THE CLASSICAL MASSIVE THIRRING SYSTEM REVISITED

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Dedicated with great pleasure to Sergio Albeverio on the occasion of his 60th birthday

Abstract. We provide a complete treatment of algebro-geometric solutions of the classical massive Thirring system. In particular, we study Dubrovin-type equations for auxiliary divisors, consider the corresponding algebro-geometric initial value problem, and derive the theta function representations of algebro-geometric solutions of the Thirring system.

1. Introduction

Ever since its publication in 1958, the Thirring model [40] kept its fascination as is witnessed by the incredible amount of attention paid to it since then and by the interest it continues to generate (see, e.g., [26] for a recent review). In the present paper we are not concerned with its importance as a solvable quantum field theory model but rather restrict our attention to its complete integrability aspects from a classical point of view. Thirring's classical $(1+1)$-dimensional model equations in appropriate light cone coordinates, and after appropriate rescaling of the mass and coupling constant parameters, etc., can be cast in the form

$$\begin{aligned} -iu_x + 2v + 2|v|^2 u &= 0, \\ -iv_t + 2u + 2|u|^2 v &= 0. \end{aligned} \tag{1.1}$$

Formal integrability of (1.1) was originally established by Mikhailov [34] in 1976 by establishing a corresponding commutator representation (cf. (2.8) and (2.9)). In fact, one can replace (1.1) by a more general system, without identifying u^* and v^* with the complex conjugates $\bar{u}$ and $\bar{v}$ of u and v, respectively,

$$\begin{aligned} -iu_x + 2v + 2vv^*u &= 0, \\ iu^*_x + 2v^* + 2vv^*u^* &= 0, \\ -iv_t + 2u + 2uu^*v &= 0, \\ iv^*_t + 2u^* + 2uu^*v^* &= 0, \end{aligned} \tag{1.2}$$

without losing formal integrability, and we will actually investigate (1.2) rather than (1.1). Both (1.1) and (1.2) have been studied by numerous authors, who derived the inverse scattering approach [27], [29], [30], [43], considered soliton solutions [2], [3], [4], [10], [11], [12], [39], [42], investigated Bäcklund transformations and close

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connections with other integrable equations (especially, the sine-Gordon equation) [1], [28], [31], [32], [36], [37], [38], [41], [44], and considered monodromy deformations [8].

In the present paper we focus on algebro-geometric solutions of the classical massive Thirring system (1.2). The first attempt to derive algebro-geometric solutions of (1.1) is due to Date [9] in 1978 and almost simultaneously to Prikarpatskii and Holod [38] (see also [25]). Both papers are remarkably similar in strategy, in fact, they are nearly identical. In particular, both discuss theta function representations for symmetric functions of appropriate symmetric functions associated with auxiliary divisors, but neither derives explicit theta function representations of u and v. The first theta function representations of u, v, u^*, v^* for the general massive Thirring system (1.2) were derived by Bikbaev [6], however, with insufficient care paid to details. (In fact, his terms e^w and $e^{\hat{w}}$ on p. 581 are not defined, and in his formula (29), (x,t)-dependent terms are missing.) More recently, algebro-geometric solutions of (1.1) were also briefly considered by Wisse [45], again without explicitly deriving theta function representations for u and v.

In Section 2 we follow Date's [9] explicit realization of Mikhailov's commutator representation in terms of polynomials in the spectral parameter. In Section 3 we develop the basic algebro-geometric formalism for (1.2), and from that point on we deviate from previous investigations and focus on a different approach based on the solution ϕ of a Riccati-type equation associated with the Thirring system (1.2). We consider Dubrovin-type equations for auxiliary divisors and define the Baker–Akhiezer vector associated with the system (1.2) in terms of the fundamental function ϕ on $\mathcal{K}_n$, the underlying hyperelliptic curve of genus $n \in \mathbb{N}_0$. We also study the algebro-geometric initial value problem in detail. Our principal results, the theta function representations of u, v, u^*, v^*, and ϕ are derived in detail in Section 4. Finally, Appendix A collects some basic results on compact Riemann surfaces and introduces the terminology freely used in Sections 3 and 4.

2. The basic polynomial setup

In this section we start from Mikhailov's [34] commutator representation of the classical massive Thirring system in a form used by Date [9] (see also [25], [38], which contain similar material) in his analysis of quasi periodic solutions of this model.

Assuming $u, v, u^*, v^* \colon \mathbb{R}^2 \to \mathbb{C}$ to satisfy

$$\begin{aligned}
&u(\,\cdot\,,t), u^*(\,\cdot\,,t) \in C^1(\mathbb{R}), \quad v(\,\cdot\,,t), v^*(\,\cdot\,,t) \in C^\infty(\mathbb{R}), \quad t \in \mathbb{R},\\
&u(x,\,\cdot\,), u^*(x,\,\cdot\,) \in C^1(\mathbb{R}), \quad \partial_x^k v(x,\,\cdot\,), \partial_x^k v^*(x,\,\cdot\,) \in C^1(\mathbb{R}), \;\; k \in \mathbb{N}_0,\; x \in \mathbb{R}, \quad (2.1)
\end{aligned}$$

we introduce the 2×2 matrices

$$U(\zeta,x,t) = i\begin{pmatrix} z - v(x,t)v^*(x,t) & 2\zeta v(x,t) \\ 2\zeta v^*(x,t) & -z + v(x,t)v^*(x,t)\end{pmatrix}, \tag{2.2}$$

$$V_{n+1}(\zeta,x,t) = i\begin{pmatrix} -G_{n+1}(z,x,t) & \zeta F_n(z,x,t) \\ -\zeta H_n(z,x,t) & G_{n+1}(z,x,t)\end{pmatrix}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

$$\widetilde{V}(\zeta,x,t) = i\begin{pmatrix} z^{-1} - u(x,t)u^*(x,t) & 2\zeta^{-1} u(x,t) \\ 2\zeta^{-1} u^*(x,t) & -z^{-1} + u(x,t)u^*(x,t)\end{pmatrix}, \tag{2.4}$$

$$\zeta \in \mathbb{C}\setminus\{0\},\; z = \zeta^2,\; (x,t) \in \mathbb{R}^2,$$

where $F_n(z,x,t)$, $H_n(z,x,t)$, and $G_{n+1}(z,x,t)$ are polynomials with respect to z of degree n and $n+1$, respectively, that is, they are of the type

$$F_n(z,x,t) = \sum_{j=0}^{n} f_{n-j}(x,t) z^j = f_0(x,t) \prod_{j=1}^{n} (z - \mu_j(x,t)), \tag{2.5}$$

$$G_{n+1}(z,x,t) = \sum_{j=0}^{n+1} g_{n+1-j}(x,t) z^j, \quad g_0(x,t) = 1, \tag{2.6}$$

$$H_n(z,x,t) = \sum_{j=0}^{n} h_{n-j}(x,t) z^j = h_0(x,t) \prod_{j=1}^{n} (z - \nu_j(x,t)). \tag{2.7}$$

The classical massive Thirring system is then defined by demanding the zero-curvature representation

$$-V_{n+1,x}(\zeta,x,t) + [U(\zeta,x,t), V_{n+1}(\zeta,x,t)] = 0, \quad (\zeta,x,t) \in \mathbb{C} \setminus \{0\} \times \mathbb{R}^2, \tag{2.8}$$

$$-V_{n+1,t}(\zeta,x,t) + [\widetilde{V}(\zeta,x,t), V_{n+1}(\zeta,x,t)] = 0, \quad (\zeta,x,t) \in \mathbb{C} \setminus \{0\} \times \mathbb{R}^2. \tag{2.9}$$

Explicitly, equations (2.8) and (2.9) yield

$$F_{n,x}(z,x,t) = -2i(v(x,t)v^*(x,t) - z)F_n(z,x,t) + 4iv(x,t)G_{n+1}(z,x,t), \tag{2.10}$$

$$G_{n+1,x}(z,x,t) = 2izv^*(x,t)F_n(z,x,t) + 2izv(x,t)H_n(z,x,t), \tag{2.11}$$

$$H_{n,x}(z,x,t) = 2i(v(x,t)v^*(x,t) - z)H_n(z,x,t) + 4iv^*(x,t)G_{n+1}(z,x,t), \tag{2.12}$$

$$\begin{aligned} F_{n,t}(z,x,t) &= -2i(u(x,t)u^*(x,t) - z^{-1})F_n(z,x,t) \\ &\quad + 4iz^{-1}u(x,t)G_{n+1}(z,x,t), \end{aligned} \tag{2.13}$$

$$G_{n+1,t}(z,x,t) = 2iu^*(x,t)F_n(z,x,t) + 2iu(x,t)H_n(z,x,t), \tag{2.14}$$

$$\begin{aligned} H_{n,t}(z,x,t) &= 2i(u(x,t)u^*(x,t) - z^{-1})H_n(z,x,t) \\ &\quad + 4iz^{-1}u^*(x,t)G_{n+1}(z,x,t). \end{aligned} \tag{2.15}$$

By (2.10)–(2.14) one infers that

$$\left(G_{n+1}^2 - zF_nH_n\right)_x = \left(G_{n+1}^2 - zF_nH_n\right)_t = 0 \tag{2.16}$$

and hence

$$G_{n+1}(z,x,t)^2 - zF_n(z,x,t)H_n(z,x,t) = R_{2n+2}(z), \tag{2.17}$$

where the integration constant $R_{2n+2}(z)$ is a monic polynomial in z of degree $2n+2$, that is,

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}, \tag{2.18}$$

since we chose $g_0 = 1$. Moreover, (2.17) implies

$$g_{n+1}(x,t)^2 = \prod_{m=0}^{2n+1} E_m \tag{2.19}$$

and we will choose

$$g_{n+1} \neq 0, \text{ that is, } E_m \neq 0, \; m = 0, \dots, 2n+1. \tag{2.20}$$

The actual sign of g_{n+1} will be determined later (cf. (3.9), (3.10)). A comparison of coefficients of z^k in (2.10)–(2.15) then yields

$$\begin{aligned}
f_0 &= -2v,\\
f_1 &= iv_x + 2v^2 v^* + c_1(-2v),\\
f_n &= -2g_{n+1}u,\\
g_0 &= 1,\\
g_1 &= -2vv^* + c_1,\\
g_{n+1} &= \left(\prod_{m=0}^{2n+1} E_m\right)^{1/2},\\
h_0 &= 2v^*,\\
h_1 &= iv^*_x - 2v(v^*)^2 + c_1 2v^*,\\
h_n &= 2g_{n+1}u^*, \text{ etc.},
\end{aligned} \tag{2.21}$$

where $\{c_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}$ denote integration constants, and

$$-iu_x(x,t) + 2v(x,t) + 2v(x,t)v^*(x,t)u(x,t) = 0, \tag{2.22}$$
$$iu^*_x(x,t) + 2v^*(x,t) + 2v(x,t)v^*(x,t)u^*(x,t) = 0, \tag{2.23}$$
$$-iv_t(x,t) + 2u(x,t) + 2u(x,t)u^*(x,t)v(x,t) = 0, \tag{2.24}$$
$$iv^*_t(x,t) + 2u^*(x,t) + 2u(x,t)u^*(x,t)v^*(x,t) = 0. \tag{2.25}$$

Equations (2.22)–(2.25) represent the classical massive Thirring system in light cone coordinates. It should be emphasized that the original Thirring equations are given by (2.22), (2.24) imposing the constraints

$$u^*(x,t) = \overline{u(x,t)}, \quad v^*(x,t) = \overline{v(x,t)}, \tag{2.26}$$

where the bar denotes the operation of complex conjugate. In this paper, however, we will not impose the constraints (2.26) but rather study the system (2.22)–(2.25).

Given (2.22)–(2.25), a straightforward computation verifies the commutator relation

$$U_t(\zeta,x,t) - \widetilde{V}_x(\zeta,x,t) + [U(\zeta,x,t), \widetilde{V}(\zeta,x,t)] = 0, \quad (\zeta,x,t) \in \mathbb{C}\setminus\{0\}\times\mathbb{R}^2, \tag{2.27}$$

complementing (2.8) and (2.9).

This concludes our brief review of the polynomial setup by Date [9], and for the remainder of this paper we will deviate from his strategy and focus on an approach based on the solution ϕ of a Riccati-type equation associated with the Thirring system. This will enable us to employ a formalism previously applied to the KdV, AKNS, Toda, Boussinesq, and the combined sine-Gordon–mKdV hierarchies [7], [13], [14], [18], [19], [22], [23].

We conclude this section by mentioning the elementary fact that the Thirring system (2.22)–(2.25) is invariant under the scaling transformation,

$$(u,v,u^*,v^*) \to (Au, Av, A^{-1}u^*, A^{-1}v^*), \quad A \in \mathbb{C}\setminus\{0\}. \tag{2.28}$$

In the special case where $u^* = \bar{u}, v^* = \bar{v}$, A in (2.28) is further constrained by $|A| = 1$.

3. The basic algebro-geometric formalism

Introducing the (possibly singular) hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$,

$$\mathcal{K}_n \colon \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0, \tag{3.1}$$

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C} \setminus \{0\}, \tag{3.2}$$

we denote points P on $\mathcal{K}_n$ by $P = (z,y)$ and compactify $\mathcal{K}_n$ by joining two points at infinity P_{∞_+}, P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, still denoting the compactified curve by $\mathcal{K}_n$. Moreover, we recall the hyperelliptic involution (sheet exchange map) $*$ on $\mathcal{K}_n$,

$$* \colon \mathcal{K}_n \to \mathcal{K}_n, \quad P = (z,y) \mapsto P^* = (z,-y),\, {P_{\infty_+}}^* = P_{\infty_-}. \tag{3.3}$$

For additional facts on $\mathcal{K}_n$ and further notation freely employed throughout this paper, the reader may want to consult Appendix A.

Next, we define the fundamental meromorphic function $\phi(\,\cdot\,,x,t)$ on $\mathcal{K}_n$ by

$$\phi(P,x,t) = \frac{y(P) + G_{n+1}(z,x,t)}{F_n(z,x,t)} \tag{3.4}$$

$$= \frac{-zH_n(z,x,t)}{y(P) - G_{n+1}(z,x,t)}, \quad P = (z,y) \in \mathcal{K}_n,\, (x,t) \in \mathbb{R}^2, \tag{3.5}$$

where we used (2.17) to obtain (3.5). Introducing

$$\hat{\mu}_j(x,t) = (\mu_j(x,t), G_{n+1}(\mu_j(x,t),x,t)) \in \mathcal{K}_n, \quad j = 1,\dots,n,\, (x,t) \in \mathbb{R}^2, \tag{3.6}$$

$$\hat{\nu}_j(x,t) = (\nu_j(x,t), -G_{n+1}(\nu_j(x,t),x,t)) \in \mathcal{K}_n, \quad j = 1,\dots,n,\, (x,t) \in \mathbb{R}^2, \tag{3.7}$$

and

$$P_{0,\pm} = (0, \pm G_{n+1}(0)) = (0, \pm g_{n+1}) \in \mathcal{K}_n, \tag{3.8}$$

we fix the branch of $y(P)$ near $P_{\infty_\pm}$ according to

$$\lim_{|z|\to\infty} \frac{y(P)}{G_{n+1}(z,x,t)} = \lim_{|z|\to\infty} \frac{y(P)}{z^{n+1}} = \mp 1 \text{ as } P \to P_{\infty_\pm} \tag{3.9}$$

and consequently determine the sign of g_{n+1},

$$g_{n+1} = \left(\prod_{m=0}^{2n+1} E_m \right)^{1/2}, \tag{3.10}$$

by compatibility of all local charts on $\mathcal{K}_n$. We note that $P_{0,\pm}$ and $P_{\infty_\pm}$ are not necessarily on the same sheet of $\mathcal{K}_n$. The actual sheet on which $P_{0,\pm}$ lie depends on the sign of g_{n+1} and hence on the location of all E_m.

