Solutions to the prescribed mean curvature equation

with Dirichlet conditions by variational methods

P. Amster, P. De Nápoli and M. C. Mariani
Universidad de Buenos Aires

Abstract:

We apply variational methods in order to prove that the nonlinear system (1) admits at least one regular solution.

Introduction

We consider the problem of finding $u \in H^1(\Omega, I\!\!R^3)$ such that

$$(1) \begin{cases} \triangle u = 2H(u)u_x \wedge u_y \text{ in } \Omega \\ u = \gamma \text{ in } \partial\Omega \end{cases}$$

where H is a given continuously differentiable function and Ω is an open subset of $I\!\!R^2$ with C^1 boundary.

The system of differential equations above is the Dirichlet problem for the prescribed mean curvature equation in isothermal coordinates, also called H-system, which arises in the generalized Plateau's problem (see [O]).

Hildebrant, Wente (see [W1,W2]) and others have studied the problem for constant H. For this case, Brezis-Coron and Struwe have shown by variational methods (see [BC], [S]) that if the boundary data is small and non constant, then there are at least two weak solutions. The problem has been also studied by fixed point methods (see [AMR]).

The goal of this work is to show that also for non constant H, the problem may be solved by variational methods, under rather general conditions on H and the boundary data.

Our main result is the following:

Theorem 1 *Let $\gamma \in H^{1/2}(\partial\Omega, I\!\!R^3) \cap L^\infty(\partial\Omega, I\!\!R^3)$, $R = \|\gamma - c\|_{L^\infty(\partial\Omega, I\!\!R^3)}$ for some constant c, and $B \subset I\!\!R^3$ the closed ball of radius R centered at c. Then, if $H \in C^1(B)$ satisfies*

$$\|H\|_{L^\infty(B)} < \frac{1}{R}$$

then problem (1) admits a weak solution $u \in H^1(\Omega, I\!\!R^3) \cap L^\infty(\Omega, I\!\!R^3)$, *with* $\|u - c\|_{L^\infty(B)} \leq R$.

Remark: By a result of Bethuel ([B]) weak solutions of the mean curvature equation are $C^{2,\alpha}(\Omega)$ for H bounded and Lipschitz continuous. Moreover, by a result of Chanillo and Li ([CL]) the solutions are continuous up to the boundary.

Variational formulation of the problem

For simplicity, we may assume that $c = 0$ and extend H to a function of class $C^1(I\!\!R^3)$ such that $\|H\|_{L^\infty(I\!\!R^3)} < \frac{1}{R}$.

We define the vector field $q : I\!\!R^3 \to I\!\!R^3$:

$$q(x,y,z) = \frac{4}{3}\left(\int_0^x H(t,y,z)dt, \int_0^y H(x,t,z)dt, \int_0^z H(x,y,t)dt\right)$$

It is immediate that $\operatorname{div} q = 4H$ and $|q(x,y,z)| \leq \frac{4}{3}\|H\|_{L^\infty}|(x,y,z)|$.

We'll work in the Banach space $H^1(\Omega, I\!\!R^3) \cap L^\infty(\Omega, I\!\!R^3)$, with the norm:

$$\|u\| = \|u\|_{H^1} + \|u\|_{L^\infty}$$

and consider energy functional

$$E(u) = D(u) + Q(u)$$

where D is the Dirichlet's integral and Q is the functional

$$Q(u) = \int_\Omega q(u) \cdot (u_x \wedge u_y)$$

We'll use the estimatives

(2) $$|Q(u)| \leq \int_\Omega |q(u)| \cdot |u_x \wedge u_y)| \leq \frac{2}{3}\|H\|_{L^\infty}\|u\|_{L^\infty}\int_\Omega |\nabla u|^2$$

(3) $$E(u) \geq \left(1 - \frac{2}{3}\|H\|_{L^\infty}\|u\|_{L^\infty}\right)\int_\Omega |\nabla u|^2$$

Technical lemmas

In this section we present some technical facts.

