Existence and uniqueness of H-systems's solutions with Dirichlet conditions

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ABSTRACT: *We give conditions on the boundary data in order to obtain a solution for the problem (1) below and we prove that in certain cases the solutions are isolated. The condition* $\|H\|_\infty \|g\|_\infty \leq 1$ *is not necessary.*

Introduction

We consider the Dirichlet problem in the unit disc $B = \{(u,v) \in R^2; u^2+v^2 < 1\}$ for a vector function $X : \overline{B} \longrightarrow R^3$ which satisfies the equation of prescribed mean curvature

$$(1) \begin{cases} \Delta X = 2H(u,v) X_u \wedge X_v & \text{in} \quad B \\ X = g \quad \text{in} \quad \partial B \end{cases}$$

where $X_u = \dfrac{\partial X}{\partial u}$, $X_v = \dfrac{\partial X}{\partial v}$, $\wedge$ denotes the exterior product in R^3 and $H : \overline{B} \longrightarrow R$ is a given continuous function.

The problem above arises in the Plateau and Dirichlet problems for the prescribed mean curvature equation that has been studied in [1-3-4-5-6-7-8].

The main results are the following theorems:

THEOREM 1 *Let be* $2 < p < \infty$, *and* $g \in W^{2,p}(B, R^3)$ *such that* $g - a$ *is small enough for a certain* $a \in R$, *then there exists a solution* $X \in W^{2,p}$ *of (1).*

THEOREM 2 *Let* X *be a solution of (1) in* $W^{2,p}$, $1 < p < \infty$. *Then:*

i) If $p \leq 2$ *and* $X \in W^{1,\infty}$ *then* X *is isolated in* $(W^{1,\infty} \cap W^{2,p}, \|\cdot\|_{1,\infty})$.

ii) If $p > 2$ *then* X *is isolated in* $W^{1,p}$. *Moreover, if* $\overline{\Omega}$ *is a bounded subset of* $W^{1,p}$ *the number of solutions of (1) in* $\overline{\Omega}$ *is finite.*

Remark

For H constant Hildebrandt found a solution of (1) in $W^{1,2}$, for the case $|H|\|g\|_\infty \leq 1$ [3]. Theorem 1 gives a solution for $p > 2$ when g is close to a constant a. Since the equation in B depends only on the derivatives of X, we may suppose $a = 0$

1. Solution by fixed point methods

The systems (2) and (3) are equivalent to (1) with $X = X_0 + Y$

$$(2)\quad \begin{cases} \Delta X_0 = 0 \text{ in } & B \\ X_0 = g & \text{in} \quad \partial B \end{cases}$$

$$(3)\quad \begin{cases} \Delta Y = F(X_0, Y) & \text{in} \quad B \\ Y = 0 & \text{in} \quad \partial B \end{cases}$$

where F is given by

$$F(X_0, Y) = 2H(u,v)[(X_{0u} \wedge Y_v + Y_u \wedge X_{0v}) + (Y_u \wedge Y_v + X_{0u} \wedge X_{0v})]$$

For $g \in W^{2,p}\left(B, R^3\right)$, (2) admits a unique solution in $W^{2,p}\left(B, R^3\right)$. We can rewrite (3) as:

$$\begin{cases} L(X_0)Y = F_1(X_0, Y) & \text{in} \quad B \\ Y = 0 & \text{in} \quad \partial B \end{cases}$$

where $L(X_0)$ is the linear operator

$$L(X_0)Y = \Delta Y - 2H(u,v)(X_{0u} \wedge Y_v + Y_u \wedge X_{0v})$$

and F_1 is defined by

$$F_1(X_0, Y) = 2H(u,v)(Y_u \wedge Y_v + X_{0u} \wedge X_{0v}).$$

Remark *$L(X_0)$ is strictly elliptic and for $p > 2$ its coefficients are bounded since $X_0 \in W^{1,\infty}\left(B, R^3\right)$ and $H \in C(\overline{B})$.*

