Solutions to general quasilinear elliptic second order problems

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Abstract:

We study the second order quasilinear equation (1) $Qu = f(x, u, D_1u, ..., D_nu)$ *with Dirichlet conditions. We obtain existence and uniqueness results in the Sobolev Space* $W^{2,p}$. *We also prove that, in certain cases, the set of solutions of (1) is a connected subset of* $W^{2,p}$.

Statement of the problem and known results

We consider the Dirichlet problem in a bounded $C^{1,1}$ domain $\Omega \subset R^n$ for $u : \overline{\Omega} \longrightarrow R$ satisfying

$$(2) \begin{cases} Qu = f(x, u, D_1u, ..., D_nu) \text{ in } \Omega \\ u = g \quad \text{ in } \partial\Omega \end{cases}$$

where Q is the quasilinear elliptic operator given by $Qu = \sum a_{ij}D_{ij}u + \sum b_jD_ju + cu$, with $a_{ij}(x, u, D_1u, ..., D_nu)$ continuous and C^1 in $u, D_1u, ..., D_nu$, $b_j(x, u)$ continuous and C^1 in u, and $c \in L^\infty(\Omega)$. We'll assume that Q is strictly elliptic, i.e. if λ_i and λ_s are respectively the minimum and the maximum eigenvalues of the matrix $a_{ij}(x, u, D_1u, ..., D_nu)$, then $\lambda_i \geq \lambda > 0$. We'll assume also that $f : \overline{\Omega} \times R \times R^n \longrightarrow R$ is continuous and $g \in W^{2,p}(\Omega)$ for some $n < p < \infty$.

Existence results for (1) have been obtained in certain cases. For example, for Q uniformly elliptic, i.e. such that λ_s/λ_i is bounded, see [GT]. A well known example of a non-uniformly elliptic problem is the mean curvature equation for non parametric surfaces (here $\lambda_i = 1$, $\lambda_s = 1 + |\nabla u|^2$) which has been studied by Gilbarg, Trudinger, Simon, Serrin, among other authors [GT], [Se], [Si], [G]. A more general situation is contemplated in [AMR],[DST],[W].

We linearize the problem in the following way: for fixed $\overline{u} \in C^1(\overline{\Omega})$, we consider the linear elliptic operator

$$L_{\overline{u}}v = \sum a_{ij}(x, \overline{u}, D_1\overline{u}, ..., D_n\overline{u})D_{ij}v + \sum b_j(x, \overline{u})D_jv + cv$$

For $c \leq 0$ we recall the apriori bound for $L_{\overline{u}}$ [GT]: there exists a constant $c = c(\overline{u})$ such that

$$\|w\|_{2,p} \leq c\|L_{\overline{u}}w\|_p$$

for any $w \in W^{2,p} \cap W_0^{1,p}$. It is immediate that we can choose $c(\overline{u})$ minimum, then we obtain the following result:

LEMMA 1

$c : C^1 \longrightarrow R^+$ is upper semicontinuous.

<u>*Proof*</u>

Let $t > c(\overline{u})$, $w \in W^{2,p} \cap W_0^{1,p}$. For $\|\overline{v} - \overline{u}\|_{1,\infty} \leq R$, we have:

$$\|L_{\overline{u}}w\|_p \geq \frac{1}{c(\overline{u})}\|w\|_{2,p} - \|(L_{\overline{u}} - L_{\overline{v}})w\|_p$$

As

$$\|(L_{\overline{u}} - L_{\overline{v}})w\|_p \leq \| \sum (a_{ij}(x, \overline{u}, D_1\overline{u}, ..., D_n\overline{u}) - a_{ij}(x, \overline{v}, D_1\overline{v}, ..., D_n\overline{v}))D_{ij}w +$$

$$\sum (b_j(x, \overline{u}) - b_j(x, \overline{v}))D_jw\|_p \leq kR\|w\|_{2,p}$$

for some constant k, we obtain:

$$\|L_{\overline{u}}w\|_p \geq (\frac{1}{c(\overline{u})} - kR)\|w\|_{2,p}$$

and choosing R such that $kR < \frac{1}{c(\overline{u})}$ it follows that $\frac{1}{c(\overline{v})} \geq \frac{1-kRc(\overline{u})}{c(\overline{u})}$. Letting $R \longrightarrow 0$ we conclude that $c(\overline{v}) \leq \frac{c(\overline{u})}{1-kRc(\overline{u})} < t$ and the result holds.

