + \sigma^+_2\right) + t\bar\lambda\left(\sigma^-_1 + \sigma^-_2\right) - \mu\left(\sigma^+_1\sigma^-_1 + \sigma^+_2\sigma^-_2\right) + U\sigma^+_1\sigma^-_1\sigma^+_2\sigma^-_2 \tag{3}$$

and

$$\lambda = \operatorname{tr} \rho_{\beta\lambda} \sigma_1^- = \operatorname{tr} \rho_{\beta\lambda} \sigma_2^- . \tag{4}$$

In fact equation (2) for $\rho_{\beta\lambda}$ is equivalent to the equation (4) for λ. Remark that (4) also results from the assumption of symmetry in the particle types. Looking for equilibrium states breaking this symmetry would violate (4).

Moreover, we will limit here our study to the case of a half filled lattice. Mathematically this means that we require:

$$\omega_{\beta\lambda}(\sigma_{i\alpha}^z) = 0 \quad \text{for all } i \in \mathbb{Z}, \quad \alpha = 1, 2.$$

This fixes the chemical potential $\mu = U/2$. In order to simplify the notations, we adapt our Hamiltonian by adding a constant term and by scaling the constant U to $U/4$. In this case one gets for (3) a more simple expression:

$$h_\lambda = t\lambda \left(\sigma_1^+ + \sigma_2^+\right) + t\bar{\lambda} \left(\sigma_1^- + \sigma_2^-\right) + U\sigma_1^z \sigma_2^z. \tag{5}$$

On a finite subset $\Lambda \subset \mathbb{Z}$ we define Fourier-transformed operators:

$$a_{k\alpha} = \frac{1}{\sqrt{|\Lambda|}} \sum_{j,i \in \Lambda} \sigma_j^- \exp\left[\mathrm{i}\frac{2\pi}{|\Lambda|} kj\right]; \quad k \in \Lambda^* \text{ the dual of } \Lambda,$$

the annihilation operator of the α-particle with quasi-momentum k.

Then,

$$n_k = \sum_{\alpha=1}^{2} a_{k\alpha}^* a_{k\alpha}$$

is the number operator of particles with quasi-momentum k.

Compute in a state $\omega_{\beta\lambda}$ (2):

$$\rho_0 = \lim_{|\Lambda| \to \infty} \omega_{\beta\lambda}\left(\frac{n_0}{|\Lambda|}\right) = 2\omega_{\beta\lambda}(\sigma^+)\omega_{\beta\lambda}(\sigma^-) = 2|\lambda|^2; \tag{6}$$

i.e. the 0-mode has a density ρ_0 equal to $2|\lambda|^2$. A solution $\lambda \neq 0$ of (4) yields a macroscopic occupation of the 0-mode and the density of the condensate ρ_0 equals $2|\lambda|^2$. Therefore, if one finds solutions of (2) or (4) such that $\lambda \neq 0$, one proved Bose-Einstein condensation.

2.2 Equilibrium States

Now we proceed to look for the solution of (4) and its properties. It is clear that based on general results [8] we reduced our problem to a one-site problem, namely looking for the density matrix state $\operatorname{tr} \rho_{\beta\lambda}$. with $\rho_{\beta\lambda} \in M_2 \otimes M_2$.

In general $M_2 \otimes M_2 = M_4$ and we use the following isomorphism:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{12} & a_{12}b_{12} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12} \\ a_{11}b_{21} & a_{12}b_{21} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{21} & a_{22}b_{21} & a_{21}b_{22} & a_{22}b_{22} \end{pmatrix}.$$

This yields the following matrix representation for h_λ:

$$h_\lambda = \begin{pmatrix} U & t\lambda & t\lambda & 0 \\ t\bar{\lambda} & -U & 0 & t\lambda \\ t\bar{\lambda} & 0 & -U & t\lambda \\ 0 & t\bar{\lambda} & t\bar{\lambda} & U \end{pmatrix}. \tag{7}$$

Then a straightforward diagonalisation of h_λ yields the following spectrum:

$$\epsilon_0 = -U, \quad \epsilon_1 = U, \quad \epsilon_2 = \eta, \quad \epsilon_3 = -\eta, \tag{8}$$

where $\eta = \sqrt{U^2 + 4|t\lambda|^2}$.

