Nonlinear problems for a second order ODE

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ABSTRACT:

We study the general class of semilinear second order ordinary differential equations $u''(t) + r(t)u'(t) + g(t, u(t)) = f(t)$ *with a fixed constraint* $u(0) = u_0$. *Under a growth condition on* g *we prove the existence of solutions satisfying the nonlinear condition* $u(T) = h(u'(T))$. *Moreover, we give conditions in order to assure that any solution satisfying a Cauchy condition* $u(0) = u_0, \quad u'(0) = v_0$ *is defined over* $[0, T]$.

Introduction

We'll study the second order ODE

$$(*) \qquad u''(t) + r(t)u'(t) + g(t, u(t)) = f(t)$$

with a fixed initial data $u(0) = u_0$.

In the first section we'll state the basic assumptions and results concerning the Dirichlet problem associated to (*), which will be applied in the following sections.

In the second section we'll define a fixed point operator in order to solve a problem for the final position $u(T)$ depending on the speed at time T. Furthermore, we'll prove that if g satisfies a growth condition (which holds, for example, if g is *sublinear*) then there exist a class of $h's$ such that equation (*) admits at least one solution u with $u(0) = u_0$, $u(T) = h(u'(T))$.

Finally, in the third section we'll prove that the non-existence of a blow-up in the interval $[0, T]$ for a solution of (*) with initial value u_0 is equivalent to the solvability of the equation $\psi(s) = u'(0) \in I$, where $\psi : \mathbb{R} \to \mathbb{R}$ is a continuous function which depends on u_0. Furthermore, if g is locally Lipschitz on $0 \times \mathbb{R}^2$ then the disjoint union over u_0 of the sets $\{u_0\} \times I(u_0)$ is an open subset of $\mathbb{R}^2$.

1. Basic assumptions and unique solvability of the Dirichlet problem

Let $S : H^2(0,T) \to L^2(0,T)$ be the semilinear operator given by $Su = u'' + ru' + g(t,u)$. We'll assume throughout the paper that g is continuous and satisfies the condition

$$\text{(G1)} \qquad \frac{g(t,u) - g(t,v)}{u - v} \le c < \left(\frac{\pi}{T}\right)^2 \qquad \text{for any } t \in [0,T], u, v \in \mathbb{R}, u \neq v$$

Moreover, we'll assume that the friction coefficient $r \in H^1(0,T)$ is nondecreasing.

We recall the following results, which establish the existence of an apriori bound for S and the solvability of the problem $Su = f$ for arbitrary $f \in L^2(0,T)$ and Dirichlet conditions. Furthermore, the set $S^{-1}(f) \subset H^2(0,T)$ of all the solutions of $Su = f$ is homeomorphic to $\mathbb{R}^2$, thoght as the set of all possible Dirichlet boundary data. The proofs may be found in [AM]:

Lemma 1

Let $u, v \in H^2(I)$ with $u = v$ on ∂I. Then

$$\|Su - Sv\|_2 \ge \left((\frac{\pi}{T})^2 - c \right) \|u - v\|_2$$

and

$$\|Su - Sv\|_2 \ge \frac{(\frac{\pi}{T})^2 - c}{\frac{\pi}{T}} \|u' - v'\|_2$$

Theorem 2

The Dirichlet problem

$$\begin{cases} Su = f(t) & \text{in } (0,T) \\ u(0) = u_0, \quad u(T) = u_T \end{cases}$$

is uniquely solvable in $H^2(I)$ for any $f \in L^2(I)$.

Theorem 3

Let $f \in L^2(0,T)$ and $Tr : S^{-1}(f) \to \mathbb{R}^2$ the restriction of the usual trace function, i.e. $Tr(u) = (u(0), u(T))$. Then Tr is an homeomorphism.

2. Nonlinearities at the endpoint

In this section we'll study the problem

$$(1) \begin{cases} u'' + ru' + g(t,u) = f & \text{in } (0,T) \\ u(0) = u_0, \quad u(T) = h(u'(T)) \end{cases}$$

for fixed $f \in L^2(0,T)$ and continuous h. First we'll transform the problem in a one-dimensional fixed point problem: indeed, for $s \in \mathbb{R}$ theorem 2 allows us to define u_s as the unique solution of the problem

$$\begin{cases} u'' + ru' + g(t,u) = f & \text{in } (0,T) \\ u(0) = u_0, \quad u(T) = h(s) \end{cases}$$

Hence, if $\varphi_s(t) = \frac{h(s)-u_0}{T}t + u_0$, we have that

$$u_s(t) - \varphi_s(t) = \int_0^T (f - ru_s' - g(\theta), u_s'))G(t,\theta)d\theta$$

where G is the Green function asociated to the second-derivative operator ∂^2, namely

$$G(t,\theta) = \begin{cases} \frac{t(\theta-T)}{T} & \text{if } \theta \geq t \\ \frac{\theta(t-T)}{T} & \text{if } \theta \leq t \end{cases}$$

By a simple computation we obtain that

$$u_s'(T) = \frac{h(s)-u_0}{T} + \int_0^T (f - ru_s' - g(\theta), u_s'))\frac{\theta}{T}d\theta$$

and by theorem 3 we have:

Lemma 4

Let $\xi : \mathbb{R} \to \mathbb{R}$ be given by

$$\xi(s) = \frac{h(s)-u_0}{T} + \int_0^T (f - ru_s' - g(\theta, u_s'))\frac{\theta}{T}d\theta$$

Then ξ is continuous. Moreover, u is a solution of (1) if and only if there exists $s \in \mathbb{R}$ such that s is a fixed point of ξ.

