SOLUTIONS TO THE MEAN CURVATURE EQUATION FOR NONPARAMETRIC SURFACES BY FIXED POINT METHODS

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ABSTRACT

We study the existence of solutions for the equation of prescribed mean curvature when the surface is the graph of $u:\overline{\Omega}\longrightarrow R$, with mean curvature $H(x,y,u(x,y))$. We give conditions on the boundary data in order to obtain at least one solution for the quasilinear Dirichlet problem (1) below, with H a given continuous function.

INTRODUCTION

We consider the quasilinear Dirichlet problem in a bounded domain $\Omega\subset R^2$ with $\partial\Omega\in C^2$

$$(1)\begin{cases}(1+u_y^2)u_{xx}+(1+u_x^2)u_{yy}-2u_xu_yu_{xy}=2H(x,y,u)\left(1+|\nabla u|^2\right)^{\frac{3}{2}} & \text{in} \quad \Omega\\ u(x,y)=\varphi(x,y)\quad \text{on } \partial\Omega\end{cases}$$

where $H:\overline{\Omega}\times[\epsilon,\epsilon]\longrightarrow R$ is continuous for some $\epsilon>0$ and $\varphi\in W^{2,p}(\Omega)$ is the boundary data.

The problem above is the mean curvature equation for nonparametric surfaces which has been studied in general for hypersurfaces in R^{n+1} by Gilbarg, Trudinger, Simon, Serrin, Díaz, Saa and Thiel among other authors. For H independent of u it has been proved [GT] that there exists a solution for any smooth boundary data if the mean curvature H' of $\partial\Omega$ satisfies:

$$H'(x_1,...,x_n)\geq\frac{n}{n-1}|H(x_1,...,x_n)|$$

for any $(x_1,...,x_n)\in\partial\Omega$, and $H\in C^1(\overline{\Omega},R)$ satisfying the inequality:

$$|\int_\Omega H\varphi|\leq\frac{1-\epsilon}{n}\int_\Omega|D\varphi|$$

for any $\varphi\in C_0^1(\Omega,R)$ and some $\epsilon>0$. The sharpness of the geometric condition on the curvature of $\partial\Omega$ is shown by a non-existence result ([GT], corollary 14.13): if $H'(x_1,...,x_n)<\frac{n}{n-1}|H(x_1,...,x_n)|$ for some $(x_1,...,x_n)$ and the sign of H is constant, then for any $\epsilon > 0$ there exists $g \in C^\infty(\overline{\Omega})$ such that $\|g\|_\infty \leq \epsilon$ for which the Dirichlet's problem is not solvable.

On the other hand, Díaz, Saa and Thiel [DST] studied the general quasilinear elliptic equation $div(Q(|\nabla u|)\nabla u) + f(u) = g(x_1, ..., x_n)$ in R^n under Dirichlet and Neumann conditions. They studied existence and uniqueness of the problem for nonincreasing f by finding apriori bounds for ∇u. The case $Q(r) = (1 + r^2)^{-1/2}$ corresponds to the mean curvature equation (1), and the condition on f becomes: $h'(u) \geq 0$.

In the present paper we study the problem by topological methods, obtaining a solution under some restrictions on $\|H\|_\infty$ and $\|\varphi\|_{2,p}$ but avoiding the conditions on the curvature of $\partial\Omega$. The condition $\frac{\partial h}{\partial u} \geq 0$ will not be necessary either.

The general Plateau problem and the Dirichlet asociated problem, have been studied in [AMR],[BC],[H],[LD-M],[MR] [S1],[S2],[WG], etc.

The quasilinear operator associated to problem (1) is strictly elliptic since its eigenvalues are $\lambda = 1$ and $\Lambda = 1 + |p|^2$, where $p = (u_x, u_y)$ (see [GT] chapter 10).

The main result is the following theorem

THEOREM 1

Let $p > 2$ and assume that $\|\varphi\|_{2,p}$ and $\|H\|_{L^\infty(\overline{\Omega}\times[\epsilon,\epsilon])}$ are small enough with respect to $|\Omega|$, the Sobolev's constant and the apriori bounds for Δ in Ω. Then there exists at least one solution $u \in W^{2,p}(\Omega)$ of (1).

SOLUTIONS BY FIXED POINT METHODS

First we note that u is a solution of (1), if and only if $w = u - \varphi$ is a solution of the following equation:

$$(2) \begin{cases} (1 + (w_y + \varphi_y)^2)w_{xx} + (1 + (w_x + \varphi_x)^2)w_{yy} - 2(w_x + \varphi_x)(w_y + \varphi_y)w_{xy} \\ \qquad = 2H(x, y, w + \varphi)\left(1 + |\nabla(w + \varphi)|^2\right)^{\frac{3}{2}} - (1 + (w_y + \varphi_y)^2)\varphi_{xx} \\ \qquad\quad -(1 + (w_x + \varphi_x)^2)\varphi_{yy} + 2(w_x + \varphi_x)(w_y + \varphi_y)\varphi_{xy} \qquad \text{in } \Omega \\ w = 0 \qquad \text{on } \partial\Omega \end{cases}$$