The corresponding eigenvectors are:

$$\begin{aligned} |\phi_0\rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \qquad & |\phi_1\rangle &= \frac{1}{\sqrt{2}|t\lambda|} \begin{pmatrix} t\lambda \\ 0 \\ 0 \\ -t\bar{\lambda} \end{pmatrix}, \\ |\phi_2\rangle &= \frac{1}{\sqrt{\eta(\eta - U)}} \begin{pmatrix} t\lambda \\ \frac{\eta-U}{2} \\ \frac{\eta-U}{2} \\ t\bar{\lambda} \end{pmatrix}, \qquad & |\phi_3\rangle &= \frac{1}{\sqrt{\eta(\eta + U)}} \begin{pmatrix} t\lambda \\ -\frac{\eta+U}{2} \\ -\frac{\eta+U}{2} \\ t\bar{\lambda} \end{pmatrix}. \end{aligned} \tag{9}$$

Using (8) and (9) one gets for equation (4) the following explicit and nonlinear equation for λ:

$$\lambda \left[1 + \frac{t}{\eta} \frac{\sinh(\beta\eta)}{\cosh(\beta U) + \cosh(\beta\eta)} \right] = 0. \tag{10}$$

Remark that $\lambda = 0$ is always a solution. It corresponds to a state without condensation (see (6)).

Looking for solutions $\lambda \neq 0$ amounts to look for a solution η satisfying:

$$\eta = f_\beta(\eta); \tag{11}$$

where

$$f_\beta(\eta) = -t\frac{\sinh(\beta\eta)}{\cosh(\beta U) + \cosh(\beta\eta)}.$$

Since the condensate density $\rho_0 = 2|\lambda|^2$ (cfr. (6)) cannot exceed the total particle density ρ (which is equal to one in the half filled lattice), $\sqrt{U^2 + 2t^2}$ is an upper bound on $\eta = \sqrt{U^2 + 4|t\lambda|^2}$. The minimum value of η is U. This defines I, the range of accessible values for η:

$$\eta \in I = \left[U, \sqrt{U^2 + 2t^2}\right].$$

Remark that $f_\beta(\eta)$ is unaffected by the sign of U. On the other hand, equation (11) has no solutions if $t \geq 0$. Therefore we take $t < 0$ and $U > 0$.

Discussion of the solutions of (10)

We are looking for solutions yielding a second order phase transition.

Remark first that for $\eta \in I$:

$$\frac{\partial f_\beta(\eta)}{\partial \eta} > 0 \tag{12}$$

$$\frac{\partial^2 f_\beta(\eta)}{\partial \eta^2} < 0 \tag{13}$$

$$\frac{\partial f_\beta(\eta)}{\partial \beta} > 0 \tag{14}$$

i.e. $\eta \to f_\beta(\eta)$ is a monotonically increasing concave function on I; also $\beta \to f_\beta(\eta)$ is monotonically increasing for T decreasing.

The problem here is that, due to the presence of the potential U, these properties are not valid outside the interval I, such that the situation is more complicated than for the usual mean field case.

First we consider temperature $T > 0$, and we distinghuish two cases:

case a):

$$f_\beta(U) > U \tag{15}$$

yielding

$$\frac{1}{2U}\ln\left(\frac{-t + 2U}{-t - 2U}\right) < \beta; \tag{16}$$

and implying the condition on the interparticle interaction:

$$2U < -t \tag{17}$$

i.e. the interparticle interaction must be relatively small compared with the hopping term (1); this condition is a meaningful situation for this kind of systems, if the interaction becomes to strong, the typical quantum effects like BEC disappear.