Proof

Continuity of ξ follows immediately from the continuity of $Tr^{-1} : \mathbb{R}^2 \to S^{-1}(f)$. Moreover, if $\xi(s) = s$, then $u_s(T) = h(u_s'(T))$, proving that u_s is a solution of (1). Conversely, if u is a solution of (1), then $u = u_s$ for $s = u'(T)$. □

In order to state an existence result for (1), we'll give a different formulation for ξ. Indeed, as

$$\int_0^T r(\theta)u_s'(\theta)\theta d\theta = r(\theta)\theta u_s(\theta)\Big|_0^T - \int_0^T [r(\theta)\theta]'u_s(\theta)d\theta$$

$$= r(T)Th(s) - \int_0^T [r(\theta) + \theta r'(\theta)]u_s(\theta)d\theta$$

we obtain:

$$\xi(s) = (\frac{1}{T} - r(T))h(s) + \frac{1}{T}\left[\int_0^T \theta f(\theta)d\theta - u_0\right] + \frac{1}{T}\int_0^T (r + \theta r')u_s - \theta g(\theta, u_s)d\theta$$

Hence, s is a fixed point of ξ if and only if

$$\text{(2)} \qquad sT = (1 - r(T)T)h(s) + \int_0^T (r + \theta r')u_s - \theta g(\theta, u_s)d\theta + \int_0^T \theta f(\theta)d\theta - u_0$$

LEMMA 5

Let g satisfy the growth condition

$$\text{(G2)} \qquad |g(t,x)| \leq \alpha|x| + \beta$$

for some positive constants α and β. Then

$$\|u_s\|_2 \leq \sqrt{\frac{T}{3}}c(s) + \frac{T^2}{\pi^2 - cT^2}\left(\alpha\sqrt{\frac{T}{3}}c(s) + |h(s) - u_0|\frac{\|r\|_2}{T} + \|f\|_2 + \beta\right) := A(s)$$

where $c(s) = \sqrt{h(s)^2 + h(s)u_0 + u_0^2}$.

In particular,

$$\|u_s\|_2 \leq c_T|h(s)| + \gamma|h(s)|^{1/2} + \delta$$

for some constants γ, δ and

$$c_T = \sqrt{\frac{T}{3}} + \frac{T^2}{\pi^2 - cT^2}\left(\alpha\sqrt{\frac{T}{3}} + \frac{\|r\|_2}{T}\right)$$

<u>*Proof*</u>

By lemma 1, we have that

$$\|u_s - \varphi_s\|_2 \leq \frac{T^2}{\pi^2 - cT^2}\|Su_s - S\varphi_s\|_2 = \|f - r\varphi_s' - g(\cdot, \varphi_s)\|_2$$

A simple computation shows that $\|\varphi_s\|_2 = \sqrt{\frac{T}{3}}c(s)$, and as $\varphi_s' = \frac{h(s)-u_0}{T}$, and $\|g(\cdot,\varphi_s)\|_2 \leq \alpha\|\varphi_s\|_2 + \beta$, the result follows. □

THEOREM 6

Let us assume that (G2) holds, and define the functions $\xi^{\pm} : \mathbb{R} \to \mathbb{R}$ *as*

$$\xi^{\pm}(s) = \frac{1}{T}\left((1 - r(T)T)h(s) \pm \|r + \theta r'\|_2 A(s) + \sqrt{\frac{T^3}{3}}(\alpha A(s) + \beta) + \int_0^T \theta f(\theta) d\theta - u_0\right)$$

Moreover, assume that there exist $s_{\pm} \in \mathbb{R}$ *such that* $s_- \leq \xi^-(s_-)$ *and* $\xi^+(s_+) \leq s_+$. *Then (1) admits at least one solution* $u \in H^2(0,T)$.

Proof

¿From the previous computations it follows that

$$|\int_0^T (r + \theta r')u_s - \theta g(\theta, u_s) d\theta| \leq \|r + \theta r'\|_2 \|u_s\|_2 + \sqrt{\frac{T^3}{3}}(\alpha \|u_s\|_2 + \beta) \leq$$

$$\|r + \theta r'\|_2 A(s) + \sqrt{\frac{T^3}{3}}(\alpha A(s) + \beta)$$

and then $\xi^- \leq \xi \leq \xi^+$. Thus, the result follows from lemma 4. □

Remarks and Examples

We also have that

$$|\int_0^T (r + \theta r')u_s - \theta g(\theta, u_s) d\theta| \leq (\|r + \theta r'\|_2 + \sqrt{\frac{T^3}{3}}\alpha) c_T |h(s)| + B(s)$$

where $B(s)$ *is a small-order term.*

Corollary 7

Let us assume that (G1) holds and that h *changes sign on* $\mathbb{R}$. *Then (1) admits a solution for any* T *small enough.*

Proof

As $c_T \to 0$ for $T \to 0$, then for small T we have from the previous remark that

$$T\xi(s) \simeq \kappa(T)h(s) + B(s),$$

where $B(s)$ is a small-order term and $\kappa \to 1$. Choosing $s_{\pm} \in \mathbb{R}$ such that $h(s_+) > 0 > h(s_-)$, it suffices to take T such that $h(s_+) > s_+ T$, $h(s_-) < s_- T$.

