Existence and stability of traveling fronts in the extended Fisher-Kolmogorov equation.

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Abstract

We study traveling wave solutions to a general fourth-order differential equation that is a singular perturbation of the Fisher-Kolmogorov equation. We apply the geometric method for singularly perturbed systems to show that for every positive wavespeed there exists a traveling wave. Also, we find that there exists a critical wavespeed c^* which divides these solutions into monotonic ($c \geq c^*$) and oscillatory ($c < c^*$) solutions. We show that the monotonic fronts are locally stable under perturbations in appropriate weighted Sobolev spaces by using various energy functionals.

1 Introduction

We study the following equation

$$B_t = B + DB_{xx} - B_{xxxx} + \mathcal{N}(B), \tag{1.1}$$

where $x \in \mathbf{R}^n$, $t \in \mathbf{R}_+$, $D > 0$ and $\mathcal{N}$ is a nonlinear, analytic function that satisfies $\mathcal{N}(0) = 0$, $\mathcal{N}(1) = -1$, $\frac{d\mathcal{N}}{dB}(B) < 0$ for $0 < B < 1$ and $\frac{d\mathcal{N}}{dB}(B)_{|B=1} < -1$. Also, since $\mathcal{N}$ is a nonlinear term we know that $\frac{d\mathcal{N}}{dB}(B)_{|B=0} = 0$ holds. The choice of $D > 0$ is dictated by the physical requirement that the model is stable at short wavelengths. We will study the regime in which D is very large. The amplitude B is chosen to be real. We are interested in the existence and stability of traveling front solutions to this equation.

Equation (1.1) is found, with a quadratic ($\mathcal{N}(B) = -B^2$) or a cubic nonlinearity ($\mathcal{N}(B) = -B^3$), to describe the behaviour of small solutions of a reaction-diffusion system near a co-dimension 2 point, see [28, 30, 29]. There, a general reaction-diffusion system is studied for small perturbations of the zero-solution where an amplitude equation approach is used. Near criticality, the behaviour of these solutions is described by the Ginzburg-Landau (GL) equation. However, moving towards a co-dimension 2 point in parameter-space, this GL-equation cannot be used anymore to describe the behaviour of the solutions and it is shown that it must be replaced by the extended Fisher-Kolmogorov (eFK) equation (1.1) with a quadratic or a cubic nonlinearity (depending on the nonlinear terms in the reaction-diffusion system). In [30], the relation between the GL-equation and the eFK-equation that comes from this amplitude approach is studied asymptotically. Also, the full reaction-diffusion system is studied numerically. It is observed that the stable, stationary, periodic patterns predicted by the GL-equation evolve towards various different families of stable stationary (but not necessarily periodic) so-called 'multi-bump' solutions. These multi-bump patterns are shown to exist as solutions to the (stationary) eFK-equation with a quadratic [6] or with a cubic nonlinearity [20, 26, 19]. When studying the stationary part of the reaction-diffusion system by a normal form approach, solutions are found (to the normal form) that have the same multi-bump structure as the solutions that were found in the numerical simulations, see [29]. Existence and stability of traveling fronts in an autocatalytic reaction-diffusion system is studied in [14].

When we rescale x with ε, i.e. introduce $\tilde{x} = \varepsilon x$ in equation (1.1) we obtain

$$B_t = B + \varepsilon^2 D B_{\tilde{x}\tilde{x}} - \varepsilon^4 B_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \mathcal{N}(B). \tag{1.2}$$

Therefore, we choose $D = \frac{1}{\varepsilon^2}$ where ε is introduced to be a small parameter. This implies that we take D large. Taking D large is interesting because this choice corresponds to studying a reaction-diffusion system where the rates of diffusion are very different, see [30] for a expression of D in terms of these diffusion coefficients. The choice $D = \frac{1}{\varepsilon^2}$ implies that equation (1.1) is a singular perturbation of the so-called Fisher-Kolmogorov (FK) equation

$$u_t = u_{xx} + u - u^2,$$

associated with the names of Fisher [13] and Kolmogorov, Petrovsky and Piskunov [23]. Equation (1.2) is an interesting perturbation of the FK-equation since the fourth order term would come into the equation as one of the higher order terms when doing an amplitude equation approach. The FK-equation arose in the context of a genetic model for the spread of an advantageous gene through a population. These types of reaction-diffusion equations have also been used as mathematical models that describe the motion and interaction of different types or species in chemistry and biology [12]. It is shown in [2] that, for an appropriate nonlinearity, the FK-equation has a continuous family of traveling fronts describing the invasion of one species by another.

For certain choices of the nonlinearity $\mathcal{N}$, equation (1.1) has already been studied, especially for $\mathcal{N}(B) = -B^3$. With this choice the equation is generally referred to as the extended Fisher-Kolmogorov (eFK) equation. It served as a model for the study of bi-stable systems arising in a variety of situations [7, 8]. An important application of the eFK-equation is found in the theory of instability in nematic liquid crystals [5, 33]. So far mainly stationary solutions have been studied and it was shown that the eFK-equation admits a standing traveling wave connecting $B = -1$ and $B = 1$ for all $D > 0$. This 'kink' solution has been obtained by variational methods [27] and by ODE methods, see [25] and the references given there. Also, periodic solutions , solitons (homoclinic/heteroclinic orbits) and chaotic patterns have been studied for the stationary part of this equation, see [26, 20]. For the nonlinearity $\mathcal{N} = a(1 - B^2) - B^3$ $(0 < a < 1)$ the existence of traveling waves connecting $B = \pm 1$ has been studied in [1] and [3]. Finally, in [3] it is also shown that for $\mathcal{N}(B) = -B^2$ there exists a traveling wave from $B = 1$ to $B = 0$ for every positive wavespeed and $D \in \mathbf{R}$.

We start our study by demonstrating the existence of traveling fronts to equation (1.1). Traveling front solutions are of the form

$$B(x,t) = \phi(x - \tilde{c}t), \tag{1.3}$$

where $\tilde{c}$ is the wavespeed. We are looking for solutions where $\lim_{\xi\to\infty} \phi(\xi) = 0$ and $\lim_{\xi\to-\infty} \phi(\xi) = 1$ and $\xi = x - \tilde{c}t$. Here $\phi \equiv 1$ is a stable stationary state of (1.2) and $\phi = 0$ is an unstable state. One could also study fronts which connect two stable solutions of (1.1), if any exist. On the basis of extensive studies of fronts in second order equations we expect the properties of these two types of fronts to be quite different. For instance stable-stable fronts typically exist for a unique speed while unstable-stable fronts usually exist for a range of speeds, [32]. For questions of pattern formation unstable-stable fronts seem to be most relevant and we will focus exclusively on them in this article. The prior studies of fronts in the eFK-equation, [27, 25, 1] and [3] focussed on the existence of fronts connecting $B = +1$ to $B = -1$ which are stable-stable fronts. As far as we know the only other rigorous work on unstable-stable fronts in the eFK-equation is [3], which proves that such fronts exist if one chooses a nonlinear term $\mathcal{N}(B) = -B^2$ and which proves the existence of fronts connecting $B = -a$ to $B = \pm 1$ for $\mathcal{N}(B) = a(1 - B^2) - B^3$ where $0 < a < 1$. Our work extends that of [3] in several ways. First of all, we consider a general class of nonlinear terms and we develop a geometrical proof of the existence of these fronts as opposed to the variational approach in [3]. More importantly, we also analyse the nonlinear stability of these fronts which does not yet appear to have been discussed.

As is shown in section 2.1 the equation for the front profile can be reduced, after substituting (1.3) into (1.1), to

$$\varepsilon^4 \phi'''' = \phi'' + c\phi' + \phi + \mathcal{N}(\phi), \tag{1.4}$$

where we use that $D = \frac{1}{\varepsilon^2}$ and we introduce $\zeta = \varepsilon\xi$ and $c = \varepsilon\tilde{c}$. The equation (1.4) can be written as the following 4-dimensional singularly perturbed system by introducing $u = \phi, u' = v, v' = w$ and $\varepsilon^2 w' = z$

$$\left\{\begin{array}{rcl} u' &=& v \\ v' &=& w \\ \varepsilon^2 w' &=& z \\ \varepsilon^2 z' &=& w + cv + u + \mathcal{N}(u). \end{array}\right. \tag{1.5}$$

Then, this 4-dimensional system is analysed by the geometric theory for singularly perturbed systems originally developed by Fenichel [11]. Here we use a form of this theorem due to Jones [18] to show that, for ε sufficiently small, there exists a solution which connects the two fixed points $(0,0,0,0)$ and $(1,0,0,0)$. Also, we find that there exists a critical wavespeed c^* such that for $c \geq c^*$ the solution is monotonic decreasing and for $c < c^*$, the solution oscillates around the origin. In section 4 we determine this c^* explicitly.

In section 5, we will prove that the solutions are locally stable in appropriate function spaces for all $c \geq c^*$. For $c < c^*$, the solutions are expected to be unstable, however, we do not prove that in this paper. The stability of traveling waves of the FK-equation and similar parabolic equations has been studied extensively, where various methods were used. For example, [23, 13, 2] used comparison theorems based on the Maximum Principle. Using functional analytic techniques, renormalisation group methods or energy functionals, a local stability analysis in suitable weighted spaces has been studied by Sattinger [31], Kirchgässner [22], Kapitula [21], Bricmont and Kupiainen [4], Eckmann and Wayne [10] and Gallay and Raugel [15, 16, 17].

The front solutions we construct are only expected to be stable if we restrict ourselves to perturbations which decay to 0 sufficiently fast as $x \to \infty$. To achieve this, we look for perturbations of the front solution of the form

$$B(x,t) = \phi(\zeta) + a(\zeta)w(\zeta,t), \tag{1.6}$$

where $\zeta = \varepsilon x - ct$ and $a(\zeta) = e^{-\gamma\zeta}$ for some $\gamma > 0$ that will be chosen later on.

