The prescribed mean curvature equation with Dirichlet conditions

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ABSTRACT:

In this paper we study H-systems with a Dirichlet boundary data g *. We find solutions under different conditions on* H *and* g *, and prove that weak solutions in* $W^{1,\infty}$ *are classic.*

Introduction

We consider the Dirichlet problem in a bounded $C^{1,1}$ domain $\Omega \subset R^2$ for a vector function $X : \overline{\Omega} \longrightarrow R^3$ which satisfies the equation of prescribed mean curvature

$$(1) \begin{cases} \Delta X = 2H(u, v, X, X_u, X_v) X_u \wedge X_v & \text{in} \quad \Omega \\ X = g \quad \text{in} \quad \partial\Omega \end{cases}$$

where $X_u = \dfrac{\partial X}{\partial u}$, $X_v = \dfrac{\partial X}{\partial v}$, $\wedge$ denotes the exterior product in R^3 and $H : \overline{\Omega} \times (R^3)^3 \longrightarrow R$ is a given continuous function. The problem above arises in the Plateau and Dirichlet problems for the prescribed mean curvature equation that has been studied in [1-2-3-4-5-6-7-10].

In order to study problem (1) we assume that $g \in W^{2,p}(\Omega, R^3)$ for $2 < p < \infty$ and define an operator $T : C^1(\overline{\Omega}, R^3) \longrightarrow C^1(\overline{\Omega}, R^3)$, $T\overline{X} = X$, where $X \in W^{2,p} \hookrightarrow C^1$ is the only solution of the linear problem

$$\begin{cases} \Delta X = 2H(u, v, \overline{X}, \overline{X}_u, \overline{X}_v) \overline{X}_u \wedge \overline{X}_v & \text{in} \quad \Omega \\ X = g \quad \text{in} \quad \partial\Omega \end{cases}$$

Thus a strong solution of (1) can be regarded as a fixed point of T .

THEOREM 1

Let $2 < p < \infty$ and $g \in W^{2,p}(\Omega, R^3)$. Then the operator T defined above is compact.

<u>*Proof*</u>

For $X = T(\overline{X})$, $Y = T(\overline{Y})$, it holds that $X = Y$ in $\partial\Omega$. Moreover, by Sobolev imbedding and lemma 9.17 in [8] we obtain that

$$\|X - Y\|_{1,\infty} \leq c_1\|X - Y\|_{2,p} \leq c_1 c\|\Delta(X - Y)\|_p =$$

$$2c_1 c\|H(u, v, \overline{X}, \overline{X}_u, \overline{X}_v)\overline{X}_u \wedge \overline{X}_v - H(u, v, \overline{Y}, \overline{Y}_u, \overline{Y}_v)\overline{Y}_u \wedge \overline{Y}_v\|_p$$

and continuity of T follows. On the other hand, for $X_0 = T(0)$, $\|\overline{X}\|_{1,\infty} \leq R$, it holds that

$$\|X - X_0\|_{2,p} \leq c\|\Delta(X - X_0)\|_p = 2c\|H(u, v, \overline{X}, \overline{X}_u, \overline{X}_v)\overline{X}_u \wedge \overline{X}_v\|_p \leq$$

$$2c\|H(u, v, \overline{X}, \overline{X}_u, \overline{X}_v)\|_p\|\overline{X}_u\|_\infty\|\overline{X}_v\|_\infty \leq kR^2$$

As the imbedding $W^{2,p}(\Omega, R^3) \hookrightarrow C^1(\overline{\Omega}, R^3)$ is compact, the result follows.

THEOREM 2

The following properties are equivalent:

i) (1) admits a solution in $W^{2,p}(\Omega, R^3)$

ii) There exists $R > 0$ such that

$$\inf_{\|\Delta\overline{Y}\|_p \leq R, \overline{Y}=g \text{ in } \partial\Omega} \|\Delta\overline{Y} - 2H(u, v, \overline{Y}, \overline{Y}_u, \overline{Y}_v)\overline{Y}_u \wedge \overline{Y}_v\|_p = 0$$

iii) There exists $R > 0$ such that

$$\inf_{\|\overline{Y}\|_{1,\infty} \leq R, \overline{Y}=g \text{ en } \partial\Omega} \|\overline{Y} - T(\overline{Y})\|_{1,\infty} = 0$$

