How fast does Langton's ant move?

Jean Pierre Boon *
Center for Nonlinear Phenomena and Complex Systems
Université Libre de Bruxelles, 1050 - Bruxelles, Belgium

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Abstract

The automaton known as 'Langton's ant' exhibits a dynamical transition from a disordered phase to an ordered phase where the particle dynamics (the ant) produces a regular periodic pattern (called 'highway'). Despite the simplicity of its basic algorithm, Langton's ant has remained a puzzle in terms of analytical description. Here I show that the highway dynamics obeys a discrete equation where from the speed of the ant ($c = \sqrt{2}/52$) follows exactly.

Keywords: Langton's ant; propagation; lattice models.

Langton's ant has been a recurring theme in the mathematical and physical literature [1]. There are two reasons. The first is of physical relevance: the automaton known as Langton's ant (which I describe below) offers a prototype of complexity out of simplicity, and, in particular, can be viewed as a model of dynamical transition from a disordered phase to an ordered phase [2]. The second reason is mathematical: despite the simplicity of the basic algorithm, the spatio-temporal dynamics generated by the automaton (see Fig.1) has so far resisted analytical treatment.

The basic process governing the automaton dynamics follows a simple rule. The automaton universe is the square lattice with checker board parity so defining H sites and V sites. A particle moves from site to site (by one lattice unit length) in the direction given by an indicator. One may think of

*E-mail address: `jpboon@ulb.ac.be`

the indicator as a 'spin' (up or down) defining the state of the site. When the particle arrives at a site with spin up (down), it is scattered to the right (left) making an angle of $+\pi/2$ ($-\pi/2$) with respect to its incoming velocity vector. But the particle modifies the state of the visited site (up $\Longleftrightarrow$ down) so that on its next visit, the particle is deflected in the direction opposite to the scattering direction of its former visit. Thus the particle entering from below a H site with spin up is scattered East, and on its next visit to that same site (now with spin down), if it arrives from above, it will be scattered East again, while if it arrives from below, it will be scattered West. Similar reasoning shows how the particle is scattered North or South on V sites.

At the initial time, all sites are in the same state (all spins up or all spins down), and the position and velocity direction of the particle are fixed, but arbitrary. So if we paint the sites black or white according to their spin state, we start initially with say an all white universe. Then as the particle moves, the visited sites turn alternately black and white depending on whether they are visited an odd or even number of times. This color coding offers a way to observe the evolution of the automaton universe. The particle starts exploring the universe by first creating centrally symmetric transient patterns (see figures in references [1]), then after about 10 000 time steps (9977 to be precise), it leaves a seemingly 'random territory'[1] to enter a 'highway' (see Fig.1) showing a periodic pattern: in the highway, the particle travels with constant propagation speed.

A theorem by Bunimovich and Troubetzkoy [3] demonstrates that the automaton fulfills the conditions for unboundedness of the trajectory of the particle (see the highway in Fig.1). But, to the best of our knowledge, the dynamical transition from the disordered phase to the ordered phase remains unexplained to date.[2] Here, I consider the ordered phase and show analytically that the propagation speed is $c = \sqrt{2}/52$ (in lattice units) as measured in automaton simulations [1].

Because of the complexity of the dynamics on the square lattice, Grosfils, Boon, Cohen, and Bunimovich [4] developed a one-dimensional version of

[1]The disordered phase is not what a random walk would produce: the automaton is deterministic and its rules create correlations between successive states of the substrate, so also between successive positions of the particle. The power spectrum computed from the particle position time correlation function measured over the first 9977 time steps goes like $\sim \nu^{-\zeta}$ with $\zeta \simeq 4/3$. In the ordered phase ('highway'), the power spectrum shows a peak at $\nu = 1/104$ with harmonics.