Hypothesis We assume that the nonlinearity $\mathcal{N}$ is analytic and satisfies $\mathcal{N}(0) = \mathcal{N}'(0) = 0$, $\mathcal{N}(1) = -1$, $\mathcal{N}'(B) < 0$ for $B \in (0,1)$ and $\mathcal{N}'(1) < -1$ as stated above, and in addition that the coefficient of either the quadratic or cubic term in the power series for $\mathcal{N}$ is non-zero.

Remark 1.1 *In particular, the above assumptions hold for either of the common choices of* $\mathcal{N}(u) = -u^2$ *or* $\mathcal{N}(u) = -u^3$.

We substitute the above expression (1.6) into equation (1.1) and use the fact that $\phi(\varepsilon x - ct)$ satisfies equation (1.1). Then, using that $D = \frac{1}{\varepsilon^2}$, leads to

$$\begin{aligned} \frac{\partial w}{\partial t} &= (1 - 6\varepsilon^4\gamma^2)w_{\zeta\zeta} + (c - 2\gamma + 4\varepsilon^4\gamma^3)w_\zeta + (1 - c\gamma + \gamma^2 - \varepsilon^4\gamma^4)w + 4\varepsilon^4\gamma w_{\zeta\zeta\zeta} \\ &\quad -\varepsilon^4 w_{\zeta\zeta\zeta\zeta} + wg, \end{aligned} \tag{1.7}$$

where

$$wg = \frac{1}{a}(\mathcal{N}(\phi + aw) - \mathcal{N}(\phi)). \tag{1.8}$$

Note that for $1 - c\gamma + \gamma^2 - \varepsilon^4\gamma^4 > 0$, the solution $w = 0$ will be linearly unstable. Therefore we choose γ to be the smallest positive root of $1 - c\gamma + \gamma^2 - \varepsilon^4\gamma^4 = 0$, this gives us the biggest perturbation space in which we can hope for stability. In this discussion we follow the intuition provided by Gallay and Raugel's [16] study of the stability of fronts in damped hyperbolic equations. The choice of γ implies that we must take $c \geq c^*$ and that γ equals the exponential decay rate of ϕ as $\zeta \to \infty$. With this choice of γ, equation (1.7) reduces to

$$\frac{\partial w}{\partial t} = (1 - 6\varepsilon^4\gamma^2)w_{\zeta\zeta} + (c - 2\gamma + 4\varepsilon^4\gamma^3)w_\zeta + 4\varepsilon^4\gamma w_{\zeta\zeta\zeta} - \varepsilon^4 w_{\zeta\zeta\zeta\zeta} + wg. \tag{1.9}$$

Using a Taylor expansion, the term wg can be rewritten as

$$wg = \mathcal{N}'(\phi)w + aw^2R,$$

where we know that there exists a $0 < \eta < aw$ such that $R = \frac{1}{2}\mathcal{N}''(\phi + \eta)$. From the fact that the nonlinearity $\mathcal{N}$ is analytic, it follows that R is bounded by a constant $K_R > 0$.

The main results of section 5 can be summarised in the following theorem

Theorem 1.2 *Let $\mathcal{N}$ satisfy the above assumptions. There exists an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ and $c \geq c^*$, there exists a $\delta > 0$ such that if we choose $\psi \in X^5$, the function space defined below, with $||\psi||_{X^3} \leq \delta$ and, let w be the corresponding solution of (1.9) such that $w(0) = \psi$ and let B be as in (1.6), then*

$$\lim_{t\to\infty} ||B(\cdot, t) - \phi(\varepsilon \cdot -ct)||_{L^\infty} = 0$$

i.e. the solution ϕ is stable.

We define the following function spaces. By $H^j = H^j(\mathbf{R}^n)$ for $j \in \mathbf{N}$ we denote the usual (real) Sobolev space of order j over $\mathbf{R}^n$, with $H^0(\mathbf{R}^n) = L^2(\mathbf{R}^n)$. We denote by $H^j_a = H^j_a(\mathbf{R}^n)$ the weighted Sobolev space defined by the norm $||w||_{H^j_a} = ||aw||_{H^j}$. We also set $H^0_a(\mathbf{R}^n) = L^2_a(\mathbf{R}^n)$. Finally, we write X^j for the intersection $H^j \cap H^j_a$ equipped with the norm $||w||^2_{X^j} = ||w||^2_{H^j} + ||w||^2_{H^j_a}$

To prove Theorem 1.2 we introduce the following functionals

$$\begin{aligned}
E(t) &= \int_{-\infty}^{\infty} \frac{1}{2}Aw^2 + \frac{1}{2}Bw_\zeta^2 + \frac{1}{2}Cw_{\zeta\zeta}^2 + \frac{1}{2}\varepsilon^6 w_{\zeta\zeta\zeta}^2 d\zeta, &(1.10)\\
F(t) &= \int_{-\infty}^{\infty} \frac{1}{2}\tilde{A}(aw)^2 + \frac{1}{2}\tilde{B}((aw)_\zeta)^2 + \frac{1}{2}\tilde{C}((aw)_{\zeta\zeta})^2 + \frac{1}{2}\varepsilon^6((aw)_{\zeta\zeta\zeta})^2 d\zeta, &(1.11)\\
F_1(t) &= \beta E(t) + F(t), &(1.12)
\end{aligned}$$

where $A, B, C, \tilde{A}, \tilde{B}, \tilde{C}$ and β are positive constants that are chosen such that they satisfy Lemma 5.7 and Lemma 5.9 (these functionals are modelled on those of [16]).

We will show in section 5 that when w is sufficiently small -this is satisfied because w is a perturbation of the traveling wave solution- E and F_1 are non-increasing functions of time. From this the convergence of w to zero follows and therefore the solution ϕ is stable.

Our results also shed some light on the very interesting ideas of Ebert and van Saarloos [9]. Although their arguments are formal these authors predict the nonlinear stability of what they refer to as 'pulled fronts' and also predict the rate and form of convergence towards such fronts. In the context of our model, pulled fronts are precisely those with $c = c*$ and in accordance with the predictions of [9], we show that these fronts are stable. Furthermore, [9] predict that if the front solution $\phi(\zeta) \approx \alpha e^{-\gamma\zeta}$ as $\zeta \to \infty$ (or $\phi(\zeta) \approx \alpha\zeta e^{-\gamma\zeta}$ for $c = c^*$), then the front should be stable with respect to perturbations which obey a steepness condition which implies that the perturbation goes to zero as ζ goes to infinity faster than $e^{-\gamma\zeta}$. This should be compared with our result which proves that the front is stable so long as the perturbation $p(\zeta)$ is in X^5, and the function $e^{-\gamma\zeta}p(\zeta)$ is also in X^5. While the condition that $e^{-\gamma\zeta}p(\zeta)$ be in X^5 might be weakened to a requirement that it be in X^s for some $s < 5$, the point we wish to emphasise is that the stability requires that the decay rate of the perturbation as $\zeta \to \infty$ is linked to that of the decay rate of the front solution itself, again as predicted by [9]. Indeed, if the perturbation decays with a rate $e^{-\delta\zeta}$, with $\delta < \gamma$, then we see from the discussion just prior to (1.9) that the front will be linearly unstable.

Remark 1.3 *The outline of this paper is as follows: in sections 2 and 3 we prove the existence of traveling fronts and in section 4 we determine the critical speed. In section 5 the stability of these traveling fronts is studied.*

2 The existence of traveling fronts

2.1 The equation for traveling fronts.

In this section we will derive the equation which describes the behaviour of traveling fronts of equation (1.1). Traveling fronts are solutions of the form

$$B(x,t) = \phi(x - \tilde{c}t), \tag{2.1}$$

where $\tilde{c}$ is the wavespeed. We are interested in solutions where $\lim_{\xi\to\infty}\phi(\xi) = 0$ and $\lim_{\xi\to-\infty}\phi(\xi) = 1$.

Substituting (2.1) into equation (1.1) gives

$$-\tilde{c}\phi_\xi = \phi + D\phi_{\xi\xi} - \phi_{\xi\xi\xi\xi} + \mathcal{N}(\phi), \tag{2.2}$$

where we introduce $\xi = x - \tilde{c}t$. As we described in the introduction, we are interested in the regime in which D is very large and hence we look for solutions to this equation for $D = \frac{1}{\varepsilon^2}$ where $\varepsilon \ll 1$. Taking $D = \frac{1}{\varepsilon^2}$ and introducing $\zeta = \varepsilon\xi$ into equation (2.2) leads to

$$\varepsilon^4\phi'''' = \phi'' + c\phi' + \phi + \mathcal{N}(\phi), \tag{2.3}$$

where the primes mean differentiation with respect to ζ and $c = \varepsilon\tilde{c}$.

We will show below that (2.3) has solutions with $c = \mathcal{O}(1)$. This corresponds to very 'fast' ($\tilde{c} = \mathcal{O}(\frac{1}{\varepsilon})$) fronts in the original variables.

Remark 2.1 *Note that if we define* $\tilde{x} = xD^{-\frac{1}{2}}$, *then (1.1) can be written as*

$$B_t = B + B_{\tilde{x}\tilde{x}} - \frac{1}{D^2} B_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \mathcal{N}(B).$$

The existence of front solutions of this form of the extended Fisher-Kolmogorov equation has been studied in [25, 27] for a cubic nonlinearity and in [1, 3] for $\mathcal{N}(B) = a(1-B^2) - B^3$ $(0 < a < 1)$.