We will use the following technical lemmas:

LEMMA 4 *Let be* $X_0 \in W^{2,p}(B, R^3)$, *then*

i) If $2 < p < \infty$,

$$\|F_1(X_0, Y_1)\|_{p/2} \leq 2\|H\|_\infty(\|X_0\|_{1,p}^2 + \|Y_1\|_{1,p}^2) \tag{4}$$

$$\|F_1(X_0, Y_1) - F_1(X_0, Y_2)\|_{p/2} \leq 4\|H\|_\infty R\left(\|Y_1 - Y_2\|_{1,p}\right) \tag{5}$$

for $Y_1, Y_2 \in B_R(0) \subset W^{1,p}$.

ii) If $p > 1$, X_0, Y_1, $Y_2 \in W^{1,p} \cap W^{1,\infty}$

$$\|F_1(X_0, Y_1)\|_p \leq 2\|H\|_\infty(\|X_0\|_{1,\infty}.\|X_0\|_{1,p} + \|Y_1\|_{1,\infty}.\|Y_1\|_{1,p}) \tag{6}$$

$$\|F_1(X_0, Y_1) - F_1(X_0, Y_2)\|_p \leq 4\|H\|_\infty R\left(\|Y_1 - Y_2\|_{1,p}\right) \tag{7}$$

for $Y_1, Y_2 \in B_R(0) \subset W^{1,\infty}$.

<u>*Proof*</u>

$$\|F_1(X_0, Y_1)\|_{p/2} \leq 2\|H\|_\infty(\|Y_{1u} \wedge Y_{1v}\|_{p/2} + \|X_{0u} \wedge X_{0v}\|_{p/2})$$

and

$$\|Y_{1u} \wedge Y_{1v}\|_{p/2}^{p/2} \leq \int |Y_{1u}|^{p/2}|Y_{1v}|^{p/2} \leq \|Y_{1u}\|_p^{p/2}\|Y_{1v}\|_p^{p/2}$$

The same inequality holds for X_0, and in order to prove (5), we also have that

$$\|Y_{1u} \wedge Y_{1v} - Y_{2u} \wedge Y_{2v}\|_{p/2} \leq \|Y_{1u} \wedge (Y_{1v} - Y_{2v})\|_{p/2} + \|Y_{2v} \wedge (Y_{1u} - Y_{2u})\|_{p/2}$$

and the proof follows.

In the same way we obtain (6) and (7), using that $\|a \wedge b\|_p \leq \|a\|_\infty.\|b\|_p$ ♣

LEMMA 5 *If* $\|X_0\|_{1,p} \leq \delta$, $2 < p < \infty$ *with* δ *small enough, then there exists a constant* C *independent of* X_0 *verifying*

$$\|Y\|_{2,p/2} \leq C\|L(X_0)Y\|_{p/2}$$

for all $Y \in W^{2,p/2} \cap W_0^{1,p/2}$

<u>*Proof*</u>

Writing $L(X_0)Y = \Delta Y - R(X_0)Y$ it follows that

$$\|L(X_0)Y\|_{p/2} \geq \|\Delta Y\|_{p/2} - \|R(X_0)Y\|_{p/2} \geq \frac{1}{C_1}\|Y\|_{2,p/2} - \|R(X_0)Y\|_{p/2}$$

where C_1 is the constant provided by Lemma 9.17 in [2].

As $\|R(X_0)Y\|_{p/2} \leq 4\|H\|_\infty\|X_0\|_{1,p}\|Y\|_{1,p} \leq 4\overline{c}\|H\|_\infty\|X_0\|_{1,p}\|Y\|_{2,p/2}$ (where $\overline{c}$ is the constant of Sobolev inmersion $W^{2,p/2} \hookrightarrow W^{1,p}$), the proof follows.♣

REMARK

The same Lemma 9.17 in [2] shows that if $\|g\|_{2,p}$ *is small, then* $\|X_0 - g\|_{2,p}$ *is small, and then also* $\|X_0\|_{1,p}$.

