**Nonlinear two-point boundary value problems and a Duffing equation**

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Abstract:

*In this paper we study a general semilinear second order ODE*

$$(*) \qquad (pu')' + g(t,u,u') = f$$

*Under an appropiate growth condition on $g$ we prove that the Dirichlet problem for (*) is uniquely solvable. Moreover, the set of $H^2$-solutions of (*) is homeomorphic to the two-dimensional real space. We also establish conditions for the existence of periodic solutions of (*).*

## Introduction

The two-point boundary value problem for a general semilinear second order ODE, namely

$$(**) \qquad u'' + g(t,u,u') = 0 \qquad , u(0) = u_0, \quad u(\alpha) = u_\alpha$$

has been studied by many authors from the pioneering work of Picard [P], who proved the existence of a solution of $(**)$ by an application of the well known method of successive approximations under a Lipschitz condition on $g$ and a smallness condition on $\alpha$. Sharper results were obtained by Hamel [H] in the special case of a forced pendulum equation (see also [M1], [M2]). The existence of periodic solutions for this case has been first considered by Duffing [Du] in 1918. Variational methods have been also applied when $g = g(t,u)$ by Lichtenstein [L], who considered the functional

$$I(u) = \int_0^\alpha \frac{u'^2}{2} - (\int_0^{u(t)} g(t,s)ds)dt$$

Finally, we mention also the topological approach, introduced in 1905 by Severini [S] with a shooting method, which gave a survey of results by the use of Leray-Schauder techniques and degree theory. For a complete overview of the problem and further results we refer the reader to [M3].

Let $I=(0,\alpha)$ and $S:H^2(I)\to L^2(I)$ be the semilinear operator given by

$$Su=(pu')'+g(t,u,u')$$

where $p\in C^1(\overline{I})$, $p>0$ and $g:\overline{I}\times \mathbb{R}^2\to\mathbb{R}$ is continuous. We'll assume that

$$\text{(G1)}\qquad [g(t,u,x)-g(t,v,y)](u-v)\le c_1(u-v)^2+c_2(t)(x-y)^2$$
$$+[r(t)(u-v)+\psi(u-v)](x-y)$$

holds, for some $\psi\in L^1_{loc}(\mathbb{R})$, $r\in H^1(I)$, $r'\ge 0$ a.e., and $c_1\in\mathbb{R}$, $c_2\in L^\infty(I)$ satisfying the condition

$$\sup_{t\in\overline{I}}\frac{c_2(t)}{p(t)}<1-\frac{c_1}{\lambda_1}$$

where $\lambda_1$ is the first eigenvalue of the problem

$$-(pu')'=\lambda u, \qquad u|_{\partial I}=0.$$

We remark that (G1) implies a growing condition on $u$:

$$\frac{g(t,u,x)-g(t,v,x)}{u-v}\le c_1$$

In particular, if $g$ is differentiable with respect to $u$, then $\frac{\partial g}{\partial u}\le c_1$. Moreover, considering $u=v$ we also obtain that $\psi(0)=0$ and $c_2\ge 0$.

For simplicity, we'll consider $\varepsilon=1-\frac{c_1}{\lambda_1}-\sup_{t\in\overline{I}}\frac{c_2(t)}{p(t)}$ and $\delta=\lambda_1\varepsilon\sup_t p(t)$. Then we have:

Lemma 1

*Let $u,v\in H^2(I)$ with $u=v$ on $\partial I$. Then*

$$\|Su-Sv\|_2\ge\delta\|u-v\|_2$$

*and*

$$\|Su-Sv\|_2\ge\frac{\delta}{\sqrt{\lambda_1}}\left(\int_I p(u'-v')^2\right)^{1/2}$$