The equation (2.3) can be written as the following 4-dimensional system by introducing, $u = \phi$, $u' = v$, $v' = w$ and $\varepsilon^2 w' = z$,

$$\begin{cases} u' &= v \\ v' &= w \\ \varepsilon^2 w' &= z \\ \varepsilon^2 z' &= w + cv + u + \mathcal{N}(u). \end{cases} \tag{2.4}$$

Clearly (2.4) is a singularly perturbed problem and as usual in such cases we introduce a second system of equations by defining $\zeta = \varepsilon^2 \eta$ into system (2.4). This leads to

$$\begin{cases} \dot{u} &= \varepsilon^2 v \\ \dot{v} &= \varepsilon^2 w \\ \dot{w} &= z \\ \dot{z} &= w + cv + u + \mathcal{N}(u), \end{cases} \tag{2.5}$$

where the dot means differentiation with respect to η. Generally (2.4) is referred to as the slow system, since the time scale ζ is slow, and (2.5) as the fast system, since the time scale η is fast. Here w and z are called the fast (=$\mathcal{O}(1)$) variables and u and v are called the slow (=$\mathcal{O}(\varepsilon)$) variables.

Remark 2.2 *In [1], a similar reduction is used to study traveling wave solutions of a fourth order semi-linear diffusion equation. In that reference, a special choice of the non-linear term is made. More importantly, the traveling wave solutions which are constructed there connect two stable states whereas here we are interested in fronts that connect an unstable state* ($\phi = 0$) *with a stable state* ($\phi = 1$). *One expects, on the basis of physical arguments and the examples studied in second order partial differential equations that the properties of these two sorts of fronts will be quite different, as is explained in the introduction.*

2.2 The unperturbed system.

First, we will study the $\epsilon \to 0$ limit of the fast system. Setting $\epsilon = 0$ in (2.5), we find that the resulting system has a two dimensional manifold of fixed points. In this limit, $u = u_0$ and $v = v_0$ are obviously constants of the motion. The two remaining equations are:

$$\begin{cases} \dot{w} &= z \\ \dot{z} &= w + cv_0 + u_0 + \mathcal{N}(u_0). \end{cases} \tag{2.6}$$

The critical points of this system (using the fact that u_0 and v_0 are time independent) are clearly $z = 0$ and $w = -(cv_0 + u_0 + \mathcal{N}(u_0))$, so that the invariant manifold of (2.5) is given by

$$M_0 = \{(u, v, w, z) | u = u_0, v = v_0, z = 0, w = -(cv_0 + u_0 + \mathcal{N}(u_0))\}.$$

If we linearise (2.5) about any point in M_0, the linearised equations have, for $\varepsilon = 0$, two zero eigenvalues (corresponding to directions tangent to M_0), and two nonzero eigenvalues $\lambda_\pm = \pm 1$. Therefore, the manifold M_0 is normally hyperbolic.

As is well known (see, for example, [18],) if ϵ is small, but nonzero, the flow on this manifold is given to a first approximation by setting $\varepsilon = 0$ in (2.4), from which we obtain

$$\left\{ \begin{array}{rcl} u' & = & v \\ v' & = & -(cv + u + \mathcal{N}(u)), \end{array} \right. \tag{2.7}$$

since $w = -(cv + u + \mathcal{N}(u))$. Combining these equations into a single second order equation that approximately describes this flow we find

$$u'' + cu' + u + \mathcal{N}(u) = 0. \tag{2.8}$$

Remark 2.3 *This ordinary differential equation is exactly the same one that describes traveling fronts for the extensively studied second order PDE, the FK-equation*

$$u_t = u + u_{xx} + \mathcal{N}(u). \tag{2.9}$$

In particular, from the results of [2], we know that provided $\mathcal{N}$ satisfies $\mathcal{N}(0) = 0$, $\mathcal{N}(1) = -1$ and $u + \mathcal{N}(u) > 0$ for $u \in (0, 1)$, (2.9) has traveling front solutions, and hence (2.8) has a solution connecting the stationary points $u = 1$ with $u = 0$.

We now state precisely the results on invariant manifolds which allow us to study (2.4) and (2.5) for ϵ small but nonzero. The first use of normally hyperbolic invariant manifolds to treat such singularly perturbed problems was due to Fenichel [11]. We use a form of this theorem due to C. Jones [18]. The following theorem is a combination of Theorems 1,2 and 3 of [18].

Theorem 2.4 *Given a C^∞ vector field of the form*

$$\begin{array}{rcl} x' & = & f(x, y, \epsilon) \\ y' & = & \epsilon g(x, y, \epsilon) \end{array} \tag{2.10}$$

such that when $\epsilon = 0$, the system of equations has a compact, normally hyperbolic, manifold of critical points, M_0, with M_0 given as the graph of a C^∞ function $h^0(y)$. Then for every $r > 0$, there exists an $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$, there exists a manifold M_ϵ which is:

1. *Locally invariant under the flow defined by (2.10).*

2. C^r *in* x, y, *and* ϵ.

3. $M_\epsilon = \{(x,y) \mid x = h^\epsilon(y)\}$, *for some* C^r *function* h^ϵ, *and* y *in some compact set* K.

4. *There are local stable and unstable manifolds,* $W^s(M_\epsilon)$ *and* $W^u(M_\epsilon)$, *that lie within* $\mathcal{O}(\epsilon)$ *of and are diffeomorphic to* $W^s(M_0)$ *and* $W^u(M_0)$, *respectively. Here,* $W^s(M_0)$ *and* $W^u(M_0)$ *are the stable and unstable manifolds of the unperturbed manifold* M_0.

As a corollary of Theorem 2.4, we have

Corollary 2.5 *Fix* $r > 0$. *There exists an* $\epsilon_0 > 0$ *such that for* $|\epsilon| < \epsilon_0$, *the system of equations (2.5) has a* C^r *invariant manifold,* M_ϵ *which is a small perturbation of* M_0. *Furthermore, the flow on this invariant manifold is given by a small perturbation of (2.8). Finally, any solution of (2.5) which remains in a neighbourhood of the fixed points of system (2.5) for all* $t \in \mathbf{R}$ *must lie in* M_ϵ.

Proof: The existence of the invariant manifold follows immediately from Jones' theorem 2.4. Although the manifold of fixed points M_0 in our case is not compact, we can consider its intersection with a ball of radius $R > 1$. This gives a normally hyperbolic manifold of critical points. The fact that the flow on the manifold is a small perturbation of (2.8) follows from the fact if we denote the parameterisation of the manifold M_ϵ as $w = h^\epsilon_w(u,v)$, $z = h^\epsilon_z(u,v)$, we can derive the restriction of the vector field to the manifold substituting these expressions for w and z into (2.5). If we then rescale the 'time' parameter as we did to go from (2.4) to (2.5) and note that for ϵ small, $h^\epsilon_z \approx 0$ and $h^\epsilon_w \approx -(cv + u + \mathcal{N}(u))$, we find that the resulting equations are close to (2.8). Finally, if any solution in a neighbourhood of a fixed point remains in that neighbourhood for all time, it cannot lie in either $W^s(M_\epsilon)$ or $W^u(M_\epsilon)$, so it must lie in M_ϵ itself. □

Remark 2.6 *We can write the function whose graph gives the invariant manifold more precisely as* $(h^\epsilon_w(u,v), h^\epsilon_z(u,v)) = (-(cv + u + \mathcal{N}(u)) + H^\epsilon_w(u,v), H^\epsilon_z(u,v))$ *where* H^ϵ_w *and* H^ϵ_z *are both* $\mathcal{O}(\epsilon)$.

3 The perturbed manifold, M_ϵ

We will show, by using the results of the previous section, that for ε sufficiently small there exists a solution which connects $(0,0,0,0)$ to $(1,0,0,0)$. We now discuss the flow on the perturbed manifold M_ϵ. First note that as a consequence of Corollary 2.5, the two fixed points $(0,0,0,0)$ and $(1,0,0,0)$ of (2.5) must lie in the invariant manifold M_ϵ. Furthermore, both of the fixed points are hyperbolic and hence for ϵ sufficiently small, they will be hyperbolic fixed points of the flow on M_ϵ.

We examine each fixed point in turn where we from now on focus on $c > 0$. The fixed point at the origin is a sink for all $c > 0$ (there exists a critical wavespeed denoted by c^* such that the origin is a node for $c > c^*$ and a focus for $0 < c < c^*$). This implies that if we fix $c_0 > 0$ (arbitrarily small), then there exist $s > 0$ and $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, and $c_0 < c < c_0^{-1}$, all solutions within a circle of radius s from the origin are

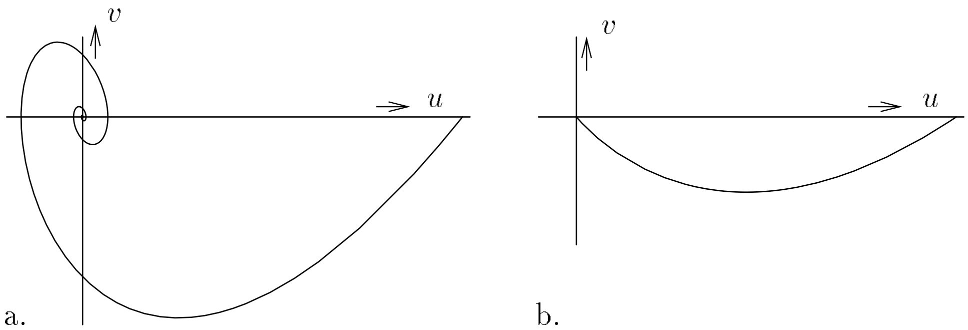

Figure 1: The solutions in the (u,v)-plane for a. $c < c^*$ and b. $c \geq c^*$.

attracted to the origin by the flow on M_ϵ.

