LIEB-THIRRING INEQUALITIES – Inequalities concerning the negative eigenvalues of the **Schrödinger operator**

$$H = -\Delta + V(x)$$

on $L^2(\mathbf{R}^n), n \geq 1$. With $e_1 \leq e_2 \leq \cdots < 0$ denoting the negative eigenvalues of H (if any), the Lieb-Thirring inequalities state that for suitable $\gamma \geq 0$ and constants $L_{\gamma,n}$

$$\sum_{j\geq 1} |e_j|^\gamma \leq L_{\gamma,n} \int_{\mathbf{R}^n} V_-(x)^{\gamma+n/2}\ \mathrm{d}x \qquad (1)$$

with $V_-(x) := \max\{-V(x), 0\}$. When $\gamma = 0$ the left side is just the number of negative eigenvalues. Such an inequality (1) can hold if and only if

$$\begin{aligned} \gamma \geq \frac{1}{2} &\quad \text{for} \quad \mathrm{n} = 1 \\ \gamma > 0 &\quad \text{for} \quad \mathrm{n} = 2 \\ \gamma \geq 0 &\quad \text{for} \quad \mathrm{n} \geq 3\ . \end{aligned} \qquad (2)$$

The cases $\gamma > \frac{1}{2}, n = 1, \gamma > 0, n \geq 2$, were established by E.H. Lieb and W.E. Thirring [13] in connection with their proof of **stability of matter**. The case $\gamma = \frac{1}{2}, n = 1$ was established by T. Weidl [16]. The case $\gamma = 0$, $n \geq 3$ was established independently by M. Cwikel [4], Lieb [9] and G.V. Rosenbljum [14] by different methods and is known as the CLR bound; the smallest known value for $L_{0,n}$ is in [9], [11].

Closely associated with the inequality (1) is the *semi-classical approximation* for $\sum |e|^\gamma$, which serves as a heuristic motivation for (1). It is (*cf.* [13]).

$$\begin{aligned} \sum_{j\geq 1} |e|^\gamma &\approx (2\pi)^{-n} \int_{\mathbf{R}^n \times \mathbf{R}^n} \left[p^2 + V(x)\right]_-^\gamma \mathrm{d}p\mathrm{d}x \\ &= L^c_{\gamma,n} \int_{\mathbf{R}^n} V_-(x)^{\gamma+n/2}\ \mathrm{d}x \end{aligned}$$

with

$$L^c_{\gamma,n} = 2^{-n}\pi^{-n/2}\Gamma(\gamma+1)/\Gamma(\gamma+1+n/2)\ .$$

Indeed, $L^c_{\gamma,n} < \infty$ for all $\gamma \geq 0$ whereas (1) holds only for the range given in (2). It is easy to prove (by considering $V(x) = \lambda W(x)$ with W smooth and $\lambda \to \infty$) that

$$L_{\gamma,n} \geq L^c_{\gamma,n}$$

An interesting, and mostly open problem is to determine the sharp value of the constant $L_{\gamma,n}$, especially to find those cases in which $L_{\gamma,n} = L^c_{\gamma,n}$. M. Aizenman and Lieb [1] proved that the ratio $R_{\gamma,n} = L_{\gamma,n}/L^c_{\gamma,n}$ is a monotonically non-increasing function of γ. Thus, if $R_{\Gamma,n} = 1$ for some Γ then $L_{\gamma,n} = L^c_{\gamma,n}$ for all $\gamma \geq \Gamma$. The equality $L_{\frac{3}{2},n} = L^c_{\frac{3}{2},n}$ was proved for $n = 1$ in [13] and for $n > 1$ in [8] by A. Laptev and T. Weidl. (See also [2].)

The following sharp constants are known:

$$\begin{aligned} L_{\gamma,n} &= L^c_{\gamma,n} && \text{all } \gamma \geq 3/2,\ [13],\ [1],\ [8] \\ L_{1/2,1} &= 1/2 && [7] \end{aligned}$$

There is strong support for the conjecture [13] that

$$L_{\gamma,1} = \frac{1}{\sqrt{\pi}(\gamma - \frac{1}{2})} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)} \left(\frac{\gamma - \frac{1}{2}}{\gamma + \frac{1}{2}}\right)^{\gamma+1/2} \qquad (3)$$

for $\frac{1}{2} < \gamma < \frac{3}{2}$.

Instead of considering all the negative eigenvalues as in (1), one can consider just e_1. Then for γ as in (2)

$$|e_1|^\gamma \leq L^1_{\gamma,n} \int_{\mathbf{R}^n} V_-(x)^{\gamma+n/2}\mathrm{d}x\ .$$

Clearly, $L^1_{\gamma,n} \leq L_{\gamma,n}$, but equality can hold, as in the cases $\gamma = 1/2$ and $3/2$ for $n = 1$. Indeed, the conjecture in (3) amounts to $L^1_{\gamma,1} = L_{\gamma,1}$ for $1/2 < \gamma < 3/2$. The sharp value (3) of $L^1_{\gamma,n}$ is obtained by solving a differential equation [13]. It has been conjectured that for $n \geq 3, L_{0,n} = L^1_{0,n}$. In any case, B. Helffer and D. Robert showed that for all n and all $\gamma < 1$, $L_{\gamma,n} > L^c_{\gamma,n}$.