REMARK: *In the situation of lemma 1, if $kR < \frac{1}{c(\overline{u})}$ then it is possible to fix a constant $\overline{c}$ such that $\overline{c} \geq c(\overline{v})$ for any $\overline{v} \in B_R(\overline{u})$. We also remark that for a bounded set $C \subset W^{2,p} \hookrightarrow C^1$ the constant $c(\overline{u})$ may be chosen uniformly in C.*

In order to solve (2), we'll apply a coefficient freezing method that will allow us to define an operator $T : C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega})$ and find solutions by a fixed point theorem. Indeed, for $\overline{u} \in C^1$ we call $T\overline{u}$ the unique solution in $W^{2,p} \hookrightarrow C^1$ of the linear problem

$$\begin{cases} L_{\overline{u}} u = f(x, \overline{u}, D_1\overline{u}, ..., D_n\overline{u}) \text{ in } \Omega \\ u = g \quad \text{ in } \partial\Omega \end{cases}$$

LEMMA 2

T is continuous and locally compact.

<u>*Proof*</u>

Let $\overline{u} \in C^1$ and $R > 0$ such that $kR < \frac{1}{c(\overline{u})}$ where k is the constant of lemma 1. Fixing $\overline{c}$ such that $\overline{c} \geq c(\overline{v})$ for any $\overline{v} \in B_R(\overline{u})$, we obtain:

$$\|T\overline{u} - T\overline{v}\|_{1,\infty} \leq c_1 \|T\overline{u} - T\overline{v}\|_{2,p} \leq \overline{c} c_1 \|L_{\overline{v}}(T\overline{u} - T\overline{v})\|_p \leq$$

$$\overline{c} c_1 (\|L_{\overline{u}} T\overline{u} - L_{\overline{v}} T\overline{u}\|_p + \|(L_{\overline{u}} - L_{\overline{v}}) T\overline{u}\|_p) \leq$$

$$\overline{c} c_1 (\|f(x, \overline{u}, D_1\overline{u}, ..., D_n\overline{u}) - f(x, \overline{v}, D_1\overline{v}, ..., D_n\overline{v})\|_p + k\|T\overline{u}\|_{2,p}\|\overline{u} - \overline{v}\|_{1,\infty}).$$

Then continuity follows. Furthermore, as $T(B_R(\overline{u})) \subset W^{2,p}$ is bounded, by the compactness of the imbedding $W^{2,p} \hookrightarrow C^1$, we conclude that the closure of $T(B_R(\overline{u}))$ is compact.

The next result establishes the existence of a solution in a neighborhood of any u satisfying $Qu = f(x, u, D_1 u, ..., D_n u)$, if the boundary data g is close enough to u. More generally, we can also allow small perturbations of f:

THEOREM 3

Let $g_0 \in W^{2,p}$, $u_0 \in W^{2,\infty}$ satisfy $Qu_0 = f_0(x, u_0, D_1 u_0, ..., D_n u_0)$ in Ω, $u_0 = g_0$ on $\partial\Omega$. We define

$$\psi(x) := \sum_{i,j} \frac{\partial a_{ij}}{\partial u}(x, u_0, D_1 u_0, ..., D_n u_0) D_{ij} u_0 + \sum_j \frac{\partial b_j}{\partial u}(x, u_0) D_j u_0$$

and assume that one of the following conditions holds:

i) f_0 is Lipschitz in $u, D_1 u, ..., D_n u$ with constant small enough and $\psi + c \leq 0$.

ii) f_0 is C^1 in $u, D_1 u, ..., D_n u$ and $\psi + c \leq \frac{\partial f_0}{\partial u}(\cdot, u_0, D_1 u_0, ..., D_n u_0)$.

Then problem (2) is solvable in $W^{2,p}$ for any (f, g) close enough to (f_0, g_0) in $C(\overline{\Omega} \times R \times R^n) \times W^{2,p}(\Omega)$.