One derives also

$$f_\beta\left(\sqrt{U^2+2t^2}\right) \leq \sqrt{U^2+2t^2}. \tag{18}$$

Together, condition (15) and (18) yield a unique solution of $f_\beta(\eta) = \eta \in I$, and because of (14) a second order phase transition with $\lambda \neq 0$ as an order parameter. Equation (16) learns that there exists a critical β_c determinated by:

$$\frac{1}{k_B T_c} = \beta_c = \frac{1}{2U} \ln\left(\frac{-t+2U}{-t-2U}\right). \tag{19}$$

This defines the critical temperature T_c.

case b):

$$f_\beta(U) < U \tag{20}$$

This case is somewhat more complicated, a careful analysis learns that apart from a second order phase transition, a first order one can occur, even if (17) is satisfied. In order to avoid non-natural first order transitions, one has to impose:

$$\left.\frac{\partial f_{\beta_c}(\eta)}{\partial \eta}\right|_U \leq 1; \quad f_{\beta_c}(U) = U.$$

The analysis of these yields a stronger condition on U than (17):

$$U < (-t)\kappa,$$

where κ can be computed numerically as the solution of the equation:

$$\ln\left(\frac{1+2\kappa}{1-2\kappa}\right) = \frac{4\kappa}{1-2\kappa^2};$$
$$\kappa \approx 0.461292.$$

Finally we consider the ground state ($T = 0$) situation. The self-consistency equation (10) becomes:

$$\eta = \lim_{\beta\to\infty} f_\beta(\eta) = f_\infty(\eta);$$

with:

$$f_\infty(\eta) = \begin{cases} 0 & \text{if } \eta < U \\ -t & \text{if } \eta > U. \end{cases}$$

If $U > -t$, then there are no non-trivial solutions $\lambda \neq 0$. On the other hand, if $U < -t$, then there exists always a unique solution $\lambda \neq 0$. It is obtained as the solution of:

$$-t = \sqrt{U^2 + 4t^2|\lambda|^2}$$

given by:

$$|\lambda| = \frac{1}{2t}\left(t^2 - U^2\right)^{1/2} < \frac{1}{2}.$$

Since $f_\beta(\eta)$ is increasing with T decreasing (14), this also implies an expression for a supremum on the condensate density ρ_0 (6):

$$\rho_0 \leq \frac{(t^2 - U^2)}{2t^2} < \rho. \tag{21}$$

The condensate density is always strictly smaller than the total particle density, which is compatible with former results on this topic [4,5]. This finishes the study of the equilibrium properties of the model (1).

3 Spontaneous Symmetry Breaking

The model H_Λ (1) and its equilibrium states have a discrete symmetry consisting of the interchange of the two particle types into each other. It is implemented by the following unitary matrix: $u = \sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^- + \sigma_1^-\sigma_1^+\sigma_2^-\sigma_2^+ + \sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+$, then for any local observable A_{12} depending on the types 1 and 2, one gets:

$$\gamma_{1\leftrightarrow 2}(A_{12}) = \left(\prod_{i\in\Lambda}(u_{12})_i\right) A_{12} \left(\prod_{i\in\Lambda}(u_{12})_i\right) = A_{21}.$$

The eigenvectors (9) of h_λ satisfy

$$u_{12}|\phi_0\rangle = -|\phi_0\rangle; \quad u_{12}|\phi_1\rangle = |\phi_1\rangle; \quad u_{12}|\phi_2\rangle = |\phi_2\rangle; \quad u_{12}|\phi_3\rangle = |\phi_3\rangle.$$

A second discrete symmetry is the particle-hole ($p \leftrightarrow h$) symmetry. This symmetry interchanges the creation and annihilation operators (σ_i^+ and σ_i^-) and is implemented by the unitaries $\sigma_{12}^x = \sigma_1^x\sigma_2^x$ (σ^x is the x-component Pauli matrix). The invariance of H_Λ (1) is expressed by:

$$\begin{aligned}\gamma_{p\leftrightarrow h}(H_\Lambda) &= \left(\prod_{i\in\Lambda}(\sigma_{12}^x)_i\right) H_\Lambda \left(\prod_{i\in\Lambda}(\sigma_{12}^x)_i\right) \\ &= H_\Lambda.\end{aligned}$$

The eigenfunctions (9) of h_λ satisfy

$$\sigma_{12}^x|\phi_0\rangle = -|\phi_0\rangle; \quad \sigma_{12}^x|\phi_1\rangle = -|\phi_1\rangle; \quad \sigma_{12}^x|\phi_2\rangle = \ |\phi_2\rangle; \quad \sigma_{12}^x|\phi_3\rangle = \ |\phi_3\rangle.$$