[2]As well as the transient symmetrical patterns in the early stages of the dynamics.

the automaton for which they provided a complete mathematical analysis also applicable to the two-dimensional triangular lattice. One of their main results is the mean-field equation describing the microscopic dynamics of the particle subject to the more general condition that the spins at the initial time are randomly distributed on the lattice. The equation reads, for the one-dimensional lattice,

$$f(r+1,t) \,=\, q\,f(r,t-1) \,+\, (1-q)\,f(r,t-3)\,, \tag{1}$$

and, for the two-dimensional triangular lattice,

$$\begin{aligned} f(r+1,t) \;&=\; q(1-q)\,f(r,t-2) \\ &+\; [q^2+(1-q)^2]\,f(r,t-8) \\ &+\; (1-q)q\,f(r,t-14)\,, \end{aligned} \tag{2}$$

where $f(r,t)$ is the single particle distribution function, i.e. the probability that the particle visits site r *for the first time* at time t, and q is the probability that the immediately previously visited site along the propagation strip (the highway) has initially spin up, i.e. the probability that the particle be scattered, in the one-dimensional case, along the direction of its velocity vector when arriving at the scattering site at $r-1$, and, in the two-dimensional triangular case, along the direction forming clockwise an angle of $+2\pi/3$ with respect to the incoming velocity vector of the particle. Equations (1) and (2) express the probability of a first visit at a site along the propagation strip in terms of the probability of an earlier visit at the previous site along the strip. Note that in the two-dimensional case, the equation describes the one-dimensional propagation motion along the edge of the strip. The equations were shown to yield exact solutions for propagative behavior (corresponding to an ordered phase of the lattice) in the two classes of models considered by Grosfils *et al.* [4].

Equations (1) and (2) are generalized as follows

$$\begin{gathered} f(r+\rho,t) \,=\, \sum_{j=0}^{n} p_j(q)\; f(r,t-\tau_j) \;\; ; \\ \tau_j \,=\, (1+\alpha j)\,m\,\tau \;\; ; \;\; \sum_{j=0}^{n} p_j = 1 \;\; ; \;\; n \leq r/\rho\,, \end{gathered} \tag{3}$$

where, as above, $f(r,t)$ is the *first visit* distribution function. Here ρ denotes the elementary space increment of the dynamics along the propagation strip.

p_j is the probability that the particle propagates from r to $r+\rho$ in τ_j time steps, i.e. τ_j is the time delay between two successive first visits on the strip (more precisely on the one-dimensional edge of the strip) for the path with probability p_j, and m is the corresponding minimum number of automaton time steps ($\tau_0 \;=\; m\tau$, where τ is the automaton time step, $\tau = 1$). The sum is over all possible time delays, weighted by the probability p_j (a polynomial function of q). α denotes the number of lattice unit lengths in an 'elementary loop', i.e. the minimum number of lattice unit lengths necessary to return to a site.[3]

Now from the expectation value of the time delay, computed with (3),

$$E[\tau(q)] \;=\; \sum_{j=0}^{n} \tau_j \, p_j(q) \;=\; (1+\alpha\,\langle j\rangle)\, m\,\tau\,, \tag{4}$$

where $\langle j\rangle = \sum_{j=0}^{n} j\, p_j(q)$, one obtains immediately the average propagation speed of the particle in the ordered phase: $c(q) = \rho/E[\tau(q)]$.

It is straightforward to verify that Eqs. (1) and (2) are particular cases of the general equation (3):
for 1-D : $\alpha = 2$, $m = 1$, $\rho = 1$, $n = 1$, with $p_0 = q$, $p_1 = 1-q$;
for 2-D (triangular lattice) : $\alpha = 3$, $m = 2$, $\rho = 1$, $n = 2$, with $p_0 = q\,(1-q)$, $p_1 = q^2 + (1-q)^2$, $p_2 = (1-q)\,q$.
The corresponding propagation speeds are then readily obtained from (4); for the one-dimensional case one finds $c(q) = 1/\langle\tau(q)\rangle = [1+2(1-q)]^{-1} = 1/(3-2q)$, and for the triangular lattice: $\langle\tau(q)\rangle = [1+3(q^2+(1-q)^2+2q(1-q))]\times 2$, so that $c = 1/8$. These results are in exact agreement with those obtained in [4].

For the 2-D square lattice : $\alpha = 4$, $m = 2\times 4$, $\rho = 2\sqrt{2}$. The value of ρ is easily checked by inspection of the highway path shown in the upper box of Fig.1 : it is the length of the elementary increment along the edge of the propagation strip. Correspondingly m is 2×4 (because the minimum number of time steps necessary to move one elementary space increment must be counted on each edge of the strip). For the square lattice, one does not know the value of n, but from the structure of the p_j's for the 1-D and 2-D triangular lattices given above, one can infer a plausible value: $n = 6$, with $p_0 = p_6 = q^2\,(1-q)^2$, $p_1 = p_5 = q\,(1-q)\,[q^2 + (1-q)^2]$, $p_2 = p_4 = p_0 + p_1$, $p_3 = [q^2 + (1-q)^2]^2$. However the precise expressions are