Given these conventions, the divisor $(\phi(\,\cdot\,,x,t))$ of $\phi(\,\cdot\,,x,t)$ then reads

$$(\phi(\,\cdot\,,x,t)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(x,t)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(x,t)}, \quad (x,t) \in \mathbb{R}^2. \tag{3.11}$$

Next we collect a few characteristic properties of ϕ.

Lemma 3.1. *Assume* (2.1), (2.8), (2.9), *and* (3.2) *and let* $P = (z,y) \in \mathcal{K}_n$, $(x,t) \in \mathbb{R}^2$. *Then ϕ satisfies the Riccati-type equations*

$$\begin{aligned} &\phi_x(P,x,t) + 2iv(x,t)\phi(P,x,t)^2 \\ &\qquad + 2i(z - v(x,t)v^*(x,t))\phi(P,x,t) = 2izv^*(x,t), \end{aligned} \tag{3.12}$$

$$\begin{aligned}\phi_t(P,x,t) + 2iz^{-1}u(x,t)\phi(P,x,t)^2 \\ + 2i(z^{-1} - u(x,t)u^*(x,t))\phi(P,x,t) = 2iu^*(x,t).\end{aligned} \tag{3.13}$$

Moreover,

$$\phi(P,x,t)\phi(P^*,x,t) = zH_n(z,x,t)/F_n(z,x,t), \tag{3.14}$$

$$\phi(P,x,t) + \phi(P^*,x,t) = 2G_{n+1}(z,x,t)/F_n(z,x,t), \tag{3.15}$$

$$\phi(P,x,t) - \phi(P^*,x,t) = 2y(P)/F_n(z,x,t). \tag{3.16}$$

Proof. Equation (3.12) follows from (2.10)–(2.11), (2.17), and (3.4). Similarly, (3.13) follows from (2.13)–(2.14), (2.17), and (3.4). Relations (3.14)–(3.16) are obvious from (2.17) and (3.4). □

Given $\phi(P,x,t)$, we can define the Baker–Akhiezer vector $\Psi(P,\zeta,x,x_0,t,t_0)$ by

$$\begin{aligned}\Psi(P,\zeta,x,x_0,t,t_0) = \begin{pmatrix} \psi_1(P,x,x_0,t,t_0) \\ \psi_2(P,\zeta,x,x_0,t,t_0) \end{pmatrix}, \\ P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_\pm}\},\ z = \zeta^2,\ (x,t),(x_0,t_0) \in \mathbb{R}^2,\end{aligned} \tag{3.17}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t,t_0) \\ &\quad = \exp\bigg(i\int_{t_0}^t ds\,\big(z^{-1} - u(x_0,s)u^*(x_0,s) + 2z^{-1}u(x_0,s)\phi(P,x_0,s)\big) \\ &\qquad + i\int_{x_0}^x dx'\,\big(z - v(x',t)v^*(x',t) + 2v(x',t)\phi(P,x',t)\big)\bigg),\end{aligned} \tag{3.18}$$

$$\psi_2(P,\zeta,x,x_0,t,t_0) = \zeta^{-1}\psi_1(P,x,x_0,t,t_0)\phi(P,x,t). \tag{3.19}$$

Properties of Ψ are summarized in the following result.

Lemma 3.2. *Assume* (2.1), (2.8), (2.9), *and* (3.2) *and let* $P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_\pm}\}$, $(x,t),(x_0,t_0) \in \mathbb{R}^2$. *Then* $\Psi(P,\zeta,x,x_0,t,t_0)$ *satisfies*

$$\Psi_x(P,\zeta,x,x_0,t,t_0) = U(\zeta,x,t)\Psi(P,\zeta,x,x_0,t,t_0), \tag{3.20}$$

$$\Psi_t(P,\zeta,x,x_0,t,t_0) = \widetilde{V}(\zeta,x,t)\Psi(P,\zeta,x,x_0,t,t_0), \tag{3.21}$$

$$iy(P)\Psi(P,\zeta,x,x_0,t,t_0) = V_{n+1}(\zeta,x,t)\Psi(P,\zeta,x,x_0,t,t_0). \tag{3.22}$$

Moreover, if the zeros of $F_n(\,\cdot\,,x,t)$ *are all simple for* $(x,t) \in \Omega$, $\Omega \subseteq \mathbb{R}^2$ *open and connected, then* $\psi_1(\,\cdot\,,x,x_0,t,t_0)$, $(x,t),(x_0,t_0) \in \Omega$, *is meromorphic on* $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$. *In addition,*

$$\begin{aligned}\psi_1(P,x,x_0,t,t_0) = \bigg(\frac{F_n(z,x,t)}{F_n(z,x_0,t_0)}\bigg)^{1/2} \times \\ \times \exp\bigg(2iy(P)z^{-1}\int_{t_0}^t ds\,\frac{u(x_0,s)}{F_n(z,x_0,s)} + 2iy(P)\int_{x_0}^x dx'\,\frac{v(x',t)}{F_n(z,x',t)}\bigg),\end{aligned} \tag{3.23}$$

$$\psi_1(P,x,x_0,t,t_0)\psi_1(P^*,x,x_0,t,t_0) = F_n(z,x,t)/F_n(z,x_0,t_0), \tag{3.24}$$

$$\psi_2(P,\zeta,x,x_0,t,t_0)\psi_2(P^*,\zeta,x,x_0,t,t_0) = H_n(z,x,t)/F_n(z,x_0,t_0), \tag{3.25}$$

$$\begin{aligned}\psi_1(P,x,x_0,t,t_0)\psi_2(P^*,\zeta,x,x_0,t,t_0) + \psi_1(P^*,x,x_0,t,t_0)\psi_2(P,\zeta,x,x_0,t,t_0) \\ = 2\zeta^{-1}G_{n+1}(z,x,t)/F_n(z,x_0,t_0).\end{aligned} \tag{3.26}$$

Proof. Equations (3.20), (3.21) are verified using (2.10)–(2.14), (3.12), (3.13), (3.18), and (3.19). (3.22) follows by combining (2.3), (3.4), (3.5), (3.18), and (3.19). Clearly ψ_1 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}, \hat\mu_1(x,t), \dots, \hat\mu_n(x,t)\}$ by (3.18). Since

$$2iv(x',t)\phi(P,x',t) \underset{P\to\hat\mu_j(x',t)}{=} \frac{\partial}{\partial x'} \ln\big(F_n(z,x',t)\big) + O(1) \text{ as } z \to \mu_j(x',t), \tag{3.27}$$

$$2iz^{-1}u(x_0,s)\phi(P,x_0,s) \underset{P\to\hat\mu_j(x_0,s)}{=} \frac{\partial}{\partial s} \ln\big(F_n(z,x_0,s)\big) + O(1) \text{ as } z \to \mu_j(x_0,s), \tag{3.28}$$

one infers that ψ_1 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ if the zeros of $F_n(\,\cdot\,,x,t)$ are all simple. This follows from (3.18) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j(x_0)$ of $\{\hat\mu_j(x_0,s) \in \mathcal{K}_n \,|\, (x_0,s) \in \Omega,\ s \in [t_0,t]\}$ such that $\hat\mu_k(x_0,s) \notin \mathcal{U}_j(x_0)$ for all $s \in [t_0,t]$ and all $k \in \{1,\dots,n\} \setminus \{j\}$, and similarly, by restricting P to a sufficiently small neighborhood $\mathcal{U}_j(t)$ of $\{\hat\mu_j(x',t) \in \mathcal{K}_n \,|\, (x',t) \in \Omega,\ x' \in [x_0,x]\}$ such that $\hat\mu_k(x',t) \notin \mathcal{U}_j(t)$ for all $x' \in [x_0,x]$ and all $k \in \{1,\dots,n\} \setminus \{j\}$. Equation (3.23) follows from (3.18) after replacing ϕ by the right-hand side of (3.4) and utilizing (2.10) in the x'-integral and (2.13) in the s-integral. Equations (3.24)–(3.26) immediately follow from (3.14)–(3.16), and (3.19). ☐

Next we discuss the asymptotic behavior of $\phi(P,x,t)$ as $P \to P_{0,\pm}, P_{\infty_\pm}$ in some detail since this will turn out to be a crucial ingredient for the theta function representation to be derived in Section 4.

Lemma 3.3. *Assume* (2.1), (2.8), (2.9), *and* (3.2). *Then*

$$\phi(P,x,t) \underset{z\to\infty}{=} -\frac{1}{v(x,t)}z + \frac{i}{2}\bigg(\frac{1}{v(x,t)}\bigg)_x + O\bigg(\frac{1}{z}\bigg) \text{ as } P=(z,y)\to P_{\infty_-}, \tag{3.29}$$

$$\phi(P,x,t) \underset{z\to\infty}{=} v^*(x,t) + \frac{i}{2}v^*_x(x,t)\frac{1}{z} + O\bigg(\frac{1}{z^2}\bigg) \text{ as } P=(z,y)\to P_{\infty_+}, \tag{3.30}$$

$$\phi(P,x,t) \underset{z\to 0}{=} u^*(x,t)z + \frac{i}{2}u^*_t(x,t)z^2 + O(z^3) \text{ as } P=(z,y)\to P_{0,-}, \tag{3.31}$$

$$\phi(P,x,t) \underset{z\to 0}{=} -\frac{1}{u(x,t)} + \frac{i}{2}\bigg(\frac{1}{u(x,t)}\bigg)_t z + O(z^2) \text{ as } P=(z,y)\to P_{0,+}. \tag{3.32}$$

Proof. The existence of these asymptotic expansions (in terms of local coordinates $\zeta = 1/z$ near $P_{\infty_\pm}$ and local coordinate $\zeta = z$ near $P_{0,\pm}$) is clear from the explicit form of ϕ in (3.4). Insertion of the polynomials F_n, H_n, and G_{n+1}, then in principle, yields the explicit expansion coefficients in (3.29)–(3.32). However, this is a cumbersome procedure, especially with regard to the next to leading coefficients in (3.29)–(3.32). Much more efficient is the actual computation of these coefficients utilizing the Riccati-type equations (3.12) and (3.13). Indeed, inserting the ansatz

$$\phi \underset{z\to\infty}{=} z\phi_{-1} + \phi_0 + O\bigg(\frac{1}{z}\bigg) \tag{3.33}$$

into (3.12) and comparing the first two leading powers of z immediately yields (3.29). Similarly, the ansatz

$$\phi \underset{z\to\infty}{=} \phi_0 + \phi_1\frac{1}{z} + O\bigg(\frac{1}{z^2}\bigg) \tag{3.34}$$

inserted into (3.12) immediately produces (3.30). In exactly the same manner, inserting the ansatz

$$\phi \underset{z\to 0}{=} \phi_1 z + \phi_2 z^2 + O(z^3) \tag{3.35}$$

and the ansatz

$$\phi \underset{z\to 0}{=} \phi_0 + \phi_1 z + O(z^2) \tag{3.36}$$

into (3.13) immediately yields (3.31) and (3.32), respectively. □

We follow up with a similar asymptotic analysis of $\psi_1(P,x,x_0,t,t_0)$.

Lemma 3.4. *Assume* (2.1), (2.8), (2.9), *and* (3.2). *Then*

$$\psi_1(P,x,x_0,t,t_0) \underset{z\to\infty}{=} \exp\big(\mp iz(x-x_0)+O(1)\big) \text{ as } P=(z,y)\to P_{\infty_\mp}, \tag{3.37}$$

$$\psi_1(P,x,x_0,t,t_0) \underset{z\to 0}{=} \exp\big(\pm iz^{-1}(t-t_0)+O(1)\big) \text{ as } P=(z,y)\to P_{0,\mp}. \tag{3.38}$$

Proof. Equations (3.37) and (3.38) follow from (3.18) noting

$$\begin{aligned} i(z-v(x,t)v^*(x,t))+2iv(x,t)\phi(P,x,t) &\underset{z\to\infty}{=} \mp iz+O(1)\\ &\text{as } P=(z,y)\to P_{\infty_\mp}, \end{aligned} \tag{3.39}$$

$$\begin{aligned} i(z^{-1}-u(x_0,s)u^*(x_0,s))+2iz^{-1}u(x_0,s)\phi(P,x_0,s) &\underset{z\to\infty}{=} O(1)\\ &\text{as } P=(z,y)\to P_{\infty_\mp}, \end{aligned} \tag{3.40}$$

$$\begin{aligned} i(z-v(x,t)v^*(x,t))+2iv(x,t)\phi(P,x,t) &\underset{z\to 0}{=} O(1)\\ &\text{as } P=(z,y)\to P_{0,\mp}, \end{aligned} \tag{3.41}$$

$$\begin{aligned} i(z^{-1}-u(x_0,s)u^*(x_0,s))+2iz^{-1}u(x_0,s)\phi(P,x_0,s) &\underset{z\to 0}{=} \pm iz^{-1}+O(1)\\ &\text{as } P=(z,y)\to P_{0,\mp}. \end{aligned} \tag{3.42}$$

□

In some of the following considerations it is appropriate to assume that $\mathcal{K}_n$ is nonsingular and hence we then assume

$$E_m \neq E_{m'} \text{ for } m\neq m',\ m,m'=0,\dots,2n+1 \tag{3.43}$$

in addition to (3.2).

Next, we turn to Dubrovin-type equations for $\mu_j(x,t)$, $\nu_j(x,t)$, $j=1,\dots,n$, that is, we derive the nonlinear first-order system of partial differential equations governing their (x,t)-variation.

Lemma 3.5. *Let* $n\in\mathbb{N}$. *Assume* (2.1), (2.8), (2.9), *and* (3.2) *and suppose that the zeros* $\{\mu_j(x,t)\}_{j=1,\dots,n}$ *of* $F_n(\,\cdot\,,x,t)$ *remain distinct for* $(x,t)\in\widetilde\Omega_\mu$, *where* $\widetilde\Omega_\mu\subseteq\mathbb{R}^2$ *is open and connected. Then* $\{\mu_j(x,t)\}_{j=1,\dots,n}$ *satisfies the following system of differential equations*

$$\mu_{j,x}(x,t) = 2iy(\hat\mu_j(x,t))\prod_{\substack{\ell=1\\ \ell\neq j}}^n (\mu_j(x,t)-\mu_\ell(x,t))^{-1}, \tag{3.44}$$

$$\mu_{j,t}(x,t) = (-1)^n g_{n+1}^{-1}\bigg(\prod_{\substack{k=1\\ k\neq j}}^n \mu_k(x,t)\bigg) 2iy(\hat\mu_j(x,t))\prod_{\substack{\ell=1\\ \ell\neq j}}^n (\mu_j(x,t)-\mu_\ell(x,t))^{-1},$$

$$j=1,\dots,n,\ (x,t)\in\widetilde\Omega_\mu. \tag{3.45}$$

Next, assume $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\{\hat\mu_j(x_0,t_0)\}_{j=1,\dots,n}\subset\mathcal{K}_n, \tag{3.46}$$

where $\{\mu_j(x_0,t_0)\}_{j=1,\dots,n}$ remain distinct and distinct from zero. Then there exists an open and connected set $\Omega_\mu\subseteq\mathbb{R}^2$, with $(x_0,t_0)\in\Omega_\mu$, such that the initial value problem (3.44)–(3.46) *has a unique solution $\{\hat\mu_j(x,t)\}_{j=1,\dots,n}$ satisfying*

$$\hat\mu_j\in C^\infty(\Omega_\mu,\mathcal{K}_n),\quad j=1,\dots,n. \tag{3.47}$$