*Proof*

Let $w=u-v$, then

$$\|Su-Sv\|_2\|w\|_2\ge-\int(Su-Sv).w=\int p(w')^2-\int[g(t,u,u')-g(t,v,v')](u-v)$$

$$\geq \int (p-c_2)(w')^2 - c_1\|w\|_2^2 - \int (rw+\psi(w))w'$$

By Poincaré's inequality $\|w\|_2^2 \leq \frac{1}{\lambda_1}\int p(w')^2$. Moreover, if $\widetilde{\psi}(t) = \int_0^t \psi$ then $\int \psi(w)w' = \left[\widetilde{\psi}(w)\right]_0^\alpha = 0$ and as $-\int rww' = \int r'\frac{w^2}{2} \geq 0$, we obtain:

$$\|Su-Sv\|_2\|w\|_2 \geq \int ((1-\frac{c_1}{\lambda_1})p - c_2)(w')^2 \geq \frac{\delta}{\lambda_1}\int p(w')^2$$

Hence, the result follows. □

REMARKS:

*i) For simplicity and by the previous lemma, we may denote by $c_0$ the best constant such that $\|u-v\|_{1,2} \leq c_0\|Su-Sv\|_2$ for any $u,v \in H^2(I)$ such that $u=v$ on $\partial I$.*

*ii) It's easy to prove that if $g$ satisfies a Lipschitz condition on $(u,u')$ then there exists an apriori inferior bound for $S$ on $H^2(I)$, i.e. a constant $c$ such that*

$$c\|Su-Sv\|_2 \geq \|u-v\|_{2,2}$$

*for any $u,v \in H^2(I)$ with $u=v$ on $\partial I$.*

We'll prove an existence and uniqueness result assuming a weaker condition on $g$, namely:

(G2) $$|g(t,u,x)| \leq g_0(u)k|x|$$

for any $t \in \overline{I}$, $u,x \in \mathbb{R}$, with $g_0$ such that $g_R := g_0|_{[-R,R]} < \infty$ for any $R$.

THEOREM 2

*Let us assume that $g$ satisfies (G1) and (G2). Then the Dirichlet problem*

$$(1)\begin{cases} Su = f(t) & \text{in } (0,\alpha) \\ u(0) = u_0, \quad u(\alpha) = u_\alpha \end{cases}$$

*is uniquely solvable in $H^2(I)$ for any $f \in L^2(I)$.*

<u>*Proof*</u>

Let $\sigma \leq 1$ and $S_\sigma u := (pu')' + \sigma g(t,u,u')$, then the result is obviously true for $S_0$. If the result holds for $S_\sigma$, for any $0 \leq \sigma < \sigma_0$, then for $\varepsilon > 0$ and fixed $\overline{u} \in H^1(I)$ we may define $u = T\overline{u}$ as the unique solution of the problem

$$\begin{cases} S_{\sigma_0-\varepsilon}u = f(t) - \varepsilon g(t,\overline{u},\overline{u}') & \text{in } (0,\alpha) \\ u(0) = u_0, \qquad u(\alpha) = u_\alpha \end{cases}$$

For $\overline{u} \in H^1(I)$, $g(\cdot,\overline{u},\overline{u}') \in L^2(I)$ with $\|g(\cdot,\overline{u},\overline{u}')\|_2 \leq \|g_0(\overline{u})\|_2 + k\|\overline{u}'\|_2$. This shows that $T : H^1(I) \to H^1(I)$ is well defined. On the other hand, if $c_0(\sigma)$ is the constant of lemma 1 for $S_\sigma$, we may assume that $c_0(\sigma) \leq c_0(1) = c_0$ and for $u = T\overline{u}$, $v = T\overline{v}$ we have:

$$\|u - v\|_{1,2} \leq c_0\|S_{\sigma_0-\varepsilon}u - S_{\sigma_0-\varepsilon}v\|_2 = \varepsilon c_0\|g(\cdot,\overline{u},\overline{u}') - g(\cdot,\overline{v},\overline{v}')\|_2$$