Next consider the point $(1,0,0,0)$. Note that because of the fact that we assumed that $\frac{d\mathcal{N}}{du}(u)_{|u=1} < -1$, this point is a saddle for all $c > 0$. Since the flow on M_ϵ is a small perturbation of the flow defined by (2.8), any finite piece of the unstable manifold of $(1,0,0,0)$, restricted to M_ϵ will vary smoothly with ϵ. In particular, since the unstable manifold of the fixed point $(1,0)$ of (2.8) is asymptotic to the origin for $c > 0$, we can choose $\epsilon_0 > 0$ (possibly smaller than the ϵ_0 chosen in the previous paragraph) such that for c_0 and s as above, the unstable manifold of $(1,0,0,0)$ for the flow defined by (2.5), restricted to M_ϵ, passes within a distance s of the origin. But then by the results of the previous paragraph, it must be attracted to the origin – i.e. the unstable manifold of $(1,0,0,0)$ is asymptotic to $(0,0,0,0)$ as $\xi \to \infty$, and we have proven:

Proposition 3.1 *Fix $c_0 > 0$ (arbitrarily small). There exists an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and $c_0 < c < c_0^{-1}$, there exists a solution $(u(\xi), v(\xi), w(\xi), z(\xi))$ which is asymptotic as $\xi \to -\infty$ to $(1,0,0,0)$ and as $\xi \to \infty$ to $(0,0,0,0)$. Furthermore, this solution lies in M_ϵ.*

Corollary 3.2 *Fix $c_0 > 0$ (arbitrarily small). There exists an $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$ and $c_0 < c < c_0^{-1}$, equation (1.1) has a front solution with speed c.*

Remark 3.3 *We leave it as an (easy) exercise for the reader to show that $\phi(\zeta) \approx c_1 e^{-\gamma\zeta}$ as $\zeta \to \infty$ for $c > c^*$ and that γ satisfies $1 - c\gamma + \gamma^2 - \varepsilon^4\gamma^4 = 0$ as claimed following (1.7).*

4 The critical speed

For the stability results of the next section it will be important to know whether the fronts we constructed above are monotonic or not. For the 'classical' Fisher-Kolmogorov equation, fronts with speeds $c \geq 2$ are monotonic (and stable) whereas for $0 < c < 2$ the fronts oscillate. This value of c corresponds to the value at which the eigenvalues of the linearisation of the flow (2.8) at the origin become complex. We denote this critical wavespeed by c^*. The solutions can be given in the (u,v)-plane as in figure 1. We consider the flow of (2.5) restricted to M_ϵ and ask for what speed the eigenvalues of the linearisation of this flow at the origin become complex.

As noted in Remark 2.6, M_ϵ can be written as the graph of a function

$$(w,z) = (-(cv+u+\mathcal{N}(u)) + H_w^\epsilon(u,v), H_z^\epsilon(u,v)) \,, \tag{4.1}$$

where H_w^ϵ and H_z^ϵ are both $\mathcal{O}(\epsilon)$. The flow on M_ϵ can be obtained by inserting these expressions for w and z into the equations of motion for u and v in (2.4), leading to

$$\begin{aligned} u' &= v \\ v' &= -(cv+u+\mathcal{N}(u)) + H_w^\epsilon(u,v). \end{aligned} \tag{4.2}$$

Linearising about the origin, we see that the matrix of the linearisation is

$$\begin{pmatrix} 0 & 1 \\ -1+\partial_u H_w^\epsilon(0,0) & -c+\partial_v H_w^\epsilon(0,0) \end{pmatrix},$$

(recall that $\frac{\partial\mathcal{N}}{\partial u}{}_{|u=0} = 0$). Thus, the eigenvalues of the linearisation of the vector field on M_ϵ at the origin are

$$\lambda_\pm = \frac{1}{2}\{-c+\partial_v H_w^\epsilon(0,0) \pm \sqrt{(c-\partial_v H_w^\epsilon(0,0))^2 - 4(1-\partial_u H_w^\epsilon(0,0))}\} \,.$$

From this expression, we see that both of the eigenvalues of the linearisation about the fixed point at the origin will be real, and the front solution monotonic, provided

$$c \geq 2\sqrt{1-\partial_u H_w^\epsilon(0,0)} + \partial_v H_w^\epsilon(0,0). \tag{4.3}$$

A standard calculation involving the invariant manifold theorem allows one to compute $H_w^\epsilon(0,0)$ and $H_w^\epsilon(0,0)$, and as demonstrated in Appendix A, one finds $\partial_v H_w^\epsilon(0,0) = \epsilon^4(1-c^2) + \text{hot}$, while $\partial_u H_w^\epsilon(0,0) = \epsilon^4(2c-c^3) + \text{hot}$. Inserting these into the previous relation we find:

Proposition 4.1 *The front solutions of the eFK-equation (1.1) constructed above are monotonic if their speed c satisfies (4.3). For small ϵ, this implies that the speed satisfies $c \geq 2-\epsilon^4+\ldots$.*

Remark 4.2 *The critical speed c^* can also be calculated by using another method. As can be seen from equation (2.3) the eigenvalues of the linearisation of the 4-dimensional system (2.4) at the origin must satisfy $\varepsilon^4\lambda^4 = \lambda^2 - c\lambda + 1$. From this it can be determined for which speed the eigenvalues become complex, this also gives the critical wavespeed c^*.*

5 The stability of the traveling wave solutions.

In this section we will study the stability of the traveling wave solutions that we found in the previous sections. We will prove that for $c \geq c^*$ the solutions are stable.

First, we substitute all the rescalings back in the traveling wave solution $\phi(y)$ that we found. In the previous section we introduced the rescalings $\zeta = \varepsilon\xi = \varepsilon(x - \tilde{c}t)$ and $c = \varepsilon\tilde{c}$. So,

$$\phi(y) = \phi(\varepsilon(x - \tilde{c}t)) = \phi(\varepsilon x - ct). \tag{5.1}$$

Recall that we also have that $\lim_{\zeta\to-\infty}\phi(\zeta) = 1$ and $\lim_{\zeta\to\infty}\phi(\zeta) = 0$. We shall assume without loss of generality that $\phi(0) = \frac{1}{2}$ and that $\phi(\zeta) \geq \alpha a(\zeta)$ for all $\zeta \geq 0$ and some $\alpha > 0$. This can always be achieved by replacing $\phi(\zeta)$ with $\phi(\zeta - \zeta_0)$ for some sufficiently large $\zeta_0 > 0$.

We assume that the nonlinearity $\mathcal{N}$ is an analytic function and satisfies $\mathcal{N}(0) = \mathcal{N}'(0) = 0$, $\mathcal{N}(1) = -1$, $\mathcal{N}'(\phi) < 0$ for $\phi \in (0,1)$ and $\mathcal{N}'(1) < -1$ and that the coefficient of either the quadratic or cubic term in the power series for $\mathcal{N}$ is non-zero. Since $\mathcal{N}$ is a nonlinear function where $\mathcal{N}'(\phi) < 0$ for all $0 < \phi < 1$ we can find a function $\tilde{F}$ such that $\mathcal{N}'(\phi) := \mathcal{F}(\phi) = -\phi\tilde{F}(\phi)$ and $\tilde{F}(\phi) > 0$ for all $0 < \phi < 1$.

As explained in the introduction we will study a perturbation of these traveling wave solutions:

$$B(x,t) = \phi(\zeta) + a(\zeta)w(\zeta,t), \tag{5.2}$$

where $a(\zeta) = e^{-\gamma\zeta}$ and $\gamma > 0$ is the smallest positive root of $1 - c\gamma + \gamma^2 - \varepsilon^4\gamma^4 = 0$. We will assume that w is a function that is small. Also, we know that w satisfies equation (1.9). We will first show the local existence of solutions to (1.9).

Recall that the function spaces were defined as follows. By $H^j = H^j(\mathbf{R}^n)$ for $j \in \mathbf{N}$ we denote the usual (real) Sobolev space of order j over $\mathbf{R}^n$, with $H^0(\mathbf{R}^n) = L^2(\mathbf{R}^n)$. We denote by $H_a^j = H_a^j(\mathbf{R}^n)$ the weighted Sobolev space defined by the norm $||w||_{H_a^j} = ||aw||_{H^j}$. We also set $H_a^0(\mathbf{R}^n) = L_a^2(\mathbf{R}^n)$. Finally, we write X^j for the intersection $H^j \cap H_a^j$ equipped with the norm $||w||_{X^j}^2 = ||w||_{H^j}^2 + ||w||_{H_a^j}^2$.

The main results of this section can be summarised in the following theorem

Theorem 5.1 *Let $\mathcal{N}$ satisfy the above assumptions. There exists an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ and $c \geq c^*$, there exists a $\delta > 0$ such that if we choose $\psi \in X^5$ with $||\psi||_{X^3} < \delta$ and let w be the corresponding solution $w \in C^0([0,T], X^5) \cap C^1((0,T), X^5)$ such that $w(0) = \psi$, then*

$$\lim_{t\to\infty} ||B(\cdot,t) - \phi(\varepsilon\cdot - ct)|| = 0$$

i.e. the solution ϕ is stable.

5.1 Existence of solutions to equation (1.9).

In this section, we prove local existence and uniqueness of solutions to (1.9). This is a standard result and we include it only for completeness.

