SEMICLASSICAL RESOLVENT ESTIMATES FOR TRAPPING PERTURBATIONS

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Abstract. We study the semiclassical estimates of the resolvent $R(\lambda + i\tau)$, $\lambda \in J \subset\subset \mathbb{R}^+$, $\tau \in]0, 1]$ of a self-adjoint operator $L(h)$ in the space of bounded operators $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$, $s > 1/2$. In the general case of long-range trapping "black-box" perturbations we prove that the estimate of the cut-off resolvent $\|\chi(x)R(\lambda+i0)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \leq C\exp(Ch^{-p})$, $\chi(x) \in C_0^\infty(\mathbb{R}^n)$, $p \geq 1$ implies the estimate $\|R(\lambda + i\tau)\|_{s,-s} \leq C_1 \exp(C_1 h^{-p})$.

1. Introduction

The purpose of this paper is to obtain semiclassical estimates for the resolvent

$$R(z) = (L(h) - z)^{-1},\ z \in \mathbb{C} \setminus \mathbb{R}$$

of a self-adjoint operator $L = L(h)$ depending on $h \in]0, h_0]$ for

$$z \in \mathcal{B}_\pm = \{z = \lambda + i\tau \in \mathbb{C} : \lambda \in J,\ \pm\tau \in]0, 1]\},$$

$J =]\mu_0, \mu_1[\subset\subset \mathbb{R}^+$. The operator L is defined in a domain $\mathcal{D} \subset \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)),\ B(0, R_0) = \{x \in \mathbb{R}^n : |x| \leq R_0\},\ n \geq 2$$

and L satisfies the long-range "black box" assumptions (1.7)-(1.15) described below. Introduce the spaces

$$\mathcal{H}^{0,s} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0), \langle x\rangle^s dx),\ \langle x\rangle = (1 + |x|^2)^{1/2}.$$

First we show that for $s > 1/2$ and fixed $h > 0$ the limits

$$\lim_{\epsilon\to 0,\ \epsilon>0} R(\lambda \pm i\epsilon) = R(\lambda \pm i0),\ \ \lambda \in J \tag{1.1}$$

exist in the space of bounded operators $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$, where here and below we omit in $R(z)$ the dependence of h. Next we give semiclassical resolvent estimates

$$\|R(\lambda \pm i\tau)\|_{s,-s} \leq Cr(h),\ \lambda \in J,\ \ \tau \in]0, 1],\ \ h \in]0, h_0], \tag{1.2}$$

where $\|.\|_{s,-s}$ denotes the norm in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$. Such estimates have been obtained by many authors in great generality in the case when every $\lambda \in J$ is a *non-trapping energy level* for the principal symbol $l_0(x, \xi)$ of a differential operator L (see [18], [5], [19], [8], [25], [15], [16], [24] and the survey article [17] for other references). The proofs of these estimates are essentially based on Mourre method [12], [3], [13] and the main point is to construct a conjugate operator $A(h)$ for which the Mourre inequality (5.2) holds (see Section 5.1). Roughly speaking we call these operators non-trapping perturbations and in this case we have $r(h) = h^{-1}$ in (1.2). Moreover, for such perturbations there are no resonances converging sufficiently fast to the real axis as $h \to 0$.

The case of *trapping energy levels* is more complicated. First, without the non-trapping assumption, in general, a Mourre type inequality is not true. Secondly, resonances converging exponentially fast to the real axis could exist and, consequently, we must have $r(h) = e^{Ch^{-p}}$ with some $p \geq 1$. In the special case of a Schrödinger operator $-h^2\Delta + V(x)$, with potential $V(x)$ having the form of a "well in an island", the results in [6], [7] imply the estimate (1.2) with $r(h) = e^{Ch^{-1}}$. Recently, for long-range trapping perturbations in the exterior of an obstacle, Burq [2] established the estimate

$$\|\chi(x)R(\lambda + i0)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \leq C\exp(Ch^{-p}),\ \lambda \in J,\ 0 < h \leq h_0, \tag{1.3}$$

with $p = 1$ and some constants $C > 0$, $h_0 > 0$, provided $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on $\overline{B(0, R_0)}$. In fact, Burq [2] obtained the estimate

$$\|(L_\theta(h) - \lambda)^{-1}\|_{L^2(\mathcal{H}_\theta)} \leq Ce^{C_1 h^{-p}},\ \lambda \in J, \tag{1.4}$$

with $p = 1$, $L_\theta(h)$ being the operator given by complex scaling (see [22], [20]). On the other hand, the estimate (1.4) implies immediately (1.3) since we have

$$\chi(x)R(\lambda + i0)\chi(x) = \chi(x)(L_\theta(h) - \lambda)^{-1}\chi(x)$$

for suitably chosen $\chi(x)$ (see Lemma 3.5 in [22]). Next, denote by Res $L(h)$ the set of resonances of $L(h)$ and assume that there exist $\epsilon > 0$, $c > 0$ and $q \geq 1$ so that

$$\text{dist}\,\{\text{Res}\,L(h),\ [\mu_0, \mu_1]\} \geq \epsilon \exp(-ch^{-q}),\ 0 < h \leq h_0.$$

Then an application of Lemma 1 in [23] with $g(h) = \exp(-\frac{c}{2}h^{-q})$ yields the estimate (1.4) with $p \geq q$. Thus we have several sufficient conditions leading to the estimate (1.3).

In this work we show that an estimate of the cut-off resolvent $\|\chi(x)R(\lambda + i0)\chi(x)\|_{\mathcal{H}}$, $\lambda \in J$, implies an estimate of $\|R(\lambda + i\tau)\|_{s,-s}$ with the same order p in the bound $\exp(Ch^{-p})$. Consequently, we extend the result of [2] replacing the cut-off factor $\chi(x)$ by a function $\psi(x) \in C^\infty(\mathbb{R}^n)$ which is equal to $\langle x\rangle^{-s}$, $s > 1/2$ for $|x| \geq R_1 > R_0$. Moreover, for long-range perturbations in $\mathbb{R}^n$ studied in [2], we prove the conjecture of Robert [16] who claimed that (1.2) with $r(h) = C\exp(Ch^{-p})$ holds for general trapping perturbations. Under this assumption, Robert [16] obtained a Weyl type asymptotic for the scattering phase related to two h-admissible pseudodifferential operators $L_1(h)$, $L_2(h)$. Our main result is the following.

Theorem 1. *Let $L(h)$ be a self-adjoint operator satisfying the "black box" assumptions $(1.7)-(1.15)$. Then for fixed $s > 1/2$ and for sufficiently small fixed h the limits*

$$R(\lambda \pm i0) = \lim_{\text{Im}\, z \to 0,\ \pm \text{Im}\, z > 0} R(z), \quad \text{Re}\, z = \lambda \in J,$$

exist in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$ uniformly with respect to $\lambda \in J$. Moreover, if there exist $\delta > 0$ and $p \geq 1$ such that

$$\|\chi(x)R(\lambda \pm i0)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \leq C\exp(Ch^{-p}),\ \lambda \in]\mu_0 - \delta, \mu_1 + \delta[,\ 0 < h \leq h_0, \tag{1.5}$$

holds, then for each $s > \frac{1}{2}$ there exist $C_s > 0$, $h_1 > 0$ such that for $(\lambda, \tau) \in J \times]0, 1]$ we have

$$\|R(\lambda \pm i\tau)\|_{s,-s} \leq C_s \exp(C_s h^{-p}),\ \ 0 < h \leq h_1. \tag{1.6}$$

The idea of our proof is to construct a self-adjoint operator $\tilde{L} = \tilde{L}(h)$ on $L^2(\mathbb{R}^n)$ such that each $\lambda \in J$ is a non-trapping energy level for $\tilde{L}$ and $L(h)\psi = \tilde{L}(h)\psi$ for any $\psi \in C_0^\infty(\mathbb{R}^n)$ supported away from a ball $B(0, R)$. For this purpose we show that the bounded trajectories related to the symbol of $L(h)$ and to the energy levels $\lambda \in J$ are included in a compact set. Next we represent $R(z)$ by a sum of terms involving the resolvent $\tilde{R}(z)$ of $\tilde{L}$ (see Section 3) and an application of the semiclassical estimates for non-trapping energy levels reduces the problem to the estimation of $\|\chi(x)R(z)\chi(x)\|_{\mathcal{H}}$ (see Proposition 3). In Section 2 our approach is motivated by the proof of Lemma 1.2 in [7] concerning the operator $L(h) = -h^2\Delta + V(x)$, where the potential $V(x)$ involves a "well in an island". Having an "island" around the trapped trajectories, Gérard, Martinez and Robert have introduced a perturbation $W_0(x) \in C_0^\infty(\mathbb{R}^n)$ such that the perturbed symbol $|\xi|^2 + V(x) + W_0(x)$ becomes a non-trapping Hamiltonian for energy levels $\lambda \in J$. We prove a similar result in the general case by constructing a perturbation $\tilde{l}(x, \xi)$ of $l_0(x, \xi)$ (see Proposition 2) so that all $\lambda \in J$ are non-trapping energy levels for $\tilde{l}(x, \xi)$. The existence of $\tilde{l}(x, \xi)$ implies immediately the existence of an operator $\tilde{L}(h)$ with the properties mentioned above. Finally, let us mention that the estimate (1.6) combined with the argument of Robert [16] lead to a Weyl asymptotic of the scattering phase for general long-range perturbations in $\mathbb{R}^n$ when we have not a "black box".