If $\overline{u} \to \overline{v}$ in $H^1(I) \hookrightarrow C(\overline{I})$ by dominated convergence it's easy to prove that $g(\cdot,\overline{u},\overline{u}') \to g(\cdot,\overline{v},\overline{v}')$ for the $L^2$-norm and continuity of $T$ follows. Furthermore, if $\|\overline{u}\|_{1,2} \leq R$ and $\overline{v} = 0$,

$$\|u\|_{1,2} \leq \|v\|_{1,2}+\varepsilon c_0(\|g(\cdot,\overline{u},\overline{u}')\|_2+\|g(\cdot,0,0)\|_2) \leq \|v\|_{1,2}+\varepsilon c_0(g_{k_1R}\alpha^{1/2}+\|g(\cdot,0,0)\|_2)$$

where $k_1$ is the constant of the imbedding $H^1(I) \hookrightarrow L^\infty(I)$. Moreover, as

$$\|(u-v)''\|_2 \leq \frac{1}{inf p(t)}(\|p'(u'-v') + (\sigma_0-\varepsilon)(g(t,u,u') - g(t,v,v'))\|_2$$

$$+\varepsilon\|g(t,\overline{u},\overline{u}') - g(t,0,0)\|_2)$$

we conclude that $T(B_R(0))$ is bounded for $\|\cdot\|_{2,2}$. By the compactness of the imbedding $H^2(I) \hookrightarrow H^1(I)$, $T(B_R(0))$ is precompact.

Fixing $\varphi \in C^1(\overline{I})$ such that $\varphi(0) = u_0$, $\varphi(\alpha) = u_\alpha$, we have that

$$\|v - \varphi\|_{1,2} \leq c_0\|f - \varepsilon g(t,0,0) - S_{\sigma_0-\varepsilon}\varphi\|_2 \leq k_0$$

for some constant $k_0$ independent of $\varepsilon$. Then, for $R > k_0 + \|\varphi\|_{1,2}$ and $\varepsilon < \frac{R-(k_0+\|\varphi\|_{1,2})}{c_0(g_{k_1R}\alpha^{1/2}+\|g(\cdot,0,0)\|_2)}$ we obtain that $T(B_R(0)) \subset B_R(0)$ and by Schauder Theorem $T$ has a fixed point. As uniqueness follows immediately from lemma 1, the result holds for $\sigma_0$.

In the same way, if the result holds for $S_\sigma$, with $0 \leq \sigma < 1$, then for $\varepsilon > 0$ and fixed $\overline{u} \in H^1(I)$ we may define $u = T\overline{u}$ as the unique solution of the problem

$$\begin{cases} S_\sigma u = f(t) - \varepsilon g(t,\overline{u},\overline{u}') & \text{in } (0,\alpha) \\ u(0) = u_0, \qquad u(\alpha) = u_\alpha \end{cases}$$

proving that the result holds for $S_{\sigma+\varepsilon}$ if $\varepsilon$ is small enough. Hence, the proof is complete. $\square$

REMARK: *In the previous proof one may have assumed that $f = 0$, since for $g$ satisfying (G1)-(G2) it suffices to take $\overline{g} = g(t,u,u') - f(t)$, even for noncontinuous $f \in L^2$.*

As a consequence of Theorem 2, we obtain a version of a Green-type operator for the nonlinear case:

COROLLARY 3

*Let $g$ satisfy (G1) and (G2), and consider the operator $G : L^2(I) \to H^1(I)$ given by $Gf = u$, where $u$ is the unique solution of (1). Then $G$ is compact.*

<u>*Proof*</u>

For $f, f_0 \in L^2(I)$ we have that

$$\|Gf - Gf_0\|_{1,2} \leq c_0\|SGf - SGf_0\|_2 = c_0\|f - f_0\|_2$$

and the continuity of $G$ follows. Moreover, setting $u = G(f)$ and $v = G(0)$ it follows that

$$\|(u-v)''\|_2 \leq \frac{1}{inf p(t)}(\|f - p'(u'-v') - (g(t,u,u') - g(t,v,v'))\|_2$$

and the result follows from the compactness of the imbedding $H^2(I) \hookrightarrow H^1(I)$. □

We'll apply theorem 2 in order to find solutions for the general Dirichlet problem

$$(2)\begin{cases} Su = f(t,u,u') & \text{in } (0,\alpha) \\ u(0) = u_0, \quad u(\alpha) = u_\alpha \end{cases}$$

