Resonances and Chaos in the Collective Oscillations of a Trapped Bose Condensate

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Abstract

We investigate the collective oscillations of a Bose condensate in an anisotropic harmonic trap. We analytically calculate the values of the trap deformation for which the normal modes are in resonance. In presence of low-order resonances, the amplitude dependence of frequencies is strongly enhanced. We also study the transition to chaos of the collective modes as a function of the trap anisotropy.

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The Bose-Einstein condensation (BEC) is the macroscopic occupation of the single-particle ground-state of a system of bosons. In the 1995 BEC has been achieved with dilute vapors of confined alkali-metal atoms [1-3]. The experiments with alkali-metal atoms generally consist of a laser cooling and confinement in an external potential (magnetic or magneto-optical trap) and an evaporative cooling [1-3]. Nowadays a dozen of experimental groups have obtained Bose condensates by using different atomic species and geometries of the confining trap.

The theoretically predicted low-energy collective oscillations of the Bose condensate [4] have been experimentally confirmed by laser imaging techniques [5]. Moreover, the dynamics of the collective oscillations of the condensate has been theoretically studied beyond the linear regime, showing strong enhancement of the amplitude dependence of frequencies in presence of resonances [6].

In this paper we derive analytical formulas that give the values of trap anisotropy for which the normal modes of the collective oscillations are in resonance. We discuss also the transition to chaos of the collective modes by increasing the energy of the system, namely by considering oscillations of very large amplitude. Note that the chaoticity of the high-energy single-particle excitations of the condensate has been shown in the classical limit by Fliesser *et al* [7]. Here, we consider instead the low-energy collective excitations: they have a classical-like behavior but satisfy the quantum mechanical equations of a superfluid in an external potential [4].

In current experiments alkali vapors are quite dilute and at zero temperature the atoms are practically all in the condensate [1-3]. The dynamics of the Bose condensate can be accurately described by the Gross-Pitaevskii (GP) equation [8] of mean-field approximation

$$i\hbar \frac{\partial}{\partial t}\phi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + g|\phi(\mathbf{r},t)|^2 \right] \phi(\mathbf{r},t) \; , \tag{1}$$

where $\phi(\mathbf{r},t)$ is the macroscopic wavefunction (order parameter) of the condensate, normalized to the number N of condensed atoms. The external trap is well approximated by an anisotropic harmonic potential

$$U(\mathbf{r}) = \frac{m}{2}(\omega_1 x^2 + \omega_2 y^2 + \omega_3 z^2) \ . \qquad (2)$$

and the parameter of the nonlinear term $g = 4\pi\hbar^2 a_s/m$ is the scattering amplitude with a_s the s-wave scattering length. It is useful to introduce the geometric average $\omega_H = (\omega_1\omega_2\omega_3)^{1/3}$ of the trap frequencies and the characteristic length $a_H = (\hbar/m\omega_H)^{1/2}$ of the trap. In previous papers we have studied the properties of the Bose condensate with attractive interaction, in particular for ^{7}Li atoms [9,10]. Here, instead, we consider only the case of a repulsive interaction.

The complex wavefunction $\phi(\mathbf{r},t)$ of the condensate can be written in terms of a modulus and a phase, as follows

$$\phi(\mathbf{r},t) = \sqrt{\rho(\mathbf{r},t)}\; e^{iS(\mathbf{r},t)} \ . \qquad (3)$$

The phase S fixes the velocity field $\mathbf{v} = (\hbar/m)\nabla S$. The GP equation can hence be rewritten in the form of two coupled hydrodynamic equations for the density and the velocity field [4]. If the repulsive interaction among atoms is strong enough, i.e. $Na_s/a_H >> 1$, then one can safely neglect the kinetic pressure term in the equation for the velocity field. In this way, the hydrodynamic equations read

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\mathbf{v}\rho) = 0 \ ,$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \nabla\left(U + g\rho + \frac{mv^2}{2}\right) = 0 \ . \qquad (4)$$

These equations are those of a superfluid in an external potential [4]. The ground-state solution ($\mathbf{v} = 0$) of the hydrodynamic equations is $\rho(\mathbf{r}) = g^{-1}[\mu - U(\mathbf{r})]$, in the region where $\mu > U(\mathbf{r})$, and $\rho = 0$ outside. The normalization condition on $\rho(\mathbf{r})$ provides $\mu = (\hbar\omega_H/2)(15Na_s/a_H)^{2/5}$.