3. Blow-up results

In this section we'll study the behavior of the solutions of the Cauchy problem

$$(3) \begin{cases} u'' + ru' + g(t,u) = f & \text{in } (0,T) \\ u(0) = u_0, \quad u'(0) = v_0 \end{cases}$$

As in the previous section, for every $s \in \mathbb{R}$ we'll define u_s as a solution of a two-point boundary value problem. In this case, we'll consider $\varphi_s(t) = k(s)t + u_0$ for some $k : \mathbb{R} \to \mathbb{R}$ and u_s the unique solution of

$$\begin{cases} u'' + ru' + g(t,u) = f & \text{in } (0,T) \\ u(0) = u_0, \quad u(T) = \varphi_s(T) \end{cases}$$

Then

$$u_s(t) - \varphi_s(t) = \int_0^T (f - ru_s' - g(\theta), u_s'))G(t,\theta)d\theta$$

and it's easy to conclude that

$$u_s'(0) = k(s) + \int_0^T (f - ru_s' - g(\theta), u_s'))\frac{\theta - T}{T}d\theta := \psi(s)$$

If we define $I(u_0) = Range(\psi)$, then

Theorem 8

Let us assume that $v_0 \in I(u_0)$. Then at least one solution of (3) is defined over $[0,T]$. Conversely, if (3) admits a solution defined over $[0,T]$ then $v_0 \in I(u_0)$.

Proof

It's clear that if $v_0 = \psi(s)$, then $u_s'(0) = v_0$. The converse is also immediate. □

Remarks

a) In particular, if g is Lipschitz with respect to u in a neighborhood of $(0, u_0)$ the unique solution of (3) lasts up to T. In this case, we'll see in next theorem that $I(u_0)$ is open.

b) The existence of an interval $I(u_0)$ such that theorem 8 holds may be deduced in a direct way, by proving that the set of initial data v_0 such that the solution of (3) does not blow-up in $[0,T]$ is connected. Indeed, it suffices to see that two solutions of $Su = f, u(0) = u_0$ do not intersect before the endpoint T, but this is a immediate consequence of the uniqueness of theorem 2, applied to any interval $[0,T_0] \subset [0,T]$.

Theorem 9

Let us assume that g is locally Lipschitz on $\{0\} \times A$, where A is an open subset of $\mathbb{R}$. Then

$$\bigcup_{u_0 \in A} \{u_0\} \times I(u_0)$$

is open in $\mathbb{R}^2$ *.*

Proof

Let $S_A = \{u \in H^2(0,T) : Su = f, u(0) \in A\}$, and consider the continuous mapping $\rho : S_A \to \mathbb{R}^2$ given by $rho(u) = (u(0), u'(0))$. As g is locally Lipschitz, κ is injective, and hence $Tr^{-1}o \quad rho : S_A \to S^{-1}(f)$ is open.

As a final remark, we'll give a result which avoids our basic assumption (G1) if g grows at most linearly at infinity:

THEOREM 10

Let us assume that $|g(t,u)| \leq \alpha|u| + \beta$, *with* $\alpha < \left(\frac{\pi}{T}\right)^2$. *Then any solution of (3) is defined over* $[0,T]$.

Proof

Let u be a solution of (3) such that $u(t) \to +\infty$ for $t \to t_0 \leq T$ and consider the linear operator $Lw := w'' + rw' + \alpha w$. Then $Lu \geq f - \beta$. Choosing $v : \mathbb{R} \to \mathbb{R}$ such that

$$Lv = f - \beta, \qquad v(0) = u_0, \quad v'(0) \geq v_0$$

then $v > u$ in a maximal interval $(0, A)$ with $A < T$. As $Lu \geq Lv$, we obtain:

$$0 \leq \int_0^A (Lv - Lu)(v - u) = -\int_0^A (v' - u')^2 + \int_0^A r(v' - u')(v - u) + \int_0^A \alpha(v - u)^2$$

As $\int_0^A r(v' - u')(v - u) = -\frac{1}{2}\int_0^A r'(v - u)^2 \leq 0$ and $\int_0^A (v - u)^2 \leq \left(\frac{A}{\pi}\right)^2 \int_0^A (v' - u')^2$, we have that

$$0 \leq \left(\alpha(\frac{A}{\pi})^2 - 1\right) \int_0^A (v' - u')^2,$$

a contradiction. The case $u \to -\infty$ is analogous. $\square$

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