<u>*Proof*</u>

Let (f, g) be in a neighborhood of (f_0, g_0). We look for a solution of (2) for (f, g), which is equivalent to find $u \in W^{2,p}(\Omega)$ such that $u = g$ in $\partial\Omega$ and

$$L_u u - L_{u_0} u_0 = f(x, u, D_1 u, ..., D_n u) - f(x, u_0, D_1 u_0, ..., D_n u_0) \quad \text{in } \Omega$$

For $z = u - u_0$, we have that

$$L_u u - L_{u_0} u_0 = \sum a_{ij}(x, u, D_1 u, ..., D_n u) D_{ij} z + \sum b_j(x, u) D_j z + cz +$$

$$\sum (a_{ij}(x, u, D_1 u, ..., D_n u) - a_{ij}(x, u_0, D_1 u_0, ..., D_n u_0)) D_{ij} u_0 + \sum (b_j(x, u) - b_j(x, u_0)) D_j u_0$$

We may write

$$a_{ij}(x, u, D_1 u, ..., D_n u) - a_{ij}(x, u_0, D_1 u_0, ..., D_n u_0) = \frac{\partial a_{ij}}{\partial u}(x, u_0, D_1 u_0, ..., D_n u_0) z$$

$$+ \sum_k \frac{\partial a_{ij}}{\partial D_k u}(x, u_0, D_1 u_0, ..., D_n u_0) D_k z - \rho_{ij}(x, z, D_1 z, ..., D_n z)$$

$$b_j(x, u) - b_j(x, u_0) = \frac{\partial b_j}{\partial u}(x, u_0) - \rho_j(x, z)$$

and define the quasilinear operator

$$Mz = \sum a_{ij}(x, z + u_0, D_1(z + u_0), ..., D_n(z + u_0)) D_{ij} z +$$

$$\sum_k \left(b_k(x, z + u_0) + \sum_{i,j} \frac{\partial a_{ij}}{\partial D_k u}(x, u_0, D_1 u_0, ..., D_n u_0) D_{ij} u_0 \right) D_k z +$$

$$\left(\sum_{i,j} \frac{\partial a_{ij}}{\partial u}(x, u_0, D_1 u_0, ..., D_n u_0) D_{ij} u_0 + \sum_j \frac{\partial b_j}{\partial u}(x, u_0) D_j u_0 + c \right) z$$

Thus, the problem becomes

$$\begin{cases} Mz = \sum \rho_{ij} D_{ij} u_0 + \sum \rho_j D_j u_0 + F \text{ in } \Omega \\ z = g - g_0 \quad \text{in } \partial\Omega \end{cases}$$

where $F(x, z, D_1 z, ..., D_n z) = f(x, u, D_1 u, ..., D_n u) - f_0(x, u_0, D_1 u_0, ..., D_n u_0)$. As the operator M satisfies the required conditions, lemma 1 holds, and we may define a continuous and locally compact operator $T : \overline{z} \longrightarrow z$ as in lemma 2. Then, for R small we fix $\overline{c}$ such that $\overline{c} \geq c(\overline{z})$ for any $\overline{z} \in B_R(0) \subset C^1$. If i) holds, we obtain:

$$\|T(\overline{z})\|_{1,\infty} \leq \|g - g_0\|_{1,\infty} + \|T(\overline{z}) - (g - g_0)\|_{1,\infty} \leq$$

$$\|g-g_0\|_{1,\infty}+\overline{c}c_1\|L_{\overline{z}}(T(\overline{z})-(g-g_0))\|_p \le \|g-g_0\|_{1,\infty}+$$

$$\overline{c}c_1(\|\sum \rho_{ij}D_{ij}(u_0)+\sum \rho_j D_j(u_0)+F(x,\overline{z},D_1\overline{z},...,D_n\overline{z})\|_p+\|L_{\overline{z}}(g-g_0))\|_p).$$

Moreover,

$$\|F(x,\overline{z},D_1\overline{z},...,D_n\overline{z})\|_p \le$$

$$\|f(x,\overline{z}+u_0,D_1(\overline{z}+u_0),...,D_n(\overline{z}+u_0))-f_0(x,\overline{z}+u_0,D_1(\overline{z}+u_0),...,D_n(\overline{z}+u_0))\|_p+$$

$$\|f_0(x,\overline{z}+u_0,D_1(\overline{z}+u_0),...,D_n(\overline{z}+u_0))-f_0(x,u_0,D_1u_0,...,D_nu_0)\|_p$$

and then taking (f,g) close enough to (f_0,g_0) it's easy to conclude that

$$\|T(\overline{z})\|_{1,\infty}\le R$$

for some R small enough. Hence, $T(B_R(0))\subset B_R(0)$ and the result follows from Schauder fixed point theorem.