The Hamiltonian H_Λ (1) has also a two-dimensional continuous gauge symmetry group given by the actions:

$$\gamma_{\phi_1\phi_2}(\sigma_k^+) = \mathrm{e}^{\mathrm{i}\phi_k}\sigma_k^+, \quad \phi_k \in [0, 2\pi], \quad k = 1, 2. \tag{22}$$

For a finite volume Λ, it is implemented by the unitaries:

$$U_{\Lambda(\phi_1,\phi_2)} = \exp\frac{\mathrm{i}}{2}\Big(\phi_1 Q_{1,\Lambda} + \phi_2 Q_{2,\Lambda}\Big);$$

where

$$Q_{k,\Lambda} = \sum_{j\in\Lambda}(\sigma_k^z)_j; \quad k = 1, 2.$$

Clearly, $\gamma_{\phi_1\phi_2}(H_\Lambda) = U_{\Lambda(\phi_1,\phi_2)}H_\Lambda U^*_{\Lambda(\phi_1,\phi_2)} = H_\Lambda$. Hence, per lattice site one has a commutative two-dimensional Lie-algebra of this symmetry group, generated by σ_1^z and σ_2^z. However, the equilibrium states $\omega_{\beta\lambda}$ with $\lambda \neq 0$ (see section 2) break this symmetry down. Indeed we have here for $k = 1, 2$:

$$\omega_{\beta\lambda}\left(\gamma_{\phi_1\phi_2}\left(\sigma_k^+\right)\right) = \mathrm{e}^{-\mathrm{i}\phi_k}\omega_{\beta\lambda}(\sigma_k^+) \neq \omega_{\beta\lambda}(\sigma_k^+)$$

In contrast to all other models for which we have been studying the Goldstone phenomenon [9, 10], here we have the spontaneous symmetry breakdown of a two-dimensional group. In the following, we construct again the canonical normal modes of the Goldstone bosons together with their corresponding plasmon frequencies. We take a symmetry breaking state $\omega_{\beta\lambda}$ with $\lambda \neq 0$ and we consider fluctuation mode operators. Before constructing the relevant canonical normal coordinates of the Goldstone particles in the next section, we introduce here the notion of *fluctuation operators*.

For the mathematical details we refer to [11, 12]. Denote by $M_4^{sa} = (M_2 \otimes M_2)^{sa}$ the vectorspace of the Hermitian 4×4 matrices, i.e. these are one-site observables.
For any $A \in M_4^{sa}$, its fluctuation operator $F_\Lambda(A)$ for the volume Λ in the state $\omega_{\beta\lambda}$ is as usual defined by:

$$F_{\Lambda,\lambda}(A) = \frac{1}{\sqrt{|\Lambda|}}\sum_{x\in\Lambda}\left(A_x - \omega_{\beta\lambda}(A)\right),$$

where A_x is a copy of A on site $x \in \mathbb{Z}$. In [11] it is proved that the limit:

$$\lim_{\Lambda\to\mathbb{Z}} F_{\Lambda,\lambda}(A) \equiv F_\lambda(A)$$

exists in the sense of a non-commutative central limit as an operator on a Hilbert space $\mathcal{H}_\lambda$, generated by a normalised vector $\tilde{\Omega}_\lambda$, and vectors $F_\lambda(A_1)\dots F_\lambda(A_n)\tilde{\Omega}_\lambda$, with $A_i \in M_4^{sa}$, n is arbitrary and a scalar product:

for $A_i, B_j \in M_4^{sa}$,

$$\left(F_\lambda(A_1)\dots F_\lambda(A_n)\tilde{\Omega}_\lambda, F_\lambda(B_1)\dots F_\lambda(B_m)\tilde{\Omega}_\lambda\right) = \delta_{nm}\mathrm{Perm.}\left((\tilde{\Omega}_\lambda, F_\lambda(A_i)F_\lambda(B_j)\tilde{\Omega}_\lambda)\right)_{i,j=1,\dots n} \tag{23}$$

where the two-point function is given by:

$$\left(\tilde{\Omega}_\lambda, F_\lambda(A_i)F_\lambda(B_j)\tilde{\Omega}_\lambda\right) = \omega_{\beta\lambda}(AB) - \omega_{\beta\lambda}(A)\,\omega_{\beta\lambda}(B). \tag{24}$$

i.e. by the truncated two-point functions of the given state $\omega_{\beta\lambda}$. Furthermore, as already could be understood from (23), the fluctuation operators $F_\lambda(A), F_\lambda(B), \dots$ satisfy the canonical commutation relations:

$$[F_\lambda(A), F_\lambda(B)] = \omega_{\beta\lambda}([A,B])\mathbf{1} \tag{25}$$

i.e. the fluctuation operators are quantum canonical variables (compare $[q,p] = \mathrm{i}\hbar$) with quantisation parameter $\omega_{\beta\lambda}([A,B])$ (instead of $\mathrm{i}\hbar$) and the state $\left(\tilde{\Omega}_\lambda, .\,\tilde{\Omega}_\lambda\right)$ is a generalised free or quasi-free state of a boson field algebra of observables. In section 4, we look for the relevant operators $A, B, \dots$ in order to obtain the normal canonical variables for the Goldstone particles, i.e. the particles generated by the spontaneous symmetry breaking. We derive the dynamical equations for these modes and we solve them.

4 Spontaneous Symmetry Breaking and Plasma Oscillations

We will consider fluctuations of the generators

$$Q_\pm = \sigma_1^z \pm \sigma_2^z$$

of the broken symmetry.

Let P_i, $(i = 0,1,2,3)$ be the spectral projections on the eigenfunctions $|\phi_i\rangle$ corresponding to the eigenvalues ϵ_i (8) of h_λ (7), i.e. $h_\lambda = \sum_{i=0}^3 \epsilon_i P_i$. Consider for any local observable $A \in M_4^{sa}$ the Hermitian operators:

$$E_{ij}(A) = P_i A P_j + P_j A P_i; \qquad i < j; \quad i,j = 0,1,2,3; \tag{26}$$

and the adjoints:

$$JE_{ij}(A) = \mathrm{i}\left(P_i A P_j - P_j A P_i\right). \tag{27}$$

An immediate calculation leads to:

$$\begin{aligned} [h_\lambda, E_{kl}(A)] &= -\mathrm{i}(\epsilon_k - \epsilon_l) J E_{kl}(A); \\ [h_\lambda, J E_{kl}(A)] &= \mathrm{i}(\epsilon_k - \epsilon_l) E_{kl}(A). \end{aligned} \tag{28}$$

Now we take $A = Q_+$, respectively $A = Q_-$. Due to the particle-hole and type-exchange symmetries (section 3), the only non-zero operators of the type (26-27) are:

$$E_{0i}(Q_\pm),\ JE_{0i}(Q_\pm),\ E_{1i}(Q_\pm),\ JE_{1i}(Q_\pm) \quad i = 2, 3. \tag{29}$$

It is interesting to write out these operators in terms of the Pauli matrices $\sigma_i^\pm, \sigma_i^z,\ i = 1, 2$. As an example we give the expression for the first pair of these operators $(E_{02}(Q_-), JE_{02}(Q_-))$, the others are left as an exercise for the reader:

$$\begin{aligned} E_{02}(Q_-) &= \frac{\xi_-}{2\eta} Q_- + \frac{t\lambda}{\eta}(\sigma_1^z \sigma_2^+ - \sigma_2^z \sigma_1^+) + \frac{t\bar{\lambda}}{\eta}(\sigma_1^z \sigma_2^- - \sigma_2^z \sigma_1^-); \\ JE_{02}(Q_-) &= \mathrm{i}\frac{\xi_-}{2\eta}(\sigma_1^+ \sigma_2^- - \sigma_2^+ \sigma_1^-) + \mathrm{i}\frac{t\lambda}{\eta}(\sigma_1^+ - \sigma_2^+) - \mathrm{i}\frac{t\bar{\lambda}}{\eta}(\sigma_1^- - \sigma_2^-). \end{aligned}$$

$E_{02}(Q_-)$ is a component of the infintesimal generator Q_- and $JE_{02}(Q_-)$ is the adjoint order parameter operator to $E_{02}(Q_-)$.