An analytic class of solutions of Eq. (4) has been found found by Dalfovo *et al* [6] writing the density in the form $\rho(\mathbf{r},t) = a_0(t) - a_1(t)x^2 - a_2(t)y^2 - a_3(t)z^2$ in the region where $\rho(\mathbf{r},t)$ is positive, and the velocity field as $\mathbf{v}(\mathbf{r},t) = \frac{1}{2}\nabla[b_1(t)x^2 + b_2(t)y^2 + b_3(t)z^2]$. The coefficient a_0 is fixed by the normalization of the density $a_0 = (15N/8\pi)^{2/5}(a_1a_2a_3)^{1/5}$. By inserting these expressions in the hydrodynamic equations one finds 6 coupled differential equations for the time-dependent parameters $a_i(t)$ and $b_i(t)$. By introducing new variables q_i, defined by $a_i = m\omega_i^2(2gq_i^2q_1q_2q_3)^{-1}$, the hydrodynamic equations give $b_i = \dot{q}_i/q_i$ and

$$\ddot{q}_i + \omega_i^2 q_i = \frac{\omega_i^2}{q_i q_1 q_2 q_3} \ , \tag{5}$$

with $i = 1, 2, 3$. It is important to observe that, using the new variables q_i, the equations of motion do not depend on the value of the coupling constant g. In terms of q_i the mean square radii of the condensate are $< x_i^2 >= (2\mu/m\omega_i)q_i^2$ and the velocities are $< v_i^2 >= (2\mu/m\omega_i)\dot{q}_i^2$ (see also [6]).

The Eq. (5) are the classical equations of motion of a system with coordinates q_i and Lagrangian given by

$$L = \frac{1}{2}\sum_{i=1}^{3}(\omega_i^{-2}\dot{q}_i^2 - q_i^2) - \frac{1}{q_1 q_2 q_3} \ . \tag{6}$$

This Lagrangian describes the collective modes of the Bose condensate for $Na_s/a_H >> 1$. In such a case the collective dynamics of the condensate does not depend on the number of atoms or the scattering length. The low-energy collective excitations of the condensate are the small oscillations of variables q_i's around the equilibrium point, i.e. the minimum of the effective potential $\sum_{i=1}^{3} q_i^2 + 1/(q_1q_2q_3)$. The normal mode frequencies Ω of the condensate are given by the following equation

$$\Omega^6 - 3\left(\omega_1^2 + \omega_2^2 + \omega_3^2\right)\Omega^4 + 8\left(\omega_1^2\,\omega_2^2 + \omega_1^2\,\omega_3^2 + \omega_2^2\,\omega_3^2\right)\Omega^2 - 20\,\omega_1^2\,\omega_2^2\,\omega_3^2 = 0 \ . \tag{7}$$

Note that this formula has been recently obtained by using a variational approach [11] and also by studying hydrodynamic density fluctuations of the condensate [12]. For an axially symmetric trap, where $\omega_1 = \omega_2 = \omega_\perp$, the previous equation gives

$$\Omega_{1,2} = \sqrt{2 + \frac{3}{2}\lambda^2 \pm \frac{1}{2}\sqrt{16 + 9\lambda^4 - 16\lambda^2}}\;\omega_\perp \ ,$$

$$\Omega_3 = \sqrt{2}\,\omega_\perp \ , \tag{8}$$

where $\lambda = \omega_3/\omega_\perp$ is the anisotropy parameter of the trap. The frequencies $\Omega_{1,2}$ are the $m_z = 0$ monopole and quadrupole modes, where m_z is the third component of the angular momentum. Instead the frequency Ω_3 is the $m_z = 2$ quadrupole mode [4]. It is important to observe that the experimental results obtained with sodium vapors at MIT ($\lambda = \sqrt{8}$) are in good agreement with these theoretical values [13].

The normal modes are functions of the trap frequencies, i.e. of the trap deformation. Beyond the linear regime there is a strong enhancement of nonlinear effects when frequencies of different modes, or of their harmonics, coincide, i.e. when the normal modes are in resonance [6,14]. By using a perturbative approach, Dalfovo *et al* [6] have identified 3 resonant values of the trap anisotropy. For such values, the numerical solution of Eq. (5) shows that the collective frequencies deviate from the predictions of Eq. (7) or (8) also for small amplitude oscillations.