Under assumption ii), the proof is similar, writing

$$f_0(x,u,D_1u,...,D_nu)-f_0(x,u_0,D_1u_0,...,D_nu_0)=\frac{\partial f_0}{\partial u}(x,u_0,D_1u_0,...,D_nu_0)z$$

$$+\sum_k \frac{\partial f_0}{\partial D_k u}(x,u_0,D_1u_0,...,D_nu_0)D_kz+\rho(x,u,D_1u,...,D_nu)$$

and defining the operator

$$M'z=Mz-\frac{\partial f_0}{\partial u}(\cdot,u_0,D_1u_0,...,D_nu_0)z-\sum_k \frac{\partial f_0}{\partial D_k u}(\cdot,u_0,D_1u_0,...,D_nu_0)D_kz.$$

REMARKS:

a) In i) and ii) the condition on ψ may be replaced by the assumption that $\|\psi+c\|_p$ (respectively $\|\psi+c-\frac{\partial f_0}{\partial u}(\cdot,u_0,D_1u_0,...,D_nu_0)\|_p$) is small enough.

b) Theorem 3 can be combined with the non-existence conditions in [GT] (see e.g. Theorem 14.12, and corollary 14.13 for the mean curvature equation). For instance, if $H>0$ satisfies the existence conditions of [GT, theorem 16.10] with $H'(y)=\frac{n}{n-1}H(y)$ at some point $y\in\partial\Omega$ where H' is the mean curvature of $\partial\Omega$, there is a sequence $H_n\longrightarrow H$ satisfying the conditions of [GT, corollary 14.13]. Thus, there exists a sequence g_n such that $\|g_n\|_\infty\longrightarrow 0$ and the Dirichlet problem is not solvable. From theorem 3 we conclude that no subsequence of g_n converges in $W^{2,p}$.

In the same way we can prove the local uniqueness of the solutions of (2). More precisely:

THEOREM 4

Let $u_0 \in W^{2,\infty}$ *be a solution of (2),* ψ *as in the previous theorem and assume that one of the following conditions holds:*

i) f *is Lipschitz in* $u, D_1u, ..., D_nu$ *with constant small enough and* $\psi + c \leq 0$.

ii) f *is* C^1 *in* $u, D_1u, ..., D_nu$ *and* $\psi + c \leq \frac{\partial f}{\partial u}(\cdot, u_0, D_1u_0, ..., D_nu_0)$.

Then u_0 *is isolated in* C^1.

Proof

As in the previous theorem, for $f = f_0$ and $g = g_0$, we obtain for small R and $\overline{z} \neq 0$ that

$$\|T(\overline{z})\|_{1,\infty} < \|\overline{z}\|_{1,\infty}$$

and the result holds.

Furthermore, in certain cases we can prove the global uniqueness:

THEOREM 5

Let $u_0 \in W^{2,\infty}$ *be a solution of (2) and assume that* a_{ij}, b_j *do not depend of* u *and* f *is* C^1 *in* $u, D_1u, ..., D_nu$. *Then, if* $c \leq \frac{\partial f}{\partial u}(y)$ *for any* $y \in R^{n+1}$, u_0 *is unique in* C^1.

Proof

As in theorem 3, if u is another solution of (2), we have that $L_u u - L_{u_0} u_0 = f(x, u, D_1u, ..., D_nu) - f(x, u_0, D_1u_0, ..., D_nu_0)$ and $u = u_0$ on $\partial\Omega$. We can choose φ_{ij}^k, φ, $\varphi^k \in L^\infty$ such that

$$a_{ij}(x, D_1u, ..., D_nu) - a_{ij}(x, D_1u_0, ..., D_nu_0) = \sum_k \frac{\partial a_{ij}}{\partial D_k u}(x, \varphi_{ij}^1, ..., \varphi_{ij}^n) D_k z$$

and

$$f(x, u, D_1u, ..., D_nu) - f(x, u_0, D_1u_0, ..., D_nu_0) = \frac{\partial f}{\partial u}(x, \varphi, \varphi^1, ..., \varphi^n) z$$

$$+ \sum_k \frac{\partial f}{\partial D_k u}(x, \varphi^1, ..., \varphi^n) D_k z$$

where $z = u - u_0$. Thus, we obtain:

$$Lz = 0 \text{ in } \Omega \qquad z = 0 \text{ in } \partial\Omega$$

for some linear elliptic operator satisfying the conditions of lemma 9.17 in [GT] and the result follows.