Now we will recall the abstract "black box" assumptions given in [22], [20] and [21]. Suppose that $\mathcal{D}$ satisfies

$$\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}\mathcal{D} = H^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{1.7}$$

uniformly with respect to h in the sense of [20]. More precisely, equip $H^2(\mathbb{R}^n \setminus B(0,R_0))$ with the norm $\| < hD >^2 u\|_{L^2}$, $< hD >^2 = 1 + (hD)^2$ and equip $\mathcal{D}$ with the norm $\|(L+i)u\|_{\mathcal{H}}$. Then we require that $\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)} : \mathcal{D} \longrightarrow H^2(\mathbb{R}^n \setminus B(0,R_0))$ is uniformly bounded with respect to h and this map has a uniformly bounded right inverse.

Assume that

$$\mathbb{1}_{B(0,R_0)}(L+i)^{-1}\text{is compact} \tag{1.8}$$

and

$$(Lu)|_{\mathbb{R}^n\setminus\overline{B(0,R_0)}} = Q\Big(u|_{\mathbb{R}^n\setminus\overline{B(0,R_0)}}\Big), \tag{1.9}$$

where Q is a formally self-adjoint differential operator

$$Qu = \sum_{|\alpha|\leq 2} a_\alpha(x;h)(hD_x)^\alpha u, \tag{1.10}$$

with $a_\alpha(x;h) = a_\alpha(x)$ independent of h for $|\alpha| = 2$ and $a_\alpha \in C_b^\infty(\mathbb{R}^n)$ uniformly bounded with respect to h.

Next we assume the following properties:

There exists $C > 0$ such that

$$l_0(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq C\langle\xi\rangle^2,\ \ \langle\xi\rangle = (1+|\xi|^2)^{1/2}. \tag{1.11}$$

There exists $\gamma > 0$ such that for every $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, we have

$$|\partial_x^\alpha\partial_\xi^\beta\Big(\sum_{|\alpha|\leq 2} a_\alpha(x;h)\xi^\alpha - |\xi|^2\Big)| \leq C_{\alpha,\beta}\langle x\rangle^{-\gamma-|\alpha|}\langle\xi\rangle^2 \tag{1.12}$$

uniformly with respect to h.

There exist $\theta_0 \in]0,\pi[$, $\epsilon > 0$ and $R_1 > R_0$ so that the coefficients $a_\alpha(x;h)$ of Q can be extended holomorphically in x to

$$\{r\omega;\ \omega \in \mathbb{C}^n,\ \text{dist}\,(\omega, S^{n-1}) < \epsilon,\ r \in \mathbb{C},\ |r| > R_1,\ \arg r \in [-\epsilon, \theta_0 + \epsilon)\} \tag{1.13}$$

and (1.12) extend to this larger set.

Let $R > R_0$, $T = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$, $\tilde{R} > 2R$. Set

$$\mathcal{H}^\# = \mathcal{H}_{R_0} \oplus L^2(T \setminus B(0,R_0))$$

and consider a self-adjoint operator $L^\# : \mathcal{H}^\# \longrightarrow \mathcal{H}^\#$ with domain

$$\mathcal{D}^\# = \{u \in \mathcal{H}^\# :\ \chi u \in \mathcal{D},\ (1-\chi)u \in H^2\},$$

where $\chi \in C_0^\infty(B(0,R);[0,1])$ is equal to 1 near $\overline{B(0,R_0)}$. Take a differential operator

$$Q^\# = \sum_{|\alpha|\leq 2} a_\alpha^\#(x;h)(hD)^\alpha$$

with $a_\alpha^\#(x;h) = a_\alpha(x;h)$ for $|x| < R$, put

$$L^\# u = L(\chi u) + Q^\#((1-\chi)u),\ u \in \mathcal{D}^\#$$

and denote by $N(L^\#, [-\lambda^2,\lambda^2])$ the number of eigenvalues of $L^\#$ in the interval $[-\lambda^2,\lambda^2]$. Then we assume that

$$N(L^\#,[-\lambda^2,\lambda^2]) = \mathcal{O}(\Big(\frac{\lambda}{h}\Big)^{n^\#}),\ n^\# \geq n,\ \lambda \geq 1. \tag{1.14}$$

Finally, we suppose that

$$\sigma_{pp}(L(h)) \cap [\mu_0 - \delta, \mu_1 + \delta] = \emptyset,\ h \in]0, h_0], \tag{1.15}$$

where $\delta > 0$ is the constant in (1.5). A typical example for an operator satisfying the conditions (1.7)-(1.15) is an elliptic self-adjoint Schrödinger operator

$$P = \sum_{|\alpha| \leq 2} a_\alpha(x; h)(hD_x)^\alpha$$

with C^∞ coefficients in the connex exterior $\Omega = \mathbb{R}^n \setminus \mathcal{O}$ of a bounded obstacle $\mathcal{O}$ with C^∞ boundary $\partial\Omega$ with Dirichlet or Neumann boundary condition on $\partial\Omega$ (see [2]). In this case $\mathcal{D} = H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H_0^1(\mathbb{R}^n \setminus \mathcal{O})$ and P is a positive operator in $L^2(\mathbb{R}^n \setminus \mathcal{O})$.

Under the above assumptions the resonances close to the real axis can be defined by the method of complex scaling [22], [20] and they coincide with the poles of the meromorphic continuation of the resolvent $(L(h) - z)^{-1} : \mathcal{H}_{\text{comp}} \longrightarrow \mathcal{D}_{\text{loc}}$ from $\operatorname{Im} z > 0$ to a conic neighborhood of the positive real axis in the lower half plane. The set of resonances will be denoted by Res $L(h)$.

The paper is organized as follows. Section 2 is devoted to the analysis of the non-trapping energy levels and to the construction of the symbol $\tilde{l}(x, \xi)$. In Section 3 we prove Theorem 1 introducing the operator $\tilde{L}(h)$ and exploiting the semiclassical estimates for non-trapping energy levels. In Section 4 we obtain some microlocal estimates closely related to the bounds of $\|R(z)\|_{s,-s}$. In a further work we will apply the microlocal estimates given in Proposition 5 for the analysis of the semiclassical asymptotic of the scattering phase.

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2. Classical trajectories and non-trapping energy levels

Let $l(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n : \mathbb{R})$ be a classical Hamiltonian, and let $H_l = (\partial_\xi l, -\partial_x l)$ be the associated Hamiltonian vector field:

$$H_l(f)(x, \xi) = \{l, f\}(x, \xi) = (\partial_\xi l \partial_x f - \partial_x l \partial_\xi f)(x, \xi),\ \ f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n).$$

Denote by Φ_l^t the Hamiltonian flow of l given by

$$\Phi_l^t = \exp(tH_l)\ :\ (x_0, \xi_0) \mapsto \Big(x(t, x_0, \xi_0), \xi(t, x_0, \xi_0)\Big).$$

Therefore for each $\lambda \in \mathbb{R}$, the energy surface $\Sigma_\lambda = l^{-1}(\lambda)$ is stable by Φ_l^t. In the exposition below we denote $J =]\mu_0, \mu_1[\subset\subset \mathbb{R}^+$. We will say that (x, ξ) is a non-critical point for $l(x, \xi)$ if $\nabla_{x,\xi} l(x, \xi) \neq 0$ and that λ is a non-critical energy level for $l(x, \xi)$, if any $(x, \xi) \in \Sigma_\lambda = l^{-1}(\lambda)$ is a non-critical point for $l(x, \xi)$. We start by an examination of the bounded trajectories.