For the zeros $\{\nu_j(x,t)\}_{j=1,\dots,n}$ of $H_n(\,\cdot\,,x,t)$ identical statements hold with μ replaced by ν, $\widetilde\Omega_\mu$ by $\widetilde\Omega_\nu$, etc. In particular, $\{\hat\nu_j(x,t)\}_{j=1,\dots,n}$ satisfies

$$\nu_{j,x}(x,t)=2iy(\hat\nu_j(x,t))\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\nu_j(x,t)-\nu_\ell(x,t))^{-1}, \tag{3.48}$$

$$\nu_{j,t}(x,t)=(-1)^n g_{n+1}^{-1}\bigg(\prod_{\substack{k=1\\ k\neq j}}^n\nu_k(x,t)\bigg)2iy(\hat\nu_j(x,t))\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\nu_j(x,t)-\nu_\ell(x,t))^{-1},$$

$$j=1,\dots,n,\ (x,t)\in\widetilde\Omega_\nu. \tag{3.49}$$

Proof. Equations (2.5), (2.10), and (3.6) imply

$$F_{n,x}(\mu_j)=f_0(-\mu_{j,x})\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell)=4ivG_{n+1}(\mu_j)=4ivy(\hat\mu_j). \tag{3.50}$$

Using $f_0=-2v$ by (2.21), one concludes (3.44). Similarly, one derives from (2.5), (2.13), and (3.6),

$$F_{n,t}(\mu_j)=f_0(-\mu_{j,t})\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell)=(4iu/\mu_j)G_{n+1}(\mu_j)=(4iu/\mu_j)y(\hat\mu_j). \tag{3.51}$$

Since

$$-4iu/f_0=2if_n/(f_0g_{n+1})=2i(-1)^n\Big(\prod_{k=1}^n\mu_k\Big)/g_{n+1} \tag{3.52}$$

by (2.21) and (2.5), one arrives at (3.45). Equations (3.48) and (3.49) are derived analogously. In order to conclude (3.47), one first needs to investigate the case where $\hat\mu_j(x,t)$ hits one of the branch points $(E_m,0)\in\mathcal{B}(\mathcal{K}_n)$ and hence the right-hand sides of (3.44) and (3.45) vanish. Thus we suppose that

$$\mu_{j_0}(x,t)\to E_{m_0}\text{ as }(x,t)\to(\tilde x_0,\tilde t_0) \tag{3.53}$$

for some $j_0\in\{1,\dots,n\}$, $m_0\in\{0,\dots,2n+1\}$ and some $(\tilde x_0,\tilde t_0)\in\Omega_\mu$. Introducing

$$\zeta_{j_0}(x,t)=(\mu_{j_0}(x,t)-E_{m_0})^{1/2},\quad \mu_{j_0}(x,t)=E_{m_0}+\zeta_{j_0}(x,t)^2 \tag{3.54}$$

for (x,t) in an open neighborhood of $(\tilde x_0,\tilde t_0)\in\Omega_\mu$, equations (3.44) and (3.45) become

$$\zeta_{j_0,x}(x,t)\underset{(x,t)\to(\tilde x_0,\tilde t_0)}{=}2i\bigg(\prod_{\substack{m=0\\ m\neq m_0}}^{2n+1}(E_{m_0}-E_m)\bigg)^{1/2}\bigg(\prod_{\substack{k=1\\ k\neq j_0}}^n(E_{m_0}-\mu_k(x,t))^{-1}\bigg)\times$$

$$\times (1 + O(\zeta_{j_0}(x,t)^2)), \tag{3.55}$$

$$\zeta_{j_0,t}(x,t) \underset{(x,t)\to(\tilde{x}_0,\tilde{t}_0)}{=} 2i\bigg(\prod_{\substack{m=0\\ m\neq m_0}}^{2n+1}(E_{m_0}-E_m)\bigg)^{1/2}\bigg(\prod_{\substack{k=1\\ k\neq j_0}}^{n}(E_{m_0}-\mu_k(x,t))^{-1}\bigg)\times$$
$$\times\bigg(\prod_{\substack{\ell=1\\ \ell\neq j_0}}^{n}\mu_\ell(x,t)\bigg)(1+O(\zeta_{j_0}(x,t)^2)). \tag{3.56}$$

Since by hypothesis the right-hand sides of (3.55) and (3.56) are nonvanishing, one arrives at (3.47). □

Next we derive a few trace formulas involving u, v, u^*, v^* and some of their x-derivatives in terms of $\mu_j(x,t)$ and $\nu_j(x,t)$.

Lemma 3.6. *Let $n \in \mathbb{N}$ and assume* (2.1), (2.8), (2.9), *and* (3.2). *Then*

$$i\frac{v_x(x,t)}{v(x,t)} + 2v(x,t)v^*(x,t) - 2c_1 = 2\sum_{j=1}^n \mu_j(x,t), \tag{3.57}$$

$$i\frac{v_x(x,t)}{v(x,t)} - 2v(x,t)v^*(x,t) = -i\sum_{j=1}^n \frac{\mu_{j,x}(x,t)}{\mu_j(x,t)} + \frac{2(-1)^n g_{n+1}}{\prod_{j=1}^n \mu_j(x,t)}, \tag{3.58}$$

$$\frac{v(x,t)}{u(x,t)} = \frac{(-1)^n g_{n+1}}{\prod_{j=1}^n \mu_j(x,t)}, \tag{3.59}$$

$$i\frac{v^*_x(x,t)}{v^*(x,t)} - 2v(x,t)v^*(x,t) + 2c_1 = -2\sum_{j=1}^n \nu_j(x,t), \tag{3.60}$$

$$i\frac{v^*_x(x,t)}{v^*(x,t)} + 2v(x,t)v^*(x,t) = -i\sum_{j=1}^n \frac{\nu_{j,x}(x,t)}{\nu_j(x,t)} - \frac{2(-1)^n g_{n+1}}{\prod_{j=1}^n \nu_j(x,t)}, \tag{3.61}$$

$$\frac{v^*(x,t)}{u^*(x,t)} = \frac{(-1)^n g_{n+1}}{\prod_{j=1}^n \nu_j(x,t)}. \tag{3.62}$$

Here

$$c_1 = -\frac{1}{2}\sum_{m=0}^{2n+1} E_m \tag{3.63}$$

and $g_{n+1} = \big(\prod_{m=0}^{2n+1} E_m\big)^{1/2}$ *has been introduced in* (3.9) *and* (3.10).

Proof. Equations (3.57) and (3.60) follow from (2.5), (2.7) by comparing powers of z^n and z^{n-1}, using (2.21). (3.58) and (3.61) follow from taking $z = 0$ in (2.10) and (2.12), using again (2.21). Finally, (3.59) and (3.62) follow from $f_n = f_0 \prod_{j=1}^n(-\mu_j)$, $h_n = h_0\prod_{j=1}^n(-\nu_j)$ and (2.21). □

While we are not explicitly introducing the hierarchy of massive Thirring equations in this paper, we note that Dubrovin-type equations such as (3.44), (3.45) combined with trace formulas for u, v, u^*, v^* in terms of $\mu_j(x,t)$, enable one to discuss such a hierarchy following the approach outlined in [21].

Up to this point we assumed the zero curvature equations (2.8) and (2.9), or equivalently, (2.10)–(2.15) and as a consequence, derived the corresponding algebro-geometric formalism. In the remainder of this section we will study the algebro-geometric initial value problem, that is, starting from the Dubrovin equations (3.44)–(3.46) and the trace formulas (3.57)–(3.59), derive (2.10)–(2.15), and hence the zero curvature equations (2.8) and (2.9).

We start with an elementary result extending the scaling transformation mentioned in (3.29).

Lemma 3.7. *Assume* (2.1) *and suppose* u, v, u^*, v^* *satisfy the Thirring system* (2.22)–(2.25). *Assume* $A(t) = \exp\big(\int^t ds\, a(s)\big)$, $t \in \mathbb{R}$, *with* $a \in C(\mathbb{R})$ *and consider the time-dependent scaling transformation*

$$(u, v, u^*, v^*) \to (\breve{u}, \breve{v}, \breve{u}^*, \breve{v}^*) = (Au, Av, A^{-1}u^*, A^{-1}v^*). \tag{3.64}$$

Then $\breve{u}, \breve{v}, \breve{u}^*, \breve{v}^*$ *satisfy the corresponding extended massive Thirring system*

$$-i\breve{u}_x(x,t) + 2\breve{v}(x,t) + 2\breve{v}(x,t)\breve{v}^*(x,t)\breve{u}(x,t) = 0, \tag{3.65}$$

$$i\breve{u}^*_x(x,t) + 2\breve{v}^*(x,t) + 2\breve{v}(x,t)\breve{v}^*(x,t)\breve{u}^*(x,t) = 0, \tag{3.66}$$

$$-i\breve{v}_t(x,t) + 2\breve{u}(x,t) + 2\breve{u}(x,t)\breve{u}^*(x,t)\breve{v}(x,t) + ia(t)\breve{v}(x,t) = 0, \tag{3.67}$$

$$i\breve{v}^*_t(x,t) + 2\breve{u}^*(x,t) + 2\breve{u}(x,t)\breve{u}^*(x,t)\breve{v}^*(x,t) - ia(t)\breve{v}^*(x,t) = 0. \tag{3.68}$$

Proof. It suffices to insert (3.64) into the system (2.22)–(2.25). □

In the special case where $u^*(x,t) = \overline{u(x,t)}, v^*(x,t) = \overline{v(x,t)}$, $A(t)$ in Lemma 3.7 is further constrained by $|A(t)| = 1$, $t \in \mathbb{R}$.

Next we provide the basic setup for the algebro-geometric initial value problem. We start from the following assumptions.

Hypothesis 3.8. *Given the hyperelliptic curve* $\mathcal{K}_n$ *in* (3.2), *and the proper choice of the branch of* g_{n+1} *defined by* $g_{n+1} = \big(\prod_{m=0}^{2n+1} E_m\big)^{1/2}$, *according to* (3.9) (*i.e., according to* $\lim_{|z|\to\infty} y(P)z^{-n-1} = \mp\infty$ *as* $P \to P_{\infty_\pm}$), *consider the Dubrovin-type system of differential equations* (3.44), (3.45) *on* Ω_μ, *for some intitial conditions* (3.46). *Here* $\Omega_\mu \subseteq \mathbb{R}^2$ *is assumed to be open and connected, and such that the projections* $\mu_j(x,t)$ *of* $\hat{\mu}_j(x,t)$ *onto* $\mathbb{C}$ *remain distinct and distinct from zero for* $(x,t) \in \Omega_\mu$, *that is,*

$$\mu_j(x,t) \neq \mu_{j'}(x,t) \text{ for } j \neq j',\ j, j' = 1, \dots, n,\ (x,t) \in \Omega_\mu, \tag{3.69}$$

$$\{\mu_j(x,t)\}_{j=1,\dots,n} \cap \{0\} = \emptyset, \quad (x,t) \in \Omega_\mu. \tag{3.70}$$

Assuming Hypothesis 3.8 in the following, we will next define u, v, u^*, v^* and the polynomials F_n, G_{n+1}, H_n in the following steps (S1)–(S4).

(S1). Use the trace formulas (3.57)–(3.59) on Ω_μ, that is,

$$i\frac{v_x(x,t)}{v(x,t)} + 2v(x,t)v^*(x,t) - 2c_1 = 2\sum_{j=1}^n \mu_j(x,t), \tag{3.71}$$

$$i\frac{v_x(x,t)}{v(x,t)} - 2v(x,t)v^*(x,t) = -i\sum_{j=1}^n \frac{\mu_{j,x}(x,t)}{\mu_j(x,t)} + \frac{2(-1)^n g_{n+1}}{\prod_{j=1}^n \mu_j(x,t)}, \tag{3.72}$$

$$u(x,t) = (-1)^n g_{n+1}^{-1} v(x,t) \prod_{j=1}^n \mu_j(x,t), \quad (x,t) \in \Omega_\mu, \tag{3.73}$$

to define $u(x,t), v(x,t), v^*(x,t)$ on Ω_μ up to a possibly t-dependent multiple factor according to the scale transformation described in Lemma 3.7.

(S2). Define the polynomial $F_n(z,x,t)$ on $\mathbb{C}\times\Omega_\mu$ of degree n with respect to z by

$$F_n(z,x,t) = -2v(x,t)\prod_{j=1}^n (z-\mu_j(x,t)), \quad (z,x,t)\in\mathbb{C}\times\Omega_\mu \tag{3.74}$$

and define the polynomial $G_{n+1}(z,x,t)$ on $\mathbb{C}\times\Omega_\mu$ of degree $n+1$ with respect to z by

$$F_{n,x}(z,x,t) = -2i(v(x,t)v^*(x,t)-z)F_n(z,x,t) + 4iv(x,t)G_{n+1}(z,x,t), \quad (z,x,t)\in\mathbb{C}\times\Omega_\mu. \tag{3.75}$$

One then verifies from

$$2iy(\hat\mu_j) = \mu_{j,x}\prod_{\substack{k=1\\k\neq j}}^n (\mu_j-\mu_k) = \frac{F_{n,x}(\mu_j)}{2v}, \quad j=1,\dots,n \tag{3.76}$$

and (3.75) that

$$y(\hat\mu_j(x,t)) = \frac{F_{n,x}(\mu_j(x,t)),x,t)}{4iv(x,t)} = G_{n+1}(\mu_j(x,t),x,t), \quad j=1,\dots,n,\ (x,t)\in\Omega_\mu \tag{3.77}$$

and hence

$$\big(G_{n+1}(z,x,t)^2 - R_{2n+2}(z)\big)\big|_{z=\mu_j(x,t)} = 0, \quad j=1,\dots,n,\ (x,t)\in\Omega_\mu. \tag{3.78}$$

(S3). Taking $z=0$ in (3.75), using (3.74), results in

$$\frac{2(-1)^nG_{n+1}(0)}{\prod_{j=1}^n\mu_j} = i\frac{v_x}{v} - 2vv^* + i\sum_{j=1}^n\frac{\mu_{j,x}}{\mu_j} \tag{3.79}$$

and hence a comparison with (3.72) yields

$$G_{n+1}(0,x,t) = g_{n+1} = \bigg(\prod_{m=0}^{2n+1}E_m\bigg)^{1/2} \tag{3.80}$$

and thus,

$$\big(G_{n+1}(z,x,t)^2 - R_{2n+2}(z)\big)\big|_{z=0} = 0, \quad (x,t)\in\Omega_\mu. \tag{3.81}$$

Because of (3.78) and (3.81) we can define a polynomial $H_n(z,x,t)$ on $\mathbb{C}\times\Omega_\mu$ of degree n with respect to z by

$$G_{n+1}(z,x,t)^2 - R_{2n+2}(z) = zF_n(z,x,t)H_n(z,x,t), \quad (z,x,t)\in\mathbb{C}\times\Omega_\mu. \tag{3.82}$$

(S4). Given $H_n(z,x,t)$ we finally define $u^*(x,t)$ on Ω_μ by

$$u^*(x,t) = \frac{H_n(0,x,t)}{2g_{n+1}}, \quad (x,t)\in\Omega_\mu, \tag{3.83}$$

Again $u^*(x,t)$ is defined up to a possibly t-dependent factor in accordance with Lemma 3.7.

The algebro-geometric initial value problem now can be solved as follows.