<u>*Proof*</u>

i) $\Longrightarrow$ *ii)* is obvious. In order to see *ii)* $\Longrightarrow$ *iii)*, it suffices to note that if $\overline{Y} \in W^{2,p}(\Omega, R^3)$ verifies $\|\Delta\overline{Y}\|_p \leq R, \overline{Y} = g$ in $\partial\Omega$ and g is harmonic then

$$\|\overline{Y}\|_{1,\infty} \leq c_1\|\overline{Y}\|_{2,p} \leq c_1(\|\overline{Y} - g\|_{2,p} + \|g\|_{2,p}) \leq c_1(c\|\Delta(\overline{Y} - g)\|_p + \|g\|_{2,p}) \leq$$

$$\leq c_1(cR + \|g\|_{2,p}) = R'$$

Moreover,

$$\|\overline{Y} - T(\overline{Y})\|_{1,\infty} \leq c_1\|\overline{Y} - T(\overline{Y})\|_{2,p} \leq c_1 c\|\Delta(\overline{Y} - T(\overline{Y}))\|_p =$$

$$= c_1 c\|\Delta\overline{Y} - 2H(u,v,\overline{Y},\overline{Y}_u,\overline{Y}_v)\overline{Y}_u \wedge \overline{Y}_v\|_p$$

and the result follows. Finally, we prove $iii) \Longrightarrow i)$: let us consider a sequence $\overline{Y}_n$ bounded in $C^1(\overline{\Omega}, R^3)$ such that $\|\overline{Y}_n - T(\overline{Y}_n)\|_{1,\infty} \longrightarrow 0$. Being T compact we may assume that $T(\overline{Y}_n)$ converges to $\overline{Y}$ in $C^1(\overline{\Omega}, R^3)$. Then $\overline{Y}_n \longrightarrow \overline{Y}$, and $\overline{Y}$ is a fixed point of T.

THEOREM 3

Let c_1, c be the constants of the previous theorems, $2 < p < \infty$, $g \in W^{2,p}(\Omega, R^3)$ harmonic, and assume that for some $R > 0$ it holds, for any $\overline{X} \in B_R(g)$:

$$2c_1 c\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_u\|_p \leq \frac{R}{R + \|\nabla g\|_\infty}$$

or

$$2c_1 c\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_v\|_p \leq \frac{R}{R + \|\nabla g\|_\infty}$$

Then there is at least one solution of (1) in $B_R(g)$.

Proof

We have, for $\|\overline{X} - g\|_{1,\infty} \leq R$:

$$\|X - g\|_{1,\infty} \leq 2c_1 c\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_u \wedge \overline{X}_v\|_p$$

Then

$$\|X - g\|_{1,\infty} \leq 2c_1 c\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_u\|_p\|\overline{X}_v\|_\infty$$

$$\leq 2c_1 c\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_u\|_p(R + \|\nabla g\|_\infty)$$

and also

$$\|X - g\|_{1,\infty} \leq 2c_1 c\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_v\|_p\|\overline{X}_u\|_\infty$$

$$\leq 2c_1 c\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_v\|_p(R + \|\nabla g\|_\infty)$$

Thus $T(B_R) \subset B_R$ and the result follows by Schauder Theorem.

COROLLARY 4

Let c_1, c be the constants of the previous theorems, $2 < p < \infty$, and suppose that

$$H = \sup_{\overline{X} \in C^1(\overline{\Omega}, R^3)} \|H(u, v, \overline{X}, \overline{X}_u, \overline{X}_v)\|_p < \infty$$

Then (1) has a solution for any harmonic $g \in W^{2,p}(\Omega, R^3)$ such that $\|\nabla g\|_\infty \leq \dfrac{1}{8c_1cH}$.

COROLLARY 5

Let c_1, c be the constants of the previous theorems, $2 < p < \infty$, $g \in W^{2,p}(\Omega, R^3)$, and suppose that

$$0 < H' = \sup_{\overline{X} \in C^1(\overline{\Omega}, R^3), \overline{X} \neq g} \frac{\|H(u, v, \overline{X}, \overline{X}_u, \overline{X}_v) - H(u, v, g, g_u, g_v)\|_p}{\|\overline{X} - g\|_{1,\infty}} < \infty$$

We define $h = \|H(u, v, g, g_u, g_v)\|_p$, $R_0 = \dfrac{-h + \sqrt{h^2 + 8hH'\|\nabla g\|_\infty}}{4H'}$, and the function

$$f(R) = (H' + \frac{h}{R})(R + \|\nabla g\|_\infty)^2$$

Then, if $f(R_0) \leq \dfrac{1}{2c_1c}$, (1) admits a solution in $W^{2,p}(\Omega, R^3)$.