Lemma 1. *Let $\mu_1 > \mu_0 > 0$ be fixed. Assume there exist $\mu \in]0, \mu_0[$, $R > 0$ such that*

$$|x| \geq R,\ (x, \xi) \in l^{-1}(]\mu_0, \mu_1[)\ \Rightarrow\ |\frac{1}{4}\{l, x \cdot \partial_\xi l\}(x, \xi) - l(x, \xi)| \leq \mu. \tag{2.1}$$

Then for each $\lambda \in J$ and each $(x_0, \xi_0) \in \Sigma_\lambda$ with $|x_0| > R$ the function $x(\cdot, x_0, \xi_0)$ is non-bounded.

Proof. By definition, $|x(t)|^2$ satisfies

$$\begin{cases} (|x|^2)' = \left\{l, x^2\right\}(x, \xi) = 2x \cdot \partial_\xi l(x, \xi), \\ (x \cdot \partial_\xi l(x, \xi))' = \left\{l, x \cdot \partial_\xi l\right\}(x, \xi). \end{cases} \tag{2.2}$$

For $(x_0,\xi_0)\in\Sigma_\lambda$ we have $(x(t),\xi(t))\in\Sigma_\lambda$ and this yields

$$(x\cdot\partial_\xi l(x,\xi))' = 4\lambda + \{l, x\cdot\partial_\xi l\}(x,\xi) - 4l(x,\xi). \tag{2.3}$$

Let $|x_0|>R$ and $(x_0,\xi_0)\in l^{-1}(]\mu_0,\mu_1[)$. Denote by $[T_m,T_M]$ the maximal interval containing $t_0=0$ so that

$$|\Big(\{l,x\cdot\partial_\xi l\}-4l\Big)(x(t),\xi(t))|\le 4\mu,\ \ t\in[T_m,T_M]. \tag{2.4}$$

By the continuity of $x(t)$ and $|\{l,x\cdot\partial_\xi l\}-4l|(x(t),\xi(t))$ a such interval exists and it is not reduced to $\{0\}$. From the relation (2.3) we deduce

$$(x\cdot\partial_\xi l(x,\xi))'(t)\ge 4(\lambda-\mu)\ge 4(\mu_0-\mu),\ \ \forall t\in[T_m,T_M],$$

that is

$$\begin{cases} (x\cdot\partial_\xi l(x,\xi))(t)\ge x_0\cdot\partial_\xi l(x_0,\xi_0)+4(\mu_0-\mu)t, & \forall t\in[0,T_M],\\ (x\cdot\partial_\xi l(x,\xi))(t)\le x_0\cdot\partial_\xi l(x_0,\xi_0)+4(\mu_0-\mu)t, & \forall t\in[T_m,0].\end{cases}$$

Then applying (2.2), we get

$$|x(t)|^2\ge|x_0|^2+2x_0\cdot\partial_\xi l(x_0,\xi_0)t+4(\mu_0-\mu)t^2,\ \ \forall t\in[T_m,T_M].$$

Consequently, if $\pm x_0\cdot\partial_\xi l(x_0,\xi_0)\ge 0$, for $\pm t\ge 0$, we have

$$|x(t)|^2\ge|x_0|^2+4(\mu_0-\mu)t^2 \tag{2.5}$$

which implies immediately that $T_M=+\infty$ (if $x_0\cdot\partial_\xi l(x_0,\xi_0)\ge 0$) or $T_m=-\infty$ (if $x_0\cdot\partial_\xi l(x_0,\xi_0)\le 0$) and (2.5) holds for any $t\ge 0$ (or $t\le 0$). Thus, as soon as $|x_0|>R$, and $l(x_0,\xi_0)\in]\mu_0,\mu_1[$, the function $x(\cdot,x_0,\xi_0)$ is non-bounded.

□

Proposition 1. *Let $\mu_1>\mu_0>0$ and let $\lim_{|\xi|\to\infty} l(x,\xi)=+\infty$. Assume there exists $\gamma>0$ such that for every $(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n$, $|\alpha|\le 1$, $|\beta|\le 2$ we have*

$$|\partial_x^\alpha\partial_\xi^\beta\Big(l(x,\xi)-|\xi|^2\Big)|\le C_{\alpha,\beta}\langle x\rangle^{-\gamma-|\alpha|}\langle\xi\rangle^2.$$

Then there exists a compact set $K(J)\subset\mathbb{R}^{2n}$ such that for each $\lambda\in J$, the set $K(J)\cap\Sigma_\lambda$ contains all bounded trajectories of the Hamiltonian flow Φ_l^t on Σ_λ and all critical points of $l(x,\xi)$ in Σ_λ.

Remark 1. *In a more restrictive setting, this result is stated and exploited by the first author* [1] *to study the scattering phase for short-range trapping perturbations of the Laplacian.*

Proof. First, as $\lim_{|\xi|\to\infty} l(x,\xi)=+\infty$, it is easy to see that there exists $M_{\mu_1}>0$ such that

$$(x,\xi)\in\Sigma_\lambda,\ \forall\lambda\le\mu_1\ \Rightarrow\ |\xi|\le M_{\mu_1}. \tag{2.6}$$

Now we will prove that the x-component of the bounded trajectories and the critical points are included in a compact set. This means that outside a fixed compact set the x-component of the Hamiltonian trajectories are non-bounded and $\nabla_{x,\xi}l(x,\xi)\neq 0$.

Using Lemma 1 it is sufficient to establish the existence of $R>0$ such that (2.1) holds with $\mu<\mu_0$. We write

$$\begin{aligned}\{l,x\cdot\partial_\xi l\}&=\partial_\xi l\cdot\partial_x(x\cdot\partial_\xi l)-\partial_x l\cdot\partial_\xi(x\cdot\partial_\xi l)\\ &=|\partial_\xi l|^2+\textstyle\sum_{j=1}^n\partial_{\xi_j}l\ x\cdot\partial_\xi\partial_{x_j}l-\partial_x l\cdot\partial_\xi(x\cdot\partial_\xi l).\end{aligned} \tag{2.7}$$

Set $r(x,\xi)=l(x,\xi)-|\xi|^2$ and observe that

$$|\partial_\xi l|^2=4|\xi|^2+4\xi\cdot\partial_\xi r+|\partial_\xi r|^2,$$

which implies

$$\frac14|\partial_\xi l|^2-l(x,\xi)=\xi\cdot\partial_\xi r+\frac14|\partial_\xi r|^2-r(x,\xi). \tag{2.8}$$

Moreover, since $\partial_{x_j} l = \partial_{x_j} r$, we deduce the following representation

$$\frac{1}{4}\{l, x\cdot\partial_\xi l\} - l(x,\xi)$$
$$= \xi\cdot\partial_\xi r + \frac{1}{4}|\partial_\xi r|^2 - r(x,\xi) + \frac{1}{4}\sum_{j=1}^n \partial_{\xi_j} l\; x\cdot\partial_\xi\partial_{x_j} r - \partial_x r\cdot\partial_\xi(x\cdot\partial_\xi l).$$

From the assumption on r we obtain

$$|\frac{1}{4}\{l, x\cdot\partial_\xi l\} - l(x,\xi)| \leq C\langle x\rangle^{-\gamma}\langle\xi\rangle^4.$$

Thus we can find $R = R(\mu_0, \mu_1, \gamma)$ such that (2.1) holds. Moreover taking R such that the RHS of (2.8) is bounded by $\mu < \mu_0$, we obtain $|\partial_\xi l| \neq 0$ on $l^{-1}(J)\cap\{(x,\xi); |x| > R\}$. Consequently, each $(x,\xi) \in l^{-1}(J)\cap\{(x,\xi); |x| > R\}$ is non-critical for l and for each $\lambda \in J$ every trajectory with initial data $| x_0 |> R$ is non-bounded. $\square$

Now we pass to the analysis of the non-trapping energy levels in the sense given in [18]. Recall that $\lambda\in\mathbb{R}$ is a *non-trapping energy level* for $l(x,\xi)$ if for every $R > 0$ there exists $T_R > 0$ such that for $(x_0,\xi_0)\in\Sigma_\lambda$, $|x_0| < R$, we have

$$|x(t,x_0,\xi_0)| > R,\ \forall |t| > T_R.$$

Lemma 2. *Let $\mu_1 > \mu_0 > 0$ and let $\lim_{|\xi|\to\infty} l(x,\xi) = +\infty$. If there exists $\mu\in]0,\mu_0[$ such that*

$$(x,\xi)\in l^{-1}(]\mu_0,\mu_1[) \ \Rightarrow\ |\frac{1}{4}\{l, x\cdot\partial_\xi l\}(x,\xi) - l(x,\xi)| \leq \mu, \tag{2.9}$$

then each $\lambda\in J$ is a non-trapping energy level for $l(x,\xi)$.