Theorem 3.9. *Assume Hypothesis* 3.8*, define* u, v, u^*, v^* *and* F_n, G_{n+1}, H_n *as in* $(S1)-(S4)$ *and let* $(x,t) \in \Omega_\mu$*. Then there exists a function* $a \in C^\infty(\Omega_\mu)$*, independent of* x $(a_x|_{\Omega_\mu} = 0)$*, such that*

$$F_{n,x}(z,x,t) = -2i(v(x,t)v^*(x,t) - z)F_n(z,x,t) + 4iv(x,t)G_{n+1}(z,x,t), \tag{3.84}$$

$$G_{n+1,x}(z,x,t) = 2izv^*(x,t)F_n(z,x,t) + 2izv(x,t)H_n(z,x,t), \tag{3.85}$$

$$H_{n,x}(z,x,t) = 2i(v(x,t)v^*(x,t) - z)H_n(z,x,t) + 4iv^*(x,t)G_{n+1}(z,x,t), \tag{3.86}$$

$$\begin{aligned} F_{n,t}(z,x,t) &= -2i(u(x,t)u^*(x,t) - z^{-1})F_n(z,x,t) + a(t)F_n(z,x,t) \\ &\quad + 4iz^{-1}u(x,t)G_{n+1}(z,x,t), \end{aligned} \tag{3.87}$$

$$G_{n+1,t}(z,x,t) = 2iu^*(x,t)F_n(z,x,t) + 2iu(x,t)H_n(z,x,t), \tag{3.88}$$

$$\begin{aligned} H_{n,t}(z,x,t) &= 2i(u(x,t)u^*(x,t) - z^{-1})H_n(z,x,t) - a(t)H_n(z,x,t) \\ &\quad + 4iz^{-1}u^*(x,t)G_{n+1}(z,x,t). \end{aligned} \tag{3.89}$$

In particular, u, v, u^*, v^* *satisfy the extended massive Thirring system* (3.65)–(3.68) *on* Ω_μ*,*

$$-iu_x(x,t) + 2v(x,t) + 2v(x,t)v^*(x,t)u(x,t) = 0, \tag{3.90}$$

$$iu^*_x(x,t) + 2v^*(x,t) + 2v(x,t)v^*(x,t)u^*(x,t) = 0, \tag{3.91}$$

$$-iv_t(x,t) + 2u(x,t) + 2u(x,t)u^*(x,t)v(x,t) + ia(t)v(x,t) = 0, \tag{3.92}$$

$$iv^*_t(x,t) + 2u^*(x,t) + 2u(x,t)u^*(x,t)v^*(x,t) - ia(t)v^*(x,t) = 0. \tag{3.93}$$

Proof. Define the polynomial

$$\begin{aligned} P_n(z,x,t) = 2izv^*(x,t)F_n(z,x,t) + 2izv(x,t)H_n(z,x,t) - G_{n+1,x}(z,x,t), \\ (z,x,t) \in \mathbb{C} \times \Omega_\mu. \end{aligned} \tag{3.94}$$

Using (3.77) and $2G_{n+1}G_{n+1,x} = z(F_{n,x}H_n + F_nH_{n,x})$ (by differentiating (3.82) with respect to x) one then computes

$$\begin{aligned} G_{n+1}(\mu_j)P_n(\mu_j) &= 2i\mu_j vH_n(\mu_j)G_{n+1}(\mu_j) - G_{n+1}(\mu_j)G_{n+1,x}(\mu_j) \\ &= \frac{1}{2}\mu_j H_n(\mu_j)F_{n,x}(\mu_j) - \frac{1}{2}\mu_j F_{n,x}(\mu_j)H_n(\mu_j) = 0, \\ &\qquad j = 1, \dots, n. \end{aligned} \tag{3.95}$$

In order to investigate the leading-order term with respect to z of $P_n(z)$ we first study the leading-order z-behavior of $F_n(z), G_{n+1}(z)$, and $H_n(z)$. Writing (cf. (2.5)–(2.7))

$$F_n(z) = \sum_{j=0}^{n} f_{n-j}z^j, \quad H_n(z) = \sum_{j=0}^{n} h_{n-j}z^j, \quad G_{n+1}(z) = \sum_{j=0}^{n+1} g_{n+1-j}z^j, \; g_0 = 1, \tag{3.96}$$

a comparison of leading powers with respect to z in (3.74), (3.75), and (3.82) yields

$$f_0 = -2v, \tag{3.97}$$

$$g_0 = 1, \tag{3.98}$$

$$v_x + 2iv^2v^* + if_1 + 2ig_1 = 0, \tag{3.99}$$

$$2g_1 + 2vh_0 + \sum_{m=0}^{2n+1} E_m = 0. \tag{3.100}$$

Since (3.71) can be rewritten in the form

$$f_1 = iv_x + 2v^2v^* + v\sum_{m=0}^{2n+1} E_m, \tag{3.101}$$

a comparison of (3.99) and (3.101) then yields

$$g_1 = -2vv^* - \frac{1}{2}\sum_{m=0}^{2n+1} E_m \tag{3.102}$$

and hence

$$h_0 = 2v^*. \tag{3.103}$$

Insertion of (3.97), (3.98), and (3.103) into (3.94) then yields

$$P_n(z,x,t) = O(z^n) \text{ as } |z| \to \infty. \tag{3.104}$$

Thus, (3.95) and (3.104) prove

$$P_n(z,x,t) = b(x,t)F_n(z,x,t), \quad (z,x,t) \in \mathbb{C} \times \Omega_\mu \tag{3.105}$$

for some $b \in C^\infty(\Omega_\mu)$ (independent of z), implying

$$G_{n+1,x}(z,x,t) = 2izv^*(x,t)F_n(z,x,t) + 2izv(x,t)H_n(z,x,t) - b(x,t)F_n(z,x,t), \quad (z,x,t) \in \mathbb{C} \times \Omega_\mu. \tag{3.106}$$

Taking $z = 0$ in (3.106), observing that $G_{n+1}(0,x,t)$ is independent of $(x,t) \in \Omega_\mu$ by (3.80), then shows that

$$0 = -b(x,t)F_n(0,x,t), \quad (x,t) \in \Omega_\mu, \tag{3.107}$$

and hence $b = 0$ on Ω_μ because of (3.70). Thus,

$$G_{n+1,x}(z,x,t) = 2izv^*(x,t)F_n(z,x,t) + 2izv(x,t)H_n(z,x,t), \quad (z,x,t) \in \mathbb{C} \times \Omega_\mu. \tag{3.108}$$

Differentiating (3.82) with respect to x, inserting (3.75) and (3.108), then yields

$$H_{n,x}(z,x,t) = 2i(v(x,t)v^*(x,t) - z)H_n(z,x,t) + 4iv^*(x,t)G_{n+1}(z,x,t), \quad (z,x,t) \in \mathbb{C} \times \Omega_\mu \tag{3.109}$$

and we proved (3.84)–(3.86).

Next, combining (3.45), (3.73), and (3.77) one computes

$$\begin{aligned} F_{n,t}(\mu_j) &= 2v\frac{(-1)^n}{g_{n+1}}\bigg(\prod_{\substack{k=1\\k\neq j}}^n \mu_k\bigg)2iy(\hat\mu_j) = \frac{(-1)^n}{g_{n+1}}\bigg(\prod_{\substack{k=1\\k\neq j}}^n \mu_k\bigg)\frac{4iv}{\mu_j}G_{n+1}(\mu_j) \\ &= \frac{4iu}{\mu_j}G_{n+1}(\mu_j), \quad j = 1,\dots,n. \end{aligned} \tag{3.110}$$

Since clearly

$$F_{n,t}(z) - \big(-2i(uu^* - z^{-1})F_n(z) + 4iz^{-1}uG_{n+1}(z)\big) = O(z^n) \text{ as } |z| \to \infty, \tag{3.111}$$

a comparison of (3.110) and (3.111) yields

$$\begin{aligned} &F_{n,t}(z,x,t) - \big(-2i(u(x,t)u^*(x,t) - z^{-1})F_n(z,x,t)) + 4iz^{-1}uG_{n+1}(z,x,t)\big) \\ &= a(x,t)F_n(z,x,t), \quad (z,x,t) \in \mathbb{C} \times \Omega_\mu \end{aligned} \tag{3.112}$$

for some $a \in C^\infty(\Omega_\mu)$ (independent of z), and hence (3.87) (except for $a_x = 0$). A comparison of powers of z^n in (3.112) then yields (3.92).

Next, we restrict Ω_μ a bit further and introduce $\widetilde{\Omega}_\mu \subseteq \Omega_\mu$ by the requirement that $\mu_j(x,t)$ remain distinct and also distinct from $\{E_m\}_{m=0,\dots,2n+1} \cup \{0\}$ for $(x,t) \in \widetilde{\Omega}_\mu$, that is, we suppose

$$\mu_j(x,t) \neq \mu_{j'}(x,t) \text{ for } j \neq j', \ j,j' = 1,\dots,n, \ (x,t) \in \widetilde{\Omega}_\mu, \tag{3.113}$$

$$\{\mu_j(x,t)\}_{j=1,\dots,n} \cap \{\{E_m\}_{m=0,\dots,2n+1} \cup \{0\}\} = \emptyset, \quad (x,t) \in \widetilde{\Omega}_\mu. \tag{3.114}$$

Differentiating (3.82) with respect to t inserting (3.112) then yields

$$\begin{aligned} 2G_{n+1}(z)G_{n+1,t}(z) = {} & zF_n(z)\big(-2i(uu^* - z^{-1})H_n(z) + aH_n(z) + H_{n,t}(z)\big) \\ & + 4iuG_{n+1}H_n(z). \end{aligned} \tag{3.115}$$

Since the zeros of F_n and G_{n+1} are disjoint by hypothesis (3.114) (cf. also (3.77)), $zH_{n,t}(z)$ necessarily must be of the form

$$\begin{aligned} zH_{n,t}(z,x,t) = {} & 2i(zu(x,t)u^*(x,t) - 1)H_n(z,x,t) - a(x,t)zH_n(z,x,t) \\ & + 4id(x,t)G_{n+1}(z,x,t), \quad (z,x,t) \in \mathbb{C} \times \widetilde{\Omega}_\mu \end{aligned} \tag{3.116}$$

for some $d \in C^\infty(\widetilde{\Omega}_\mu)$ (independent of z) and (3.116) inserted into (3.115) then yields

$$G_{n+1}(z,x,t) = 2iu(x,t)H_n(z,x,t) + 2id(x,t)F_n(z,x,t), \quad (z,x,t) \in \mathbb{C} \times \widetilde{\Omega}_\mu. \tag{3.117}$$

Since

$$u(x,t) = -\frac{F_n(0,x,t)}{2g_{n+1}}, \quad (x,t) \in \widetilde{\Omega}_\mu, \tag{3.118}$$

combining (3.73) and (3.74), taking $z = 0$ in (3.117), observing (3.80) and (3.83), results in

$$0 = 2iu2g_{n+1}u^* + 2id(-2g_{n+1}u) \tag{3.119}$$

and hence in

$$d(x,t) = u^*(x,t), \quad (x,t) \in \widetilde{\Omega}_\mu. \tag{3.120}$$

Using property (3.47), (3.116)–(3.120) then extend by continuity from $\widetilde{\Omega}_\mu$ to Ω_μ. This proves (3.88) and (3.89) (except for $a_x = 0$). A comparison of powers of z^n in (3.89) then yields (3.93). Taking $z = 0$ in (3.84) and (3.86), observing (3.83) and (3.118), then proves (3.90) and (3.91). Finally, computing the partial t-derivative of $F_{n,x}$ and separately the partial x-derivative of $F_{n,t}$, utilizing (3.84), (3.85), (3.87), (3.88), and (3.90)–(3.93) then shows

$$F_{n,xt}(z,x,t) - F_{n,tx}(z,x,t) = -a_x(x,t)F_n(z,x,t), \quad (z,x,t) \in \mathbb{C} \times \Omega_\mu \tag{3.121}$$

and hence

$$a_x(x,t) = 0, \quad (x,t) \in \Omega_\mu. \tag{3.122}$$

□

Remark 3.10. (i) The fact that the system of Dubrovin equations (3.44)–(3.46) cannot uniquely determine the solutions u, v, u^*, v^* of the massive Thirring system (2.22)–(2.25), as is evident from the occurrence of $a(t)$ in (3.92), (3.93), is of course due to the scale covariance displayed explicitly in Lemma 3.7. In particular, once a certain $a(t)$ has been identified, a scaling transformation of the type (3.64) (with $A(t)$ replaced by $1/A(t)$) will restore the extended massive Thirring system (3.90)–(3.93) to it's original form in (2.22)–(2.25).
(ii) For simplicity we formulated Theorem 3.9 in terms of $\{\hat{\mu}_j\}_{j=1,\dots,n}$ and (3.44)–(3.46) only. Of course there exists a completely analogous approach starting with $\{\hat{\nu}_j\}_{j=1,\dots,n}$ and the system (3.48), (3.49) instead.
(iii) Invoking the explicit theta function representations for u, v, u^*, v^* to be proven in Section 4 next (this approach is independent of that used to prove Theorem 3.10), one can extend the principal assertions (3.84)–(3.93) of Theorem 3.10 by continuity to (x,t) lying in a larger set $\Omega \subseteq \mathbb{R}^2$ as long as the divisors $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ remain nonspecial for $(x,t) \in \Omega$ (cf. Theorem 4.4 and Theorem A.7).

4. Theta function representations

In our final section we now derive theta function representations for the principal objects of Section 3, including ϕ, ψ_1, u, v, u^*, v^*. These representations complement the papers by Date [9] and Prikarpatskii and Golod [38], where theta function representations were derived for appropriate symmetric functions associated with auxiliary divisors, but not explicitly for u, v, u^*, v^*. Moreover, we correct some inaccuracies of such formulas in a paper by Bikbaev [6] (which follows a different strategy than ours).

According to our shift in emphasis from the Baker–Akhiezer vector Ψ to our fundamental meromorphic function ϕ on $\mathcal{K}_n$, we next aim at the theta function representation of ϕ.

Assuming $\mathcal{K}_n$ to be nonsingular for the remainder of this section (i.e., $E_m \neq E_{m'}$ for $m \neq m'$, $m, m' = 0, \dots, 2n+1$) and $n \in \mathbb{N}$ for simplicity (to avoid repeated case distinctions), we next recall the formula for a normal differential of the third kind, which has simple poles at $P_{0,-}$ and P_{∞_-}, corresponding residues $+1$ and -1, vanishing a-periods, and is holomorphic otherwise on $\mathcal{K}_n$. One computes

$$\omega^{(3)}_{P_{0,-},P_{\infty_-}} = \frac{y + y_{0,-}}{2z}\frac{dz}{y} + \frac{\prod_{j=1}^n (z-\lambda_j)dz}{2y}, \quad P_{0,-} = (0, y_{0,-}) = (0, -g_{n+1}), \tag{4.1}$$

where $\{\lambda_j\}_{j=1,\dots,n}$ are uniquely determined by the normalization

$$\int_{a_j} \omega^{(3)}_{P_{0,-},P_{\infty_-}} = 0, \quad j = 1, \dots, n. \tag{4.2}$$

The explicit formula (4.1) then implies (using the local coordinate $\zeta = z$ near $P_{0,\mp}$)

$$\omega^{(3)}_{P_{0,-},P_{\infty_-}}(P) \underset{\zeta\to 0}{=} \begin{Bmatrix} \zeta^{-1} \\ 0 \end{Bmatrix} d\zeta \pm \left(\sum_{q=0}^\infty (q+1)\omega^0_{q+1}\zeta^q\right) d\zeta \text{ as } P \to P_{0,\mp}, \tag{4.3}$$

and similarly (using the local coordinate $\zeta = 1/z$ near $P_{\infty_\mp}$),

$$\omega^{(3)}_{P_{0,-},P_{\infty_-}}(P) \underset{\zeta\to 0}{=} \begin{Bmatrix} -\zeta^{-1} \\ 0 \end{Bmatrix} d\zeta \pm \left(\sum_{q=0}^\infty (q+1)\omega^\infty_{q+1}\zeta^q\right) d\zeta \text{ as } P \to P_{\infty_\mp}. \tag{4.4}$$

In particular,

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} \ln(\zeta) \\ 0 \end{Bmatrix} + \omega_0^{0,\mp} \pm \omega_1^0 \zeta \pm \omega_2^0 \zeta^2 + O(\zeta^3) \text{ as } P \to P_{0,\mp}, \tag{4.5}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} -\ln(\zeta) \\ 0 \end{Bmatrix} + \omega_0^{\infty_\mp} \pm \omega_1^\infty \zeta \pm \omega_2^\infty \zeta^2 + O(\zeta^3) \text{ as } P \to P_{\infty_\mp}. \tag{4.6}$$

Here $Q_0 \in \mathcal{B}(\mathcal{K}_n)$ is an appropriate base point and we agree to choose the same path of integration from Q_0 to P in all Abelian integrals in this section.