Proposition 5.2 *There exists an $\varepsilon_0 > 0$ such that if we fix $\varepsilon < \varepsilon_0$ and choose $\psi \in X^5$, then there exists a time $T = T(\varepsilon, \psi) > 0$ such that (1.9) has a unique solution $w \in C^0([0,T], X^5) \cap C^1((0,T), X^5)$.*

Proof: We rewrite (1.9) as

$$w_t = Lw + M(w)$$

where L is the linear operator

$$L = (1 - 6\varepsilon^4\gamma^2)\partial_\zeta^2 + (c - 2\gamma + 4\varepsilon^4\gamma^3)\partial_\zeta + 4\varepsilon^4\gamma\partial_\zeta^3 - \varepsilon^4\partial_\zeta^4 + \mathcal{N}'(\phi),$$

and $M(w) = aw^2R$. The operator L is a bounded perturbation of the constant coefficient operator

$$L_0 = (1 - 6\varepsilon^4\gamma^2)\partial_\zeta^2 + (c - 2\gamma + 4\varepsilon^4\gamma^3)\partial_\zeta + 4\varepsilon^4\gamma\partial_\zeta^3 - \varepsilon^4\partial_\zeta^4.$$

L_0 is the generator of an analytic semigroup of operators on X^5 as one can verify by direct computation, and hence by Theorem 3.2.1 of [24], L also generates an analytic semigroup.

It is also easy to check that $M : X^5 \to X^5$ is Lipschitz-continuous. Thus, standard results in semigroup theory imply that there exists a T such that,

$$w(t) = e^{Lt}\psi + \int_0^T e^{L(t-\tau)}M(w(\tau))d\tau$$

has an unique solution $w \in C^0([0,T], X^5) \cap C^1((0,T), X^5)$. □

5.2 The energy functionals.

To prove stability we use the three functionals E, F and F_1 defined in (1.10), (1.11) and (1.12). We will show that E and F_1 are non-increasing functions of time and as explained in the introduction, this implies that the solution is stable. These results can be summarised in the following theorems

Theorem 5.3 *Let $\mathcal{N}$ satisfy the assumptions above. There exists an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ and $c \geq c^*$, there exists a $\delta > 0$ such that if $w \in X^5$ and*

$$||w||_{X^3} \leq \delta \tag{5.3}$$

for $t = 0$, then for all $t \geq 0$

$$\frac{\partial E}{\partial t} \leq 0 \tag{5.4}$$

and there exists a positive constant d such that

$$\frac{\partial F_1}{\partial t} \leq -dF(t) \tag{5.5}$$

Because of the fact that the inequalities (5.4) and (5.5) imply that $E+F_1$ is monotonically decreasing as a function of t, the bound (5.3) holds for all times $t > 0$. The theorem is proved by explicitly calculating $\frac{\partial E}{\partial t}$ and $\frac{\partial F}{\partial t}$ and using the fact that $\frac{\partial w}{\partial t}$ satisfies the equation (1.9). Then all the different integrals can be separately approximated. We will give the proof of this theorem in the next subsection. Note that from (5.3) it follows that $||w||_{L^\infty} \leq \delta$, $||aw||_{L^\infty} \leq \delta$ and $||(aw)_\zeta||_{L^\infty} \leq \delta$; we will use this in the following sections. From this theorem we obtain the following convergence results

Theorem 5.4 *Under the assumptions of Theorem 5.3 one has*

$$\lim_{t\to\infty} ||aw||_{L^\infty} = 0.$$

Proof: From the fact that $\frac{dE}{dt} < 0$ and $E(t) \geq 0$ it follows that $\lim_{t\to\infty} E(t) = E_\infty$ exists. In the same way, $\lim_{t\to\infty} F_1(t) = F_{1\infty}$ exists. This implies that $\lim_{t\to\infty} F(t) = F_\infty$ exists. Integrating the inequality (5.5) that we found for $\frac{dF_1}{dt}$ and reversing inequalities leads to

$$d\int_0^\infty F(t)dt \leq -\int_0^\infty \frac{dF_1}{dt}dt = F_1(0) - F_{1\infty} \leq F_1(0),$$

which is finite. Thus, since we know that $\lim_{t\to\infty} F(t)$ exists, this implies that $\lim_{t\to\infty} F(t) = 0$. From the fact that $||aw||^2_{L^2} < \frac{2}{B}F(t)$ it then follows that $\lim_{t\to\infty} ||aw||_{L^2} = 0$. But now, by Sobolev's Theorem

$$||aw||^2_{L^\infty} \leq ||aw||_{L^2}||(aw)_\zeta||_{L^2}.$$

We know that $||(aw)_\zeta||_{L^2} \leq C$ since $||(aw)_\zeta||_{L^2} \leq F(t) \leq C$. Thus, since $\lim_{t\to\infty} ||aw||_{L^2} = 0$, we see that $\lim_{t\to\infty} ||aw||_{L^\infty} = 0$. □

Corollary 5.5 *Under the assumptions of Theorem 5.3 one has convergence*

$$\lim_{t\to\infty} ||B(\frac{1}{\varepsilon}(\cdot + ct), t) - \phi(\cdot)||^2_{L^\infty} = 0.$$

Proof: We substitute the expression for the perturbation of ϕ (5.2) in the above limit. Then we use that from Theorem 5.4 it follows that $\lim_{t\to\infty} ||aw||_{L^\infty} = 0$ to prove the statement. □

Note that Theorem 5.1 follows immediately from this corollary.

5.3 The proof of Theorem 5.3.

In this section we will give the proof of Theorem 5.3. Note that for $w_{|t=0} \in X^5$, Proposition 5.2 guarantees that $\frac{\partial w}{\partial t}$ is well defined.

Differentiating $\frac{1}{2}\int w^2$ with respect to t gives

$$\begin{aligned} \frac{1}{2}\frac{\partial(\int w^2)}{\partial t} &= \int w(\zeta,t)\frac{\partial w}{\partial t}(\zeta,t)d\zeta \\ &= \int w((1-6\varepsilon^4\gamma^2)w_{\zeta\zeta} + (c - 2\gamma + 4\varepsilon^4\gamma^3)w_\zeta + 4\varepsilon^4\gamma w_{\zeta\zeta\zeta} - \varepsilon^4 w_{\zeta\zeta\zeta\zeta} \\ &\quad + w\mathcal{F}(\phi) + aw^2R) \\ &= -(1-6\gamma^2\varepsilon^4)\int w_\zeta^2 - \varepsilon^4\int w_{\zeta\zeta}^2 - \int \phi\tilde{F}(\phi)w^2 + \int aw^3R, \end{aligned}$$

where we used (1.9). We integrated by parts and used that w is a smooth function. Here and from now on we denote $\int_{-\infty}^{\infty}$ by $\int$. We will bound $\int Raw^3$ by using the following Lemma:

Lemma 5.6 *There exists a $K_1 > 0$ such that*

$$\int |Raw^3| \leq \delta(\frac{2K_R}{c_f} + \frac{\tilde{c_f}}{\alpha}) \int |\tilde{F}(\phi)|\phi w^2 = \delta K_1 \int |\tilde{F}(\phi)|\phi w^2.$$

Proof: We will estimate $\int Raw^3$ by separating it into two parts for $\zeta \geq 0$ and $\zeta < 0$. Recall that $R = \frac{1}{2}\mathcal{N}''(\phi + \eta)$ for some $0 \leq \eta \leq aw$. If $\zeta < 0$ we know that $\phi(\zeta) \geq \frac{1}{2}$ and that there exists a positive constant c_f such that $\tilde{F}(\phi) \geq c_f$, so

$$|R| \leq \frac{K_R}{c_f}|\tilde{F}(\phi)|.$$

Thus

$$\int_{\zeta<0} |Raw^3| \leq \frac{K_R}{c_f}\int_{\zeta<0} |\tilde{F}(\phi)|aw^3 \leq \frac{2\delta K_R}{c_f}\int_{\zeta<0} |\tilde{F}(\phi)|\phi w^2,$$

where we use that $\phi(\zeta) \geq \frac{1}{2}$ for $\zeta \leq 0$ and $||aw||_{L^\infty} \leq \delta$. Next, suppose that $\zeta \geq 0$, then $\phi(\zeta) \geq \alpha a(\zeta)$ for some $\alpha > 0$ and $||w||_{L^\infty} \leq \delta$. Therefore $(1 - \frac{\delta}{\alpha})\phi \leq \phi + \eta \leq (1 + \frac{\delta}{\alpha})\phi$. Now consider

$$|\frac{R}{\tilde{F}(\phi)}| = \frac{1}{2}|\frac{\phi\mathcal{N}''(\phi + \eta)}{\mathcal{N}'(\phi)}|.$$

Since $\mathcal{N}(\phi)$ is analytic and $\mathcal{N}'(\phi) < 0$ for $0 < \phi < 1$, there exists a positive constant $\tilde{c_f}$ such that $|\frac{R}{\tilde{F}(\phi)}| \leq \tilde{c_f}$ for $\zeta \geq 0$. This gives that

$$\int_{\zeta\geq 0} |Raw^3| \leq \frac{\delta\tilde{c_f}}{\alpha}\int_{\zeta\geq 0} |\tilde{F}(\phi)|\phi w^2,$$

where we again use that $\phi(\zeta) \geq \alpha a(\zeta)$ for $\zeta \geq 0$ and $||w||_{L^\infty} \leq \delta$. Combining these estimates gives us the bound as in the Lemma. □

Now we differentiate $\frac{1}{2}\int w_\zeta^2$ with respect to t

$$\frac{1}{2}\frac{\partial(\int w_\zeta^2)}{\partial t} = -\int w_{\zeta\zeta}w_t = -(1 - 6\gamma^2\varepsilon^4)\int w_{\zeta\zeta}^2 - \varepsilon^4 w_{\zeta\zeta\zeta}^2 - \int w_{\zeta\zeta}wg.$$