Proof. Our aim is to prove that $x(\cdot,x_0,\xi_0)$ is non-bounded in both directions $t\to\pm\infty$ for any $(x_0,\xi_0)\in\Sigma_\lambda$. Following the proof of Lemma 1, we have

$$|\Big(\{l, x\cdot\partial_\xi l\} - 4l\Big)(x(t),\xi(t))| \leq 4\mu,\ \ \forall t\in\mathbb{R}, \tag{2.10}$$

(there is no restriction on t by the assumption on l). Then we deduce

$$|x(t)|^2 \geq |x_0|^2 + 2x_0\cdot\partial_\xi l(x_0,\xi_0)t + 4(\mu_0-\mu)t^2,\ \ \forall t\in\mathbb{R}.$$

On the other hand the set $\{\xi_0 : (x_0,\xi_0)\in\Sigma_\lambda\ ,\ \lambda\leq\mu_1\}$ is bounded. Consequently, for $R > 0$ and $|x_0| < R$, we conclude that $x_0\cdot\partial_\xi l(x_0,\xi_0)$ is bounded, and there exists $C_R > 0$ such that

$$|x(t)|^2 \geq 4|t|\Big((\mu_0-\mu)|t| - C_R\Big).$$

Thus, as soon as $|t| > 2C_R + 2\sqrt{C_R^2 + (\mu_0-\mu)R^2}$, we will have $|x(t)| > R$, hence λ is a non-trapping energy level. $\square$

Introduce a function $\chi\in C_0^\infty(\{x\in\mathbb{R}^n;\ |x|\leq 2\})$, $0\leq\chi\leq 1$ such that $\chi(x) = 1$ for $|x|\leq 1$ and set $\chi_R(x) = \chi(\frac{x}{R})$. We have the following

Proposition 2. *Let $\mu_1 > \mu_0 > 0$. Assume there exists $\gamma > 0$ such that for every $(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n$, $|\alpha|\leq 1$, $|\beta|\leq 2$ we have*

$$|\partial_x^\alpha\partial_\xi^\beta\Big(l(x,\xi) - |\xi|^2\Big)| \leq C_{\alpha,\beta}\langle x\rangle^{-\gamma-|\alpha|}\langle\xi\rangle^2.$$

Then there exists $R_c > 0$ such that for $R\geq R_c$, each $\lambda\in J$ is a non-trapping and non-critical energy level for the Hamiltonian

$$\tilde{l}(x,\xi) = l(x,\xi) - \chi_R(x)\Big(l(x,\xi) - |\xi|^2\Big).$$

Proof. We will take $R > 0$ large enough in order to satisfy the conditions of Lemma 2 for

$$\tilde{l}(x,\xi) = l(x,\xi) - \chi(\frac{x}{R})\Big(l(x,\xi) - |\xi|^2\Big).$$

First, we have

$$\tilde{l}(x,\xi) = |\xi|^2 + \Big(1 - \chi(\frac{x}{R})\Big)\Big(l(x,\xi) - |\xi|^2\Big),$$

and by the assumption on l, we obtain

$$\tilde{l}(x,\xi) \geq \langle\xi\rangle^2\Big(1 - C_{0,0}\langle x\rangle^{-\gamma}\Big) - 1.$$

Thus there exists R_1 such that for any $R \geq R_1$, $\lim_{|\xi|\to\infty} \tilde{l}(x,\xi) = +\infty$ for $|x| \geq R$ and this holds also for $|x| \leq R$, where $\tilde{l}(x,\xi) = |\xi|^2$. Now, as in the proof of the Proposition 1, we have

$$\{\tilde{l}, x\cdot\partial_\xi\tilde{l}\} = |\partial_\xi\tilde{l}|^2 + \sum_{j=1}^n \partial_{\xi_j}\tilde{l}\, x\cdot\partial_\xi\partial_{x_j}\tilde{l} - \partial_x\tilde{l}\cdot\partial_\xi(x\cdot\partial_\xi\tilde{l}). \tag{2.11}$$

Next notice that for $r(x,\xi) = \tilde{l}(x,\xi) - |\xi|^2$, the derivative of $\tilde{l} = |\xi|^2 + (1-\chi_R)r$ satisfies

$$|\partial_\xi\tilde{l}|^2 = 4|\xi|^2 + 4(1-\chi_R)\xi\cdot\partial_\xi r + (1-\chi_R)^2|\partial_\xi r|^2$$

and we deduce

$$\frac{1}{4}|\partial_\xi\tilde{l}|^2 - \tilde{l}(x,\xi) = (1-\chi_R)\xi\cdot\partial_\xi r + \frac{1}{4}(1-\chi_R)^2|\partial_\xi r|^2 - (1-\chi_R)r(x,\xi). \tag{2.12}$$

On the other hand, it is clear that

$$\partial_{x_j}\tilde{l} = \partial_{x_j}(1-\chi_R)r = (1-\chi_R)\partial_{x_j}r - \frac{1}{R}\partial_{x_j}\chi(\frac{x}{R})r,$$

and, consequently,

$$\begin{aligned} &\sum_{j=1}^n \partial_{\xi_j}\tilde{l}\, x\cdot\partial_\xi\partial_{x_j}\tilde{l} - \partial_x\tilde{l}\cdot\partial_\xi(x\cdot\partial_\xi\tilde{l}) \\ &= (1-\chi_R)\left(\sum_{j=1}^n \partial_{\xi_j}\tilde{l}\, x\cdot\partial_\xi\partial_{x_j}r - \partial_x r\cdot\partial_\xi(x\cdot\partial_\xi\tilde{l})\right) \\ &\quad - \sum_{j=1}^n \frac{1}{R}\partial_{x_j}\chi(\frac{x}{R})\left(\partial_{\xi_j}\tilde{l}\, x\cdot\partial_\xi r - r\cdot\partial_{\xi_j}(x\cdot\partial_\xi\tilde{l})\right). \end{aligned} \tag{2.13}$$

The function $(1-\chi_R)$ (resp. $\partial_x\chi_R$) is supported in $\{|x| \geq R\}$, (resp. $\{2R \geq |x| \geq R\}$) and the assumptions on $r(x,\xi)$ combined with the relations (2.11), (2.12), (2.13) lead to

$$|\frac{1}{4}\{\tilde{l}, x\cdot\partial_\xi\tilde{l}\} - \tilde{l}(x,\xi)| \leq C\langle R\rangle^{-\gamma}\langle\xi\rangle^4.$$

Finally, ξ being bounded on the the energy level Σ_λ, we can find $R_c = R_c(\mu_0,\mu_1,\gamma) \geq R_1$ such that for any $R \geq R_c$ the function $\tilde{l}$ satisfies (2.9). Thus each $\lambda \in J$ is a non-trapping energy level for $\tilde{l}(x,\xi)$. Moreover, taking R_c such that the right hand side of (2.12) is bounded by $\mu < \mu_0$, we obtain $|\partial_\xi\tilde{l}| \neq 0$ on $\tilde{l}^{-1}(J)$ and, consequently, each $\lambda \in J$ is a non-critical energy level for $\tilde{l}(x,\xi)$. This completes the proof. $\square$

Remark 2. *For a more complete analysis of the qualitative aspects of classical trajectories, we refer to the recent work of A. Knauf* [11]. *For the Hamiltonian $l(x,\xi) = \frac{1}{2}|\xi|^2 + V(x)$, he obtained a topological condition for the existence of trapped trajectories.*

3. Resolvent estimates

We start with the following proposition reducing the proof of the resolvent estimates to that for the cut-off resolvent.

Proposition 3. *Let $L(h)$ satisfy the "black box" assumptions $(1.7)-(1.12)$ and let $J =]\mu_0, \mu_1[\subset\subset \mathbb{R}^+$. Then there exists a self-adjoint differential operator $\tilde{L} = \tilde{L}(h)$ on $L^2(\mathbb{R}^n)$, satisfying the assumptions $(1.7)-(1.12)$ such that each $\lambda \in J$ is a non-trapping and non-critical energy level for $\tilde{L}(h)$ and*

$$L(h)\psi = \tilde{L}(h)\psi$$

for any $\psi \in C^\infty(\mathbb{R}^n)$ supported away from $B(0, R_0)$. Moreover, for every $s > \frac{1}{2}$ there exists $C > 0$, $h_0 > 0$ and $\rho > R_0$ such that for $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ for $|x| \leq \rho$, we have

$$\|R(z)\|_{s,-s} \leq Ch^{-2}\Big(1 + \|\chi R(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big),$$

uniformly with respect to $z \in \mathcal{B}_\pm = \{z \in \mathbb{C};\ (\operatorname{Re} z, \pm \operatorname{Im} z) \in J \times]0, 1]\}$ and $h \in (0, h_0]$.