THEOREM 4

*Let $f$ be continuous and $g$ satisfy (G1)-(G2). We'll assume that the growing condition*

$$\text{(F)} \qquad |f(t,u,x)| \leq c|(u,x)| + d$$

*holds for some constant $c < \frac{1}{c_0}$. Then (2) is solvable in $H^2(I)$.*

<u>*Proof*</u>

Using (F), we may prove that the operator $T : H^1(I) \to H^1(I)$ given by $T\overline{u} = G(f(\cdot, \overline{u}, \overline{u}'))$ is well defined and compact. ¿From lemma 1,

$$\|T\overline{u} - T0\|_{1,2} \leq c_0\left(\|f(t,0,0)\|_2 + c\|\overline{u}\|_{1,2} + \alpha^{1/2}d\right)$$

Letting $R \to \infty$ it follows that $T(B_R(0)) \subset B_R(0)$, which completes the proof. □

In the next theorem we'll state some properties of the set of solutions of the semilinear equation $Su = f$. For $f \in L^2(I)$ and $g$ continuous let us consider the set

$$\mathcal{S}_{f,g} = \{u \in H^2(I) : (pu')' + g(t,u,u') = f\}$$

Then we have:

THEOREM 5

*Let $g, \overline{g}$ satisfy (G1) and (G2) and $f, \overline{f} \in L^2(I)$. Then $\mathcal{S}_{f,g}$ is homeomorphic to $\mathcal{S}_{\overline{f},\overline{g}}$ for the $H^2$-norm.*

REMARK:

*In particular, $\mathcal{S}_{f,g} \simeq \mathbb{R}^2$. For example, $\mathcal{S}_{0,0} = \{u = x + y\int_0^t \frac{1}{p} : x, y \in \mathbb{R}\}$.*

<u>*Proof of theorem 5*</u>

For any $u \in \mathcal{S}_{f,g}$ we consider $Tu = \varphi$, where

$$\varphi(t) = \frac{u(\alpha) - u(0)}{\alpha}t + u(0).$$

By theorem 2, $T : \mathcal{S}_{f,g} \to span\{1,t\}$ is bijective. The continuity of $T$ is clear since $H^2(I) \hookrightarrow C(\overline{I})$; on the other hand, if $\varphi_n \to \varphi$, for $u_n = T^{-1}(\varphi_n)$, $u = T^{-1}(\varphi)$ and $w_n = u_n - u$ we have:

$$0 = \int (Su_n - Su)(u_n - u) \leq [pw_n w_n']_0^\alpha + \int (c_2 - p(1 - \frac{c_1}{\lambda_1}))(w_n')^2 + \left[\widetilde{\psi}(w_n)\right]_0^\alpha$$

$$+ \int rw_n w_n' \leq -\delta \int p(w_n')^2 + [pw_n w_n']_0^\alpha + \left[\widetilde{\psi}(w_n)\right]_0^\alpha + \left[\frac{w_n^2}{2}\right]_0^\alpha$$

But $w_n(0) = \varphi_n(0) - \varphi(0) \to 0$, $w_n(\alpha) = \varphi_n(\alpha) - \varphi(\alpha) \to 0$. On the other hand, by lemma 1 $\|w_n\|_{1,2}$ is bounded and from the inequalities

$$\|w_n''\|_2 \leq \frac{1}{\inf p(t)}\|g(t,u_n,u_n') - g(t,u,u') - p'w_n'\|_2 \tag{3}$$

$$\|g(\cdot,u_n,u_n') - g(\cdot,u,u')\|_2 \leq \|g_0(u_n)\|_2 + k\|u_n'\|_2 + \|g(\cdot,u,u')\|_2 \tag{4}$$

we conclude that $w_n$ is bounded in $H^2(I)$. Hence, $\|w_n'\|_\infty$ is bounded, and

$$\delta \int p(w_n')^2 \leq [pw_n w_n']_0^\alpha + \left[\widetilde{\psi}(w_n)\right]_0^\alpha + \left[\frac{w_n^2}{2}\right]_0^\alpha \to 0.$$