We study $\int w_{\zeta\zeta}wg$ separately. Using the remark following equation (1.9) it can be written as

$$\begin{aligned}\int w_{\zeta\zeta}wg &= \int \mathcal{F}(\phi)ww_{\zeta\zeta} + \int aw^2w_{\zeta\zeta}R \\ &= -\int \mathcal{F}(\phi)w_\zeta^2 + \frac{1}{2}\int \partial_\zeta^2(\mathcal{F}(\phi))w^2 + \int aw^2w_{\zeta\zeta}R.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, the inequality $|ab| \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ and the bound for R we obtain

$$|\int aw^2 w_{\zeta\zeta} R| \le \frac{1}{2}(\int |R| a^2 w^4 + K_R \int w_{\zeta\zeta}^2) \le \frac{1}{2}(\delta \int |R| aw^3 + K_R \int w_{\zeta\zeta}^2).$$

The first integral can be estimated by Lemma 5.6 and thus

$$\int w_{\zeta\zeta} wg \le -\int \mathcal{F}(\phi) w_\zeta^2 + \frac{1}{2}\int \partial_\zeta^2(\mathcal{F}(\phi)) w^2 + \frac{1}{2}(\delta^2 K_1 \int |\tilde{F}(\phi)|\phi w^2 + K_R \int w_{\zeta\zeta}^2).$$

In like fashion

$$\frac{1}{2}\frac{\partial(\int w_{\zeta\zeta}^2)}{\partial t} = \int w_{\zeta\zeta\zeta\zeta} w_t = -(1-6\gamma^2\varepsilon^4)\int w_{\zeta\zeta\zeta}^2 - \varepsilon^4 w_{\zeta\zeta\zeta\zeta}^2 + \int w_{\zeta\zeta\zeta\zeta} wg$$

and we find that

$$\int w_{\zeta\zeta\zeta\zeta} wg = \int \mathcal{F}(\phi) w_{\zeta\zeta}^2 - 2\int \partial_\zeta^2(\mathcal{F}(\phi)) w_\zeta^2 + \frac{1}{2}\int \partial_\zeta^4(\mathcal{F}(\phi)) w^2 + \int w_{\zeta\zeta\zeta\zeta} aw^2 R,$$

where in a similar way as above

$$|\int w_{\zeta\zeta\zeta\zeta} aw^2 R| \le \int |w_{\zeta\zeta\zeta}|(|R_\zeta - \gamma R| aw^2 + 2aK_R|ww_\zeta|).$$

These two integrals can be bounded by using the Cauchy-Schwarz-inequality and $|ab| \le \frac{1}{2}(a^2+b^2)$

$$\begin{aligned} \int |R_\zeta - \gamma R| aw^2 |w_{\zeta\zeta\zeta}| \quad &\le \quad \frac{1}{2}\int |R_\zeta - \gamma R||aw^3| + \frac{1}{2}\tilde{K}_R \int a|w| w_{\zeta\zeta\zeta}^2 \\ &\le \frac{1}{2}\int (|R_\zeta| + \gamma |R|) a|w^3| + \frac{1}{2}\delta \tilde{K}_R \int w_{\zeta\zeta\zeta}^2. \end{aligned}$$

We bound the first integral by

$$\frac{1}{2}\int (|R_\zeta| + \gamma|R|) a|w^3| \le \tilde{K}_1 \delta \int |\tilde{F}(\phi)|\phi w^2, \tag{5.6}$$

using a similar argument as in obtaining the inequality in Lemma 5.6. And

$$\int a|ww_\zeta w_{\zeta\zeta\zeta}| \le \frac{1}{2}\int a|w| w_\zeta^2 + \frac{1}{2}\int a|w| w_{\zeta\zeta\zeta}^2 \le \frac{1}{2}\delta(\int w_\zeta^2 + \int w_{\zeta\zeta\zeta}^2).$$

This leads to

$$\begin{aligned} \int w_{\zeta\zeta\zeta\zeta} wg \quad &\le \quad \int \mathcal{F}(\phi) w_{\zeta\zeta}^2 - 2\int \partial_\zeta^2(\mathcal{F}(\phi)) w_\zeta^2 + \frac{1}{2}\int \partial_\zeta^4(\mathcal{F}(\phi)) w^2 \\ &+\delta(\tilde{K}_1 \int |\tilde{F}(\phi)|\phi w^2 + (K_R + \frac{1}{2}\tilde{K}_R)\int w_{\zeta\zeta\zeta}^2 + K_R \int w_\zeta^2). \end{aligned}$$

Finally

$$\frac{1}{2}\frac{\partial \int w_{\zeta\zeta\zeta}^2}{\partial t} = -\int w^{(6)} w_t = -(1-6\gamma^2\varepsilon^4)\int w_{\zeta\zeta\zeta\zeta}^2 - \varepsilon^4 w_{\zeta\zeta\zeta\zeta\zeta}^2 - \int w^{(6)} wg,$$

where

$$\begin{aligned}\int w^{(6)} wg &= -\int \mathcal{F}(\phi) w_{\zeta\zeta\zeta}^2 + \frac{3}{2}\int \partial_\zeta^2(\mathcal{F}(\phi)) w_{\zeta\zeta}^2 - 3\int \partial_\zeta^2(\mathcal{F}(\phi)) w_\zeta w_{\zeta\zeta\zeta} \\ &\quad - \int \partial_\zeta^3(\mathcal{F}(\phi)) w w_{\zeta\zeta\zeta} + \int a w^2 w^{(6)} R.\end{aligned}$$

This last term still had to be bounded, we will do this later on. Combining the estimates above leads to

$$\begin{aligned}\frac{\partial E}{\partial t} &= -A\int \phi\tilde{F}(\phi)w^2 - A(1-6\gamma^2\varepsilon^4)\int w_\zeta^2 + (-A\varepsilon^4 - B(1-6\gamma^2\varepsilon^4))\int w_{\zeta\zeta}^2 \\ &\quad +(-B\varepsilon^4 - C(1-6\gamma^2\varepsilon^4))\int w_{\zeta\zeta\zeta}^2 + (-C\varepsilon^4 - \varepsilon^6(1-6\gamma^2\varepsilon^4))\int w_{\zeta\zeta\zeta\zeta}^2 - \varepsilon^{10}\int w_{\zeta\zeta\zeta\zeta\zeta}^2 \\ &\quad +A\int w^2 g - B\int w_{\zeta\zeta} wg + C\int w_{\zeta\zeta\zeta\zeta} wg - \frac{1}{2}\varepsilon^6\int w^{(6)} wg \\ &\leq \int(-A(\tilde{F}(\phi) - \delta K_1|\tilde{F}(\phi)|) - B(\frac{\partial_\zeta^2(\mathcal{F}(\phi))}{2\phi} + \frac{1}{2}\delta^2 K_1|\tilde{F}(\phi)|) + C(\frac{\partial_\zeta^4(\mathcal{F}(\phi))}{2\phi} \\ &\quad +\delta\tilde{K}_1|\tilde{F}(\phi)|))\phi w^2 + \int(-A(1-6\gamma^2\varepsilon^4) - B\phi\tilde{F}(\phi) + C(-2\partial_\zeta^2(\mathcal{F}(\phi)) \\ &\quad +\frac{1}{2}\delta K_R))w_\zeta^2 + \int(-A\varepsilon^4 - B(1-6\gamma^2\varepsilon^4 + \frac{1}{2}K_R) - C\phi\tilde{F}(\phi) - \frac{3}{4}\varepsilon^6\partial_\zeta^2(\mathcal{F}(\phi)))w_{\zeta\zeta}^2 \\ &\quad \int(-B\varepsilon^4 - C(1-6\gamma^2\varepsilon^4 - \delta(\frac{1}{2}\tilde{K}_R + K_R)) + \frac{1}{2}\varepsilon^6\mathcal{F}(\phi))w_{\zeta\zeta\zeta}^2 - \varepsilon^{10}\int w_{\zeta\zeta\zeta\zeta\zeta}^2 \\ &\quad +\int(-C\varepsilon^4 - \varepsilon^6(1-6\gamma^2\varepsilon^4))w_{\zeta\zeta\zeta\zeta}^2 + \frac{3}{2}\varepsilon^6\int|\partial_\zeta^2(\mathcal{F}(\phi))w_\zeta w_{\zeta\zeta\zeta}| \\ &\quad +\frac{1}{2}\varepsilon^6\int|\partial_\zeta^3(\mathcal{F}(\phi))w w_{\zeta\zeta\zeta}| + \frac{1}{2}\varepsilon^6\int|a w^2 w^{(6)} R|. \end{aligned} \tag{5.7}$$

The last three integrals can all be bounded. This is done as follows

$$|\varepsilon^6\int \partial_\zeta^2(\mathcal{F}(\phi))w_\zeta w_{\zeta\zeta\zeta}| \leq \frac{1}{2}K(\varepsilon^5\int w_\zeta^2 + \varepsilon^7\int w_{\zeta\zeta\zeta}^2)$$

and

$$|\varepsilon^6\int \partial_\zeta^3(\mathcal{F}(\phi))w w_{\zeta\zeta\zeta}| \leq \frac{1}{2}(\varepsilon^5\int\frac{\partial_\zeta^3(\mathcal{F}(\phi))}{\phi}\phi w^2 + \varepsilon^7 K\int w_{\zeta\zeta\zeta}^2).$$

where we use that both $\partial_\zeta^2(\mathcal{F}(\phi))$ and $\partial_\zeta^3(\mathcal{F}(\phi))$ are bounded by a constant. The term $\varepsilon^6\int a w^2 w^{(6)} R$ must first be integrated partially, then it can be bounded in a similar way (see appendix B) as

$$\varepsilon^6|\int a w^2 w^{(6)} R| \leq K(\int|\tilde{F}(\phi)|\phi w^2 + \delta^2\varepsilon^2\int w_\zeta^2) + \frac{3}{4}\varepsilon^{10}\int {w^{(5)}}^2,$$

for some constant $K > 0$. Thus, the last three terms in (5.7) can be taken into the other terms of (5.7). We use the following Lemma to bound $\frac{dE}{dt}$ further.

