Solutions of H-systems using the Green function

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ABSTRACT:

We find a solution to the mean curvature equation with Dirichlet condition using the Green representation formula. Moreover, given H_0 and g_0 for which there exists a solution to the problem, we prove that for H and g in appropriate neighborhoods of H_0 and g_0, there still exists a solution.

1. Introduction

We consider the Dirichlet problem in a bounded smooth domain $\Omega \subset \mathbb{R}^2$ for a vector function $X : \overline{\Omega} \longrightarrow \mathbb{R}^3$ satisfying the prescribed mean curvature equation

$$(1) \begin{cases} \Delta X = 2H(u, v, X) X_u \wedge X_v & \text{in } \Omega \\ X = g & \text{on } \partial\Omega \end{cases}$$

where X_u and X_v are the partial derivatives of X, $\wedge$ denotes the exterior product in $\mathbb{R}^3$. We'll assume that $H : \overline{\Omega} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous and that g is smooth. Without loss of generality, we may extend g to a harmonic function in $C^1(\overline{\Omega})$.

Problem (1) and the general Plateau problem have been studied by variational methods for constant H and $H = H(X)$ in [BC], [H], [LDM], [S], [W], among other authors. Topological methods are applied for the case $H = H(u, v)$ in [AMR].

2. Existence of a solution

We recall the Green representation formula for the Dirichlet problem [GT], valid for $X : \overline{\Omega} \longrightarrow \mathbb{R}^3$:

$$X = \int_\Omega G \Delta X + \int_{\partial\Omega} \frac{\partial G}{\partial \nu} X$$

where $G : \overline{\Omega} \times \overline{\Omega} \longrightarrow R$ is defined by $G(w_1, w_2) = N(w_1, w_2) + h(w_1, w_2)$, with N the newtonian potential and $h(w_1, \cdot)$ harmonic such that $h(w_1, \cdot) = -N(w_1, \cdot)$ on $\partial\Omega$. Let $s_1 = supr_{w_2} \|G(\cdot, w_2)\|_1$, $s_2 = supr_{w_2} \|\nabla_{w_2} G(\cdot, w_2)\|_1$ and $s = max\{s_1, s_2\}$. For

$R > 0$, we consider the compact set $K_R = \overline{\Omega} \times (g(\overline{\Omega}) + RB_1)$, where B_1 is the closed unit ball in $\mathbb{R}^3$.

Then we can prove the following theorem:

THEOREM 1

Let $f(R) = \frac{\|H|_{K_R}\|_\infty}{R}(\|\nabla g\|_\infty + R)^2$. *Then, if* $f(R) \leq \frac{1}{s}$ *for some* $R > 0$, *problem (1) admits a solution.*

Proof

Being g harmonic, $g = \int_{\partial\Omega} \frac{\partial G}{\partial \nu} g$. Then, if we define the operator $T : C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega})$ given by

$$TX(w) = g(w) + 2\int_\Omega G(\cdot, w)H(\cdot, X)X_u \wedge X_v,$$

any solution of (1) may be regarded as a fixed point of T.

By Arzelà-Ascoli, T is compact. Moreover, for $\|X - g\|_{1,\infty} \leq R$, we have that

$$\|TX - g\|_{1,\infty} \leq s\|2H(\cdot, X)X_u \wedge X_v\|_\infty \leq sf(R)R \leq R$$

for some $R > 0$. Then, $T(B_R(g)) \subset B_R(g)$ and by Schauder's Fixed Point Theorem we conclude that T has at least a fixed point.

As a simple consequence, we see that (1) admits a solution when ∇g is small enough. Indeed, fixing $\overline{R}$ such that $\overline{R}\|H|_{K_{\overline{R}}}\|_\infty \geq \frac{1}{s}$, and calling $h = \|H|_{K_{\overline{R}}}\|_\infty$, we obtain:

COROLLARY 2

Let us assume that $\|\nabla g\|_\infty \leq \frac{1}{4sh}$. *Then (1) admits a solution in* $B_R(g)$ *for some* $R \in (0, \overline{R}]$.

Proof

For $R \leq \overline{R}$, we have that $f(R) \leq \frac{h}{R}(\|\nabla g\|_\infty + R)^2$. Then, the hypothesis of Theorem 1 is fulfilled if $(\|\nabla g\|_\infty + R)^2 \leq \frac{R}{sh}$ for some $R \in (0, \overline{R}]$, and a simple computation shows that this is equivalent to the condition $\|\nabla g\|_\infty \leq \frac{1}{4sh}$.