By Poincaré's inequality, $u_n \to u$ in $H^1(I) \hookrightarrow C(\overline{I})$. By dominated convergence we obtain from (3) that $\|w_n''\|_2 \to 0$ and the result follows. $\square$

### An application to a periodic problem

In this section we'll apply the previous results to the periodic problem

$$\text{(Per)} \begin{cases} Su = f(t) & \text{in } (0,\alpha) \\ u(0) = u(\alpha), \quad u'(0) = u'(\alpha) \end{cases}$$

Let $g$ satisfy (G1)-(G2) and assume that $p(0) = p(\alpha)$. From theorem 2, for any $s \in \mathbb{R}$ we may consider $u_s$ as the unique solution of the problem

$$\begin{cases} Su = f(t) & \text{in } (0,\alpha) \\ u(0) = u(\alpha) = s \end{cases}$$

Then $\mathcal{C} = \{u_s : s \in \mathbb{R}\} \subset \mathcal{S}_{f,g}$ is an imbedded curve for the $H^2$-norm. Clearly $\mathcal{C}$ is unbounded for $\|\cdot\|_\infty$.

Let $I : H^2(I) \to \mathbb{R}$ given by $I(u) = \int_0^\alpha g(\cdot, u, u')$. Then we have:

THEOREM 6

*Let $g$ satisfy (G1)-(G2) and $f \in L^2$. Then the following statements are equivalent:*

*i) (Per) admits (at least) a solution*

*ii) There exist $u^+, u^- \in \mathcal{C}$ such that*

$$I(u^+) \geq \int_0^\alpha f \geq I(u^-)$$

<u>*Proof*</u>

Let $\tau : H^2(I) \to \mathbb{R}$ given by $\tau(u) = \int_0^\alpha f - I(u)$. Then $\tau$ is continuous, and as $[pu']_0^\alpha = \int_0^\alpha (pu')'$, it's clear that (Per) is solvable if and only if $\tau(u) = 0$ for some $u \in \mathcal{C}$. As $\mathcal{C}$ is connected, the result follows. $\square$

REMARK:

*¿From lemma 1 we have the following apriori bound:*

$$\left(\int_I p(u_s')^2\right)^{1/2} \leq \frac{\sqrt{\lambda_1}}{\delta}\|Su_s - Ss\|_2 = \frac{\sqrt{\lambda_1}}{\delta}\|f - g(\cdot, s, 0)\|_2$$

*Furthermore, as* $u_s(t) = s + \int_0^t u_s'$ , *we obtain:*

$$\|u_s - s\|_\infty \leq \alpha^{1/2}\|u_s'\|_2 \leq \frac{(\lambda_1\alpha)^{1/2}}{\delta \inf p^{1/2}(t)}\|f - g(\cdot, s, 0)\|_2$$

*In particular, if* $g(\cdot, s, 0)$ *satisfies the growth condition*

$$\text{(G3)} \qquad \|g(\cdot, s, 0)\|_2 \leq c|s| + d \quad \text{for some} \quad c < \frac{\delta \inf p^{1/2}(t)}{(\lambda_1\alpha)^{1/2}}$$

*we obtain that* $u_s(t) \in J_s$ *for any* $t$ , *where the interval* $J_s$ *is defined by*

$$J_s = \begin{cases} [(1-\widetilde{c})s - \widetilde{d}, (1+\widetilde{c})s + \widetilde{d}] & \text{if } s \geq 0 \\ [(1+\widetilde{c})s - \widetilde{d}, (1-\widetilde{c})s + \widetilde{d}] & \text{if } s < 0 \end{cases}$$

*with*

$$\widetilde{c} = \frac{c(\lambda_1\alpha)^{1/2}}{\delta \inf p^{1/2}(t)} < 1, \qquad \widetilde{d} = \frac{(\lambda_1\alpha)^{1/2}}{\delta \inf p^{1/2}(t)}(d + \|f\|_2).$$