The sharp constant $L^1_{0,n}, n \geq 3$ is related to the sharp constant S_n in the **Sobolev inequality**

$$\| \nabla f \|_{L^2(\mathbf{R}^n)} \geq S_n \| f \|_{L^{2n/(n-2)}(\mathbf{R}^n)} \qquad (4)$$

by $L^1_{0,n} = (S_n)^{-n}$.

By a 'duality argument' [13] the case $\gamma = 1$ in (1) can be converted into the following bound for the Laplacian, Δ. This bound is referred to as a Lieb-Thirring kinetic energy inequality and its most important application is to the **stability of matter** [12], [13]. Let $f_1, f_2, \dots$ be *any* orthonormal sequence (finite or infinite) in $L^2(\mathbf{R}^n)$ such that $\nabla f_j \in L^2(\mathbf{R}^n)$ for all $j \geq 1$. Associated with this sequence is a 'density'

$$\rho(x) = \sum_{j\geq 1} |f_j(x)|^2 \ . \qquad (5)$$

Then, with $K_n := n(2/L_{1,n})^{2/n}(n+2)^{-1-2/n}$,

$$\sum_{j\geq 1} \int_{\mathbf{R}^n} |\nabla f_j(x)|^2 \mathrm{d}x \geq K_n \int_{\mathbf{R}^n} \rho(x)^{1+2/n} \mathrm{d}x \ . \qquad (6)$$

This can be extended to *antisymmetric* functions in $L^2(\mathbf{R}^{nN})$. If $\Phi = \Phi(x_1, \dots, x_N)$ is such a function we define, for $x \in \mathbf{R}^n$,

$$\rho(x) = N \int_{\mathbf{R}^{n(N-1)}} |\Phi(x, x_2, \dots, x_N)|^2 \mathrm{d}x_2 \dots \mathrm{d}x_N \ .$$

Then, if $\int_{\mathbf{R}^{nN}} |\Phi|^2 = 1$,

$$\int_{R^{nN}} |\nabla \Phi|^2 \geq K_n \int_{\mathbf{R}^n} \rho(x)^{1+2/n} \mathrm{d}x \ . \qquad (7)$$

Note that the choice $\Phi = (N!)^{-1/2} \det f_j(x_k)|_{j,k=1}^N$ with f_j orthonormal reduces the general case (7) to (6).

If the conjecture $L_{1,3} = L_{1,3}^c$ is correct then the bound in (7) equals the **Thomas-Fermi** kinetic energy *ansatz*, and hence it is a challenge to prove this conjecture. In the meantime, see [11], [3] for the best available constants to date (1998).

Of course, $\int (\nabla f)^2 = \int f(-\Delta f)$. Inequalities of the type (7) can be found for other powers of $-\Delta$ than the first power. The first example of this kind, due to I. Daubechies [5], and one of the most important physically, is to replace $-\Delta$ by $\sqrt{-\Delta}$ in H. Then an inequality similar to (1) holds with $\gamma + n/2$ replaced by $\gamma + n$ (and with a different L_{γ,n_1}, of course). Likewise there is an analogue of (7) with $1 + 2/n$ replaced by $1 + 1/n$.

All proofs of (1) (except [7] and [16])actually proceed by finding an upper bound to $N_E(V)$, the number of eigenvalues of $H = -\Delta + V(x)$ that are below $-E$. Then, for $\gamma > 0$,

$$\sum |e|^\gamma = \gamma \int_0^\infty N_E(V) E^{\gamma-1} \mathrm{d}E$$

Assuming $V = -V_-$ (since V_+ only raises the eigenvalues), $N_E(V)$ is most accessible via the positive semidefiniate *Birman-Schwinger kernel* (*cf.* [15])

$$K_E(V) = \sqrt{V_-}\,(-\Delta + E)^{-1}\sqrt{V_-} \ .$$

$e < 0$ is an eigenvalue of H if and only if 1 is an eigenvalue of $K_{|e|}(V)$. Furthermore, $K_E(V)$ is *operator* monotone decreasing in E, and hence $N_E(V)$ equals the number of eigenvalue of $K_E(V)$ that are greater than 1.

An important generalization of (1) is to replace $-\Delta$ in H by $|i\nabla + A(x)|^2$, where $A(x)$ is some arbitrary vector field in $\mathbf{R}^n$ (called a magnetic vector potential). Then (1) still holds but it is not known if the sharp value of $L_{\gamma,n}$ changes. What is known is that all *presently* known values of $L_{\gamma,n}$ are unchanged. It is also known that $(-\Delta + E)^{-1}$, as a kernel in $\mathbf{R}^n \times \mathbf{R}^n$, is pointwise greater than the absolute value of the kernel $(|i\nabla + A|^2 + E)^{-1}$.