A comparison of (4.3), (4.4) with (4.1), (A.12), and (A.14) then yields

$$\omega_1^0 = \frac{1}{4}\sum_{m=0}^{2n+1} \frac{1}{E_m} - \frac{(-1)^n}{2g_{n+1}} \prod_{j=1}^{n} \lambda_j, \tag{4.7}$$

$$\omega_1^\infty = -\frac{1}{4}\sum_{m=0}^{2n+1} E_m + \frac{1}{2}\sum_{j=1}^{n} \lambda_j. \tag{4.8}$$

Next, we intend to go a step further and derive alternative expressions for the expansion coefficients $\omega_0^{0,\pm}$, ω_1^0, $\omega_0^{\infty_\pm}$, and ω_1^∞ in (4.5) and (4.6). To begin these calculations we first recall the notion of a nonsingular odd half-period Υ defined by

$$2\underline{\Upsilon} = 0 \pmod{L_n}, \quad \theta(\underline{\Upsilon}) = 0, \quad \frac{\partial \theta(\underline{z})}{\partial z_j}\bigg|_{\underline{z}=\underline{\Upsilon}} \neq 0 \text{ for some } j \in \{1,\dots,n\}. \tag{4.9}$$

Discussions of even and odd half-periods (singular and nonsingular ones) can be found, for instance, in [16, p. 12–15], [33]. In addition, it is convenient to introduce the notation

$$\underline{\Delta}_0 = \underline{A}_{Q_0}(P_{0,+}), \quad \underline{\Delta}_\infty = \underline{A}_{Q_0}(P_{\infty_+}), \tag{4.10}$$

$$\underline{W}_1^0 = (W_{1,1}^0,\dots,W_{1,n}^0),\ W_{1,j}^0 = \frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_{0,+},0} = \frac{c_j(1)}{g_{n+1}},\ j=1,\dots,n, \tag{4.11}$$

$$\underline{W}_2^0 = (W_{2,1}^0,\dots,W_{2,n}^0),\ W_{2,j}^0 = \frac{c_j(1)}{4g_{n+1}}\sum_{m=0}^{2n+1} E_m^{-1} + \frac{c_j(2)}{2g_{n+1}},\ j=1,\dots,n, \tag{4.12}$$

$$\underline{W}_1^\infty = (W_{1,1}^\infty,\dots,W_{1,n}^\infty),\ W_{1,j}^\infty = \frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_{\infty_+},0} = c_j(n),\ j=1,\dots,n, \tag{4.13}$$

$$\underline{W}_2^\infty = (W_{2,1}^\infty,\dots,W_{2,n}^\infty),\ W_{2,j}^\infty = \frac{c_j(n)}{4}\sum_{m=0}^{2n+1} E_m + \frac{c_j(n-1)}{2},\ j=1,\dots,n. \tag{4.14}$$

Moreover, we abbreviate directional derivatives of f in the direction of $\underline{W} = (W_1,\dots,W_n) \in \mathbb{C}^n$ by

$$(\partial_{\underline{W}} f)(\underline{z}) = \sum_{j=1}^{n} W_j \frac{\partial f}{\partial z_j}(\underline{z}), \quad (\partial^2_{\underline{W}} f)(\underline{z}) = \sum_{j,k=1}^{n} W_j W_k \frac{\partial^2 f}{\partial z_j \partial z_k}(\underline{z}), \tag{4.15}$$
$$\underline{z} = (z_1,\dots,z_n) \in \mathbb{C}^n.$$

Then one obtains the following result.

Lemma 4.1. *Given* (4.1)–(4.13) *one obtains*

$$\omega_0^{0,+} = \ln\left(\frac{\theta(\underline{\Upsilon}-2\underline{\Delta}_0)\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}{\theta(\underline{\Upsilon}-\underline{\Delta}_0-\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_0)}\right), \tag{4.16}$$

$$\omega_0^{0,-} = \ln\left(\frac{(\partial_{\underline{W}_1^0}\theta)(\underline{\Upsilon})\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}{\theta(\underline{\Upsilon}+\underline{\Delta}_0-\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_0)}\right), \tag{4.17}$$

$$\omega_1^0 = -\partial_{\underline{W}_1^0}\ln(\theta(\underline{\Upsilon}+\underline{\Delta}_0-\underline{\Delta}_\infty)) + \frac{(\partial_{\underline{W}_2^0}\theta)(\underline{\Upsilon}) + 2^{-1}(\partial^2_{\underline{W}_1^0}\theta)(\underline{\Upsilon})}{(\partial_{\underline{W}_1^0}\theta)(\underline{\Upsilon})} \tag{4.18}$$

$$= \partial_{\underline{W}_1^0}\ln\left(\frac{\theta(\underline{\Upsilon}-2\underline{\Delta}_0)}{\theta(\underline{\Upsilon}-\underline{\Delta}_0-\underline{\Delta}_\infty)}\right), \tag{4.19}$$

$$\omega_0^{\infty+} = \ln\left(\frac{\theta(\underline{\Upsilon}-\underline{\Delta}_0-\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_\infty}{\theta(\underline{\Upsilon}-2\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_0)}\right), \tag{4.20}$$

$$\omega_0^{\infty-} = -\ln\left(\frac{(\partial_{\underline{W}_1^\infty}\theta)(\underline{\Upsilon})\theta(\underline{\Upsilon}-\underline{\Delta}_0)}{\theta(\underline{\Upsilon}-\underline{\Delta}_0+\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}\right), \tag{4.21}$$

$$\omega_1^\infty = \partial_{\underline{W}_1^\infty}\ln(\theta(\underline{\Upsilon}-\underline{\Delta}_0+\underline{\Delta}_\infty)) - \frac{(\partial_{\underline{W}_2^\infty}\theta)(\underline{\Upsilon}) + 2^{-1}(\partial^2_{\underline{W}_1^\infty}\theta)(\underline{\Upsilon})}{(\partial_{\underline{W}_1^\infty}\theta)(\underline{\Upsilon})} \tag{4.22}$$

$$= \partial_{\underline{W}_1^\infty}\ln\left(\frac{\theta(\underline{\Upsilon}-\underline{\Delta}_0-\underline{\Delta}_\infty)}{\theta(\underline{\Upsilon}-2\underline{\Delta}_\infty)}\right). \tag{4.23}$$

Proof. Abbreviating

$$\underline{w}(P,Q_0) = \underline{\Upsilon} - \underline{A}_{Q_0}(P) + \underline{A}_{Q_0}(Q) \pmod{L_n}, \tag{4.24}$$

one infers from

$$\underline{A}_{Q_0}(P) \underset{\zeta\to 0}{=} \underline{A}_{Q_0}(P_{0,\pm}) \pm \underline{W}_1^0\zeta \pm \underline{W}_2^0\zeta^2 + O(\zeta^3) \text{ as } P\to P_{0,\pm}, \tag{4.25}$$

$$\underline{A}_{Q_0}(P) \underset{\zeta\to 0}{=} \underline{A}_{Q_0}(P_{\infty_\pm}) \pm \underline{W}_1^\infty\zeta \pm \underline{W}_2^\infty\zeta^2 + O(\zeta^3) \text{ as } P\to P_{\infty_\pm} \tag{4.26}$$

(cf. (A.11) and (A.13)), and (4.11), (4.13), that

$$\begin{aligned}\theta(\underline{w}(P,Q)) \underset{\zeta\to 0}{=} &\theta(\underline{w}(P_{0,\pm},Q)) \mp (\partial_{\underline{W}_1^0}\theta)(\underline{w}(P_{0,\pm},Q))\zeta \mp (\partial_{\underline{W}_2^0}\theta)(\underline{w}(P_{0,\pm},Q))\zeta^2\\ &+ 2^{-1}(\partial^2_{\underline{W}_1^0}\theta)(\underline{w}(P_{0,\pm},Q))\zeta^2 + O(\zeta^3) \text{ as } P\to P_{0,\pm},\end{aligned} \tag{4.27}$$

$$\begin{aligned}\theta(\underline{w}(P,Q)) \underset{\zeta\to 0}{=} &\theta(\underline{w}(P_{\infty_\pm},Q)) \mp (\partial_{\underline{W}_1^\infty}\theta)(\underline{w}(P_{\infty_\pm},Q))\zeta \mp (\partial_{\underline{W}_2^\infty}\theta)(\underline{w}(P_{0,\pm},Q))\zeta^2\\ &+ 2^{-1}(\partial^2_{\underline{W}_1^\infty}\theta)(\underline{w}(P_{0,\pm},Q))\zeta^2 + O(\zeta^3) \text{ as } P\to P_{\infty_\pm}.\end{aligned} \tag{4.28}$$

Next, observing the fact that

$$\omega^{(3)}_{P_{0,-},P_{\infty_-}} = d\log\left(\frac{\theta(\underline{w}(\,\cdot\,,P_{0,-}))}{\theta(\underline{w}(\,\cdot\,,P_{\infty_-}))}\right), \tag{4.29}$$

it becomes a straightforward matter deriving (4.16)–(4.23). For simplicity we just focus on the expansion of $\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}$ as $P\to P_{0,\pm}$, the rest is completely analogous. Using

$$\begin{aligned}&\underline{w}(Q_0,P_{0,\pm}) = \underline{\Upsilon}\pm\underline{\Delta}_0, \quad \underline{w}(Q_0,P_{\infty_\pm}) = \underline{\Upsilon}\pm\underline{\Delta}_\infty, \quad \underline{w}(Q,Q) = \underline{\Upsilon},\ Q\in\mathcal{K}_n,\\ &\underline{w}(P_{\infty_\sigma},P_{0,\sigma'}) = \underline{\Upsilon}+\sigma'\underline{\Delta}_0-\sigma\underline{\Delta}_\infty, \quad \underline{w}(P_{0,\sigma'},P_{\infty_\sigma}) = \underline{\Upsilon}-\sigma'\underline{\Delta}_0+\sigma\underline{\Delta}_\infty,\\ &\underline{w}(P_{0,\sigma},P_{0,\sigma'}) = \underline{\Upsilon}+(\sigma'-\sigma)\underline{\Delta}_0, \quad \underline{w}(P_{0,\sigma'},P_{0,\sigma}) = \underline{\Upsilon}+(\sigma-\sigma')\underline{\Delta}_0,\\ &\underline{w}(P_{\infty_\sigma},P_{\infty_{\sigma'}}) = \underline{\Upsilon}+(\sigma'-\sigma)\underline{\Delta}_\infty, \quad \underline{w}(P_{\infty_{\sigma'}},P_{\infty_\sigma}) = \underline{\Upsilon}+(\sigma-\sigma')\underline{\Delta}_\infty,\end{aligned} \tag{4.30}$$

$$\sigma, \sigma' \in \{1, -1\},$$

and (4.27)–(4.29), one computes by comparison with (4.5),

$$\begin{aligned}
&\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} = \int_{Q_0}^{P} d\log\left(\frac{\theta(\underline{w}(P',P_{0,-}))}{\theta(\underline{w}(P',P_{\infty_-}))}\right)\\
&= \ln\left(\frac{\theta(\underline{w}(P,P_{0,-}))}{\theta(\underline{w}(P,P_{\infty_-}))}\right) - \ln\left(\frac{\theta(\underline{w}(Q_0,P_{0,-}))}{\theta(\underline{w}(Q_0,P_{\infty_-}))}\right)\\
&= \ln\left(\frac{\theta(\underline{w}(P,P_{0,-}))}{\theta(\underline{w}(P,P_{\infty_-}))}\right) - \ln\left(\frac{\theta(\underline{\Upsilon}-\underline{\Delta}_0)}{\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}\right)\\
&= \ln\left(\frac{\theta(\underline{\Upsilon}-2\underline{\Delta}_0)\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}{\theta(\underline{\Upsilon}-\underline{\Delta}_0-\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_0)}\right) - \partial_{\underline{W}^0_1}\ln\left(\frac{\theta(\underline{\Upsilon}-2\underline{\Delta}_0)}{\theta(\underline{\Upsilon}-\underline{\Delta}_0-\underline{\Delta}_\infty)}\right)\zeta + O(\zeta^2)\\
&= \omega_0^{0,+} - \omega_1^0\zeta + O(\zeta^2) \text{ as } P \to P_{0,+}. \qquad (4.31)
\end{aligned}$$

This proves (4.16) and (4.19). Similarly, one calculates,

$$\begin{aligned}
&\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} = \ln\left(\frac{\theta(\underline{w}(P,P_{0,-}))}{\theta(\underline{w}(P,P_{\infty_-}))}\right) - \ln\left(\frac{\theta(\underline{\Upsilon}-\underline{\Delta}_0)}{\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}\right)\\
&= \ln(\zeta) + \ln\left(\frac{(\partial_{\underline{W}^0_1}\theta)(\underline{\Upsilon})\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}{\theta(\underline{\Upsilon}+\underline{\Delta}_0-\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_0)}\right) - \partial_{\underline{W}^0_1}\ln(\theta(\underline{\Upsilon}+\underline{\Delta}_0-\underline{\Delta}_\infty))\zeta\\
&\quad + \frac{(\partial_{\underline{W}^0_2}\theta)(\underline{\Upsilon}) + 2^{-1}(\partial^2_{\underline{W}^0_1}\theta)(\underline{\Upsilon})}{(\partial_{\underline{W}^0_1}\theta)(\underline{\Upsilon})}\zeta + O(\zeta^2)\\
&= \ln(\zeta) + \omega_0^{0,-} + \omega_1^0\zeta + O(\zeta^2) \text{ as } P \to P_{0,-}, \qquad (4.32)
\end{aligned}$$

proving (4.17) and (4.18). □

The results of Lemma 4.1 can conveniently be reformulated in terms of theta functions with characteristics associated with the vector $\underline{\Upsilon}$, but we omit further details at this point.

Combining (3.11) and Theorem A.5, the theta function representation of ϕ must be of the form

$$\phi(P,x,t) = C(x,t)\frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{0,-},P_{\infty_-}}\right),$$
$$P \in \mathcal{K}_n,\ (x,t) \in \Omega, \quad (4.33)$$

assuming $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ to be nonspecial for $(x,t) \in \Omega$, where $\Omega \subseteq \mathbb{R}^2$ is open and connected. We refer to Appendix A for our notational conventions concerning Abel maps $\underline{A}_{Q_0}$, $\underline{\alpha}_{Q_0}$ and θ-functions. Here $Q_0 \in \mathcal{K}_n \setminus \{P_{0,\pm}, P_{\infty_\pm}\}$ is a fixed base point which we will always choose among the branch points of $\mathcal{K}_n$ (e.g., $Q_0 = (E_0, 0)$). Indeed, by (3.11), (4.5), (4.6), and Theorem A.5, $\phi(P,x,t)$ and

$$\frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{0,-},P_{\infty_-}}\right) \qquad (4.34)$$

have the same singularity structure with respect to $P \in \mathcal{K}_n$. Moreover, by (A.20), (A.29), and (A.30), the expression (4.34) is single-valued and hence meromorphic on $\mathcal{K}_n$. Nonspecialty of $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ then yields (4.33).

It remains to analyze the function $C(x,t)$ in (4.33) (which is P-independent) and in the course of that we will also obtain the theta function representations of

u, u^*, v, v^*. (The strategy to follow parallels the one used in [22] in connection with algebro-geometric solutions of the AKNS hierarchy.)

In the following it will occasionally be convenient to use a short-hand notation for the arguments of the theta functions in (4.33) and hence we introduce the abbreviation

$$\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \quad \underline{Q} = (Q_1,\dots,Q_n) \in \sigma^n\mathcal{K}_n. \tag{4.35}$$

Next we show that the Abel maps linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$.