COROLLARY 7

*Let* $f \in L^2(I)$ *and assume that* $g$ *satisfies (G1), (G2) and (G3). Furthermore, we'll assume that there exist* $s^+, s^- \in \mathbb{R}$ , $\varphi^+, \varphi^- \in L^1(I)$ *and* $k_1, k_2 \in H^1(I)$ *such that*

$i)\, g(t,u,x) \geq \varphi^+(t) + k_1'(t)u + k_1(t)x$ *for any* $u \in J_{s^+}, t \in I, x \in \mathbb{R}$

$ii)\, g(t,u,x) \leq \varphi^-(t) + k_2'(t)u + k_2(t)x$ *for any* $u \in J_{s^-}, t \in I, x \in \mathbb{R}$

$iii)\, \int_0^\alpha \varphi^+ + (k_1(\alpha) - k_1(0))s^+ \geq \int_0^\alpha f \geq \int_0^\alpha \varphi^- + (k_2(\alpha) - k_2(0))s^- \geq (k_2(\alpha) - k_2(0))s^-$ .

*Then (Per) admits at least a solution.*

<u>*Proof*</u>

¿From the previous remark, $u_{s^+}(t) \in J_{s^+}$ , and then

$$\int_0^\alpha g(t, u_{s^+}, u_{s^+}') \geq \int_0^\alpha \varphi^+ + k_1'u_{s^+} + k_1u_{s^+}' = \int_0^\alpha \varphi^+ + (k_1(\alpha) - k_1(0))s^+ \geq \int_0^\alpha f$$

In the same way, $\int_0^\alpha g(t, u_{s^-}, u_{s^-}') \leq \int_0^\alpha f$ and the result follows from theorem 6. $\square$

As a consequence of corollary 7 we obtain conditions on $g$ for nonresonance:

COROLLARY 8

*Let $g$ satisfy (G1), (G2) and (G3) and assume that there exist $k_1, k_2 \in H^1(I)$ and $l_1, l_2 : \mathbb{R} \to \mathbb{R}$ such that one of the following assumptions holds:*

*Assumption A)*

$$i) g(t,u,x) \geq l_1(u) + k_1'(t)u + k_1(t)x \ \text{ for } x \in \mathbb{R}, u > 0$$

$$ii) g(t,u,x) \leq l_2(u) + k_2'(t)u + k_2(t)x \ \text{ for } x \in \mathbb{R}, u < 0$$

*and*

$$iii) l_1(u) \geq a_1 u + b_1 \ \text{ and } \ l_2(u) \leq a_2 u + b_2\,, \text{ with}$$

$$a_i(1-\widetilde{c})\alpha + k_i(\alpha) > k_i(0)$$

*Assumption B)*

$$i) g(t,u,x) \leq l_1(u) + k_1'(t)u + k_1(t)x \ \text{ for } x \in \mathbb{R}, u > 0$$

$$ii) g(t,u,x) \geq l_2(u) + k_2'(t)u + k_2(t)x \ \text{ for } x \in \mathbb{R}, u < 0$$

*and*

$$iii) l_1(u) \leq a_1 u + b_1 \ \text{ and } \ l_2(u) \geq a_2 u + b_2\,, \text{ with}$$

$$a_i(1+\widetilde{c})\alpha + k_i(\alpha) < k_i(0)$$

*Then (Per) admits at least one solution for any $f \in L^2$. Moreover, for every $f$ the set of solutions of (Per) is contained in a compact arc of $H^2(I)$.*

<u>*Proof*</u>

Under assumption A), we obtain for $s > 0$ that

$$l_1(u_s) \geq a_1 u_s(t) + b_1 \geq a_1(1-\widetilde{c})s - a_1\widetilde{d} + b_1$$

and

$$l_2(u_{-s}) \leq a_2 u_{-s}(t) + b_2 \leq -a_2(1-\widetilde{c})s + a_2\widetilde{d} + b_2$$