[3]An interesting equation follows from the continuous limit of (3); this is discussed elsewhere (J.P. Boon, to be published).

unimportant for the automaton describing Langton's ant, because all sites are initially in the same spin state; so $q = 1$, and only one p_j is non-zero: $p_3 = 1$. Equation (3) then reads:

$$
\begin{aligned}
f(r + 2\sqrt{2}\,, t) \;&=\; f(r, t - \tau_3) \;\; ; \\
\tau_3 \;&=\; (1 + 4 \times 3)\, 2 \times 4 \;=\; 104\,,
\end{aligned}
\tag{5}
$$

which describes the dynamics of the particle in the highway. This result shows that a displacement of length $2\sqrt{2}$ along the edge of the strip is performed in 104 automaton time steps. Consequently the propagation speed of Langton's ant in the highway is $c = \rho/\tau_3 = 2\sqrt{2}/104 = \sqrt{2}/52$.

The origin of particle propagation in 1-D and 2-D triangular lattices was shown to be a 'blocking mechanism' [4], and the question was raised as to whether such a mechanism also exits in the square lattice. Although the precise blocking mechanism has yet to be identified, that the same general equation, Eq.(3), describes propagation in 1-D, 2-D triangular and square lattices suggests that a similar blocking mechanism is responsible for propagative dynamics in the highway of Langton's ant.

Note: In reference [4], the 'reorganization corollary' for the 2-D triangular lattice (Corollary 3, p.599) was incorrectly stated. It should read: *All sites located on one edge of the propagation strip are in the initial state of the sites on the other edge, shifted upstream by one lattice unit length.* The particle dynamics can then be interpreted as the controller of a Turing machine which transcribes and shifts the string of characters (0's and 1's for L and R) of the input tape (on one edge) to the output tape (on the other edge). The control operator is the EXCHANGE gate of Feynman's model of a quantum computer [5]. The same corollary applies trivially to the spin states (up and down spins interchanged as 0's and 1's) on the edges of the highway of Langton's ant.

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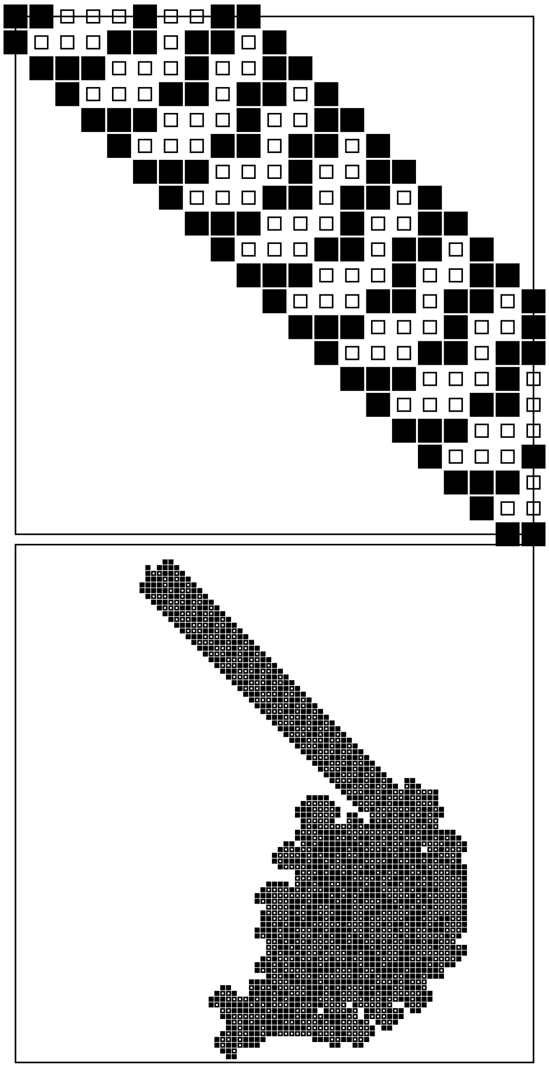

Figure 1: Langton's ant trajectory after 12,000 automaton time steps. The upper box is a blow-up of the highway showing the periodic pattern. Sites with open squares and dark squares have opposite spin states (up and down).