Lemma 4.2. *Assume* (2.1), (2.8), (2.9), *and* (3.2), *and* $(x,t),(x_0,t_0) \in \Omega$, *where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose* $\mathcal{K}_n$ *is nonsingular and* $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *are nonspecial for* $(x,t)\in\Omega$. *Then*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}) + 2i\underline{c}(n)(x-x_0) - 2i\underline{c}(1)g_{n+1}^{-1}(t-t_0), \tag{4.36}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_0)}) + 2i\underline{c}(n)(x-x_0) - 2i\underline{c}(1)g_{n+1}^{-1}(t-t_0). \tag{4.37}$$

Proof. Given the expansions (A.11) and (A.13) of ω near $P_{\infty_\pm}$ and $P_{0,\pm}$, (4.36) and (4.37) are standard facts following from Lagrange interpolation results of the type (see, e.g., [21])

$$\begin{aligned} &\sum_{j=1}^n \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell)} = \delta_{k,n}, \\ &\sum_{j=1}^n \frac{\mu_j^{k-1}\big(\prod_{\substack{m=1\\ m\neq j}}^n \mu_m\big)}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell)} = (-1)^{n+1}\delta_{k,1}, \quad k=1,\dots,n. \end{aligned} \tag{4.38}$$

□

In Lemma 3.3 we determined the asymptotic behavior of $\phi(P,x,t)$ as $P\to P_{\infty_\pm}, P_{0,\pm}$ comparing (3.4) with (3.12) and (3.13). Now we will recompute the asymtotics of ϕ starting from (4.33).

Lemma 4.3. *Assume* (2.1), (2.8), (2.9), *and* (3.2). *Moreover, suppose* $\mathcal{K}_n$ *is nonsingular and* $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *are nonspecial for* $(x,t)\in\Omega$, *where* $\Omega\subseteq\mathbb{R}^2$ *is open and connected. Then*

$$\begin{aligned} \phi(P,x,t) \underset{\zeta\to 0}{=} &\, C(x,t)e^{\omega_0^{\infty_-}} \frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)}\zeta^{-1} \\ &+ C(x,t)\omega_1^{\infty}e^{\omega_0^{\infty_-}} \frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)} \\ &- C(x,t)e^{\omega_0^{\infty_-}}\frac{i}{2}\frac{\partial}{\partial x}\left(\frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)}\right) + O(\zeta) \text{ as } P\to P_{\infty_-}, \end{aligned} \tag{4.39}$$

$$\begin{aligned}\phi(P,x,t) \underset{\zeta\to 0}{=} & \, C(x,t)e^{\omega_0^{\infty_+}}\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t)))} \\ & - C(x,t)\omega_1^{\infty}e^{\omega_0^{\infty_+}}\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t)))}\zeta \\ & + C(x,t)e^{\omega_0^{\infty_+}}\frac{i}{2}\frac{\partial}{\partial x}\left(\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t)))}\right)\zeta + O(\zeta^2) \ \textit{as } P\to P_{\infty_+},\end{aligned} \tag{4.40}$$

$$\begin{aligned}\phi(P,x,t) \underset{\zeta\to 0}{=} & \, C(x,t)e^{\omega_0^{0,-}}\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t)))}\zeta \\ & + C(x,t)\omega_1^{0}e^{\omega_0^{0,-}}\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t)))}\zeta^2 \\ & + C(x,t)e^{\omega_0^{0,-}}\frac{i}{2}\frac{\partial}{\partial t}\left(\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t)))}\right)\zeta^2 + O(\zeta^3) \ \textit{as } P\to P_{0,-},\end{aligned} \tag{4.41}$$

$$\begin{aligned}\phi(P,x,t) \underset{\zeta\to 0}{=} & \, C(x,t)e^{\omega_0^{0,+}}\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t)))} \\ & - C(x,t)\omega_1^{0}e^{\omega_0^{0,+}}\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t)))}\zeta \\ & - C(x,t)e^{\omega_0^{0,+}}\frac{i}{2}\frac{\partial}{\partial t}\left(\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t)))}\right)\zeta + O(\zeta^2) \ \textit{as } P\to P_{0,+}.\end{aligned} \tag{4.42}$$

Proof. Using (4.25) and (4.26) (cf. (A.11) and (A.13)) one obtains

$$\underline{z}(P,\underline{\hat{\mu}}(x,t)) \underset{\zeta\to 0}{=} \underline{z}(P_{\infty_\pm},\underline{\hat{\mu}}(x,t)) \mp \underline{c}(n)\zeta + O(\zeta^2) \text{ as } P\to P_{\infty_\pm}, \tag{4.43}$$

$$\underline{z}(P,\underline{\hat{\mu}}(x,t)) \underset{\zeta\to 0}{=} \underline{z}(P_{0,\pm},\underline{\hat{\mu}}(x,t)) \mp \underline{c}(1)g_{n+1}^{-1}\zeta + O(\zeta^2) \text{ as } P\to P_{0,\pm} \tag{4.44}$$

and hence

$$\begin{aligned}\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t))) \underset{\zeta\to 0}{=} & \, \theta(\underline{z}(P_{\infty_\pm},\underline{\hat{\mu}}(x,t))) \\ & \pm \frac{i}{2}\frac{\partial}{\partial x}\theta(\underline{z}(P_{\infty_\pm},\underline{\hat{\mu}}(x,t)))\zeta + O(\zeta^2) \text{ as } P\to P_{\infty_\pm},\end{aligned} \tag{4.45}$$

$$\begin{aligned}\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t))) \underset{\zeta\to 0}{=} & \, \theta(\underline{z}(P_{0,\pm},\underline{\hat{\mu}}(x,t))) \\ & \mp \frac{i}{2}\frac{\partial}{\partial t}\theta(\underline{z}(P_{0,\pm},\underline{\hat{\mu}}(x,t)))\zeta + O(\zeta^2) \text{ as } P\to P_{0,\pm}.\end{aligned} \tag{4.46}$$

Here we used (4.36) to convert the directional derivatives $\sum_{j=1}^n c_j(n)\partial/\partial w_j$ and $\sum_{j=1}^n c_j(1)\partial/\partial w_j$, $\underline{w} = (w_1,\dots,w_n) \in \mathbb{C}^n$ into $\partial/\partial x$ and $\partial/\partial t$ derivatives. Since by (4.37) exactly the same formulas (4.45) and (4.46) apply to $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$, insertion of (4.5), (4.6), (4.45), and (4.46) (and their $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ analogs) into (4.33) proves (4.39)–(4.42). $\square$

Lemma 4.3 may seem to be just another asymptotic result, however, a comparison with Lemma 3.3 reveals that in passing we have actually derived the theta function representations for u, v, u^*, and v^*.

Theorem 4.4. *Assume* (2.1), (2.8), (2.9), *and* (3.2), *and suppose* $\mathcal{K}_n$ *is nonsingular. In addition, let* $P \in \mathcal{K}_n$ *and* $(x,t) \in \Omega$, *where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ *and* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *are nonspecial for* $(x,t) \in \Omega$. *Then* $\phi(P,x,t)$ *admits the representation*

$$\phi(P,x,t) = C_0 e^{2i(\omega_1^\infty x - \omega_1^0 t)} \frac{\theta(\underline{z}(P,\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t)))} \exp\left(\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}}\right) \tag{4.47}$$

for some constant $C_0 \in \mathbb{C}\setminus\{0\}$ *and the theta function representations for the algebro-geometric solutions* u, u^*, v, *and* v^* *of the classical massive Thirring system* (2.22)–(2.25) *read*

$$u(x,t) = -C_0^{-1} e^{-\omega_0^{0,+}} \frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t)))} e^{-2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.48}$$

$$v(x,t) = -C_0^{-1} e^{-\omega_0^{\infty_-}} \frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t)))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t)))} e^{-2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.49}$$

$$u^*(x,t) = C_0 e^{\omega_0^{0,-}} \frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t)))} e^{2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.50}$$

$$v^*(x,t) = C_0 e^{\omega_0^{\infty_+}} \frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t)))} e^{2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.51}$$

with $\omega_0^{0,\pm}, \omega_1^0, \omega_0^{\infty\pm}$, *and* ω_1^∞ *given by* (4.16)–(4.23) (*cf. also* (4.5)–(4.8)).

Proof. A comparison of (3.29)–(3.32) and (4.39)–(4.42) yields

$$C_x(x,t) = 2i\omega_1^\infty C(x,t), \quad C_t(x,t) = -2i\omega_1^0 C(x,t), \tag{4.52}$$

and hence

$$C(x,t) = C_0 e^{2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.53}$$

proves (4.47). Insertion of (4.53) into the leading asymptotic term of (4.39)–(4.42) then yields (4.48)–(4.51). ☐

Remark 4.5. (i) The constant C_0 in (4.47)–(4.51) remains open due to the scaling invariance (2.28) of the Thirring system. One can rewrite (4.48)–(4.51) in the form

$$\begin{aligned} u(x,t) = u(x_0,t_0) &\frac{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t)))\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x_0,t_0)))}{\theta(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x_0,t_0)))\theta(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t)))} \times \\ &\times \exp\left(-2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\right), \end{aligned} \tag{4.54}$$

$$\begin{aligned} v(x,t) = v(x_0,t_0) &\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t)))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0,t_0)))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_0)))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t)))} \times \\ &\times \exp\left(-2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\right), \end{aligned} \tag{4.55}$$

$$\begin{aligned} u^*(x,t) = u^*(x_0,t_0) &\frac{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t)))\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x_0,t_0)))}{\theta(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x_0,t_0)))\theta(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t)))} \times \\ &\times \exp\left(2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\right), \end{aligned} \tag{4.56}$$

$$v^*(x,t) = v^*(x_0,t_0)\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)} \times \\ \times \exp\big(2i(\omega_1^\infty(x-x_0)-\omega_1^0(t-t_0))\big), \tag{4.57}$$

where

$$\underline{z}(Q,\underline{\hat{\mu}}(x,t)) = \underline{z}(Q,\underline{\hat{\mu}}(x_0,t_0)) + 2i\underline{c}(n)(x-x_0) + 2i\underline{c}(1)g_{n+1}^{-1}(t-t_0), \tag{4.58}$$

$$\underline{z}(Q,\underline{\hat{\nu}}(x,t)) = \underline{z}(Q,\underline{\hat{\nu}}(x_0,t_0)) + 2i\underline{c}(n)(x-x_0) + 2i\underline{c}(1)g_{n+1}^{-1}(t-t_0), \tag{4.59}$$

by (4.35), (4.36), and (4.37).
(ii) Since the divisors $\mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(x,t)}$ and $\mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(x,t)}$ are linearly independent by (3.11), one infers

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}) + \underline{\Delta}, \quad (x,t)\in\Omega, \quad \underline{\Delta} = \underline{A}_{P_{0,-}}(P_{\infty_-}). \tag{4.60}$$

Hence on can replace $\underline{z}(Q,\underline{\hat{\nu}}(x,t))$ in (4.47)–(4.51), (4.54)–(4.57), (4.59) in terms of $\underline{z}(Q,\underline{\hat{\mu}}(x,t))$ according to

$$\underline{z}(Q,\underline{\hat{\nu}}(x,t)) = \underline{z}(Q,\underline{\hat{\mu}}(x,t)) + \underline{\Delta}. \tag{4.61}$$

In principle, Theorem 4.4 completes the primary aim of this paper, the derivation of the theta function representation of algebro-geometric solutions of the classical massive Thirring system (2.22)–(2.25). The reader will have noticed that our approach thus far is nontraditional in the sense that we did not use the Baker–Akhiezer vector Ψ at all, but instead put all emphasis on the meromorphic ϕ on $\mathcal{K}_n$. Just for completeness we finally derive the theta function representation for ψ_1 in (3.18).

The singularity structure of $\psi_1(P,x,x_0,t,t_0)$ near $P_{\infty_\pm}$ displayed in Lemma 3.4 suggests introducing Abelian differentials $\omega_{Q,0}^{(2)}$ of the second kind, normalized by the vanishing of their a-periods,

$$\int_{a_j}\omega_{Q,0}^{(2)} = 0, \quad j=1,\dots,n, \tag{4.62}$$

with a second-order pole at Q of the type

$$\omega_{Q,0}^{(2)} \underset{\zeta\to 0}{=} (\zeta^{-2}+O(1))d\zeta \text{ as } P\to Q, \tag{4.63}$$

and holomorphic on $\mathcal{K}_n\setminus\{Q\}$. More precisely, we introduce

$$\Omega_{\infty,0}^{(2)} = \omega_{P_{\infty_+},0}^{(2)} - \omega_{P_{\infty_-},0}^{(2)}, \tag{4.64}$$

$$\Omega_{0,0}^{(2)} = \omega_{P_{0,+},0}^{(2)} - \omega_{P_{0,-},0}^{(2)}, \tag{4.65}$$

and note that

$$\int_{Q_0}^P \Omega_{\infty,0}^{(2)} \underset{\zeta\to 0}{=} \pm(\zeta^{-1}+e_{\infty,0}+e_{\infty,1}\zeta+O(\zeta^2)) \text{ as } P\to P_{\infty_\mp}, \tag{4.66}$$

$$\int_{Q_0}^P \Omega_{0,0}^{(2)} \underset{\zeta\to 0}{=} \pm(\zeta^{-1}+e_{0,0}+e_{0,1}\zeta+O(\zeta^2)) \text{ as } P\to P_{0,\mp}. \tag{4.67}$$

Theorem 4.6. *Assume* (2.1), (2.8), (2.9), (3.2), *and suppose $\mathcal{K}_n$ is nonsingular. In addition, let $P\in\mathcal{K}_n\setminus\{P_{0,\pm},P_{\infty_\pm}\}$ and $(x,t),(x_0,t_0)\in\Omega$, where $\Omega\subseteq\mathbb{R}^2$ is open and connected. Moreover, suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ are nonspecial for $(x,t)\in\Omega$. Then $\psi_1(P,x,x_0,t,t_0)$ admits the representation*

$$\psi_1(P,x,x_0,t,t_0) = \left(\frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}\right)^{1/2} \times$$
$$\times \frac{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x_0,t_0))\big)} \times$$
$$\times \exp\left(-i(x-x_0)\left(\omega_1^\infty + \int_{Q_0}^P \Omega_{\infty,0}^{(2)}\right) + i(t-t_0)\left(\omega_1^0 + \int_{Q_0}^P \Omega_{0,0}^{(2)}\right)\right), \quad (4.68)$$

or, equivalently,

$$\psi_1(P,x,x_0,t,t_0) = \left(\frac{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t))\big)}\right)^{1/2} \times$$
$$\times \frac{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x_0,t_0))\big)} \times$$
$$\times \exp\left(-i(x-x_0)\left(\omega_1^\infty + \int_{Q_0}^P \Omega_{\infty,0}^{(2)}\right) + i(t-t_0)\left(\omega_1^0 + \int_{Q_0}^P \Omega_{0,0}^{(2)}\right)\right). \quad (4.69)$$

Proof. Introducing

$$\hat{\psi}_1(P,x,x_0,t,t_0) = \frac{C(x,t)}{C(x_0,t_0)}\,\frac{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x_0,t_0))\big)} \times$$
$$\times \exp\left(-i(x-x_0)\int_{Q_0}^P \Omega_{\infty,0}^{(2)} + i(t-t_0)\int_{Q_0}^P \Omega_{0,0}^{(2)}\right),$$
$$P \in \mathcal{K}_n \setminus \{P_{0,\pm}, P_{\infty_\pm}\},\ (x,t),(x_0,t_0) \in \Omega, \quad (4.70)$$

with an appropriate normalization $C(x,t)$ (which is P-independent) to be determined later, we next intend to prove that

$$\psi_1(P,x,x_0,t,t_0) = \hat{\psi}_1(P,x,x_0,t,t_0), \quad P \in \mathcal{K}_n \setminus \{P_{0,\pm}, P_{\infty_\pm}\},\ (x,t),(x_0,t_0) \in \Omega. \quad (4.71)$$