Thus, it suffices to consider

$$\varphi^+ \equiv a_1(1-\widetilde{c})s - a_1\widetilde{d} + b_1, \qquad \varphi^- \equiv -a_2(1-\widetilde{c})s + a_2\widetilde{d} + b_2$$

since in that case $\int_0^\alpha \varphi^+ + (k_1(\alpha) - k_1(0))s = (a_1(1-\widetilde{c})\alpha + (k_1(\alpha) - k_1(0))\, s + \alpha(-a_1\widetilde{d} + b_1) \to +\infty$ $\int_0^\alpha \varphi^- - (k_2(\alpha) - k_2(0))s = -\,(a_2(1-\widetilde{c})\alpha + (k_2(\alpha) - k_2(0))\, s +$

$\alpha(a_2\widetilde{d}+b_2) \to -\infty$ and conditions of corollary 7 hold taking $s$ large enough. Furthermore, the same computations show that every solution of (Per) belongs to the arc $\{u_{\overline{s}} : -s \leq \overline{s} \leq s\}$.

The proof is analogous under assumption B). □

REMARK:

*The nonresonance (i.e. solvability of (Per) for any $f$ ) in Corollary 8 is due only to the fact that $I : \mathcal{C} \to \mathbb{R}$ is onto.*

In an similar way we can state the following existence result of periodic solutions of a Duffing equation

$$\text{(D)} \qquad u'' + g(u) = f(t, u, u')$$

THEOREM 9

*Let $g : \mathbb{R} \to \mathbb{R}$ be continuous such that $\frac{g(u)-g(v)}{u-v} \leq c_1$ , $|g(s)| \leq c_2|s| + d$ with $c_1 + c_2\pi < \left(\frac{\pi}{\alpha}\right)^2$ and $f : \overline{I} \times \mathbb{R}^2 \to \mathbb{R}$ continuous, bounded and continuously differentiable with respect to $u$ and $u'$.*

*We'll assume that $\frac{\partial f}{\partial u'} \leq 0$ , $c_1 - \frac{\partial f}{\partial u} \leq c_3$ for some constant $c_3 < \left(\frac{\pi}{\alpha}\right)^2$ and that*

$$\lim_{x \to \pm\infty} g(x)sg(x) = +\infty \qquad \text{or} \qquad \lim_{x \to \pm\infty} g(x)sg(x) = -\infty$$

*Then (D) admits at least one $\alpha$-periodic solution. Moreover, the set of $\alpha$-periodic solutions of (D) is contained in a compact arc of $H^2(I)$.*

Proof

¿From the previous theorem it will suffice to see that a version of theorem 6 holds also for this case. Indeed, from theorem 4 we have that the problem

$$(*_s) \begin{cases} Su = f(t, u, u') & \text{in } (0, \alpha) \\ u(0) = u(\alpha) = s \end{cases}$$

is solvable, and if $u, v$ are solutions of ( $*_s$ ) we have by mean value theorem that

$$Su - Sv - \frac{\partial f}{\partial u'}(t, \xi_1, \xi_2)(u - v)' - \frac{\partial f}{\partial u}(t, \xi_1, \xi_2)(u - v) = 0$$

for some $\xi \in L^\infty(I) \times L^\infty(I)$. If $\overline{S}u := Su - \frac{\partial f}{\partial u'}(t, \xi_1, \xi_2)u' - \frac{\partial f}{\partial u}(t, \xi_1, \xi_2)u$, then $\overline{S}$ satisfies the conditions of lemma 1. Hence $\|u - v\|_2 \leq c\|\overline{S}u - \overline{S}v\|_2 = 0$, and uniqueness for $(*_s)$ follows. Then the curve $\mathcal{C} = \{u_s : u_s \text{ solves } (*_s)\}$ is well defined, and its continuity can be proved in the same way of theorem 5. As in the previous corollary, $I : \mathcal{C} \to \mathbb{R}$ is onto, and $I(u_s)sg(s) \to +\infty$ (resp. $I(u_s)sg(s) \to -\infty$). As $f$ bounded, the proof is complete. $\square$

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