Proof. Choose a function $\Phi \in C_0^\infty(\{x \in \mathbb{R}^n;\ |x| \leq 2\}; \mathbb{R})$, $0 \leq \Phi \leq 1$ such that $\Phi(x) = 1$ for $|x| \leq 1$ and set $\chi_R(x) = \Phi(\frac{x}{R})$ for $R > R_0$, where R_0 is the constant in the "black box" assumptions. Introduce the self-adjoint operator on $L^2(\mathbb{R}^n)$

$$\begin{aligned} \tilde{L} &= (1-\chi_R)L\chi_R + \chi_R L(1-\chi_R) + (1-\chi_R)L(1-\chi_R) + \chi_R(-h^2\Delta)\chi_R && (3.1)\\ &= Q + \chi_R(-h^2\Delta - Q)\chi_R, && (3.2) \end{aligned}$$

where Q is the formally self-adjoint differential operator given in the Introduction.

According to Proposition 2, there exists $R_c > R_0$ such that if $R \geq R_c$, every $\lambda \in J$ is a non-trapping and non-critical energy level in the sense mentioned in Section 2 for the principal symbol $\tilde{l}_0(x, \xi)$ of $\tilde{L}(h)$ having the form

$$\tilde{l}_0(x,\xi) = l_0(x,\xi) - \chi_R^2(x)\Big(l_0(x,\xi) - |\xi|^2\Big)$$

with $l_0(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha$. Clearly, for $\psi = 0$ on the support of χ_R, we have $\tilde{L}\psi = Q\psi = L\psi$.

Let $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of χ_R and let $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of χ_1. Using $\tilde{L}(1-\chi_1) = L(1-\chi_1)$, we have

$$R(z) = R(z)\chi_2 + (1-\chi_1)\tilde{R}(z)(1-\chi_2) + R(z)[\tilde{L}, \chi_1]\tilde{R}(z)(1-\chi_2), \tag{3.3}$$

where

$$\tilde{R}(z) = (\tilde{L}(h) - z)^{-1},\ \ z \in \mathbb{C},\ \operatorname{Im} z \neq 0.$$

Thus we obtain

$$\|R(z)\|_{s,-s} \leq C\Big(\|R(z)\chi_2\|_{s,-s} + \|\tilde{R}(z)\|_{s,-s} + \|R(z)\chi_2\|_{s,-s}\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,s}\Big), \tag{3.4}$$

where here and below we denote by $C > 0$ different constants independent on h and z. Next, $[\tilde{L}, \chi_1]$ is a first order h-admissible differential operator with compactly supported coefficients, so there exists $C > 0$ such that

$$\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,s} \leq C\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,-s}.$$

Moreover, the interval J contains non-trapping energy levels for $\tilde{L}(h)$, and we can estimate $\|\tilde{R}(z)\|_{s,-s}$ and $\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,-s}$ by Lemma 4 and Corollary 1 (see Section 5.1). Consequently, there exist $C > 0$ and $h_0 > 0$ such that for any $h \in (0, h_0]$ we have

$$\|R(z)\|_{s,-s} \leq Ch^{-1}\Big(1 + \|R(z)\chi_2\|_{s,-s}\Big). \tag{3.5}$$

On the other hand, using the equality

$$\|R(z)\chi_2\|_{s,-s} = \|\chi_2 R(\overline{z})\|_{s,-s}$$

and introducing χ_2 in the estimates (3.4), (3.5), we obtain

$$\|R(z)\chi_2\|_{s,-s} \leq Ch^{-1}\Big(1 + \|\chi_2 R(z)\chi_2\|_{\mathcal{H}\to\mathcal{H}}\Big). \tag{3.6}$$

Finally, (3.5) and (3.6) yield

$$\|R(z)\|_{s,-s} \leq Ch^{-2}\Big(1 + \|\chi_2 R(z)\chi_2\|_{\mathcal{H}\to\mathcal{H}}\Big).$$

□

Proposition 4. *Assume that $L(h)$ satisfies the assumptions $(1.7)-(1.15)$. Then for small $h_0 > 0$ and fixed $s > 1/2$, $h \in (0, h_0]$ the limits*

$$R(\lambda \pm i0) = \lim_{\operatorname{Im} z\to 0,\ \pm\operatorname{Im} z>0} R(z), \quad \operatorname{Re} z = \lambda \in J,$$

exist in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$ uniformly with respect to $\lambda \in J$.

Proof. Take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $B(0, R_0)$. We shall prove that there exists $h_0 > 0$ such that for fixed $h \in (0, h_0]$ and $s > 1/2$, we have

$$\|R(z) - R(z')\|_{s,-s} \leq C_{h,s}\Big(|z-z'|^{\delta_1} + |z-z'| + \|\chi\Big(R(z) - R(z')\Big)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big),$$

with $\delta_1 \in (0, 1/2)$ uniformly with respect to $z, z' \in \mathcal{B}_\pm$.

As in the proof of the Proposition 3, we choose $R > R_c$ and introduce χ_R and an operator $\tilde{L} = \tilde{L}(h)$ with non-trapping energy levels in J. Let $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of χ_R and let $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of χ_1. Set $R(z, z') = R(z) - R(z')$ and $\tilde{R}(z, z') = \tilde{R}(z) - \tilde{R}(z')$. According to (3.3), we have

$$\begin{gathered} R(z,z') = R(z,z')\chi_2 + (1-\chi_1)\tilde{R}(z,z')(1-\chi_2) \\ +R(z,z')[\tilde{L},\chi_1]\tilde{R}(z)(1-\chi_2) + R(z')[\tilde{L},\chi_1]\tilde{R}(z,z')(1-\chi_2). \end{gathered}$$

and we deduce

$$\begin{gathered} \|R(z,z')\|_{s,-s} \leq C\|R(z,z')\chi_2\|_{s,-s}\Big(1 + \|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,s}\Big) \\ +C\|\tilde{R}(z,z')\|_{s,-s} + C\|R(z')\chi_2\|_{s,-s}\,\|[\tilde{L},\chi_1]\tilde{R}(z,z')\|_{s,s}. \end{gathered}$$

For fixed h Lemma 6 in Section 5.2 implies that the cut-off resolvent $\chi(x)R(z)\chi(x)$ is analytic in $\overline{\mathcal{B}_+}$ and the norm $\|\chi_2 R(z)\chi_2\|_{\mathcal{H}\to\mathcal{H}}$ will be uniformly bounded with respect to $z \in \mathcal{B}_\pm$. Applying Proposition 3 for $s > 1/2$, we conclude that $\|R(z')\chi_2\|_{s,-s}$ is also uniformly bounded with respect to $z \in \mathcal{B}_\pm$. On the other hand, since J is non-trapping for $\tilde{L}$, we can apply Corollary 1. Thus, as in the proof of Proposition 3, for any $s > 1/2$ there exists C_s such that

$$\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,s} \leq C_s h^{-1}$$

uniformly with respect to $z \in \mathcal{B}_\pm$. Moreover, for s and h fixed, there exists $C_{h,s}$ such that

$$\|\tilde{R}(z,z')\|_{s,-s} + \|[\tilde{L},\chi_1]\tilde{R}(z,z')\|_{s,-s} \leq C_{h,s}\Big(|z-z'| + |z-z'|^{\delta_1}\Big),$$

uniformly with respect to $z, z' \in \mathcal{B}_\pm$. Consequently, there exists $C_{h,s} > 0$, such that

$$\|R(z,z')\|_{s,-s} \quad \leq \quad C_{h,s}\Big(|z-z'|^{\delta_1} + |z-z'| + \|R(z,z')\chi_2\|_{s,-s}\Big).$$

Next, using the equality $\|R(z,z')\chi_2\|_{s,-s} = \|\chi_2 R(\overline{z}, \overline{z'})\|_{s,-s}$, and introducing the cut-off function χ_2 in the above estimates, we obtain

$$\|R(z,z')\chi_2\|_{s,-s} \leq C_{h,s}\Big(|z-z'|^{\delta_1} + |z-z'| + \|\chi_2 R(z,z')\chi_2\|_{s,-s}\Big).$$

