e to the A, in a New Way, Some More to Say

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Abstract

Expressions are given for the exponential of a hermitian matrix, A. Replacing A by iA these are explicit formulas for the Fourier transform of e^{iA}. They extend to any size A the previous results for the 2×2, 3×3, and $4\ddot{i}mes4$ cases.

In a previous paper,[1], explicit formulas were obtained for the Fourier transform of e^{iA}, making manifest the theorem of E. Nelson [2], on the support of such Fourier transformations. But this was only done for $2\times 2, 3\times 3$ and 4×4 matrices. We now treat the general case.

Let A be an $r\times r$ hermitian matrix. We write

$$\mathrm{Det}(1-A) = \sum_{j=0}^{r} P_j(A) \tag{1}$$

where $P_j(A)$ is homogeneous of degree j in the entries of A. The formulas we obtained for the exponential of A are as follows:

$$(e^A)_{\alpha\beta} = \frac{1}{\Gamma(r)}\sum_{j=0}^{r} P_j(A)\frac{d^{r-j}}{ds^{r-j}}\left(\int d\Omega e^{sTr(AW)} W_{\alpha\beta} s^r\right)\Big|_{s=1} \tag{2}$$

or

$$(e^A)_{\alpha\beta} = \frac{1}{\Gamma(r)}\sum_{j=0}^{r} P_j(A)\frac{d^{r-j}}{ds^{r-j}}\left(\int d\Omega e^{s<A\vec{n},\vec{n}>} n_\alpha \bar{n}_\beta s^r\right)\Big|_{s=1} \tag{3}$$

Here $\vec{n}$ is a unit vector in $\mathbb{C}^r$, and $W_{ij} = n_i\bar{n}_j$, a rank one hermitian matrix. $\int d\Omega$ denotes a normalized integral over all such $\vec{n}$, an integral over the unit sphere in $\mathbb{C}^r$ with unitary-invariant measure.

In fact the formulas in (2) and (3) do not coincide with formulas in [1] when $r = 2, 3$ or 4, but formulas *of such type* are not unique. We do not know the full scope of such non-uniqueness.

We sketch a derivation/proof of formulas (2) and (3), especially emphasizing the **ideas**. We note the relation between gaussian integrals in $d = 2r$ real dimensions, and integrals over the corresponding unit sphere S^{d-1}.

$$\frac{1}{\mathcal{N}}\int dx_i e^{-\Sigma x_i^2}\prod^{2N} x_{\alpha(i)} = \frac{1}{\mathcal{N}}\int r^{d-1}r^{2N}e^{-r^2}dr\int d\Omega'\prod^{2N} n_{\alpha(i)} \tag{4}$$

$$= \int_{S^{d-1}} d\Omega\prod^{2N} n_{\alpha(i)}\cdot\frac{\int dre^{-r^2}r^{2N+d-1}}{\int dre^{-r^2}r^{d-1}} \tag{5}$$

$$= \int_{S^{d-1}} d\Omega\prod^{2N} n_{\alpha(i)}\frac{\Gamma\left(\frac{2N+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{6}$$

Here n_i is the unit vector parallel to x_i, and $\int d\Omega'$ is integral over the sphere in its usual measure and $\int d\Omega$ the normalized spherical measure. From (6) we see *the integral over a unit sphere of a homogeneous polynomial o f degree* $2N$ *is "approximately"* $1/N!$ *the gaussian integral of the same polynomial.*

We note that multiplying the term in A^k by $\frac{1}{k!}$ induces a transform (formally) as follows

$$1 + A + A^2 + \cdots = \frac{1}{1-A} \longrightarrow e^A \ . \tag{7}$$

We consider the gaussian integral formula:

$$\frac{\text{Det}(1-A)}{\mathcal{N}} \int dx_i e^{-\Sigma|x_i^2| + <A\vec{x},\vec{\bar{x}}>} x_\alpha \bar{x}_\beta = \left(\frac{1}{1-A}\right)_{\alpha\beta} . \tag{8}$$

In the expansion of the integrand on the left side of (8) each power of A has associated to it two powers of x. Thus converting from a gaussian integral to an integral over a unit sphere approximately multiplies each power of A^N by $\frac{1}{N!}$, which would convert $\frac{1}{1-A}$ to e^A. The **wrong** formula we get putting these ideas together would yield:

$$\text{"Det}(1-A) \int d\Omega e^{<A\vec{n},\vec{\bar{n}}>} n_\alpha \bar{n}_\beta = (e^A)_{\alpha\beta}\text{"} \tag{9}$$

It is just a couple of hours work to correct formula (9) (since there are powers of A in $\text{Det}(1-A)$, not just in the integrand; and because the transformation to the unit sphere does not yield exactly $1/N!$) and arrive at our formulas (2) and (3).

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References

[1] P. Federbush, "e to the A, in a New Way", math-ph/9903006, to be published in the *Michigan Math. Journal.*

[2] E. Nelson, *Operants: A functional calculus for non-commuting operators*, Functional Analysis and Related Fields, Proceedings for a conference in honor of Professor Marshal Stone (Univ. of Chicago, May 1968)(F.E. Browder, ed.), Springer-Verlag, Berlin, Heidelberg, and New York, 1970, pp. 172-187. MR 54:978.