LEMMA 2 *Q is continuous in $H^1 \cap L^\infty$.*

LEMMA 3 *We consider the functional*

$$I(u,h) = \int_\Omega H(u) \cdot (u_x \wedge u_y) \cdot h$$

where $u \in H^1 \cap L^\infty$ and $h \in L^\infty$.

Then I is continuous.

Proof

Lemma 2 and Lemma 3 follow from simple computations.

LEMMA 4

The functional E is Gateaux differentiable in $H^1 \cap L^\infty$ for variations in $H_0^1 \cap L^\infty$, and

$$dE(u)(h) = 2\int_\Omega \nabla u \cdot \nabla h + \int_\Omega 4H(u) \cdot (u_x \wedge u_y) \cdot h$$

for every $u \in H^1 \cap L^\infty, h \in H_0^1 \cap L^\infty$.

Proof

Let us first take $u, h \in C^1$ and define a function $U : \Omega \times \mathbb{R} \to \mathbb{R}^3$ by:

$$U(x,y,t) = u(x,y) + t \cdot h(x,y)$$

We'll write $u^t(x,y) = U(x,y,t)$.

We recall that the field q in $\mathbb{R}^3$ is associated to the 2-form ω given by:

$$\omega(X,Y) = (X \wedge Y) \cdot q(u)$$

with

$$d\omega = \mathrm{div}(q) du^1 \wedge du^2 \wedge du^3$$

Applying Stokes theorem to the cylinder $C = \Omega \times [0, \varepsilon]$, we obtain:

$$\int_C d(U^*\omega) = \int_{\partial C} U^*\omega$$

Moreover,

$$d(U^*\omega) = U^*(d\omega) = U^*(\mathrm{div} q(u) du^1 \wedge du^2 \wedge du^3)$$

$$= 4H(U) \cdot \frac{\partial(U^1, U^2, U^3)}{\partial(x, y, t)} dx \wedge dy \wedge dt = 4H(u^t) \cdot (u_x^t \wedge u_y^t) \cdot h$$

We have:

$$U^*\omega = \omega(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}) dx \wedge dy = q(u^t) \cdot (u_x^t \wedge u_y^t)$$

on $\Omega \times \{\varepsilon\}$ and $\Omega \times \{0\}$ and

$$U^*\omega = \omega(\frac{\partial U}{\partial \sigma}, \frac{\partial U}{\partial t}) d\sigma \wedge dt = q(u^t) \cdot (\frac{\partial U}{\partial \sigma} \wedge h) d\sigma \wedge dt$$

on the lateral surface $\partial\Omega \times [0, \varepsilon]$, where σ is the unit tangent vector on $\partial\sigma$.

Thus,

$$\int_0^\varepsilon \int_\Omega 4H(u^t) \cdot (u_x^t \wedge u_y^t) \cdot h dx \wedge dy \wedge dt =$$

$$\int_{\Omega \times \{\varepsilon\}} q(u^\varepsilon) \cdot (u_x^\varepsilon \wedge u_y^\varepsilon) dx \wedge dy \wedge dt - \int_{\Omega \times \{0\}} q(u) \wedge (u_x \wedge u_y) \cdot h$$

$$+ \int_{\partial\Omega \times [0,\varepsilon]} q(u) \cdot (\frac{\partial U}{\partial \sigma} \wedge h) d\sigma \wedge dt$$

Assuming that $h = 0$ on the boundary of Ω, we get:

$$\int_0^\varepsilon \int_\Omega 4H(u^t) \cdot (u_x^t \wedge u_y^t) \cdot h = Q(u^\varepsilon) - Q(u).$$

By density, this formula holds for $u \in H^1 \cap L^\infty$ and $h \in H_0^1 \cap L^\infty$.

By dominated convergence we conclude that

$$dQ(u)(h) = \int_\Omega 4H(u) \cdot (u_x \wedge u_y) \cdot h$$

which completes the proof.