Proof

As in the previous result, for $\|\overline{X} - g\|_{1,\infty} \leq R$,

$$\|X - g\|_{1,\infty} \leq 2c_1c\|H(u, v, \overline{X}, \overline{X}_u, \overline{X}_v)\|_p \|\overline{X}_u\|_\infty \|\overline{X}_v\|_\infty \leq 2c_1cH(R + \|\nabla g\|_\infty)^2$$

In order to apply Schauder Theorem, we need that $2c_1cH(R + \|\nabla g\|_\infty)^2 \leq R$ for some R, i.e., that the parabola

$$R^2 + (2\|\nabla g\|_\infty - \frac{1}{2c_1cH})R + \|\nabla g\|_\infty^2$$

has a positive root, which is equivalent to the condition: $(2\|\nabla g\|_\infty - \dfrac{1}{2c_1cH})^2 - 4\|\nabla g\|_\infty^2 \geq 0$, or $\|\nabla g\|_\infty \leq \dfrac{1}{8c_1cH}$.

REMARK

In particular, if H is globally Lipschitz with respect to X, X_u y X_v, (1) has a solution for any g close enough to a constant. Indeed, we see that

$$f(R_0) = (H'R_0 + h)(R_0 + 2\|\nabla g\|_\infty + \frac{\|\nabla g\|_\infty^2}{R_0}) \longrightarrow 0$$

if $\|\nabla g\|_\infty \longrightarrow 0$

For example, this holds when H *has bounded derivatives with respect to* X, X_u *y* X_v *, since by mean value theorem*

$$\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v) - H(u,v,g,g_u,g_v)\|_p \leq k\|DH\|_\infty \|\overline{X} - g\|_{1,\infty}$$

EXAMPLE

Taking the maximum of both values, we may assume $h = H'$ *, and then* $R_0 = \dfrac{-1+\sqrt{1+8\|\nabla g\|_\infty}}{4}$ *. Thus, the condition we must verify is*

$$f(R_0) = h(1+\frac{1}{R_0})(R_0+1)^2 \leq \frac{1}{2c_1c}$$

For example, if $\|\nabla g\|_\infty = 1$ *it suffices to take* $h \leq \dfrac{2}{27c_1c}$ *.*

The following result eliminates any kind of condition on $\|g\|_{1,\infty}$ *:*

THEOREM 6

Let c_1, c *be the constants of the previous theorems,* $2 < p < \infty$ *,* $g \in W^{2,p}(\Omega, R^3)$ *, and suppose that there exist constants* $k, R_0 > 0$ *such that* $k < \dfrac{1}{2c_1c}$ *, and*

$$\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_u \wedge \overline{X}_v\|_p \leq k\|\overline{X}\|_{1,\infty}$$

for every $\overline{X}$ *such that* $\|\overline{X}\|_{1,\infty} \geq R_0$ *.*

Then (1) has at least one solution in $W^{2,p}(\Omega, R^3)$ *.*

Proof

In the same way as before, for harmonic g, if $T(\overline{X}) = X$ then

$$\|X - g\|_{1,\infty} \leq 2c_1c\|H(u,v,\overline{X},\overline{X}_u,\overline{X}_v)\overline{X}_u \wedge \overline{X}_v\|_p$$

Then for $\|\overline{X}\|_{1,\infty} \geq R_0$,

$$\|X - g\|_{1,\infty} \leq 2c_1ck\|\overline{X}\|_{1,\infty} \leq 2c_1ck\left(\|\overline{X} - g\|_{1,\infty} + \|g\|_{1,\infty}\right)$$

If also $\|\overline{X} - g\|_{1,\infty} \leq R$, being $2c_1ck < 1$ it follows that

$$\|X - g\|_{1,\infty} \leq 2c_1ck\left(R + \|g\|_{1,\infty}\right) \leq R$$

when R is big enough. On the other hand, if $\|\overline{X}\|_{1,\infty} \leq R_0$ then

$$\|X - g\|_{1,\infty} \leq 2c_1 c \|H(u, v, \overline{X}, \overline{X}_u, \overline{X}_v)\overline{X}_u \wedge \overline{X}_v\|_p \leq R'$$

for some R', and the result follows taking $R > R'$ big enough.

Remark

As a particular case, theorem 6 is valid when the support of $H = H(u, v, x, y, z)$ is bounded in y and z. It holds also for $H = \frac{H_1(u,v,X)}{1+|\nabla X|^2}$.

Theorem 7

Let (1_σ) be the equation given by

$$(1_\sigma) \begin{cases} \Delta X = 2\sigma H(u, v, X, X_u, X_v) X_u \wedge X_v & \text{in} \quad \Omega \\ X = \sigma g \quad \text{in} \quad \partial\Omega \end{cases}$$

and suppose there exists $M > 0$ such that for any $X \in W^{2,p}(\Omega, R^3)$ solution of (1_σ) for some $\sigma \in (0, 1)$, $\|X\|_{1,\infty} \leq M$. Then (1) has a solution in $C^1(\overline{\Omega}, R^3)$.