This implies immediately

$$\|R(z,z')\|_{s,-s} \leq C_{h,s}\Big(|z-z'|^{\delta_1} + |z-z'| + \|\chi_2 R(z,z')\chi_2\|_{s,-s}\Big).$$

From Lemma 6 we deduce the analyticity of $\chi_2 R(z,z')\chi_2$ so for fixed h and $s > 1/2$, the limits

$$R(\lambda \pm i0) = \lim_{\operatorname{Im} z \to 0,\ \pm \operatorname{Im} z > 0} R(z), \quad \operatorname{Re} z = \lambda \in J$$

exist in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$. □

For the proof of our principal result we need the following

Lemma 3. *Let $L(h)$ satisfy the assumptions (1.7) − (1.15) and assume the estimate (1.5) fulfilled. Then for every $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $B(0,R_0)$ there exist $C > 0$ such that for $(\lambda,\tau) \in J \times]0,1]$ we have*

$$\|\chi(x)R(\lambda \pm i\tau)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \le Ce^{Ch^{-p}},\ 0 < h \le h_0. \tag{3.7}$$

Proof. Set $S_\theta = \{z \in \mathbb{C} : -2\theta < \arg z \le \pi/4\}$, where $0 < \theta \le \theta_0$ is fixed and $\theta_0 \in]0,\pi[$ was introduced in the assumption (1.13) making possible a complex scaling of $L(h)$. Choose $\mu/2 > \delta > 0$ so that (1.5) holds for $\lambda \in [\mu_0 - \delta, \mu_1 + \delta]$ and consider the compact set

$$\Omega_\delta = \{z \in S_\theta : \mu_0 - \delta \le \operatorname{Re} z \le \mu_1 + \delta\}\,.$$

Then for every $m \in \mathbb{N}$, Lemma 1 in [23] yields

$$\|\chi(L(h) - z)^{-1}\chi\| \le C(\Omega_\delta, m)\exp\left(C(\Omega_\delta, m)h^{-k}\right),\ 0 < h \le h_0(\Omega_\delta, m) \tag{3.8}$$

for

$$z \in \Omega_\delta \setminus \bigcup_{z_j \in \operatorname{Res} L(h)\cap\Omega_\delta} D(z_j : h^m)$$

with $k \ge n$ and $D(z_j : h^m) = \{z \in \mathbb{C} : |z - z_j| \le h^m\}$. For the number of the scattering resonances we have the estimate (see [22], [20])

$$\sharp\{z_j : z_j \in \operatorname{Res} L(h) \cap \Omega_\delta\} \le C_1(\Omega_\delta)h^{-n^\sharp}$$

with $n^\sharp \ge n$. Next we fix an integer m satisfying $2m \ge n^\sharp + 1$ and take $0 < h_1(\Omega_\delta, m) \le h_0(\Omega_\delta, m)$ small enough to arrange

$$4C_1(\Omega_\delta)h^{2m-n^\sharp} < \delta$$

for $0 < h \le h_1(\Omega_\delta, m)$. This implies immediately the existence of $\gamma_1(h)$, $\gamma_2(h)$ so that

$$\mu_0/2 \le \mu_0 - \delta < \gamma_1(h) < \mu_0 - \frac{\delta}{2},\ \ \mu_1 + \frac{\delta}{2} < \gamma_2(h) < \mu_1 + \delta\,,$$

and

$$\gamma_j(h) + i\tau \notin \bigcup_{z_j \in \operatorname{Res} L(h)\cap\Omega_\delta} D(z_j : h^m),\ \ j = 1,2,\ \tau \ge 0\,.$$

Thus for $k > n$ fixed the estimate (3.8) holds for $z = \gamma_j(h) + i\tau,\ j = 1,2,\ \tau \ge 0$. Set $\epsilon = \frac{\delta}{10}$ and introduce the domain

$$\Pi_\delta = \{z = \lambda + i\tau : \mu_0 - \delta \le \lambda \le \mu_1 + \delta,\ 0 \le \tau \le h^M\} \subset \Omega_\delta\,,$$

where $2M > k$. By a modification of the argument in [23], Lemma 2, we may construct a function $f(z,h)$ holomorphic in Π_ϵ with the properties:

(i) $|f(z,h)| \le e$ in Π_δ,

(ii) $|f(z,h)| \ge 1/2$ in $[\mu_0 - \epsilon, \mu_1 + \epsilon] + i[0,h^M]$,

(iii) for $z \in \Pi_\delta \cap \{z : \operatorname{Re} z \le \mu_0 - 4\epsilon$, or $\operatorname{Re} z \ge \mu_1 + 4\epsilon\}$ we have

$$|f(z,h)| \le C_\epsilon \exp\Big(-\frac{\epsilon^2}{2h^{2M}}\Big),\ 0 \le h \le h(\epsilon).$$

To construct $f(z,h)$, take a function $\psi(x) \in C_0^\infty(\mathbb{R}^n)$, $0 \le \psi(x) \le 1$, such that

$$\psi(x) = \begin{cases} 0, & x \notin [\mu_0 - 3\epsilon, \mu_1 + 3\epsilon], \\ 1, & x \in [\mu_0 - 2\epsilon, \mu_1 + 2\epsilon] \end{cases}$$

and put

$$f(z,h) = (\pi\alpha^2)^{-1/2} \int \exp\Big(-\frac{(x-z)^2}{\alpha^2}\Big)\psi(x)dx, \ \alpha = h^M.$$

The check of (i) is trivial. For (ii) write $z = u + iv$ and take $u \in [\mu_0 - \epsilon, \mu_1 + \epsilon]$. Then

$$|f(z,h) - 1| \le (\pi\alpha^2)^{-1/2} \exp\Big(\frac{v^2}{\alpha^2}\Big) \int \exp\Big(-\frac{(x-u)^2}{\alpha^2}\Big)|\psi(x) - 1|dx$$

$$\le \pi^{-1/2} e \int e^{-y^2} |\psi(\alpha y + u) - 1| dy \le 2\pi^{-1/2} e \int_{\mathbb{R}\setminus B(0, \frac{\epsilon}{\alpha})} e^{-y^2} dy \le \frac{1}{2},$$

for $0 < h \le h(\epsilon)$ since $\frac{\epsilon}{\alpha} \longrightarrow \infty$ as $h \to 0$. Finally, to deal with (iii), assume for example that $u \le \mu_0 - 4\epsilon$.. We have

$$|f(z,h)| \le (\pi\alpha^2)^{-1/2} \exp\Big(\frac{v^2}{\alpha^2}\Big) \int \exp\Big(-\frac{(x-u)^2}{\alpha^2}\Big)\psi(x)dx$$

$$\le A\pi^{-1/2} h^{-M} \exp\Big(-\frac{\epsilon^2}{\alpha^2}\Big) \le C_\epsilon \exp\Big(-\frac{\epsilon^2}{2\alpha^2}\Big), \ 0 \le h \le h(\epsilon).$$

Our assumptions imply that the cut-off resolvent

$$F(z,h) = \chi(x)R(z)\chi(x)$$

has no poles on the real axis (see Appendix 5.2). Thus the function $G(z,h) = f(z,h)F(z,h)$ is holomorphic in Π_ϵ. For $\operatorname{Im} z = 0$ we have $\|G(z,h)\| \le eCe^{\frac{C}{h^p}}$, while for $\operatorname{Im} z = h^M$ we get

$$\|G(z,h)\| \le \frac{Ce}{h^M} \le e^{\frac{C_1}{h}}.$$

For $z \in \Pi_\delta$, $\operatorname{Re} z = \gamma_j(h)$, $j = 1, 2$, combining the estimates for $|f(z,h)|$ and $\|F(z,h)\|$, we obtain

$$\|G(z,h)\| \le C_\epsilon e^{\frac{C}{h^k}} e^{-\frac{\epsilon^2}{h^{2M}}} \le C_2, \ 0 < h \le h(\epsilon).$$

As in [23], by applying the maximum principle for $G(z,h)$ in Π_ϵ, we deduce the estimate (3.7) for $\lambda \in J$, $0 < \tau \le h^M$. The result for $h^M \le \tau \le 1$ follows trivially from the inequality $\|\chi(x)R(\lambda + i\tau)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \le \frac{C}{\tau}$ and the proof is complete. □

From Proposition 3 and Lemma 3 we deduce Theorem 1.