A comparison of (3.23), (3.37), (3.38), (4.66), (4.67), and (4.70) shows that ψ_1 and $\hat{\psi}_1$ share the identical essential singularity near $P_{\infty_\pm}$. Next we turn to the local bahavior of $\psi_1(P,x,x_0,t,t_0)$ with respect to its zeros and poles. We temporarily restrict Ω to $\widetilde{\Omega} \subseteq \Omega$ such that for all $(x',s) \in \widetilde{\Omega}$, $\mu_j(x',s) \neq \mu_k(x',s)$ for all $j \neq k$, $j,k = 1,\dots,n$. Then arguing as in the paragraph following (3.28) one infers from (3.18) that

$$\psi_1(P,x,x_0,t,t_0) = \begin{cases} (\mu_j(x,t)-z)O(1) & \text{as } P \to \hat{\mu}_j(x,t) \neq \hat{\mu}(x_0,t_0), \\ O(1) & \text{as } P \to \hat{\mu}_j(x,t) = \hat{\mu}(x_0,t_0), \\ (\mu_j(x_0,t_0)-z)^{-1}O(1) & \text{as } P \to \hat{\mu}_j(x_0,t_0) \neq \hat{\mu}(x,t), \end{cases}$$
$$P = (z,y) \in \mathcal{K}_n,\ (x,t),(x_0,t_0) \in \widetilde{\Omega}, \quad (4.72)$$

where $O(1) \neq 0$. Applying Lemma A.6 then proves (4.71) for $(x,t),(x_0,t_0) \in \widetilde{\Omega}$. By continuity this extends to $(x,t),(x_0,t_0) \in \Omega$ as long as $\mathcal{D}_{\underline{\hat{\mu}}(x,t)} \in \sigma^n\mathcal{K}_n$ remains nonspecial. Finally we determine $C(x,t)/C(x_0,t_0)$. A comparison of (2.5), (2.21), (3.24), (4.54), (4.55), and (4.70) yields

$$\begin{aligned}
&\psi_1(P,x,x_0,t,t_0)\psi_1(P^*,x,x_0,t,t_0)\\
&\quad\underset{z\to\infty}{=}\frac{C(x,t)^2}{C(x_0,t_0)^2}\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_0))\big)}\times\\
&\qquad\times\exp\left(-2i((x-x_0)e_{\infty,0}-(t-t_0)e_{0,0})\right)\\
&\quad=\frac{v(x,t)}{v(x_0,t_0)} \qquad (4.73)\\
&\quad=\frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}\exp\big(-2i((x-x_0)\omega_1^{\infty}-(t-t_0)\omega_1^0)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\psi_1(P,x,x_0,t,t_0)\psi_1(P^*,x,x_0,t,t_0)\\
&\quad\underset{z\to 0}{=}\frac{C(x,t)^2}{C(x_0,t_0)^2}\frac{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x_0,t_0))\big)}\times\\
&\qquad\times\exp\left(-2i((x-x_0)e_{\infty,0}-(t-t_0)e_{0,0})\right)\\
&\quad=\frac{u(x,t)}{u(x_0,t_0)} \qquad (4.74)\\
&\quad=\frac{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t))\big)}\times\\
&\qquad\times\exp\big(-2i((x-x_0)\omega_1^{\infty}-(t-t_0)\omega_1^0)\big)\,.
\end{aligned}$$

Thus, (4.73) implies

$$\begin{aligned}
\frac{C(x,t)^2}{C(x_0,t_0)^2}&=\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}\times\\
&\qquad\times\exp\big(-2i((x-x_0)\omega_1^{\infty}-(t-t_0)\omega_1^0)\big) \qquad (4.75)
\end{aligned}$$

and (4.74) yields

$$\begin{aligned}
\frac{C(x,t)^2}{C(x_0,t_0)^2}&=\frac{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t))\big)}\times\\
&\qquad\times\exp\big(-2i((x-x_0)\omega_1^{\infty}-(t-t_0)\omega_1^0)\big)\,. \qquad (4.76)
\end{aligned}$$

In order to reconcile the two expressions (4.75) and (4.76) for $C(x,t)^2/C(x_0,t_0)^2$ it suffices to recall the linear dependence of the divisors $\mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(x,t)}$ and $\mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(x,t)}$, that is,

$$\underline{A}_{Q_0}(P_{\infty_-})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)})=\underline{A}_{Q_0}(P_{0,-})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}), \qquad (4.77)$$

and

$$\underline{A}_{Q_0}(P_{0,-})=-\underline{A}_{Q_0}(P_{0,+}),\quad \underline{A}_{Q_0}(P_{\infty_-})=-\underline{A}_{Q_0}(P_{\infty_+}), \qquad (4.78)$$

to conclude that

$$\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))=\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t)),\quad \underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))=\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t)) \qquad (4.79)$$

and hence equality of the right-hand sides of (4.75) and (4.76). This proves (4.68) and (4.69). ☐

The explicit representation (4.68) for ψ_1 complements Lemma 3.2 and shows that ψ_1 stays meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ as long as $\mathcal{D}_{\hat{\underline{\mu}}(x,t)}$ and $\mathcal{D}_{\hat{\underline{\nu}}(x,t)}$ are nonspecial (assuming $\mathcal{K}_n$ to be nonsingular). An analogous theta function derivation can be performed for $\zeta\psi_2(P,\zeta,x,x_0,t,t_0)$, but we omit further details at this point.

We emphasize that $\phi(P,x,t)$ and $\psi_1(P,x,x_0,t,t_0)$ are naturally defined on the two-sheeted Riemann surface $\mathcal{K}_n$, whereas $\psi_2(P,\zeta,x,x_0,t,t_0)$ requires a four-sheeted Riemann surface due to the additional factor $1/z^{1/2}$ in (3.19). In particular, the Baker–Akhiezer vector $\Psi(P,\zeta,x,x_0,t,t_0)$ in (3.17) requires a four-sheeted Riemann surface, clearly a disadvantage when compared to our use of $\phi(P,x,t)$. Finally, we note that reality constraints of the type (2.26) and their effects on algebro-geometric quantities, such as pairs of real and complex conjugate branch points of $\mathcal{K}_n$, etc., are discussed in [6] (see also [11]).

We conclude with the elementary genus zero example (i.e., $n=0$), a case thus far excluded in this section.

Example 4.7. Assume $n=0$. Then

$$\mathcal{K}_0\colon \mathcal{F}_0(z,y)=y^2-R_2(z)=y^2-(z-E_0)(z-E_1)=0, \tag{4.80}$$

$$c_1=-(E_0+E_1)/2, \quad g_1=(E_0E_1)^{1/2}, \tag{4.81}$$

$$\omega_1^\infty=(g_1+c_1)/2, \quad \omega_1^0=-\omega_1^\infty/(E_0E_1), \tag{4.82}$$

$$\begin{aligned} v(x,t)&=v(x_0,t_0)\exp(-2i(\omega_1^\infty(x-x_0)-\omega_1^0(t-t_0)))\\ &=g_1u(x,t), \end{aligned}\tag{4.83}$$

$$\begin{aligned} v^*(x,t)&=v^*(x_0,t_0)\exp(2i(\omega_1^\infty(x-x_0)-\omega_1^0(t-t_0)))\\ &=g_1u^*(x,t), \end{aligned}\tag{4.84}$$

$$v(x,t)v^*(x,t)=(c_1-g_1)/2=g_1^2u(x,t)u^*(x,t), \tag{4.85}$$

$$\phi(P,x,t)=\frac{y(P)+z+g_1}{-2v(x,t)}=\frac{-2v^*(x,t)z}{y(P)-z-g_1}, \tag{4.86}$$

$$\psi_1(P,x,x_0,t,t_0)=\exp(-i(x-x_0)(y(P)+\omega_1^\infty)-i(t-t_0)(g_1^{-1}z^{-1}y(P)-\omega_1^0)). \tag{4.87}$$

Appendix A. Hyperelliptic curves and their theta functions

We give a brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [15] and [35], as well as monographs dedicated to integrable systems such as [5], Ch. 2, [17], App. A–C.

Fix $n\in\mathbb{N}$. The hyperelliptic curve $\mathcal{K}_n$ of genus n used in Sections 3 and 4 is defined by

$$\begin{gathered}\mathcal{K}_n:\ \mathcal{F}_n(z,y)=y^2-R_{2n+2}(z)=0, \quad R_{2n+2}(z)=\prod_{m=0}^{2n+1}(z-E_m),\\ \{E_m\}_{m=0,\dots,2n+1}\subset\mathbb{C}, \quad E_m\neq E_n \text{ for } m\neq n,\, m,n=0,\dots,2n+1. \end{gathered}\tag{A.1}$$

The curve (A.1) is compactified by adding the points P_{∞_+} and P_{∞_-}, $P_{\infty_+}\neq P_{\infty_-}$, at infinity. One then introduces an appropriate set of $n+1$ nonintersecting cuts $\mathcal{C}_j$

joining $E_{m(j)}$ and $E_{m'(j)}$. We denote

$$\mathcal{C} = \bigcup_{j\in\{1,\dots,n+1\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.2}$$

Define the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \tag{A.3}$$

and introduce the holomorphic function

$$R_{2n+2}(\,\cdot\,)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2n+1} (z - E_m)\right)^{1/2} \tag{A.4}$$

on Π with an appropriate choice of the square root branch in (A.4). Define

$$\mathcal{M}_n = \{(z, \sigma R_{2n+2}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{\pm 1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\} \tag{A.5}$$

by extending $R_{2n+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_n$ is then the set $\mathcal{M}_n$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_n$ crosswise along the cuts. The set of branch points $\mathcal{B}(\mathcal{K}_n)$ of $\mathcal{K}_n$ is given by

$$\mathcal{B}(\mathcal{K}_n) = \{(E_m, 0)\}_{m=0,\dots,2n+1} \tag{A.6}$$

and finite points P on $\mathcal{K}_n$ are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0$. Local coordinates near $P_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{\mathcal{B}(\mathcal{K}_n) \cup \{P_{\infty_+}, P_{\infty_-}\}\}$ are given by $\zeta_{P_0} = z - z_0$, near $P_{\infty_\pm}$ by $\zeta_{P_{\infty_\pm}} = 1/z$, and near branch points $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_n)$ by $\zeta_{(E_{m_0},0)} = (z - E_{m_0})^{1/2}$. The Riemann surface $\mathcal{K}_n$ defined in this manner has topological genus n.

One verifies that dz/y is a holomorphic differential on $\mathcal{K}_n$ with zeros of order $n-1$ at $P_{\infty_\pm}$ and hence

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j = 1, \dots, n \tag{A.7}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_n$. Introducing the invertible matrix C in $\mathbb{C}^n$,

$$\begin{aligned} C &= (C_{j,k})_{j,k=1,\dots,n}, \quad C_{j,k} = \int_{a_k} \eta_j, \\ \underline{c}(k) &= (c_1(k), \dots, c_n(k)), \quad c_j(k) = C^{-1}_{j,k}, \end{aligned} \tag{A.8}$$

the corresponding basis of normalized holomorphic differentials ω_j, $j = 1, \dots, n$ on $\mathcal{K}_n$ is given by

$$\omega_j = \sum_{\ell=1}^{n} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1, \dots, n. \tag{A.9}$$

Here $\{a_j, b_j\}_{j=1,\dots,n}$ is a homology basis for $\mathcal{K}_n$ with intersection matrix of the cycles satisfying

$$a_j \circ b_k = \delta_{j,k}, \quad j,k = 1, \dots, n. \tag{A.10}$$

Near $P_{\infty_\pm}$ one infers

$$\underline{\omega} = (\omega_1, \dots, \omega_n) = \pm\left(\sum_{j=1}^{n} \frac{\underline{c}(j)\zeta^{n-j}}{\left(\prod_{m=0}^{2n+1}(1 - E_m\zeta)\right)^{1/2}}\right) d\zeta$$

$$\underset{\zeta\to 0}{=} \pm\Big(\underline{c}(n) + \Big(\frac{1}{2}\underline{c}(n)\sum_{m=0}^{2n+1} E_m + \underline{c}(n-1)\Big)\zeta + O(\zeta^2)\Big)d\zeta, \quad \zeta = 1/z, \tag{A.11}$$

and

$$y(P) \underset{\zeta\to 0}{=} \mp\Big(1 - \frac{1}{2}\Big(\sum_{m=0}^{2n+1} E_m\Big)\zeta + O(\zeta^2)\Big)\zeta^{-n-1} \text{ as } P \to P_{\infty_\pm}. \tag{A.12}$$

Similarly, near $P_{0,\pm}$ one computes

$$\underline{\omega} \underset{\zeta\to 0}{=} \pm\frac{1}{g_{n+1}}\Big(\underline{c}(1) + \Big(\frac{1}{2}\underline{c}(1)\sum_{m=0}^{2n+1} E_m^{-1} + \underline{c}(2)\Big)\zeta + O(\zeta^2)\Big)d\zeta, \tag{A.13}$$

$$g_{n+1} = \Big(\prod_{m=0}^{2n+1} E_m\Big)^{1/2}, \quad \zeta = z,$$

using

$$y(P) \underset{\zeta\to 0}{=} \pm g_{n+1} + O(\zeta) \text{ as } P \to P_{0,\pm}, \tag{A.14}$$

with the sign of g_{n+1} determined by the compatibility of charts.

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,n}$ we also recall the canonical dissection of $\mathcal{K}_n$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_n$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_n$ given by

$$\partial\widehat{\mathcal{K}}_n = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n^{-1} b_n^{-1}. \tag{A.15}$$

Let $\mathcal{M}(\mathcal{K}_n)$ and $\mathcal{M}^1(\mathcal{K}_n)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_n$. The residue of a meromorphic differential $\nu \in \mathcal{M}^1(\mathcal{K}_n)$ at a point $Q \in \mathcal{K}_n$ is defined by

$$\operatorname{res}_Q(\nu) = \frac{1}{2\pi i}\int_{\gamma_Q} \nu, \tag{A.16}$$

where γ_Q is a counterclockwise oriented smooth simple closed contour encircling Q but no other pole of ν. Holomorphic differentials are also called Abelian differentials of the first kind (dfk). Abelian differentials of the second kind (dsk) $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_n)$ are characterized by the property that all their residues vanish. They are normalized, for instance, by demanding that all their a-periods vanish, that is,

$$\int_{a_j} \omega^{(2)} = 0, \quad j = 1, \dots, n. \tag{A.17}$$

If $\omega^{(2)}_{P_1,n}$ is a dsk on $\mathcal{K}_n$ whose only pole is $P_1 \in \widehat{\mathcal{K}}_n$ with principal part $\zeta^{-n-2}\,d\zeta$, $n \in \mathbb{N}_0$ near P_1 and $\omega_j = (\sum_{m=0}^{\infty} d_{j,m}(P_1)\zeta^m)\,d\zeta$ near P_1, then

$$\frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_1,m} = \frac{d_{j,m}(P_1)}{m+1}, \quad m = 0, 1, \dots \tag{A.18}$$

Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_n$ not of the first or second kind is said to be of the third kind (dtk). A dtk $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_n)$ is usually normalized by the vanishing of its a-periods, that is,

$$\int_{a_j} \omega^{(3)} = 0, \quad j = 1, \dots, n. \tag{A.19}$$

A normal dtk $\omega^{(3)}_{P_1,P_2}$ associated with two points P_1, $P_2 \in \widehat{\mathcal{K}}_n$, $P_1 \neq P_2$ by definition has simple poles at P_j with residues $(-1)^{j+1}$, $j = 1, 2$ and vanishing a-periods. If $\omega^{(3)}_{P,Q}$ is a normal dtk associated with P, $Q \in \widehat{\mathcal{K}}_n$, holomorphic on $\mathcal{K}_n \setminus \{P, Q\}$, then

$$\frac{1}{2\pi i} \int_{b_j} \omega^{(3)}_{P,Q} = \int_Q^P \omega_j, \quad j = 1, \dots, n, \tag{A.20}$$

where the path from Q to P lies in $\widehat{\mathcal{K}}_n$ (i.e., does not touch any of the cycles a_j, b_j). Explicitly, one obtains

$$\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = -\frac{\tilde\pi^n d\tilde\pi}{y} + \sum_{j=1}^n d_j\omega_j = -\frac{\prod_{j=1}^n(\tilde\pi - \lambda_j)\, d\tilde\pi}{y}, \tag{A.21}$$

$$\omega^{(3)}_{P_1,P_{\infty_+}} = \frac{1}{2}\frac{y+y_1}{\tilde\pi - z_1}\frac{d\tilde\pi}{y} - \frac{\prod_{j=1}^n(\tilde\pi - \hat\lambda_j)\, d\tilde\pi}{2y}, \tag{A.22}$$

$$\omega^{(3)}_{P_1,P_{\infty_-}} = \frac{1}{2}\frac{y+y_1}{\tilde\pi - z_1}\frac{d\tilde\pi}{y} + \frac{\prod_{j=1}^n(\tilde\pi - \tilde\lambda_j)\, d\tilde\pi}{2y}, \tag{A.23}$$

$$\omega^{(3)}_{P_1,P_2} = \frac{1}{2}\left(\frac{y+y_1}{\tilde\pi - z_1} - \frac{y+y_2}{\tilde\pi - z_2}\right)\frac{d\tilde\pi}{y}, \quad P_1, P_2 \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}, \tag{A.24}$$

where $d_j, \lambda_j, \hat\lambda_j, \tilde\lambda_j$, $j = 1, \dots, n$ are uniquely determined by the requirement of vanishing a-periods and we abbreviated $P_j = (z_j, y_j)$, $j = 1, 2$. (If $n = 0$, we use the standard convention that the product over an empty index set is replaced by 1.)