It is clear from (8) that the dynamical equations (28) for the operators (29) contain only the relative energies $\epsilon_k - \epsilon_l$ given by:

$$\xi_\pm = \eta \pm U.$$

In particular, the energy gap ξ_+ is entering in the equations:

$$[h_\lambda, E_{02}(Q_-)] = \mathrm{i}\xi_+ JE_{02}(Q_-), \quad [h_\lambda, JE_{02}(Q_-)] = -\mathrm{i}\xi_+ E_{02}(Q_-), \tag{30}$$

$$[h_\lambda, E_{13}(Q_+)] = -\mathrm{i}\xi_+ JE_{13}(Q_+), \quad [h_\lambda, JE_{13}(Q_+)] = \mathrm{i}\xi_+ E_{13}(Q_+); \tag{31}$$

and the energy gap ξ_- in:

$$[h_\lambda, E_{03}(Q_-)] = -\mathrm{i}\xi_- JE_{03}(Q_-), \quad [h_\lambda, JE_{03}(Q_-)] = \mathrm{i}\xi_- E_{03}(Q_-), \tag{32}$$

$$[h_\lambda, E_{12}(Q_+)] = \mathrm{i}\xi_- JE_{12}(Q_+), \quad [h_\lambda, JE_{12}(Q_+)] = -\mathrm{i}\xi_- E_{12}(Q_+). \tag{33}$$

All these results are on the level of the microscopic observables. In particular one discovers in (30-33) that the dynamics leaves invariant the pairs $(E_{02}(Q_-), JE_{02}(Q_-))$, $(E_{13}(Q_+), JE_{13}(Q_+))$, $(E_{03}(Q_-), JE_{03}(Q_-))$ and $(E_{12}(Q_+), JE_{12}(Q_+))$. On the other hand the different pairs do not form canonical pairs of variables. Therefore, we have to consider the macroscopic variables, namely the fluctuations of these quantities. We know from section 3 that the fluctuation operators are canonical variables, satisfying the canonical commutation relations.

We treat explicitly the first pair appearing in the equation (30), and give the results for the other pairs without technical details.

Plasmon frequency ξ_+

With the notations of section 3, denote the fluctuations:

$$X_{02} \equiv -\frac{1}{n_{02}} F_\lambda \left(E_{02}(Q_-) \right); \qquad P_{02} \equiv \frac{1}{n_{02}} F_\lambda \left(J E_{02}(Q_-) \right), \tag{34}$$

where

$$n_{02} = \left(\frac{e^{\beta U} + e^{-\beta\eta}}{Z} \right)^{1/2}; \quad Z = 2\cosh(\beta U) + 2\cosh(\beta\eta).$$

These satisfy (following (25)) the canonical commutation relation:

$$[X_{02}, P_{02}] = -\frac{1}{n_{02}^2} \omega_{\beta\lambda} \left([E_{02}(Q_-), J E_{02}(Q_-)] \right) \mathbf{1} = i\hbar_+ \tag{35}$$

with quantisation parameter:

$$\hbar_+ = \frac{4\xi_-}{\eta} \tanh\left(\beta\xi_+/2 \right). \tag{36}$$

Following (24), the calculations for the variances are straightforward:

$$\left(\tilde{\Omega}_\lambda, X_{02}^2 \tilde{\Omega}_\lambda \right) = \left(\tilde{\Omega}_\lambda, P_{02}^2 \tilde{\Omega}_\lambda \right) = 2\frac{\xi_-}{\eta}. \tag{37}$$

Now compute the dynamics of these fluctuation operators induced by the microdynamics determined by the Hamiltonian h_λ (5). In differential form the dynamics follows immediatly from the equation (30):

$$\begin{aligned} \frac{d}{idt} \left(F_\lambda \left(E_{02}(Q_-) \right) \right) &= F_\lambda \left([h_\lambda, E_{02}(Q_-)] \right) = i\xi_+ F_\lambda \left(J E_{02}(Q_-) \right); \\ \frac{d}{idt} \left(F_\lambda \left(J E_{02}(Q_-) \right) \right) &= F_\lambda \left([h_\lambda, J E_{02}(Q_-)] \right) = -i\xi_+ F_\lambda \left(E_{02}(Q_-) \right). \end{aligned}$$