For each $\overline{v} \in C^1(\overline{\Omega})$ such that $\|\overline{v} + \varphi\|_\infty \leq \epsilon$ we consider the elliptic linear Dirichlet problem associated to equation (2)

$$(3) \begin{cases} L_{\overline{v}}(v) = F(\overline{v}) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where

$$L_{\overline{v}}(v) = (1 + (\overline{v}_y + \varphi_y)^2)v_{xx} + (1 + (\overline{v}_x + \varphi_x)^2)v_{yy} - 2(\overline{v}_x + \varphi_x)(\overline{v}_y + \varphi_y)v_{xy}$$

and

$$F(\overline{v}) = 2H(x, y, \overline{v}+\varphi)\left(1 + |\nabla(\overline{v} + \varphi)|^2\right)^{\frac{3}{2}} - (1+(\overline{v}_y+\varphi_y)^2)\varphi_{xx} - (1+(\overline{v}_x+\varphi_x)^2)\varphi_{yy}$$

$$+2(\overline{v}_x + \varphi_x)(\overline{v}_y + \varphi_y)\varphi_{xy}$$

The linear equation (3) has a unique solution $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [GT], theorem 9.15). Thus, if we consider the Sobolev imbedding $W^{2,p} \hookrightarrow C^1$ with imbedding constant k (i.e. $\|u\|_{1,\infty} \leq k\,\|u\|_{2,p}$), we may define an operator $T : \overline{B_\epsilon(-\varphi)} \subset C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega})$ given by $T(\overline{v}) = v$ if v is the solution of (3) for $\overline{v}$.

We'll see that the operator T has at least one fixed point in C^1, and this will give a solution of the original problem (1).

Our main tool will be the Schauder fixed point theorem (see [GT] theorem 11.1 and corollary 11.2). We prove first the following lemma and proposition.

LEMMA 2

There exists a constant C (depending only on $|\Omega|$, p) and $R > 0$ such that if

$$\|\overline{v} + \varphi\|_{1,\infty} \leq R$$

then for every $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$\|w\|_{2,p} \leq C\,\|L_{\overline{v}}(w)\|_p$$

Proof

We can write

$$L_{\overline{v}}(w) = \Delta w + S_{\overline{v}}(w)$$

where $S_{\overline{v}}(w) = (\overline{v}_y + \varphi_y)^2 w_{xx} + (\overline{v}_x + \varphi_x)^2 w_{yy} - 2(\overline{v}_x + \varphi_x)(\overline{v}_y + \varphi_y)w_{xy}$.

The operator Δ satisfies the hypotheses of [GT], lemma 9.17, then there exists a constant C_1 (independent of w) such that

$$\|w\|_{2,p} \leq C_1\,\|\Delta w\|_p$$

for all $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Then

$$\|L_{\overline{v}}(w)\|_p \geq \|\Delta w\|_p - \|S_{\overline{v}}(w)\|_p \geq \frac{1}{C_1}\|w\|_{2,p} - \|S_{\overline{v}}(w)\|_p$$

and being

$$\|S_{\overline{v}}(w)\|_p \leq 4\,\|\overline{v}+\varphi\|_{1,\infty}^2\,\|w\|_{2,p}$$

we obtain

$$\|L_{\overline{v}}(w)\|_p \geq \left(\frac{1}{C_1} - 4\,\|\overline{v}+\varphi\|_{1,\infty}^2\right)\|w\|_{2,p}$$

The second member of the last inequality is positive if $\|\overline{v}+\varphi\|_{1,\infty} \leq R < \frac{1}{2\sqrt{C_1}}$, and setting $C = \dfrac{C_1}{1-4C_1R^2}$ the lemma holds.

In the following proposition we'll find $0 < R < \frac{1}{2\sqrt{C_1}}, \epsilon$ such that $T(\overline{B_R(-\varphi)}) \subset \overline{B_R(-\varphi)} \subset C^1(\overline{\Omega})$.

Proposition 3

Let $p > 2$ and assume that $\|\varphi\|_{2,p}$ and $\|H\|_{L^\infty(\overline{\Omega}\times[\epsilon,\epsilon])}$ are small enough. Then there exists $R \leq \epsilon$ such that if

$$\|\overline{v}+\varphi\|_{1,\infty} \leq R$$

then

$$\|T(\overline{v})+\varphi\|_{1,\infty} \leq R$$

Furthermore, the operator T is continuous in the closed ball $\overline{B_R(-\varphi)}$, and its range is a precompact set.