4. Microlocal resolvent estimates

In this section we obtain microlocal resolvent estimates in the trapping case. They are inspired by the microlocal estimates in Lemma 2.3 of [19] corresponding to the non-trapping case. In [19] (see also [16] for more general Hamiltonians), Robert-Tamura constructed an approximation of the propagator $\exp(ith^{-1}L)$ by a Fourier integral operator which is uniform with respect to the time t and h. It is convenient to localize in the outgoing or incoming regions of the phase space

$$\Gamma^\pm(R, d, \sigma_\pm) = \{(x,\xi) \in \mathbb{R}^{2n} \ : \ |x| > R, \ d^{-1} < |\xi| < d, \ \frac{\langle x.\xi\rangle}{|x||\xi|} \gtrless \sigma_\pm\}, \tag{4.1}$$

where $d > 1$, $-1 < \sigma_\pm < 1$ and $R \gg 1$ is large enough.

Consider a symbol $\omega_\pm(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ such that $\operatorname{supp} \omega_\pm \subset \Gamma^\pm(R, d, \sigma_\pm)$ and

$$|\partial_x^\alpha \partial_\xi^\beta \omega_\pm(x,\xi)| \le C_{\alpha,\beta,L} \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{-L} \tag{4.2}$$

for any $L \gg 1$. Then we have the following

Proposition 5. *Let $J \subset\subset \mathbb{R}^+$ and $\psi \in C_b^\infty(\mathbb{R}^n)$ be supported away from $B(0,R_0)$. Then there exists $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ on $B(0,R_0)$ such that for any $(\lambda,\tau) \in J \times [0,1]$ and $h \in (0,h_0]$ the following assertions hold:*

i) For any $s > 1/2$, $\delta > 1$ there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0,h_0]$ we have

$$\|\psi R(\lambda \pm i\tau)\psi\omega_\pm(x,hD_x)\|_{-s+\delta,-s} \leq Ch^{-2}(1 + \|\chi R(\lambda \pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}}). \tag{4.3}$$

ii) If $\sigma_+ > \sigma_-$, then for any $s \gg 1$, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0,h_0]$ we have

$$\|\omega_\mp(x,hD_x)\psi R(\lambda \pm i\tau)\psi\omega_\pm(x,hD_x)\|_{-s,s} \leq Ch^{-2}(1 + \|\chi R(\lambda \pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}}). \tag{4.4}$$

Proof. As in the proof of the Proposition 3, there exists a self-adjoint differential operator $\tilde{L} = \tilde{L}(h)$ on $L^2(\mathbb{R}^n)$, such that each $\lambda \in J$ is a non-trapping energy level for $\tilde{L}$ and the equality (3.3) holds. Then $i)$ is a consequence of Proposition 3 and $i)$ of Lemma 5. We use also the resolvent estimate (as in Corollary 1) and the fact that $\langle x\rangle^{-s}\omega_\pm(x,hD_x)\langle x\rangle^s$ is a bounded pseudo-differential operator for any $s \in \mathbb{R}$ (see [4], [14]).

According to (3.3) we have

$$\begin{aligned}\omega_\mp(x,hD_x)\psi R(z)\psi\omega_\pm(x,hD_x) &= \omega_\mp(x,hD_x)\psi R(z)\chi_2\Big(1 + [\tilde{L},\chi_1]\tilde{R}(z)(1-\chi_2)\Big)\psi\omega_\pm(x,hD_x)\\ &+\omega_\mp(x,hD_x)(1-\chi_1)\tilde{R}(z)(1-\chi_2)\omega_\pm(x,hD_x).\end{aligned}$$

However, $\omega_\mp(x,hD_x)\psi R(z)\chi_2 = \Big(\chi_2 R(\overline{z})\psi\omega_\mp(x,hD_x)^*\Big)^*$, then using (3.3), the Proposition 3 and Lemma 5 once more we obtain that the operator $\omega_\mp(x,hD_x)\psi R(\lambda \pm i\tau)\chi_2$ exists in $\mathcal{L}(L^2_{-s}, L^2_s)$ for $(\lambda,\tau) \in J \times [0,1]$ and following estimates hold

$$\|\omega_\mp(x,hD_x)\psi R(\lambda \pm i\tau)\chi_2\|_{-s,s} \leq Ch^{-1}(1 + \|\chi_2 R(\lambda \pm i\tau)\chi_2\|_{\mathcal{H}\to\mathcal{H}}).$$

Consequently, according to Lemma 5, $\omega_\mp(x,hD_x)\psi R(\lambda \pm i\tau)\psi\omega_\pm(x,hD_x)$ exists in $\mathcal{L}(L^2_{-s}, L^2_s)$ for $(\lambda,\tau) \in J \times [0,1]$ and these operators satisfy the claimed estimate (4.4). ☐

5. Appendix

5.1. **Resolvent estimates in the non-trapping case.** In this section we recall some resolvent estimates in a non-trapping energy interval for a self-adjoint differential operator $\tilde{L} = \tilde{L}(h)$ defined in a dense domain in $L^2(\mathbb{R}^n)$. We say that $J \subset\subset \mathbb{R}^+$ is non-trapping for $\tilde{L}(h)$ if every $\lambda \in J$ is a *non-trapping energy level* for the principal symbol of $\tilde{L}(h)$ in the sense of Section 2. Denote by $\|.\|_{s,s'}$ the norm of $\mathcal{L}(L^{2,s}, L^{2,s'})$, where $L^{2,s}$ is the weight space $L^2(\mathbb{R}^n, \langle x\rangle^s dx)$.

Lemma 4. *Let $\tilde{L} = \tilde{L}(h)$ be a self-adjoint differential operator having the form* (1.10) *and satisfying* (1.11)$-$(1.12). *Let $J \subset\subset \mathbb{R}^+$ be an open non-trapping interval for $\tilde{L}(h)$ and let $s > 1/2$. Then*

i) for $h \in (0,h_0]$ fixed, there exists $C_{h,s} > 0$ such that for any $z, z' \in \mathcal{B}_\pm$ we have

$$\|(\tilde{L} - z)^{-1} - (\tilde{L} - z')^{-1}\|_{s,-s} \leq C_{h,s}|z - z'|^{\delta_1}, \quad \delta_1 = (s - 1/2)/(2s - 1/2).$$

ii) for $\lambda \in J$, the limits

$$(\tilde{L} - \lambda \mp i0)^{-1} := \lim_{\operatorname{Im} z \to 0,\ \pm \operatorname{Im} z > 0} (\tilde{L} - z)^{-1}, \quad \operatorname{Re} z = \lambda,$$

exist in $\mathcal{L}(L^{2,s}, L^{2,-s})$, and there exists $C > 0$ and $h_0 > 0$ such that for $z \in \mathcal{B}_\pm$ and $h \in (0,h_0]$ we have

$$\|(\tilde{L} - z)^{-1}\|_{s,-s} \leq Ch^{-1}. \tag{5.1}$$

Proof. This result can be obtained following the Mourre theory. First, there exists a *conjugate operator* $A = A(h)$ to $\tilde{L}$ satisfying the following Mourre inequality: $\forall\Phi \in C_0^\infty(J)$, $\exists\gamma_0 > 0$ such that for all $h \in (0,h_0]$, we have

$$\Phi(\tilde{L})i^{-1}[\tilde{L}, A]\Phi(\tilde{L}) \geq \gamma_0 h\Phi(\tilde{L})^2. \tag{5.2}$$

For the construction of $A(h)$ we refer to [16], where a more general setup has been considered. The operator $A(h)$ has the form of a h-pseudodifferential operator with symbol

$$F(x,\xi) = K(x,\xi) + \frac{2x.\xi}{\langle\xi\rangle^2},$$

where $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is compactly supported with respect to x and K is uniformly bounded with respect to (x,ξ).