Lemma 5.7 *From the assumptions on the nonlinearity $\mathcal{N}$ it follows that for ε and δ sufficiently small, the constants A, B and C can be chosen such that the first term in $\frac{dE}{dt}$ is smaller than $-\frac{1}{2}\int A\phi\tilde{F}(\phi)w^2$ and all the other terms are negative.*

Proof: For the first part of the Lemma, we must prove that there exist positive constants A, B and C such that

$$A(-\frac{1}{2}\tilde{F}(\phi)+\delta K_1|\tilde{F}(\phi)|)-B(\frac{\partial_\zeta^2(\mathcal{F}(\phi))}{2\phi}+\frac{1}{2}\delta^2 K_1|\tilde{F}(\phi)|)+C(\frac{\partial_\zeta^4(\mathcal{F}(\phi))}{2\phi}+\delta\tilde{K}_1|\tilde{F}(\phi)|)$$
$$+\varepsilon^5\frac{\partial_\zeta^3(\mathcal{F}(\phi))}{2\phi}+\varepsilon^2 K|\tilde{F}(\phi)| \tag{5.8}$$

is negative. We prove the above in two parts, for $\phi \geq \phi_0$ and $\phi \leq \phi_0$ where ϕ_0 is sufficiently small. For $\phi \geq \phi_0$ all the terms are bounded away from 0, thus when A is chosen sufficiently large, the expression is always negative. When $\phi \to 0$, the first term in (5.8) also goes in the limit to 0 and the above argument does not hold anymore. In this case we use that from the fact that $\mathcal{N}$ is analytic we know that we can write it as $\mathcal{N}=\sum_{k=k_0}^{\infty}c_k\phi^k$ where $k_0 \geq 2$. Then $\tilde{F}=-\sum_{k=k_0}^{\infty}kc_k\phi^{k-2}$ and $\mathcal{F}(\phi)=\sum_{k=k_0}^{\infty}kc_k\phi^{k-1}$. Also, recall that $\phi \approx c_1 e^{-\gamma\zeta}$ for $\zeta \to \infty$ if $c>c^*$ (or $\phi \approx c_1\zeta e^{-\gamma\zeta}$ if $c=c^*$), see remark 3.3. The leading order term in (5.8) becomes $\approx -\frac{1}{2}A\tilde{c}e^{-\gamma\zeta(k_0-2)}$ (here we assume $c>c^*$, the modifications when $c=c^*$ are obvious). Thus, by choosing A large enough, (5.8) can be made negative for ε and δ sufficiently small. Using a similar argument we can show that the other terms in (5.7) are negative for appropriate choices of A, B and C . $\square$

From the above Lemma, we can conclude that for sufficiently small ε and δ,

$$\frac{dE}{dt} \leq -\frac{1}{2}\int A\phi\tilde{F}(\phi)w^2 < 0.$$

Remark 5.8 *Note that for $c<c^*$ the preceding argument fails since ϕ passes through zero and we can no longer control $\frac{\partial E}{\partial t}$.*

Now, we will compute the derivative of $F(t)$ with respect to t.

$$\frac{1}{2}\frac{\partial(\int(aw)^2)}{\partial t}=\int(aw)^2-\int((aw)_\zeta)^2-\varepsilon^4\int((aw)_{\zeta\zeta})^2-\int a^2w^2\phi\tilde{F}(\phi)+\int a^3w^3R.$$

Here we bound

$$|\int a^3w^3R| \leq K_R\delta\int(aw)^2.$$

Also

$$\frac{1}{2}\frac{\partial(\int((aw)_\zeta)^2)}{\partial t}=\int((aw)_\zeta)^2-\int((aw)_{\zeta\zeta})^2-\varepsilon^4\int((aw)_{\zeta\zeta\zeta})^2-\int(aw)_{\zeta\zeta}awg.$$

Here

$$\int(aw)_{\zeta\zeta}awg=\frac{1}{2}\int\partial_\zeta^2(\mathcal{F}(\phi))(aw)^2-\int\mathcal{F}(\phi)((aw)_\zeta)^2+\int R(aw)^2(aw)_{\zeta\zeta}.$$

This last integral can be bounded as

$$|\int R(aw)^2(aw)_{\zeta\zeta}| \le K_R\delta\int |aw(aw)_{\zeta\zeta}| \le \frac{1}{2}K_R\delta(\int (aw)^2 + \int ((aw)_{\zeta\zeta})^2).$$

We obtain that

$$\frac{1}{2}\frac{\partial(\int((aw)_{\zeta\zeta})^2)}{\partial t} = \int((aw)_{\zeta\zeta})^2 - \int((aw)_{\zeta\zeta\zeta})^2 - \varepsilon^4\int((aw)_{\zeta\zeta\zeta\zeta})^2 + \int(aw)_{\zeta\zeta\zeta\zeta}awg,$$

where

$$\begin{aligned}\int(aw)_{\zeta\zeta\zeta\zeta}awg &= \int\mathcal{F}(\phi)((aw)_{\zeta\zeta})^2 - 2\int\partial_\zeta^2(\mathcal{F}(\phi))((aw)_\zeta)^2 + \frac{1}{2}\int\partial_\zeta^4(\mathcal{F}(\phi))(aw)^2\\ &\quad + \int(aw)_{\zeta\zeta\zeta\zeta}(aw)^2R\end{aligned}$$

and

$$|\int(aw)_{\zeta\zeta\zeta\zeta}(aw)^2R| = |\int(aw)_{\zeta\zeta\zeta}(R_\zeta(aw)^2+2Raw(aw)_\zeta)| \le K_1\delta(\int(aw)^2+\int(aw)_\zeta^2+\int(aw)_{\zeta\zeta\zeta}^2),$$

for some positive constant K_1. Finally,

$$\frac{1}{2}\frac{\partial(\int((aw)_{\zeta\zeta\zeta})^2)}{\partial t} = \int((aw)_{\zeta\zeta\zeta})^2 - \int((aw)_{\zeta\zeta\zeta\zeta})^2 - \varepsilon^4\int((aw)^{(5)})^2 - \int(aw)^{(6)}awg,$$

where

$$\begin{aligned}\int(aw)^{(6)}awg &= -\int\mathcal{F}(\phi)((aw)_{\zeta\zeta\zeta})^2 + \frac{3}{2}\int\partial_\zeta^2(\mathcal{F}(\phi))((aw)_{\zeta\zeta})^2 - \int\partial_\zeta^3(\mathcal{F}(\phi))aw(aw)_{\zeta\zeta\zeta}\\ &\quad -3\int\partial_\zeta^2(\mathcal{F}(\phi))(aw)_\zeta(aw)_{\zeta\zeta\zeta} + \int(aw)^{(6)}(aw)^2R.\end{aligned}$$

Note that in the functional $F(t)$, the quantity $\frac{1}{2}\int((aw)_{\zeta\zeta\zeta})^2$ has a prefactor of ϵ^6. Multiplying the previous equation by ϵ^6, the last term can be bounded as in Appendix B by

$$\varepsilon^6|\int(aw)^2w^{(6)}R| \le c_6\delta(\varepsilon^2\int(aw)^2 + \varepsilon^2\int((aw)_\zeta)^2 + \varepsilon^{10}\int((aw)_{\zeta\zeta\zeta\zeta\zeta})^2),$$

for some constant $c_6 > 0$. This leads to

$$\begin{aligned}\frac{\partial F}{\partial t} &= \int\tilde{A}(1-\phi\tilde{F}(\phi))(aw)^2 + (-\tilde{A}+\tilde{B})\int((aw)_\zeta)^2 + (-\varepsilon^4\tilde{A}-\tilde{B}+\tilde{C})\int((aw)_{\zeta\zeta})^2\\ &\quad +(-\varepsilon^4\tilde{B}-\tilde{C}+\varepsilon^6)\int((aw)_{\zeta\zeta\zeta})^2 + (-\varepsilon^4\tilde{C}-\varepsilon^6)\int((aw)_{\zeta\zeta\zeta\zeta})^2 - \varepsilon^{10}\int((aw)^{(5)})^2\\ &\quad +\tilde{A}\int Ra^3w^3 - \tilde{B}\int(aw)_{\zeta\zeta}awg + \tilde{C}\int(aw)_{\zeta\zeta\zeta\zeta}awg - \varepsilon^6\int(aw)^{(6)}awg\\ &\le \int(\tilde{A}(1-\phi\tilde{F}(\phi)+K_R\delta) - \frac{1}{2}\tilde{B}(\partial_\zeta^2(\mathcal{F}(\phi))+\delta K_R) + \tilde{C}(\frac{1}{2}\partial_\zeta^4(\mathcal{F}(\phi))+K_1\delta)\\ &\quad -c_6\varepsilon^2\delta)(aw)^2 + \int(-\tilde{A}+\tilde{B}(1+\mathcal{F}(\phi)) - \tilde{C}(2\partial_\zeta^2(\mathcal{F}(\phi))-K_1\delta) - c_6\varepsilon^2\delta)((aw)_\zeta)^2\\ &\quad \int(-\varepsilon^4\tilde{A} - \tilde{B}(1+\frac{1}{2}K_R\delta) + \tilde{C}(1+\mathcal{F}(\phi)) - \frac{3}{2}\varepsilon^6\partial_\zeta^2(\mathcal{F}(\phi)))((aw)_{\zeta\zeta})^2\\ &\quad +\int(-\varepsilon^4\tilde{B} - \tilde{C}(1-\delta K_1) + \varepsilon^6(1+\mathcal{F}(\phi)))((aw)_{\zeta\zeta\zeta})^2 - \varepsilon^{10}(1+\delta c_6)\int((aw)_{\zeta\zeta\zeta\zeta\zeta})^2\\ &\quad +3\varepsilon^6\int|\partial_\zeta^2(\mathcal{F}(\phi))(aw)_\zeta(aw)_{\zeta\zeta\zeta}| + \varepsilon^6\int|\partial_\zeta^3(\mathcal{F}(\phi))aw(aw)_{\zeta\zeta\zeta}|. \qquad (5.9)\end{aligned}$$