Using the notations (34) and in integrated form one gets:

$$\begin{aligned} X_{02}(t) &= X_{02}(0)\cos(\xi_+ t) + P_{02}(0)\sin(\xi_+ t); \\ P_{02}(t) &= -\, X_{02}(0)\sin(\xi_+ t) + P_{02}(0)\cos(\xi_+ t). \end{aligned} \tag{38}$$

i.e. the pair of macroscopic variables (X_{02}, P_{02}) is a canonical pair (35), evolving in time, independently from all other degrees of freedom of the system, as a pair of quantum harmonic oscillator variables oscillating with a frequency ξ_+. Remark that if the temperature T tends to the critical temperature T_c, i.e. when $|\lambda| \to 0$, then $\xi_- \to 0$. It follows from the equations (37) that the pair disappears completely i.e. $X_{02} = P_{02} = 0$ for $T \geq T_c$.

This finishes the study of the first pair of fluctuation variables.

In a completely analogous way one defines the second pair of canonical fluctuation operators:

$$X_{13} = \frac{1}{n_{13}} F_\lambda \left(E_{13}(Q_+) \right); \qquad P_{13} = \frac{1}{n_{13}} F_\lambda \left(J E_{13}(Q_+) \right); \tag{39}$$

with

$$n_{13} = \left(\frac{\mathrm{e}^{-\beta U} + \mathrm{e}^{\beta\eta}}{Z} \right)^{1/2}.$$

One gets, comparable to (35), (37) and (38):

$$[X_{13}, P_{13}] = \mathrm{i}\hbar_+; \tag{40}$$

$$\left(\tilde{\Omega}_\lambda, X_{13}^2 \tilde{\Omega}_\lambda \right) = \left(\tilde{\Omega}_\lambda, P_{13}^2 \tilde{\Omega}_\lambda \right) = 2\frac{\xi_-}{\eta}; \tag{41}$$

$$\begin{aligned} X_{13}(t) &= X_{13}(0)\cos(\xi_+ t) + P_{13}(0)\sin(\xi_+ t); \\ P_{13}(t) &= -\, X_{13}(0)\sin(\xi_+ t) + P_{13}(0)\cos(\xi_+ t). \end{aligned} \tag{42}$$

Plasmon frequency ξ_-

Following (32), consider the third pair of canonical fluctuation operators

$$X_{03} \equiv \frac{1}{n_{03}} F_\lambda \left(E_{03}(Q_-) \right); \qquad P_{03} \equiv \frac{1}{n_{03}} F_\lambda \left(J E_{03}(Q_-) \right), \tag{43}$$

where

$$n_{03} = \left(\frac{\mathrm{e}^{\beta U} + \mathrm{e}^{\beta\eta}}{Z} \right)^{1/2}.$$

One gets

$$[X_{03}, P_{03}] = \mathrm{i}\hbar_- \tag{44}$$

with a new quantisation parameter,

$$\hbar_- = \frac{4\xi_+}{\eta} \tanh\left(\beta\xi_-/2 \right); \tag{45}$$

variances,

$$\left(\tilde{\Omega}_\lambda, X_{03}^2 \tilde{\Omega}_\lambda\right) = \left(\tilde{\Omega}_\lambda, P_{03}^2 \tilde{\Omega}_\lambda\right) = 2\frac{\xi_+}{\eta}; \tag{46}$$

and dynamics

$$\begin{aligned} X_{03}(t) &= X_{03}(0)\cos(\xi_- t) + P_{03}(0)\sin(\xi_- t); \\ P_{03}(t) &= -\,X_{03}(0)\sin(\xi_- t) + P_{03}(0)\cos(\xi_- t). \end{aligned} \tag{47}$$

Finally, following (33), consider the fourth pair of canonical fluctuation operators:

$$X_{12} \equiv -\frac{1}{n_{12}} F_\lambda\left(E_{12}(Q_+)\right); \qquad\qquad P_{12} \equiv \frac{1}{n_{12}} F_\lambda\left(JE_{12}(Q_+)\right), \tag{48}$$

where

$$n_{12} = \left(\frac{\mathrm{e}^{-\beta U} + \mathrm{e}^{-\beta\eta}}{Z}\right)^{1/2}.$$