Proof

Assume that $\|\overline{v}+\varphi\|_{1,\infty} \leq R < \frac{1}{2\sqrt{C_1}}$. Then

$$\|v+\varphi\|_{1,\infty} \leq k\,\|v\|_{2,p} + \|\varphi\|_{1,\infty} \leq Ck\,\|L_{\overline{v}}(v)\|_p + \|\varphi\|_{1,\infty} = \frac{C_1k}{1-4C_1R^2}\|F(\overline{v})\|_p + \|\varphi\|_{1,\infty}$$

and

$$\|F(\overline{v})\|_p \leq 2(1+\|\overline{v}+\varphi\|_{1,\infty}^2)^{3/2}\,\|H(x,y,\overline{v}+\varphi)\|_p + 2(1+2\,\|\overline{v}+\varphi\|_{1,\infty}^2)\,\|\varphi\|_{2,p} \leq$$

$$\leq 2(1+R^2)^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + 2(1+2R^2)\,\|\varphi\|_{2,p}$$

We look for a number R such that

$$\frac{2C_1k}{1-4C_1R^2}((1+R^2)^{3/2}|\Omega|^{1/p}\left\|H\right\|_\infty+(1+2R^2)\left\|\varphi\right\|_{2,p})+\|\varphi\|_{1,\infty}\leq R$$

or, equivalently, such that $f(R)\leq 0$, where

$$f(R)=\frac{2C_1k}{1-4C_1R^2}((1+R^2)^{3/2}|\Omega|^{1/p}\left\|H\right\|_\infty+(1+2R^2)\left\|\varphi\right\|_{2,p})+\|\varphi\|_{1,\infty}-R$$

It is clear that $f\leq\dfrac{P}{1-4C_1R^2}$ in the interval $(0,\dfrac{1}{2\sqrt{C_1}})$, where

$$P(R)=2C_1k((1+\frac{1}{4C_1})^{3/2}|\Omega|^{1/p}\left\|H\right\|_\infty+(1+\frac{1}{2C_1})\left\|\varphi\right\|_{2,p})+\|\varphi\|_{1,\infty}-R+4C_1R^3$$

P achieves a minimum in $R_0=\dfrac{1}{\sqrt{12C_1}}$ and $P(R_0)\leq 0$ if $\|\varphi\|_{2,p}$ and $\|H\|_\infty$ are small enough. Then $f(R_0)\leq 0$.

In order to complete the proof we must see that T is continuous and compact. Indeed, for $\overline{u}$, $\overline{v}\in\overline{B_R(-\varphi)}$:

$$\|u-v\|_{2,p}\leq C\left\|L_{\overline{u}}\left(u-v\right)\right\|_p\leq C\left(\left\|F(\overline{u})-F(\overline{v})\right\|_p+\left\|L_{\overline{v}}(v)-L_{\overline{u}}(v)\right\|_p\right).$$

But

$$\begin{aligned}\left\|F(\overline{u})-F(\overline{v})\right\|_p&\leq\|2H(x,y,\overline{u}+\varphi)-2H(x,y,\overline{v}+\varphi))\left(1+|\nabla(\overline{u}+\varphi)|^2\right)^{\frac{3}{2}}\|_p\\&+\|2H(x,y,\overline{v}+\varphi)(\left(1+|\nabla(\overline{u}+\varphi)|^2\right)^{\frac{3}{2}}-\left(1+|\nabla(\overline{v}+\varphi)|^2\right)^{\frac{3}{2}})\|_p\\&+\|((\overline{u}_y+\varphi_y)^2-(\overline{v}_y+\varphi_y)^2)\varphi_{xx}\|_p+\|((\overline{u}_x+\varphi_x)^2-(\overline{v}_x+\varphi_x)^2)\varphi_{yy}\|_p+2\|((\overline{u}_x+\varphi_x)(\overline{u}_y+\varphi_y)-\\&(\overline{v}_x+\varphi_x)(\overline{v}_y+\varphi_y))\varphi_{xy}\|_p\leq 2\|H(x,y,\overline{u}+\varphi)-H(x,y,\overline{v}+\varphi)\|_p(1+R^2)^{3/2}\\&+6R(1+R^2)^{\frac{1}{2}}(R+\|\varphi\|_{1,\infty})\|\overline{u}-\overline{v}\|_{1,\infty}|\Omega|^{1/p}\|H\|_\infty+8R\|\overline{u}-\overline{v}\|_{1,\infty}\|\varphi\|_{2,p}\end{aligned}$$

and

$$\left\|L_{\overline{v}}(v)-L_{\overline{u}}(v)\right\|_p\leq 8R\|\overline{u}-\overline{v}\|_{1,\infty}\|v\|_{2,p}$$

Being H uniformely continuous and $\|u-v\|_{1,\infty}\leq k\|u-v\|_{2,p}$, the continuity follows. Moreover, fixing any $\overline{v}\in C^1(\overline{\Omega})$ we see that $\overline{T(B_R(-\varphi))}$ is bounded in $W^{2,p}$, and the result follows from the compactness of the imbedding $W^{2,p}(\Omega)\hookrightarrow C^1(\overline{\Omega})$.

Remark

In the situation of proposition 3, if we write $P(R) = 4C_1R^3 - R + a$ *, the smallness of* H *and* φ *can be stated in the more precise condition* $a \leq \frac{2}{3\sqrt{12C_1}}$.

Proof of theorem 1: From proposition 3 we know that the operator T satisfies the assumptions of Schauder fixed point theorem (see [GT] corollary 11.2). Thus, we obtain a fixed point $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for the operator T, which corresponds to a solution of equation (2).

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