In a similar way as before, the last two terms can be bounded by $\varepsilon^6 c_2(\int(aw)^2+(aw)_\zeta^2+(aw)_{\zeta\zeta\zeta}^2)$ with c_2 a positive constant and these can again be taken into the other terms in (5.9). Then all the terms in (5.9), except for the first one, can be bounded by $-d(\frac{\tilde{B}}{2}\int(aw)_\zeta^2+\frac{\tilde{C}}{2}\int(aw)_{\zeta\zeta}^2+\frac{\varepsilon^6}{2}\int(aw)_{\zeta\zeta\zeta}^2)$ for some positive constant d sufficiently small. This is done is a similar way as in Lemma 5.7 by using the assumptions on $\mathcal{N}$. Thus for ε and δ sufficiently small we find that

$$\begin{aligned}\frac{dF}{dt} \;&\le\; \int(\tilde{A}(1-\phi\tilde{F}(\phi)+K_R\delta)-\frac{1}{2}\tilde{B}(\partial_\zeta^2(\mathcal{F}(\phi))+\delta K_R)+\tilde{C}(\frac{1}{2}\partial_\zeta^4(\mathcal{F}(\phi))+K_1\delta)\\ &\quad -c_6\varepsilon^2\delta+c_2\varepsilon^6+d\frac{\tilde{A}}{2})(aw)^2-dF(t).\end{aligned}$$

Now we study the derivative of F_1 with respect to t

$$\begin{aligned}\frac{dF_1}{dt}+dF(t) \;&=\; \beta\frac{dE}{dt}+\frac{dF}{dt}+dF(t)\\ &\le\; \int(-\frac{1}{2}\beta A\phi\tilde{F}(\phi)+a^2(\tilde{A}(1-\phi\tilde{F}(\phi)+\frac{d}{2}+K_R\delta)-\frac{1}{2}\tilde{B}(\partial_\zeta^2(\mathcal{F}(\phi))+\delta K_R)\\ &\quad +\tilde{C}(\frac{1}{2}\partial_\zeta^4(\mathcal{F}(\phi))+K_1\delta)-c_6\varepsilon^2\delta+\varepsilon^5c_2))w^2.\end{aligned}$$

The following Lemma leads to the inequality as stated in Theorem 5.3.

Lemma 5.9 *From the assumptions on the nonlinearity $\mathcal{N}$ it follows that for ε and δ sufficiently small, the constant β can be chosen such that*

$$\begin{aligned}&-\frac{1}{2}\beta A\phi\tilde{F}(\phi)+a^2(\tilde{A}(1-\phi\tilde{F}(\phi)+\frac{d}{2}+K_R\delta)-\frac{1}{2}\tilde{B}(\partial_\zeta^2(\mathcal{F}(\phi))+\delta K_R)\\ &+\tilde{C}(\frac{1}{2}\partial_\zeta^4(\mathcal{F}(\phi))+K_1\delta)-c_6\varepsilon^2\delta+\varepsilon^5c_2)\end{aligned}\tag{5.10}$$

is negative.

Proof: The proof is similar to that of Lemma 5.7 and we also use that $a(\zeta)=e^{-\gamma\zeta}$. We consider two cases; $\phi\ge\phi_0$ and $\phi<\phi_0$ where ϕ_0 is sufficiently small. If $\phi\ge\phi_0$, $\tilde{F}(\phi)\ge f_0$ and the result follows immediately by taking β sufficiently large. To handle the case $\phi<\phi_0$, suppose $\mathcal{N}(\phi)=\sum_{k=k_0}^\infty c_k\phi^k$. Then $\phi\tilde{F}(\phi)=-\sum_{k=k_0}^\infty kc_k\phi^{k-1}>0$. Recalling that $\phi\approx c_1e^{-\gamma\zeta}$ as $\zeta\to\infty$ if $c>c^*$ (and $\phi\approx c_1\zeta e^{-\gamma\zeta}$ if $c=c^*$) and that we assumed that either the coefficient c_2 or c_3 in the power series for $\mathcal{N}$ is non-zero, we see that the first term in (5.10) is $\approx-\frac{1}{2}\beta Ad_2e^{-\gamma\zeta}$ if $c_2\ne0$, or $-\frac{1}{2}\beta Ad_2e^{-2\gamma\zeta}$ if $c_2=0$ and $c_3\ne0$ (in both cases we assume $c>c^*$ -the modifications when $c=c^*$ are obvious). Since $a^2(\zeta)=e^{-2\gamma\zeta}$ we see again that by choosing β sufficiently large the first term in (5.10) dominates the remaining ones and the result follows. $\square$

This Lemma implies that for ε and δ sufficiently small

$$\frac{dF_1}{dt}<-dF(t),$$

for some sufficiently small constant d. This concludes the proof of theorem 5.3.

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A The function $H_w^\epsilon(u,v)$.

In this appendix, we are going to determine $H_w^\epsilon(u,v)$ and especially the expressions for $\partial_u H_w^\epsilon(0,0)$ and $\partial_v H_w^\epsilon(0,0)$ which are used in section 4. Recall the expressions that were introduced for w and z in (4.1). The leading order of the functions $H_w^\epsilon(u,v)$ and $H_z^\epsilon(u,v)$ can be given as

$$\begin{aligned} H_w^\epsilon(u,v) &= \varepsilon^{a_0} h_0 + \varepsilon^{a_1} h_{10} u + \varepsilon^{a_2} h_{01} v, \\ H_z^\epsilon(u,v) &= \varepsilon^{b_0} k_0 + \varepsilon^{b_1} k_{10} u + \varepsilon^{b_2} k_{01} v. \end{aligned}$$

Now, we substitute the expressions for w and z, (4.1), into the w' and z'-equations which are the two last equations of the slow system (2.4). Using the equations for u' and v' in system (4.2) we can sort together the constant terms. This leads to the fact that $h_0 = 0$ and $k_0 = 0$. Taking together the linear terms in u and v from the w'-equation leads to

$$\varepsilon^2(cu + (c^2-1)v) = \varepsilon^{b_1} k_{10} u + \varepsilon^{b_2} k_{01} v,$$

which gives that

$$b_1 = b_2 = 2,\ k_{10} = c,\ k_{01} = c^2 - 1.$$

Taking together the linear terms in u and v from the z'-equation yields

$$\varepsilon^4(-k_{01} u + (k_{10} - k_{01} c)v) = \varepsilon^{a_1} h_{10} u + \varepsilon^{a_2} h_{01} v.$$

This implies that

$$a_1 = a_2 = 4,\ h_{10} = -k_{01} = 1 - c^2,\ h_{01} = k_{10} - c k_{01} = 2c - c^3.$$

Finally, this gives that

$$H_w^\epsilon(u,v) = \varepsilon^4(1-c^2)u + \varepsilon^4(2c - c^3)v,$$

and

$$\begin{aligned} \partial_u H_w^\epsilon(0,0) &= \varepsilon^4(1-c^2) \\ \partial_v H_w^\epsilon(0,0) &= \varepsilon^4(2c - c^3). \end{aligned}$$

Substituting this into the condition (4.3) leads to $c \geq 2 - \varepsilon^4 + hot$.

B Appendix B

In this appendix, we bound the term $\varepsilon^6|\int Raw^2w^{(6)}| \leq \varepsilon^6(|\int R_\zeta aw^2w^{(5)}|+|\int \gamma Raw^2w^{(5)}|+|2\int Raww_\zeta w^{(5)}|)$. We study all these terms separately. Applying the Cauchy-Schwarz inequality and $|ab| \leq \frac{1}{2}a^2+\frac{1}{2}b^2$ gives that

$$\varepsilon^6|\int R_\zeta aw^2w^{(5)}| \leq \varepsilon^2\int R_\zeta^2a^2w^4+\frac{1}{4}\varepsilon^{10}\int w^{(5)^2} \leq \varepsilon^2\delta\tilde{K}\int|\tilde{F}(\phi)|\phi w^2+\frac{1}{4}\varepsilon^{10}\int w^{(5)^2}$$

where we also use inequality (5.6). Also,

$$\varepsilon^6|\int Raw^2w^{(5)}| \leq \varepsilon^2\int R^2a^2w^4+\frac{1}{4}\varepsilon^{10}\int w^{(5)^2} \leq \frac{1}{2}\varepsilon^2\delta^2(2+\frac{1}{\alpha})K_R^2\int|\tilde{F}(\phi)|\phi w^2+\frac{1}{4}\varepsilon^{10}\int w^{(5)^2}$$

and finally

$$|2\int Raww_\zeta w^{(5)}| \leq 4\varepsilon^2\int R^2a^2w^2w_\zeta^2+\frac{1}{4}\varepsilon^{10}\int w^{(5)^2} \leq 4\varepsilon^2K_R^2\delta^2\int w_\zeta^2+\frac{1}{4}\varepsilon^{10}\int w^{(5)^2}.$$

Therefore, taking together all these terms we find that

$$\varepsilon^6|\int Raw^2w^{(6)}| \leq K(\int|\tilde{F}(\phi)|\phi w^2+\delta^2\varepsilon^2\int w_\zeta^2)+\frac{3}{4}\varepsilon^{10}\int w^{(5)^2},$$

for some constant $K>0$.

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