REMARK : *in particular, for* $H = H(u, v)$ *problem (1) is solvable for* $\|\nabla g\|_\infty \leq \frac{1}{4s\|H\|_\infty}$

3. Solutions for small perturbations of H and g

In this section we'll prove under some conditions that if (1) is solvable for some (H_0, g_0), then there exists a solution for any (H, g) close enough to (H_0, g_0):

THEOREM 3

Let us assume that (1) admits a solution $X_0 \in W^{2,p}(\Omega, \mathbb{R}^3)$ for some $g_0 \in W^{2,p}$ with $p > 2$ and $H_0 = H_0(u, v)$. Then, if $2\|H_0 \nabla X_0\|_\infty < \sqrt{\lambda_1}$ where λ_1 is the first eigenvalue of $-\Delta$, problem (1) is solvable for any (H, g) close to (H_0, g_0) in $L^p \times W^{2,p}$.

<u>*Proof*</u>

Let us consider H, g such that $\|g - g_0\|_{2,p} < \delta_1$ and $\|H - H_0\|_p < \delta_2$.

We look for a solution X of (1), which is equivalent, taking $Y = X - X_0$, to find a solution of the equation

$$\begin{cases} \Delta Y = 2H(u,v)(Y + X_0)_u \wedge (Y + X_0)_v - 2H_0(u,v) X_{0_u} \wedge X_{0_v} & \text{in } \Omega \\ Y = g - g_0 \qquad \text{in} \quad \partial\Omega \end{cases}$$

If we consider the operator $LY = \Delta Y - 2H_0(X_{0_u} \wedge Y_v + Y_u \wedge X_{0_v})$, last equation becomes

$$LY = 2H_0 Y_u \wedge Y_v + 2(H - H_0)(Y + X_0)_u \wedge (Y + X_0)_v$$

By lemma 4 below and the Sobolev imbedding $W^{2,p} \hookrightarrow C^1(\overline{\Omega})$ we may define a continuous operator $T : C^1 \longrightarrow C^1$, given by $T(\overline{Y}) = Y$ where Y is the unique solution in $(g - g_0) + W^{2,p} \cap W_0^{1,p}$ of the linear problem

$$LY = 2H_0 \overline{Y}_u \wedge \overline{Y}_v + 2(H - H_0)(\overline{Y} + X_0)_u \wedge (\overline{Y} + X_0)_v$$

Moreover, as

$$\|T(\overline{Y}) - (g - g_0)\|_{2,p} \leq c(\|L(T(\overline{Y}))\|_p + \|L(g - g_0)\|_p),$$

the range of a bounded set is bounded with $\| \quad \|_{2,p}$, and by the compactness of the imbedding $W^{2,p} \hookrightarrow C^1$, we conclude that T is compact. Furthermore, for $\|\overline{Y}\|_{1,\infty} \leq R$ we obtain:

$$\|T(\overline{Y})\|_{1,\infty} \leq \|g - g_0\|_{1,\infty} + c_1 c(\|L(T(\overline{Y}))\|_p + \|L(g - g_0)\|_p)$$

$$\leq k\delta_1 + c_1 c\|2H_0 \overline{Y}_u \wedge \overline{Y}_v + 2(H - H_0)(\overline{Y} + X_0)_u \wedge (\overline{Y} + X_0)_v\|_p$$

$$\leq k\delta_1 + c_1 c(\|H_0\|_p R^2 + \delta_2(\|\nabla X_0\|_\infty + R)^2)$$

for some constant k. Then, if R, δ_1 and δ_2 are small enough, we have that $T(B_R(0)) \subset B_R(0)$ and the result follows from Schauder's Theorem.

4. A technical lemma

In this section we extend a classical result for a linear second order elliptic operator defined in $W^{2,p}(\Omega, \mathrm{I\!R})$:

Lemma 4

Let $L : W^{2,p}(\Omega, \mathrm{I\!R}^3) \longrightarrow L^p(\Omega, \mathrm{I\!R}^3)$ *be the linear elliptic operator given by* $LX = \Delta X + AX_u + BX_v + CX$, *with* $A, B, C \in L^\infty(\Omega, \mathrm{I\!R}^{3\times 3})$, $2 < p < \infty$ *and assume that* $r := \left(\frac{\| |A|^2 + |B|^2 \|_\infty}{\lambda_1}\right)^{1/2} < 1$ *and* $C \leq \delta < \lambda_1(1-r)$, *where* λ_1 *is the first eigenvalue of* $-\Delta$.

Then:

a) There exists a constant c *such that*

$$\|X\|_{2,p} \leq c\|LX\|_p$$

for every $X \in W^{2,p} \cap W_0^{1,p}(\Omega, \mathrm{I\!R}^3)$.

b) The Dirichlet problem

$$\begin{cases} LX = f & \text{in} \quad \Omega \\ u = \varphi & \text{in} \quad \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega, \mathrm{I\!R}^3)$ *for any* $f \in L^p$, $\varphi \in W^{2,p}(\Omega, \mathrm{I\!R}^3)$.