PROPOSITION 6 *Let be* $X_0 \in W^{2,p}\left(B, R^3\right)$ *with* $p > 2$ *, then the following problem*

$$(8) \quad \begin{cases} L(X_0)Y = F_1(X_0, \overline{Y}) & \text{in} \quad B \\ Y = 0 & \text{in} \quad \partial B \end{cases}$$

define a continuous map $T : \overline{Y} \longrightarrow Y$ *in* $W_0^{1,p}$ *. Furthermore, if* $\|X_0\|_{1,p}$ *is small enough, there exists a number* R *such that* T *is a contraction in* $B_R(0) \subset W^{1,p}$ *.*

<u>*Proof*</u>

¿From Theorem 9.15 in [2], problem (8) admits a unique solution $Y \in W^{2,p/2}$, so the map T is well defined. Also we have from (5) that if $\overline{Y}, \overline{Z} \in B_R \subset W^{1,p}$ then

$$\|Y - Z\|_{2,p/2} \leq C(\|L(X_0)(Y - Z)\|_{p/2} = C\|F_1(X_0, \overline{Y}) - F_1(X_0, \overline{Z})\|_{p/2} \leq 4C\|H\|_\infty R\|\overline{Y} - \overline{Z}\|_{1,p}$$

and using the Sobolev inmersion of $W^{2,p/2}$ in $W^{1,p}$ the continuity of T follows.

Also we have from (4)

$$\|Y\|_{1,p} \leq \overline{c}\,\|Y\|_{2,p/2} \leq C\overline{c}\,\|L(X_0)Y\|_{p/2} \leq 2C\overline{c}\|H\|_\infty(\|X_0\|_{1,p}^2 + \left\|\overline{Y}\right\|_{1,p}^2)$$

Choosing $\|X_0\|_{1,p}$ and R small enough, we obtain

$$4C\|H\|_\infty R < 1$$

$$\|Y\|_{1,p} \leq R$$

and T is a contraction in B_R. ♣

Remark

Note that if Y is a fixed point of T, then $Y \in W^{2,p}$, and we obtain Theorem 1 as an immediate consequence of Proposition 6.

2. Local unicity of the solutions

Now we will prove theorem 2.

Let X_0 be a solution of (1). If Y is another solution of (1) then $Z = Y - X_0$ satisfies

$$\begin{cases} L(X_0)Z = 2H(u,v)Z_u \wedge Z_v \text{ in } & B \\ Z = 0 \quad\quad \text{in} & \partial B \end{cases}$$

We consider the associated problem

$$\begin{cases} L(X_0)Z = 2H(u,v)\overline{Z}_u \wedge \overline{Z}_v \text{ in } & B \\ Z = 0 \quad\quad \text{in} & \partial B \end{cases}$$

For ii), in the same way as before, we obtain

$$\|Z\|_{1,p} \leq 2C(X_0)\overline{c}\|H\|_\infty \|\overline{Z}\|_{1,p}^2$$

and if $\overline{Z}_1, \overline{Z}_2 \in B_R(0) \subset W^{1,p}$ then

$$\|Z_1 - Z_2\|_{1,p} \leq 4C(X_0)\overline{c}\|H\|_\infty R \|\overline{Z}_1 - \overline{Z}_2\|_{1,p}$$

If we choose R verifying

$$4C(X_0)\overline{c}\|H\|_\infty R < 1$$

the map $\overline{T} : \overline{Z} \longrightarrow Z$ is a contraction in B_R, so X_0 is isolated.

Remark *¿From Sobolev inmersions $\overline{T}$ is compact in $W^{1,p}$ and being any solution of (1) a fixed point of $\overline{T}$, we conclude that the number of solutions in B_R is finite for every R.*

In the same way we prove i) using (6) and (7). ♣

REMARK *The number R in the theorems above can be estimated in terms of the Sobolev inmersions constants, $\|H\|_\infty$ and $\|g\|_{2,p}$.*

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