Remark: By lemmas 2 and 3 dE is continuous, i.e. $E \in C^1(H_0^1 \cap L^\infty)$.

Corollary 5

If u is a critical point of functional E for variations in $H_0^1 \cap L^\infty$ then u is a weak solution of (1).

Proof of Theorem 1

We fix $R' > R$ such that $\|H\|_{L^\infty} R' < 1$ and consider the nonempty set:

$$K = \{u \in H^1(\Omega, I\!R^3) : u = \gamma \text{ in } \partial\Omega, \|u\| \leq R'\}$$

Then we obtain

Lemma 6

E achieves a minimum in K.

<u>*Proof*</u>

By (3), the functional E is bounded from below in K. Let $E_0 = \inf_{v \in K} E(v)$ and consider a minimizing sequence (u^n)

By (3) and Poincaré's inequality u^n is bounded in H^1, and then we may suppose that

$$u^n \to \underline{u} \text{ in } H^1 \text{weakly}$$

and

$$u^n \to \underline{u} \text{ a.e.}$$

By the continuity of q we conclude that:

$$q(u^n) \to q(\underline{u}) \text{ a.e.}$$

and since $|q(u^n)| \leq \frac{4}{3}\|H\|_{L^\infty}$, we may also assume that:

$$q(u^n) \to q(\underline{u}) \text{ in } L^\infty \text{ weak*}$$

Set $\theta^n = u^n - u \in H_0^1$. Then $\|\theta^n\|_{L^\infty} \leq 2R'$, and

$$E(u^n) = \int_\Omega |\nabla \underline{u}|^2 + 2\int_\Omega \nabla u \cdot \nabla \theta^n$$

$$+\int_\Omega |\nabla\theta^n|^2+\int_\Omega q(u^n)\cdot(\underline{u}_x+\theta^n_x)\wedge(\underline{u}_y+\theta^n_y)$$

Thus,

$$\int_\Omega \nabla u\cdot\nabla\theta^n\to 0$$

Moreover, being

$$\int_\Omega q(u^n)\cdot(\underline{u}_x\wedge\theta^n_y)=-\int_\Omega \theta^n_y\cdot(q(u^n)\wedge\underline{u}_x)$$

then

$$\int_\Omega q(u^n)\cdot(\underline{u}_x\wedge\theta^n_y)\to 0,$$

since $\theta^n_y\to 0$ weakly in L^2, and

$$|q(u^n)\wedge\underline{u}_x|^2\le\left(\frac{4}{3}|u^n|\right)^2\cdot|\underline{u}_x|^2\le\left(\frac{4}{3}R'\right)|\underline{u}_x|^2.$$

In the same way, we conclude that

$$\int_\Omega q(u^n)\cdot(\theta^n_x\wedge\underline{u}_y)\to 0$$

We also have the estimative:

$$|\int_\Omega q(u^n)\cdot\theta^n_x\wedge\theta^n_y|\le\int_\Omega 4\|H\|_{L^\infty}\|u^n)\|_{L^\infty}|\theta^n_x\wedge\theta^n_y|$$

$$\le\frac{2}{3}\|H\|_{L^\infty}R'\int_\Omega|\nabla\theta^n|^2\le\frac{2}{3}\int_\Omega|\nabla\theta^n|^2$$

Finally as $q(u^n)\to q(\underline{u})$ weak* in L^∞:

$$\int_\Omega q(u^n)\cdot(\underline{u}_x\wedge\underline{u}_y)=\int_\Omega q(\underline{u})\cdot(\underline{u}_x\wedge\underline{u}_y)+o(1)$$

Then,

$$E(u^n)\ge E(u)+o(1)+\frac{1}{3}\int_\Omega|\nabla\theta^n|^2$$

and we can conclude that $E(u)=E_0$ and $\int_\Omega|\nabla\theta^n|^2\to 0$.