We shall always assume (without loss of generality) that all poles of dsk's and dtk's on $\mathcal{K}_n$ lie on $\widehat{\mathcal{K}}_n$ (i.e., not on $\partial\widehat{\mathcal{K}}_n$).

Define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,n}$ by

$$\tau_{j,\ell} = \int_{b_j} \omega_\ell, \quad j, \ell = 1, \dots, n. \tag{A.25}$$

Then

$$\operatorname{Im}(\tau) > 0, \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j, \ell = 1, \dots, n. \tag{A.26}$$

Associated with τ one introduces the period lattice

$$L_n = \{\underline{z} \in \mathbb{C}^n \mid \underline{z} = \underline{m} + \tau\underline{n},\ \underline{m}, \underline{n} \in \mathbb{Z}^n\} \tag{A.27}$$

and the Riemann theta function associated with $\mathcal{K}_n$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,n}$,

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^n} \exp\big(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \tau\underline{n})\big), \quad \underline{z} \in \mathbb{C}^n, \tag{A.28}$$

where $(\underline{u}, \underline{v}) = \sum_{j=1}^n \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^n$. It has the fundamental properties

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = \theta(\underline{z}), \tag{A.29}$$

$$\theta(\underline{z} + \underline{m} + \tau\underline{n}) = \exp\big(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \tau\underline{n})\big)\theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^n. \tag{A.30}$$

Next, fix a base point $Q_0 \in \mathcal{K}_n \setminus P_{0,\pm}, P_{\infty\pm}\}$, denote by $J(\mathcal{K}_n) = \mathbb{C}^n/L_n$ the Jacobi variety of $\mathcal{K}_n$, and define the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0} \colon \mathcal{K}_n \to J(\mathcal{K}_n), \quad \underline{A}_{Q_0}(P) = \Big(\int_{Q_0}^{P} \omega_1, \dots, \int_{Q_0}^{P} \omega_n \Big) \pmod{L_n}, \quad P \in \mathcal{K}_n. \tag{A.31}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0} \colon \operatorname{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P) \underline{A}_{Q_0}(P), \tag{A.32}$$

where $\operatorname{Div}(\mathcal{K}_n)$ denotes the set of divisors on $\mathcal{K}_n$. Here $\mathcal{D} \colon \mathcal{K}_n \to \mathbb{Z}$ is called a divisor on $\mathcal{K}_n$ if $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_n$. (In the main body of this paper we will choose Q_0 to be one of the branch points, i.e., $Q_0 \in \mathcal{B}(\mathcal{K}_n)$, and for simplicity we will always choose the same path of integration from Q_0 to P in all Abelian integrals.)

In connection with divisors on $\mathcal{K}_n$ we shall employ the following (additive) notation,

$$\mathcal{D}_{Q_0 \underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_n},$$
$$\underline{Q} = (Q_1, \dots, Q_n) \in \sigma^n \mathcal{K}_n, \quad Q_0 \in \mathcal{K}_n, \tag{A.33}$$

where for any $Q \in \mathcal{K}_n$,

$$\mathcal{D}_Q \colon \mathcal{K}_n \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_n \setminus \{Q\}, \end{cases} \tag{A.34}$$

and $\sigma^n \mathcal{K}_n$ denotes the nth symmetric product of $\mathcal{K}_n$. In particular, $\sigma^m \mathcal{K}_n$ can be identified with the set of nonnegative divisors $0 \le \mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$ of degree m.

For $f \in \mathcal{M}(\mathcal{K}_n) \setminus \{0\}$, $\omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E} \in \operatorname{Div}(\mathcal{K}_n)$ are called equivalent, denoted by $\mathcal{D} \sim \mathcal{E}$, if and only if $\mathcal{D} - \mathcal{E} = (f)$ for some $f \in \mathcal{M}(\mathcal{K}_n) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}] = \{\mathcal{E} \in \operatorname{Div}(\mathcal{K}_n) \mid \mathcal{E} \sim \mathcal{D}\}$. We recall that

$$\deg((f)) = 0,\ \deg((\omega)) = 2(n-1),\ f \in \mathcal{M}(\mathcal{K}_n) \setminus \{0\},\ \omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}, \tag{A.35}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P)$. It is customary to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_n) \mid f = 0 \text{ or } (f) \ge \mathcal{D}\},\ r(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}(\mathcal{D}), \tag{A.36}$$

$$\mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_n) \mid \omega = 0 \text{ or } (\omega) \ge \mathcal{D}\},\ i(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}^1(\mathcal{D}), \tag{A.37}$$

($i(\mathcal{D})$ the index of speciality of $\mathcal{D}$) one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$, $\omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}$. Then*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad n \in \mathbb{N}_0. \tag{A.38}$$

The Riemann-Roch theorem reads

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - n + 1, \quad n \in \mathbb{N}_0. \tag{A.39}$$

By Abel's theorem, $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_n)$, $n \in \mathbb{N}$ is principal if and only if

$$\deg(\mathcal{D}) = 0 \textit{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}. \tag{A.40}$$

Finally, assume $n \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n)$ is surjective (Jacobi's inversion theorem).

Next we introduce

$$\underline{W}_0 = \{0\} \subset J(\mathcal{K}_n), \quad \underline{W}_m = \underline{\alpha}_{Q_0}(\sigma^m \mathcal{K}_n),\ m \in \mathbb{N} \tag{A.41}$$

and note that while $\sigma^m \mathcal{K}_n \not\subset \sigma^n \mathcal{K}_n$ for $m < n$, one has $\underline{W}_m \subseteq \underline{W}_n$ for $m < n$. Thus $\underline{W}_m = J(\mathcal{K}_n)$ for $m \geq n$ by Jacobi's inversion theorem.

Denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,n}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, n. \tag{A.42}$$

Theorem A.2. *The set $\underline{W}_{n-1} + \underline{\Xi}_{Q_0} \subset J(\mathcal{K}_n)$ is the complete set of zeros of θ on $J(\mathcal{K}_n)$, that is,*

$$\theta(X) = 0 \textit{ if and only if } X \in \underline{W}_{n-1} + \underline{\Xi}_{Q_0} \tag{A.43}$$

(i.e., if and only if $X = (\underline{\alpha}_{Q_0}(\mathcal{D}) + \underline{\Xi}_{Q_0} \pmod{L_n}$ for some $\mathcal{D} \in \sigma^{n-1}\mathcal{K}_n$). The set $\underline{W}_{n-1} + \underline{\Xi}_{Q_0}$ has complex dimension $n-1$.

Theorem A.3. *Let $\mathcal{D}_{\underline{Q}} \in \sigma^n \mathcal{K}_n$, $\underline{Q} = (Q_1, \dots, Q_n)$. Then*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s(\leq n/2) \tag{A.44}$$

if and only if there are s pairs of the type $(P, P^*) \in \{Q_1, \dots, Q_n\}$ (this includes, of course, branch points for which $P = P^*$).

Remark A.4. While $\theta(\underline{z})$ is well-defined (in fact, entire) for $\underline{z} \in \mathbb{C}^n$, it is not well-defined on $J(\mathcal{K}_n) = \mathbb{C}^n / L_n$ because of (A.30). Nevertheless, θ is a "multiplicative function" on $J(\mathcal{K}_n)$ since the multipliers in (A.30) cannot vanish. In particular, if $\underline{z}_1 = \underline{z}_2 \pmod{L_n}$, then $\theta(\underline{z}_1) = 0$ if and only if $\theta(\underline{z}_2) = 0$. Hence it is meaningful to state that θ vanishes at points of $J(\mathcal{K}_n)$. Since the Abel map $\underline{A}_{Q_0}$ maps $\mathcal{K}_n$ into $J(\mathcal{K}_n)$, the function $\theta(\underline{A}_{Q_0}(P) - \underline{\xi})$ for $\underline{\xi} \in \mathbb{C}^n$, becomes a multiplicative function on $\mathcal{K}_n$. Again it makes sense to say that $\theta(\underline{A}_{Q_0}(\,\cdot\,) - \underline{\xi})$ vanishes at points of $\mathcal{K}_n$.

Theorem A.5. *Let $\underline{Q} = (Q_1, \dots, Q_n) \in \sigma^n \mathcal{K}_n$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*

$$\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})) = 0 \textit{ if and only if } P \in \{Q_1, \dots, Q_n\}. \tag{A.45}$$

Lemma A.6. [7, Lemmas 5.4 and 6.1] *Let $(x,t), (x_0, t_0) \in \Omega$ for some $\Omega \subseteq \mathbb{R}^2$. Assume $\psi(\,\cdot\,, x, t_r)$ to be meromorphic on $\mathcal{K}_n \backslash \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ with essential singularities at $P_{\infty_\pm}$, $P_{0,\pm}$ such that $\tilde{\psi}(\,\cdot\,, x, t)$ defined by*

$$\tilde{\psi}(P, x, t) = \psi(P, x, t) \exp\left(i(x - x_0) \int_{Q_0}^{P} \Omega_{\infty,0}^{(2)} - i(t - t_0) \int_{Q_0}^{P} \Omega_{0,0}^{(2)} \right) \tag{A.46}$$

is meromorphic on $\mathcal{K}_n$ and its divisor satisfies

$$(\tilde{\psi}(\,\cdot\,, x, t)) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x_0, t_0)}. \tag{A.47}$$

Here $\Omega^{(2)}_{\infty,0}$ and $\Omega^{(2)}_{0,0}$ are defined in (4.64) and (4.65) and the path of integration is chosen identical to that in the Abel maps (A.31) and (A.32)[1]*. Define a divisor $\mathcal{D}_0(x,t)$ by*

$$(\tilde\psi(\,\cdot\,,x,t)) = \mathcal{D}_0(x,t) - \mathcal{D}_{\underline{\hat\mu}(x_0,t_0)}. \tag{A.48}$$

Then

$$\mathcal{D}_0(x,t) \in \sigma^n\mathcal{K}_n, \quad \mathcal{D}_0(x,t) > 0, \quad \deg(\mathcal{D}_0(x,t)) = n. \tag{A.49}$$

Moreover, if $\mathcal{D}_0(x,t)$ is nonspecial for all $(x,t)\in\Omega$, that is, if

$$i(\mathcal{D}_0(x,t)) = 0, \quad (x,t)\in\Omega, \tag{A.50}$$

then $\psi(\,\cdot\,,x,t)$ is unique up to a constant multiple (which may depend on x and t).

Theorem A.7. *Suppose $\mathcal{D}_{\underline{\hat\mu}} \in \sigma^n\mathcal{K}_n$ is nonspecial, $\underline{\hat\mu} = (\hat\mu_1,\dots,\hat\mu_n)$, and $\hat\mu_{n+1}\in\mathcal{K}_n$ with $\hat\mu^*_{n+1}\notin\{\hat\mu_1,\dots,\hat\mu_n\}$. Let $\{\hat\lambda_1,\dots,\hat\lambda_{n+1}\}\subset\mathcal{K}_n$ with $\mathcal{D}_{\underline{\hat\lambda}\hat\lambda_{n+1}} \sim \mathcal{D}_{\underline{\hat\mu}\hat\mu_{n+1}}$ (i.e., $\mathcal{D}_{\underline{\hat\lambda}\hat\lambda_{n+1}} \in [\mathcal{D}_{\underline{\hat\mu}\hat\mu_{n+1}}]$). Then any n points $\hat\nu_j\in\{\hat\lambda_1,\dots,\hat\lambda_{n+1}\}$, $j=1,\dots,n$ define a nonspecial divisor $\mathcal{D}_{\underline{\hat\nu}}\in\sigma^n\mathcal{K}_n$, $\underline{\hat\nu} = (\hat\nu_1,\dots,\hat\nu_n)$.*

Proof. Since $i(\mathcal{D}_P)=0$ for all $P\in\mathcal{K}_1$, there is nothing to prove in the special case $n=1$. Hence we assume $n\ge 2$. Let $P_0\in\mathcal{B}(\mathcal{K}_n)$ be a fixed branch point of $\mathcal{K}_n$ and suppose that $\mathcal{D}_{\underline{\hat\nu}}$ is special. Then by Theorem A.3 there is a pair $\{\hat\nu,\hat\nu^*\}\subset\{\hat\nu_1,\dots,\hat\nu_n\}$ such that

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\nu}}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\underline{\hat\nu}}}), \tag{A.51}$$

where $\underline{\underline{\hat\nu}} = (\hat\nu_1,\dots,\hat\nu_n)\backslash\{\hat\nu,\hat\nu^*\}\subset\sigma^{n-2}\mathcal{K}_n$. Let $\hat\nu_{n+1}\in\{\hat\lambda_1,\dots,\hat\lambda_{n+1}\}\backslash\{\hat\nu_1,\dots,\hat\nu_n\}$ so that $\{\hat\nu_1,\dots,\hat\nu_{n+1}\} = \{\hat\lambda_1,\dots,\hat\lambda_{n+1}\}\subset\sigma^{n+1}\mathcal{K}_n$. Then

$$\begin{aligned}\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\underline{\hat\nu}}\hat\nu_{n+1}}) &= \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\nu}\hat\nu_{n+1}}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\lambda}\hat\lambda_{n+1}})\\ &= \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\mu}\hat\mu_{n+1}}) = -\underline{A}_{P_0}(\hat\mu^*_{n+1}) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\mu}}),\end{aligned} \tag{A.52}$$

and hence by Theorem A.2 and (A.52),

$$0 = \theta(\underline{\Xi}_{P_0} + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\underline{\hat\nu}}\hat\nu_{n+1}})) = \theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0}(\hat\mu^*_{n+1}) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat\mu}})). \tag{A.53}$$

Since by hypothesis $\mathcal{D}_{\underline{\hat\mu}}$ is nonspecial and $\hat\mu^*_{n+1}\notin\{\hat\mu_1,\dots,\hat\mu_n\}$, (A.53) contradicts Theorem A.5. Thus, $\mathcal{D}_{\underline{\hat\nu}}$ is nonspecial. ∎

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[1] This is to avoid multi-valued expressions and hence the use of the multiplicative Riemann–Roch theorem in the proof of Lemma A.6.

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