Regularity of the solutions

THEOREM 6

Let $u \in W^{2,p}(\Omega)$ *be a solution of (2), and assume for some* $k \geq 0$, $0 < \alpha < 1$ *that* $f \in C^{k,\alpha}$, $\partial\Omega \in C^{k+2,\beta}$, $c \in C^{k,\beta}$ *and* $g \in C^{k+2,\beta}$ *where* $\beta = \alpha(1 - \frac{n}{p})$. *Then* $u \in C^{k+2,\beta}(\overline{\Omega})$.

Proof

Case $k = 0$: by Sobolev immersion $u \in C^{1,1-n/p}(\overline{\Omega})$. Then $Lu \in C^{\beta}(\overline{\Omega})$ where the elliptic operator L has C^{β} coefficients. By theorem 6.14 in [GT], the equation $Lw = f$ en Ω, $w = g$ en $\partial\Omega$ is uniquely solvable in $C^{2,\beta}(\overline{\Omega})$, and the result follows from the uniqueness in theorem 9.15 in [GT].

The general case is now immediate, from theorem 6.19 in [8].

Properties of the set $\{Qu = f\}$

Let us assume that $\partial\Omega \in C^{2,\beta}$, $f \in C^{\alpha}$, and that f and Q satisfy the hypothesis of theorem 5.

We consider the sets:

$$M_f = \{u \in W^{2,p}(\Omega) / Qu = f(x, u, D_1 u, ..., D_n u) \text{ in } \Omega\}$$

$$S_f = \{g \in W^{2,p}(\Omega) \text{ harmonic } / \text{ (2) has a solution in } W^{2,p}(\Omega)\}$$

M_f is closed $W^{2,p}(\Omega, R)$, since $M_f = F^{-1}(0)$ for some $F : W^{2,p}(\Omega, R) \longrightarrow L^p$ continuous. Moreover, by theorem 3 and 6 S_f is open in the closed subspace $\Delta^{-1}(0) \subset W^{2,p}(\Omega)$. By theorems 5 and 6 we may define a one to one map $\lambda : M_f \cap C^{2,\beta}(\overline{\Omega}) \longrightarrow S_f \cap C^{2,\beta}(\overline{\Omega})$, where $\lambda(u)$ is the only harmonic g such that $u = g$ in $\partial\Omega$. It's immediate to see that λ is continuous, since

$$\|\lambda(u) - \lambda(v)\|_{2,p} \leq \|\lambda(u) - u - (\lambda(v) - v)\|_{2,p} + \|u - v\|_{2,p} \leq c\|\Delta u - \Delta v\|_{2,p} + \|u - v\|_{2,p}.$$

Actually, λ may be thought as the restriction of a linear operator $\overline{\lambda} : W^{2,p}(\Omega) \longrightarrow \Delta^{-1}(0)$ which is onto and continuous (and hence, open).

In order to see that λ is an homeomorfism, it suffices to consider a sequence $g_n \longrightarrow g$, and note that if $\lambda(u) = g$, we may find, following the proof of theorem 3, for every n big enough a solution $\overline{u}_n \in B_{R_n}(u)$ of the problem with boundary data g_n. By uniqueness, $\overline{u}_n = u_n$. Moreover, with the notations of theorem 3 ii) for $g = g_n$ and $f = f_0$ we have that if $\overline{z} \in B_{R_n}(0)$ then

$$\|T'(\overline{z})\|_{1,\infty} \leq k\|g_n - g_0\|_{2,p} + c_1 c\|\rho\|_p$$

As $c_1 c\|\rho\|_p$ is independent of g_n it's clear that we may take $R_n \longrightarrow 0$ when $g_n - g \longrightarrow 0$.

As a simple consequence we have the following

THEOREM 7

Let us assume that $\partial\Omega \in C^{2,\beta}$, and that Q, $f \in C^\alpha$ satisfy the hypothesis of theorem 5.

Then, if the problem (1) admits a solution for any g, $(M_f \cap C^{2,\beta}(\overline{\Omega}), \|\cdot\|_{2,p})$ is homeomorphic to the dense subspace $C^{2,\beta} \cap \Delta^{-1}(0) \subset (\Delta^{-1}(0), \|\cdot\|_{2,p})$. In particular, it is connected by arcs.

REMARK: *Theorem 7 holds for the solutions of the mean curvature equation for nonparametric surfaces with constant H, if the curvature H' of $\partial\Omega$ verifies the condition $H' \geq \frac{n|H|}{n-1}$ (see [GT], corollary 16.11).*

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