LEMMA 7

Let $u\in K$ satisfy $E(u)=\inf_{v\in K}E(v)$. Then u is a weak solution of (1).

Proof

Let us fix a nonnegative test function $\eta \in \mathcal{D}(\Omega)$. Then $u(1-\varepsilon\eta) \in K$ for $\varepsilon \geq 0$ small enough.

As the function $\varphi(\varepsilon) = E(u(1-\varepsilon\cdot\eta))$ has a minimum in $\varepsilon = 0$, then $\varphi'(0) = dE(u)(-u\cdot\eta) \geq 0$, i.e.

$$2\int_\Omega \nabla u \cdot \nabla(-\eta\cdot u) + \int_\Omega 4H(u)(u_x \wedge u_y)\cdot(-\eta u) \geq 0 \; \forall\, \eta \in \mathcal{D}(\Omega)$$

By Green's formula,

$$\int_\Omega \triangle(|u|^2)\cdot\eta = -\int_\Omega \nabla(|u|^2)\nabla(\eta) = -\int_\Omega 2(\nabla u\cdot u)\cdot\nabla\eta$$

It follows that

$$-\frac{1}{2}\triangle|u|^2 + |\nabla u|^2 + 2H(u)u\cdot u_x \wedge u_y \leq 0 \text{ in } \mathcal{D}'(\Omega)$$

Then, as

$$|2H(u)u\cdot(u_x \wedge u_y)| \leq \|H\|_{L^\infty} R' |\nabla u|^2$$

it follows that

$$-\triangle|u|^2 \leq 0 \text{ in } \mathcal{D}'(\Omega)$$

and then by Stampacchia's maximum principle (see [GT]) we get:

$$sup_\Omega |u| = sup_{\partial\Omega}|u| = R$$

Since $R < R'$, u is interior in K. It follows that $dE(u)(v) = 0$ for any $v \in H_0^1 \cap L^\infty$, and by corollary 5 u is a weak solution of (1).

References

[AMR] Amster P. Mariani, M.C, Rial, D.F: Existence and uniqueness of H-System's solutions with Dirichlet conditions. To appear in Nonlinear Analysis, Theory, Methods, and Applications.

[B] Bethuel, F: Un résultat de regularité pour les solutions de l'equation des surfaces à courboure moyenne prescrite. R.Acad. Sci. Paris Sér. I Math 314 (1992) n^o 13, 1003-100.

[B-C] Brezis, H., Coron, J., Multiple Solutions of H-Systems and Rellich's conjecture. Comm.Pure Appl. Math 37, 1984, 149-187.

[CL] Chanillo S., Li Y.: Continuity of Solutions of Uniformly Elliptic Equations in $I\!R^2$. Manuscripta math. 77,415-433 (1992)

[GT] Gilbarg, D., Trudinger N.: Elliptic Partial Differential Equations of Second Order. Springer-Verlag 1977

[O] Osserman, R: A Survey of Minimal Surfaces. Van Nostrand Reinhold Company, 1969

[S] Struwe, M.: Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag, p. 180.

[W1] Wente, H.: An Existence Theorem for Surfaces of Constant Mean Curvature. Journal of Mathematical Analysis and Applications 26, p. 318-344 (1969)

[W2] Wente, H.: The differential equation $\Delta X = 2H(X_u \wedge X_v)$ with vanishing boundary values. Proceedings of the American Mathematical Society 50 (1975), 131-7.

P.Amster*, P. De Nápoli and M. C. Mariani*

Dpto. de Matemática, Fac. de Cs. Exactas y Naturales, Universidad de Buenos Aires.

* Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)

Address for correspondence:

Prof. P. Amster and M. C. Mariani,

Dpto. de Matemática - Fac. de Cs. Exactas y Naturales, UBA

Pab. I, Ciudad Universitaria (1428)

Buenos Aires, Argentina

E-mail: pamster@dm.uba.ar - mcmarian@dm.uba.ar