Proof

Let $Z_n \in W^{2,p} \cap W_0^{1,p}(\Omega, \mathrm{I\!R}^3)$ be a sequence such that $\|LZ_n\|_p \longrightarrow 0$. Then $\|LZ_n\|_2 \longrightarrow 0$, and as $\int LZ_nZ_n \leq -\|\nabla Z_n\|_2^2 + \|(|A|^2 + |B|^2)^{1/2}\|_\infty \|\nabla Z_n\|_2 \|Z_n\|_2 + \int CZ_nZ_n \leq (r - 1 + \frac{\delta}{\lambda_1})\|\nabla Z_n\|_2^2$, we conclude that $\|\nabla Z_n\|_2 \longrightarrow 0$. By Poincaré's inequallity, we obtain that $\|Z_n\|_2 \longrightarrow 0$ and hence $\|\Delta Z_n\|_2 \longrightarrow 0$. As a) holds for Δ and $1 < p < \infty$ (see [GT]), choosing a subsequence we may assume that $\|Z_n\|_{2,2} \longrightarrow 0$, and by Sobolev imbedding $\|Z_n\|_{1,p} \longrightarrow 0$. This shows that $\|\Delta Z_n\|_p \longrightarrow 0$, a contradiction.

In order to prove b), we'll apply a continuation method: let us define for $0 \leq t \leq 1$ the operator $L_tX = \Delta X + t(AX_u + BX_v)$. We'll show that for any $f \in L^p(\Omega, R^3)$, $\varphi \in W^{2,p}(\Omega, R^3)$ the equation (*) $L_tX = f$ in Ω, $X = \varphi$ in $\partial\Omega$ is solvable for any $t \in [0,1]$. For $t = 0$ this is immediate; moreover, we can see that

1) If (*) has a solution (for any f) for every $t \in [0, t_0]$, then there exists ϵ_0 such that (*) has a solution for every $t \in [0, t_0 + \epsilon_0)$: let $t = t_0 + \epsilon$, and $\overline{X} \in W^{2,p}(\Omega, R^3)$ fixed, then $A\overline{X}_u + B\overline{X}_v\overline{X} \in L^p(\Omega, R^3)$ and the equation

$$\begin{cases} L_{t_0} X = f - \epsilon(A\overline{X}_u + B\overline{X}_v + C\overline{X}) & \text{in } \Omega \\ X = \varphi & \text{in } \partial\Omega \end{cases}$$

admits a unique solution X. Thus we define an operator $T : \overline{X} \longrightarrow X$, and for $X = T(\overline{X}), Y = T(\overline{Y})$, as $X = Y$ in $\partial\Omega$ we have by a):

$$\|X - Y\|_{2,p} \leq c(t_0)\|L_{t_0}(X - Y)\|_p = c(t_0)\epsilon\|A(\overline{X}_u - \overline{Y}_u) + B(\overline{X}_v - \overline{Y}_v) + C(\overline{X} - \overline{Y})\|_p$$

$$\leq c(t_0)\epsilon(\|A\|_\infty\|\overline{X}_u - \overline{Y}_u\|_p + \|B\|_\infty\|\overline{X}_v - \overline{Y}_v\|_p + \|C\|_\infty\|\overline{X} - \overline{Y}\|_p)$$

Thus

$$\|T(\overline{X}) - T(\overline{Y})\|_{2,p} \leq c(t_0)(\|A\|_\infty + \|B\|_\infty + \|C\|_\infty)\epsilon\|\overline{X} - \overline{Y}\|_{2,p}$$

Choosing ϵ small enough, T is a contraction and then it has a (unique) fixed point.

2) If (*) has a solution (for any f) for every $t \in [0, t_0)$, then (*) has a solution for t_0: as in the previous case, we define $T(\overline{X}) = X$, where X is the only solution of

$$\begin{cases} L_{t_0 - \epsilon} X = f - \epsilon(A\overline{X}_u + B\overline{X}_v + C\overline{X}) & \text{in } \Omega \\ X = \varphi & \text{in } \partial\Omega \end{cases}$$

Note that $\|L_t X\|_p \geq \|L_{t_0} X\|_p - \|(L_t - L_{t_0})X\|_p \geq \dfrac{1}{c(t_0)}\|X\|_{2,p} - \|(L_t - L_{t_0})X\|_p$, and

$$\|(L_t - L_{t_0})X\|_p \leq \epsilon k\|X\|_{2,p}$$

for some constant k. Thus, choosing ϵ and c such that, $\dfrac{1}{c(t_0)} - \epsilon k = \dfrac{1}{c} > 0$, we see that we may consider $c(t_0 - \epsilon) \leq c$ in a neighbourhood of t_0. As in the previous case, T is a contraction for ϵ small enough.

Remark:

Lemma 4 holds for a general linear second order elliptic operator L defined in $W^{2,p}(\Omega, \mathbb{R}^n)$, considering λ_1 the first eigenvalue of the second order part of L.

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