To find the BEC resonances we use a different and new strategy. In general, the resonance condition is given by

$$n_1\Omega_1 + n_2\Omega_2 + n_3\Omega_3 = 0 \; , \tag{9}$$

where $(n_1, n_2, n_3) \in \mathbf{Z}^3 \backslash \{\mathbf{0}\}$ is a non-zero vector of integer numbers and $R = |n_1| + |n_2| + |n_3|$ is the order of the resonance. It is possible to numerically determine the values of the trap deformation for which the resonance condition is satisfied. Nevertheless, because in most experiments the confining trap has axial symmetry [1-3], one obtains remarkable analytical results from Eq. (8) and (9) by imposing that one of the integer numbers n_i is zero. In the case $n_1 = 0$, the resonance condition (between a $m_z = 0$ mode and the $m_z = 2$ mode) gives

$$\lambda = \sqrt{\frac{2(2n_2^2 - n_3^2)n_3^2}{(5n_2^2 - 3n_3^2)n_2^2}} \; . \tag{10}$$

In the case $n_2 = 0$, one has the previous formula with n_1 instead of n_2. Finally, in the case $n_3 = 0$ one finds

$$\lambda = \frac{1}{3}\sqrt{\left[5\left(\frac{n_1^2}{n_2^2} + \frac{n_2^2}{n_1^2}\right) - 2\right] \pm \sqrt{\left[5\left(\frac{n_1^2}{n_2^2} + \frac{n_2^2}{n_1^2}\right) - 2\right]^2 - 144}} \; . \tag{11}$$

This formula gives the anisotropy parameter at the resonance condition between the two $m_z = 0$ modes.

In Table 1 are shown the values of the trap anisotropy λ for low-order resonances according to Eq. (10) and (11). Note that the values of λ for the $(1,2,0)$ and $(0,1,2)$ resonances are that previously obtained by Dalfovo *et al* [6]. As concerns resonances, the values of λ for the experimental traps of MIT ($\lambda = 0.077$) and JILA ($\lambda = \sqrt{8}$) are not of particular interest. Instead, it would be very interesting to study experimentally the system in the trap conditions given by Eq. (10) and (11). In fact, close to low-order resonances, even for relatively small amplitudes, the motion is rather complex and the dynamics can exhibit a chaotic-like behavior [14].

One expects an order-chaos transition by increasing the amplitude of oscillations, thus by increasing the energy of the system with respect to the ground-state (minimum of the effective potential) [14]. To check if such a scenario works in this context, we numerically study the chaotic dynamics of the $m_z = 0$ modes (preliminar results can be found in [15]). We put $q_1 = q_2 = q_\perp$ in the BEC lagrangian. By using the adimensional time $\tau = \omega_\perp t$, the BEC Hamiltonian of the $m_z = 0$ collective modes reads

$$H = p_\perp^2 + \frac{1}{2}\lambda^2 p_3^2 + q_\perp^2 + \frac{1}{2}q_3^2 + \frac{1}{q_\perp^2 q_3} \; , \tag{12}$$

where $p_\perp = dq_\perp/d\tau$ and $p_3 = \lambda^{-2} dq_3/d\tau$ are the conjugate momenta [15]. We use a symplectic method to numerically compute the trajectories. The conservation of energy restricts any trajectory of the four-dimensional phase space to a three-dimensional energy shell. At a particular energy, the restriction $q_\perp = 1$ defines a two-dimensional surface in the phase space, that is called Poincarè section. Each time a particular trajectory passes through the surface a point is plotted at the position of intersection (q_3, p_3). We employ a first-order interpolation process to reduce inaccuracies due to the use of a finite step length.

In Fig.1 and Fig. 2 we plot Poincarè sections of the system with $\lambda = 1.501$ (corresponding to the $(3,5,0)$ resonance) and $\lambda = \sqrt{8}$ (JILA experiment), respectively. In each panel there is a Poincarè section with a fixed value of the energy of the system. At each energy value, we have chosen different initial conditions $[q_\perp(0), q_3(0), p_\perp(0), p_3(0)]$ for the dynamics. Actually $p_\perp(0)$ has been fixed by the conservation of energy. Integration time is 400 in adimensional units, that is less than 1 second (the life-time of the condensate is about 10 seconds). Chaotic regions on the Poincarè section are characterized by a set of randomly distributed points and regular regions by dotted or solid curves.