According to Mourre type results, we deduce that $\tilde{L}$ has no eigenvalues in J for h small enough (see [12], [3]) and for $s > 1/2$ the following assertions hold (see [13], [10] and [8],[15], [16] for operators depending on h):

i') for $h \in (0, h_0]$ fixed, there exists $C_h > 0$ such that for $z, z' \in \mathcal{B}_\pm = \{z \in \mathbb{C};\ (\mathrm{Re}\, z, \pm \mathrm{Im}\, z) \in J \times]0,1]\}$ we have

$$\|\langle A\rangle^{-s}\Big((\tilde{L}-z)^{-1} - (\tilde{L}-z')^{-1}\Big)\langle A\rangle^{-s}\|_{L^2\to L^2} \leq C_h|z-z'|^{\delta_1}.$$

ii') there exist $C > 0$ and $h_0 > 0$ such that for $z \in \mathcal{B}_\pm$ and $h \in (0,h_0]$ we have

$$\|\langle A\rangle^{-s}(\tilde{L}-z)^{-1}\langle A\rangle^{-s}\|_{L^2\to L^2} \leq Ch^{-1},\ < A >= (1+A^2)^{1/2} \tag{5.3}$$

Let us note that the operators $\langle x\rangle^{-s}\langle A\rangle^{s}$, $\langle A\rangle^{s}\langle x\rangle^{-s}$, $s \geq 0$, are uniformly bounded with respect to h in $\mathcal{L}(L^2)$. This follows from the functional calculus developed by Dimassi-Sjöstrand (see Sections 7 and 8 in [4]) or by using a complexe interpolation as in Lemma 8.2 of [13]. Then, writing

$$\langle x\rangle^{-s}(\tilde{L}-z)^{-1}\langle x\rangle^{-s} = \langle x\rangle^{-s}\langle A\rangle^{s}\ \langle A\rangle^{-s}(\tilde{L}-z)^{-1}\langle A\rangle^{-s}\ \langle A\rangle^{s}\langle x\rangle^{-s},$$

we complete the proof. □

Corollary 1. *Let P be a p-order h-admissible differential operator, $p \leq 2$ and let $s > 1/2$. Then, under the assumptions of the lemma 4 the following assertions hold:*

i") for $h \in (0,h_0]$ fixed, there exists $C_{h,s} > 0$ such that for $z, z' \in \mathcal{B}_\pm$ we have

$$\|P\Big((\tilde{L}-z)^{-1} - (\tilde{L}-z')^{-1}\Big)\|_{s,-s} \leq C_{h,s}\Big(|z-z'| + |z-z'|^{\delta_1}\Big).$$

ii") there exists $C > 0$ and $h_0 > 0$ such that for $z \in \mathcal{B}_\pm$ and $h \in (0,h_0]$ we have

$$\|P(\tilde{L}-z)^{-1}\|_{s,-s} \leq Ch^{-1}. \tag{5.4}$$

Proof. The ellipticity of $\tilde{L}$ and the functional calculus developed by Dimassi-Sjöstrand (see Sections 7 and 8 in [4]) imply that the operator $\langle x\rangle^{-s}P(\tilde{L}-i)^{-1}\langle x\rangle^{s}$ is uniformly bounded in $\mathcal{L}(L^2)$ with respect to h, for $s \geq 0$. Then ii") is a direct consequence of ii) and the resolvent equation

$$(\tilde{L}-z)^{-1} = (\tilde{L}-i)^{-1} + (z-i)(\tilde{L}-i)^{-1}(\tilde{L}-z)^{-1}.$$

In the same way i") is a direct consequence of i) and the relation

$$(\tilde{L}-z)^{-1} - (\tilde{L}-z')^{-1} = (z-z')(\tilde{L}-i)^{-1}(\tilde{L}-z)^{-1} + (z'-i)(\tilde{L}-i)^{-1}\Big((\tilde{L}-z)^{-1} - (\tilde{L}-z')^{-1}\Big).$$

□

Finally, we formulate some microlocal resolvent estimates which generalize Lemma 2.3 of [19].

Lemma 5. *Let $\tilde{L} = \tilde{L}(h)$ be a self-adjoint differential operator having the form* (1.10) *and satisfying* (1.11)$-$(1.12)*. Let $J \subset\subset \mathbb{R}^+$ be an open non-trapping interval for $\tilde{L}(h)$ and let $\omega_\pm(x,\xi)$ be a symbol supported in an outgoing (incoming) region as in Section 4. Then for any $(\lambda,\tau) \in J \times [0,1]$ the following assertions hold:*

i) For any $s > 1/2$, $\delta > 1$ there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0,h_0]$ we have

$$\|(\tilde{L}-\lambda \mp i\tau)^{-1}\omega_\pm(x,hD_x)\|_{-s+\delta,-s} \leq Ch^{-1}. \tag{5.5}$$

ii) If $\sigma_+ > \sigma_-$ then for any $s \gg 1$, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0,h_0]$ we have

$$\|\omega_\mp(x,hD_x)(\tilde{L}-\lambda \mp i\tau)^{-1}\omega_\pm(x,hD_x)\|_{-s,s} \leq Ch^{N}. \tag{5.6}$$

for any $N \in \mathbb{N}$.

The proof of this lemma is analogous to the proof of Lemma 2.3 of [19] exploiting the constructions of long time approximations in section 4 of [16] for a large class of long range perturbations of the Laplacian. The essential point is the construction of the phase $\varphi_{\pm}$ (see Proposition 4.1 of [16]).

5.2. **Boundary values of the cut-off resolvent** $\chi R(z)\chi$. In this section we will show that for fixed $h \in (0, h_0]$ the limits

$$\chi(x)R(\lambda \pm i0)\chi(x) = \lim_{\operatorname{Im} z \to 0,\ \pm \operatorname{Im} z>0} \chi(x)R(z)\chi(x),\ \lambda \in J$$

exist in $\mathcal{L}(\mathcal{H},\ \mathcal{H})$, provided $\chi \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on $B(0, R_0)$ and $\chi(x) = 0$ for $|x| \geq R_0 + a_0$, $a_0 > 0$. To do this it is sufficient to show that there are no resonances z_j in the interval $]\mu_0 - \delta, \mu_1 + \delta[$.

Lemma 6. *Assume* $(1.7) - (1.15)$ *fulfilled. Then for* $0 < h \leq h_0$ *we have*

$$\operatorname{Res} L(h) \cap]\mu_0 - \delta, \mu_1 + \delta[= \emptyset.$$

Proof. The proof can be obtained from the results in [22], [20]. Here we will only give indications of how to extract it from there. We assume h fixed and let $z_0 \in J$ be a resonance for $L(h)$. Denote by $\mathcal{H}_{R_0+a_0}$ the space of elements of $\mathcal{H}$ that vanish outside $B(0, R_0 + a_0)$. Then by using a complex scaling argument and Lemma 3.5 in [22], we conclude that for $v \in \mathcal{H}_{R_0+a_0}$ and z in a neighborhood of z_0 we have

$$(L - z)^{-1}v = \sum_{j=1}^{N} (z_0 - z)^{-j} v_{-j} + G(z) \tag{5.7}$$

with $G(z)$ holomorphic in a neighborhood of z_0 and N independent of v. Consider the spectral projection

$$\pi_{\theta,z_0} = \frac{1}{2\pi i}\int_{\gamma(z_0)} (z - L_\theta)^{-1} dz,$$

L_θ being the operator obtained by complex dilation defined in a dense set $\mathcal{D}_\theta$ of the space $\mathcal{H}_\theta = \mathcal{H} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$ with $0 < \theta \leq \theta_0$ (see [20]), while $\gamma(z_0) :\ [0, 2\pi] \ni s \longrightarrow z_0 + \epsilon e^{is}$ with $\epsilon > 0$ small enough. Let F_{θ,z_0} be the image $\pi_{\theta,z_0}(\mathcal{H}_\theta)$ and let

$$\pi_{0,z_0} = \frac{1}{2\pi i}\int_{\gamma(z_0)} (z - L)^{-1} dz :\ \mathcal{H}_{\text{comp}} \longrightarrow \mathcal{D}_{\text{loc}}.$$

Then exploiting the equality (3.3) and the fact that the resolvent $\tilde{R}(z)$ is holomorphic in a small neighborhood of z_0, as in Proposition 3.6 in [22], we conclude that there exists a bijection between the spaces F_{θ,z_0} and $\pi_{0,z_0}(\mathcal{H}_{R_0+a_0})$. Since z_0 is a resonance, the space F_{θ,z_0} is not trivial and we can find $v \in \mathcal{H}_{R_0+a_0}$ so that (5.7) holds with $\sum_{j=1}^N (z_0 - z)^{-j} v_{-j} \neq 0$. For simplicity assume that $v_{-j} = 0$ for $j = 2, ..., N$ and let $v_{-1} \neq 0$. Take a cut-off function $\Phi(x) \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $B(0, R_0)$ so that $(\Phi, v_{-1})_{\mathcal{H}} \neq 0$. Therefore,

$$\lim_{\epsilon \to 0} (\Phi, i\epsilon(L - z_0 + i\epsilon)^{-1} v_{-1})_{\mathcal{H}} = (\Phi, P_{\{z_0\}} v_{-1})_{\mathcal{H}} \neq 0,$$

$P_{\{z_0\}}$ being the spectral projector of L related to z_0, and we get $z_0 \in \sigma_{pp}(L)$ which contradicts (1.15). The case of a multiple pole can be treated in the same way. ☐

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