One computes the commutator:

$$[X_{12}, P_{12}] = \mathrm{i}\hbar_-; \tag{49}$$

the variances,

$$\left(\tilde{\Omega}_\lambda, X_{12}^2 \tilde{\Omega}_\lambda\right) = \left(\tilde{\Omega}_\lambda, P_{12}^2 \tilde{\Omega}_\lambda\right) = 2\frac{\xi_+}{\eta}; \tag{50}$$

and the dynamics

$$\begin{aligned} X_{12}(t) &= X_{12}(0)\cos(\xi_- t) + P_{12}(0)\sin(\xi_- t); \\ P_{12}(t) &= -\,X_{12}(0)\sin(\xi_- t) + P_{12}(0)\cos(\xi_- t). \end{aligned} \tag{51}$$

This completes the construction of the plasmon spectrum together with all its quantum normal modes. The next question is: are these four pairs (34,39,43,48) all different ? The question is relevant, because the mode variables are all fluctuations and coarse graining might make some of them equivalent, i.e. although $A, B \in M_4^{sa}$ and $A \neq B$, $F_\lambda(A)$ might be equal to $F_\lambda(B)$. The problem of equivalence of fluctuation operators is considered before (see [11, 12]). It turns out that they are equivalent, i.e. $F_\lambda(A) = F_\lambda(B)$, if and only if

$$\left(\tilde{\Omega}_\lambda, F_\lambda(A-B)^2 \tilde{\Omega}_\lambda\right) = \omega_{\beta\lambda}\left((A - B - \omega_{\beta\lambda}(A-B))^2\right) = 0. \tag{52}$$

If we take for A, B the operators $E_{ij}(Q_\pm)$ and $JE_{ij}(Q_\pm)$ with $i < j$, then clearly

$$\omega_{\beta\lambda}(E_{ij}(Q_\pm)) = \omega_{\beta\lambda}(JE_{ij}(Q_\pm)) = 0$$

and we have to check

$$\omega_{\beta\lambda}\left((A-B)^2\right)=\omega_{\beta\lambda}\left(A^2+B^2-AB-BA\right)=0; \tag{53}$$

for A and B taken out of different pairs $(E_{02}(Q_-), JE_{02}(Q_-))$, $(E_{13}(Q_+), JE_{13}(Q_+))$, $(E_{03}(Q_-), JE_{03}(Q_-))$, $(E_{12}(Q_+), JE_{12}(Q_+))$. It is immediatly checked that in this case always $\omega_{\beta\lambda}(AB)=\omega_{\beta\lambda}(BA)=0$. Hence, (53) reduces to

$$\omega_{\beta\lambda}\left((A-B)^2\right)=\omega_{\beta\lambda}\left(A^2+B^2\right)\neq 0;$$

which follows from (37),(41),(46)and(50). Hence, the four pairs of canonical fluctuation operators are all independent, proving the two-fold degeneracy of the plasmon frequency ξ_+ and ξ_-.

Finally, remark that if the temperature T tends to the critical temperature, i.e. when $\lambda\to 0$, then the pairs (X_{02}, P_{02}) (34) and (X_{13}, P_{13}) (39) disappear completely. On the other hand the pairs (X_{03}, P_{03}) (43) and (X_{12}, P_{12}) (48) become classical variables (see (44) and (49)), and become constants of the motions (see (47) and (51)). In fact for $T\geq T_c$ the continuous symmetry group (22) is restored. The fluctuation operators X_{03} and X_{12} are the fluctuations of the generators of this symmetry. The fluctuation operators P_{03} and P_{12} are the total particle current and relative particle current fluctuations. For $T<T_c$ the physical interpretation of these operators remains the same with the basic difference that they show a definite collective or macroscopic quantum character, and a non-trivial oscillatory time behaviour. We remind that the results derived for this model are mathematically rigorous. In view of the recent interest in interferences between Bose-Einstein condensates with Josephson-like coupling [13, 14] an extension of our model including such a coupling raises over as the natural thing to do next. We leave it for a future contribution.

Acknowledgement: The authors thank Tom Michoel for helpful discussions.

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