We use the adimensional parameter χ, that is the relative increase of energy with respect to the ground-state. Fig. 1 shows that for $\lambda = 1.501$ there is an order-chaos transition by increasing χ: a chaotic sea appears at low energy ($\chi = 2/5$). On the contrary, Fig. 2 shows that for $\lambda = \sqrt{8}$ at $\chi = 3/5$ the collective oscillations are inharmonic but still not chaotic.

To estimate the chaoticity of the $m_z = 0$ modes, we calculate the fraction of initial conditions that give rise to regular trajectories. In the (λ,χ) plane we plot the configurations for which the $m_z = 0$ oscillations are regular. The numerical results are shown in Fig. 3. As previously discussed, for $\lambda = 1.501$ (or λ very close to such a value) the system remains regular only at small χ; for larger χ it becomes gradually chaotic. In general, the transition to chaos depends on χ, but for $\lambda << 1$ and $\lambda > 4$ the dynamics appears always regular. Actually, for $\lambda << 1$ the system is integrable because of an additional constant of motion [16]. The MIT trap ($\lambda = 0.077$) belongs to this stable region.

We have analyzed the Poincarè sections of the $m_z = 0$ BEC Hamiltonian for all the resonant values of λ that are shown in Tab. 1 (left). Among the resonances, the $(3, 5, 0)$ resonance ($\lambda = 0.888$ and $\lambda = 1.501$) appears the most chaotic: this is the main prediction of our paper.

An important question is the following: can BEC chaotic dynamics be experimentally detected? The answer is positive. It is important to observe that, strictly speaking, there is no dynamical chaos in quantum systems with a finite number of degrees of freedom (for a discussion of chaos and quantum chaos in field theories see [17]). Chaotic behaviour is possible only as a transient with lifetime t_H, the so-called Heisenberg time [18]. Nevertheless, t_H grows exponentially with the number of degrees of freedom and consequently the transient chaotic dynamics of quantum states and observables can be experimentally observed in Bose condensates [19]. Nowadays non-destructive images of the dynamics of the condensate can be obtained. The radius of the condensate as a function of time can be detected and its time evolution analyzed with great accuracy. Various initial conditions of the collective dynamics can be obtained by using laser beams or by modulating for a short period the magnetic fields which confine the condensate.

In conclusion, we have studied the collective oscillations of a Bose condensate in an anisotropic harmonic trap. We have found the values of trap deformation for which the normal modes of the collective oscillations are in resonance. For such values, nonlinear effects, like frequency shift and irregular patterns in the time evolution of the condensate shape, are strongly enhanced. We have analyzed the order-chaos transition of the $m_z = 0$ collective oscillations by increasing the energy, namely the amplitude of oscillation. We have obtained compelling evidence that, for certain values of trap anisotropy, chaos is reached at very low energy.

We hope that our calculations may stimulate new measurements on the collective oscillations of Bose condensates in anisotropic traps.

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Resonance	λ_1	λ_2	Resonance	λ
$(1,2,0)$	0.683	1.952	$(0,1,1)$	1
$(1,3,0)$	0.438	3.081	$(0,1,2)$	1.512
$(1,4,0)$	0.320	4.159	$(0,1,3)$	2.393
$(1,5,0)$	0.255	5.226	$(0,2,1)$	0.454
$(2,5,0)$	0.527	2.530	$(0,2,3)$	0.802
$(3,5,0)$	0.888	1.501	$(0,3,1)$	0.300

Table 1. Low-order resonances of the normal modes of the condensate and the corresponding values of the anisotropy parameter λ of the external trap.

FIGURE CAPTIONS

Figure 1. Poincarè Sections of the $m_z = 0$ modes. ¿From left to right: $\chi = 1/5$, $2/5$ and $3/5$, respectively. χ is the relative increase of energy with respect to the ground-state. Trap anisotropy $\lambda = 1.501$.

Figure 2. The same as in Figure 1. Trap anisotropy $\lambda = \sqrt{8}$ (JILA).

Figure 3. Configurations of the (λ,χ) plane for which the $m_z = 0$ oscillations are regular. λ is the trap anisotropy and χ is the relative increase of energy with respect to the ground-state. Full triangles: MIT trap. Full circles: $(3, 5, 0)$ resonance. Full squares: JILA trap. Open circles: all the other configurations.