There is another family of inequalities for orthonormal functions, which is closely related to (1) and to the CLR bound [10]. As before, let $f_1, f_2, \dots, f_N$ be N orthonormal functions in $L^2(\mathbf{R}^n)$ and set

$$\begin{aligned} u_j &= (-\Delta + m^2)^{-1/2} f_j \\ \rho(x) &= \sum_{j=1}^N |u_j(x)|^2 \ . \end{aligned}$$

u_j is a **Riesz potential** ($m = 0$) or **Bessel potential** ($m > 0$) of f_j. If $n = 1$ and $m > 0$ then, $\rho \in C^{0,1/2}(\mathbf{R}^n)$ and $\| \rho \|_{L^\infty(\mathbf{R})} \leq L/m$.

If $n = 2$ and $m > 0$ then for all $1 \leq p < \infty$
$\| \rho \|_{L^p(\mathbf{R}^2)} \leq B_p m^{-2/p} N^{1/p}$.

If $n \geq 3, p = n/(n-2)$ and $m \geq 0$ (including $m = 0$) then $\| \rho \|_{L^p(\mathbf{R}^n)} \leq A_n N^{1/p}$.

Here, L, B_p, A_n are universal constants. Without the orthogonality, $N^{1/p}$ would have to be replaced by N. Further generalizations are possible [10].

References

[1] AIZENMAN, M.A. AND LIEB, E.H.: 'On semi-classical bounds for eigenvalues of Schrödinger operators', *Phys. Lett.* **66A** (1978), 427-429.

[2] BENGURIA, R. AND LOSS, M.: 'A simple proof of a theorem of Laptev and Weidl', preprint, 1999.

[3] BLANCHARD, PH. AND STUBBE, J.: 'Bound states for Schrödinger Hamiltonians: phase space methods and applications', *Rev. Math. Phys.* **8** (1996), 503-547.

[4] CWIKEL, M.: 'Weak type estimates for singular values and the number of bound states of Schrödinger operators', *Ann. Math.* **106** (1977), 93-100.

[5] DAUBECHIES, I.: 'An uncertainty principle for fermions with generalized kinetic energy', *Commun. Math. Phys.* **90** (1983), 511-520.

[6] HELFFER, B. AND ROBERT, D.: 'Riesz means of bound states and semi-classical limit connected with a Lieb-Thirring conjecture, II', *Ann. Inst. Henri Poincaré, Sect. Physique Théorique* **53** (1990), 139-147.

[7] HUNDERTMARK, D., LIEB, E.H. AND THOMAS, L.E.: 'A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator', *Adv. Theor. Math. Phys.* **2** (1998), 719-731.

[8] LAPTEV, A. AND WEIDL, T.: 'Sharp Lieb-Thirring inequalities in high dimensions', Acta Math., in press, 1999.

[9] LIEB, E.H.: 'The numbers of bound states of one-body Schrödinger operators and the Weyl problem', Vol. 36 of *Proc. Symp. Pure Math.,* Amer. Math. Soc., 1980, pp. 241-251. (cf. [3] and [10] for improvements.)

[10] LIEB, E.H.: 'An L^p bound for the Riesz and Bessel potentials of orthonormal functions', *J. Funct. Anal.* **51** (1983), 159-165.

[11] LIEB, E.H.: 'On characteristic exponents in turbulence', *Commun. Math. Phys.* **92** (1984), 473-480.

[12] LIEB, E.H.: 'Kinetic energy bounds and their applications to the stability of matter', in H. HOLDEN and A. JENSEN (eds.): *Springer Lecture Notes in Physics* **345** (1989), 371-382.

[13] LIEB, E.H. AND THIRRING W.: 'Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities', in E. LIEB, B. SIMON, A. WIGHTMAN (eds.): *'Studies in Mathematical Physics*', Princeton University Press, 1976, pp. 269-303. See also W. Thirring (ed.) *The Stability of Matter: From Atoms to Stars, Selecta of E.H. Lieb*, Springer, 1997.

[14] ROSENBLJUM, G.V.: 'Distribution of the discrete spectrum of singular differential operators', *Dok. Akad. Nauk SSSR* **202** (1972), 1012-1015. The details are given in *Izv. Vyss. Ucebn. Zaved. Matem.* **164** (1976), 75-86. (English trans. *Sov. Math. (Iz VUZ)* **20** (1976), 63-71.

[15] SIMON, B.: 'Functional integration and quantum physics': Vol. 86 of *Pure and Applied Mathematics*, Academic Press, 1979.

[16] WEIDL, T.: 'On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$'. *Commun. Math. Phys.* **178** no. 1 (1996), 135-146.

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