Proof

For fixed $M' > M$, we define the operator T', given by

$$T'X = \begin{cases} TX & \text{if } \|TX\|_{1,\infty} \leq M' \\ \dfrac{M'TX}{\|TX\|_{1,\infty}} & \text{if } \|TX\|_{1,\infty} > M' \end{cases}$$

It is clear that T' is compact, and as $T'(B_{M'}(0)) \subset B_{M'}(0)$, T' has a fixed point X. If X is not a fixed point of T, then $X = \sigma TX$, with $\sigma = \dfrac{M'}{\|TX\|_{1,\infty}} < 1$, and X is a solution of (1_σ). Moreover, $\|X\|_{1,\infty} = M'$, a contradiction.

Theorem 8

Let $X_n \in W^{1,\infty}(\Omega, R^3)$ be a bounded sequence such that X_n is a weak solution of (1) for some $H_n \in C(\overline{\Omega} \times (R^3)^3), R)$, $g_n \in W^{2,p}(\Omega, R^3)$ such that $H_n \longrightarrow H$ and $g_n \longrightarrow g$. Then (1) has a solution in $W^{2,p}(\Omega, R^3)$ for H and g.

Proof

Let us note first that $X_n \in W^{2,p}(\Omega, R^3)$, since if Z_n is the only element in $W^{2,p}(\Omega, R^3)$ verifying

$$\begin{cases} \Delta Z_n = 2H_n(u, v, X_n, X_{n_u}, X_{n_v}) X_{n_u} \wedge X_{n_v} & \text{in} \quad \Omega \\ Z_n = g_n \quad \text{in} \quad \partial\Omega \end{cases}$$

it holds weakly that $\Delta(Z_n - X_n) = 0$ and being $Z_n - X_n = 0$ in $\partial\Omega$, we conclude that $Z_n = X_n$.

On the other hand, by compactness we may assume that $TX_n \longrightarrow X \in C^1(\overline{\Omega}, R^3)$. Moreover, $\Delta(TX_n - X_n) =$

$$2(H(u,v,X_n,X_{n_u},X_{n_v})X_{n_u}\wedge X_{n_v} - H_n(u,v,X_n,X_{n_u},X_{n_v})X_{n_u}\wedge X_{n_v}) \longrightarrow 0$$

uniformly in $\overline{\Omega}$, and then

$$\|TX_n - X_n\|_{1,\infty} \le \|g - g_n\|_{1,\infty} + c_1 c(\|\Delta(TX_n - X_n)\|_p + \|\Delta(g - g_n)\|_p) \longrightarrow 0$$

We conclude that $X_n \longrightarrow X$ en $C^1(\overline{\Omega}, R^3)$ and $TX = X$.

Remark

It is known (see e.g [9]) that if X is a weak solution of (1), then $X \in C(\overline{\Omega}, R^3) \cap C^2(\Omega, R^3)$. In the previous theorem we saw that if $X \in W^{1,\infty}(\Omega, R^3)$ then $X \in W^{2,p}(\Omega, R^3)$. We'll see now that under some extra hypothesis, $X \in C^2(\overline{\Omega}, R^3)$.

Theorem 9

Let $X \in W^{1,\infty}(\Omega, R^3)$ be a weak solution of (1), and assume, for $k \ge 0$, that $\partial\Omega \in C^{k+2,\alpha}$, $H \in C^{k,\alpha}(\overline{\Omega} \times (R^3)^3, R)$ $g \in C^{k+2,\alpha}(\overline{\Omega}, R^3)$ for some α, with $0 < \alpha \le 1 - \dfrac{2}{p}$. Then $X \in C^{k+2,\alpha}(\overline{\Omega}, R^3)$.

Proof

Case $k = 0$: by Sobolev imbedding and previous theorem, $X \in W^{2,p}(\Omega, R^3) \hookrightarrow C^{1,\alpha}(\overline{\Omega}, R^3)$. Then $\Delta X = f \in C^{\alpha}(\overline{\Omega}, R^3)$. By theorem 6.14 in [8], the equation $\Delta Z = f$ en Ω, $Z = g$ en $\partial\Omega$ is uniquely solvable in $C^{2,\alpha}(\overline{\Omega}, R^3)$, and the result follows from the uniqueness in theorem 9.15 in [8].

The general case is now immediate, from theorem 6.19 in [8].

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