SEMI-CLASSICAL ASYMPTOTICS OF THE SPECTRAL FUNCTION OF PSEUDODIFFERENTIAL OPERATORS

BY
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Abstract. We consider the asymptotic behaviour of the spectral function of a self-adjoint h pseudodifferential operator in the limit as $h \to 0$. Adapting methods developed in [SV] to the semi-classical (non-homogeneous) setting, conditions are found under which a two-term asymptotic formula for the spectral function at a point on the diagonal may be written down, or under which so-called clustering of the spectral function occurs. To illustrate the results we consider the example of a Schrödinger operator $-h^2\Delta + V$ with quadratic potential V.

1. Introduction and Main Results

1.1 Introduction.

Let $A(h) = -h^2\Delta + V(x)$ denote an h admissible Schrödinger operator. Suppose that $V \in C^\infty(\mathbb{R}^n)$ is such that $A(h)$ is essentially self-adjoint for all $h \in (0, h_0]$, $h_0 > 0$, when considered as an operator in $L^2(\mathbb{R}^n)$. Later on we will allow $A(h)$ to be an arbitrary self-adjoint h pseudodifferential operator, or h P.D.O. for short, see [R] for example, and below we give conditions under which $A(h)$ is self-adjoint for $h \in (0, h_0]$.

An h P.D.O. $P(h)$ with Weyl symbol given by the formal series $\sum_{j\geq 0} h^j p_j$ is said to have compactly supported symbol if $p_j \in C_c^\infty(\mathbb{R}^n)$ for all j. The main aim of this paper is to study the asymptotics, as $h \to 0$, of the restriction to the diagonal of the kernel of the operator,

$$P(h)\chi(A(h))P(h)^*. \tag{1.1}$$

Here $P(h)$ denotes an h P.D.O. with compactly supported symbol, $P(h)^*$ denotes the $L^2(\mathbb{R}^n)$ adjoint of $P(h)$ and χ denotes the characteristic function equal to 1 on the interval $(\lambda + rh - ch, \lambda + rh + ch]$ and zero elsewhere. Throughout, $\lambda \in \mathbb{R}$ is fixed and the real parameters r and c (the energy shift and interval size parameters) are allowed to vary in bounded intervals.

More precisely, denoting by $(P(h)\chi(A(h))P(h)^*)(x,y)$ the kernel of the operator in (1.1), we define,

$$E^P_{\lambda,y}(r,c;h) = \left. (P(h)\chi(A(h))P(h)^*)(x,y)\right|_{x=y},$$

and for $\lambda \in \mathbb{R}$ and $y \in \mathbb{R}^n$ fixed (and satisfying some additional assumptions which will be specified below), we consider the asymptotics of $E^P_{\lambda,y}(r,c;h)$ as $h \to 0$.

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We denote by a_0 the principal symbol of the operator $A(h)$, and in the case of the Schrödinger operator we note that $a_0(x,\xi) = |\xi|^2 + V(x)$. If there exists $\lambda_0 \in \mathbb{R}$ such that the set $a_0((-\infty,\lambda_0])$ is compact (and non-empty) then, according to [HR1], there exists an $h_0 > 0$ such that the spectrum of $A(h)$ to the left of λ_0 is discrete for all $h \in (0,h_0]$. In this case, from the spectral theorem, for any $\lambda < \lambda_0$ and h small enough,

$$E^P_{\lambda,y}(r,c;h) = \sum_{(\lambda+rh-ch)<\lambda_j(h)\leq(\lambda+rh+ch)} |(P(h)\phi_j(h))(y)|^2,$$

where $\lambda_j(h)$ and $\phi_j(h)$ denote eigenvalues and corresponding $L^2(\mathbb{R}^n)$ orthonormalised eigenfunctions of $A(h)$ and the sum is counted according to eigenvalue multiplicity. Hence, in the case when $A(h)$ has discrete spectrum, $E^P_{\lambda,y}$ gives information about the eigenfunctions of $A(h)$. On the other hand, we note that it makes sense to consider $E^P_{\lambda,y}$ whatever the nature of the spectrum of $A(h)$.

In the discrete case, we may choose $P(h) = f(A(h))$, where $f \in C^\infty_c(\mathbb{R})$ has $f(\mu) = \mu^{\frac{\gamma}{2}}$ for some $\gamma \geq 0$ and all μ in a neighbourhood of λ. Then, recalling that r and c lie in bounded intervals, there exists an $h_0 > 0$ such that,

$$E^P_{\lambda,y}(r,c;h) = \sum_{(\lambda+rh-ch)<\lambda_j(h)\leq(\lambda+rh+ch)} |\lambda_j(h)|^\gamma |\phi_j(h)(y)|^2, \tag{1.2}$$

for all $h \in (0,h_0]$. With $\gamma = 0$ we then have,

$$E^P_{\lambda,y}(r,c;h) = e(\lambda+rh+ch,y,y;h) - e(\lambda+rh-ch,y,y;h),$$

where $e(\lambda,x,y;h)$ denotes the spectral function of $A(h)$, defined as the integral kernel of the family of spectral projection operators associated with $A(h)$. Clearly then, the function $E^P_{\lambda,y}$ contains information about the restriction to the diagonal of the spectral function of $A(h)$.

Noting that the function in (1.2) is L^1 integrable with respect to the y variable we have, in the case $\gamma = 0$,

$$\int E^P_{\lambda,y}(r,c;h)dy = \sharp\{\lambda_j(h) | (\lambda+rh-ch) < \lambda_j(h) \leq (\lambda+rh+ch)\} =: N_{\lambda+rh,c}(h),$$

where $N_{\lambda+rh,c}(h)$ denotes the counting function considered in [PP2]. The more general object $E^P_{\lambda,y}$ is therefore related to the counting function $N_{\lambda+rh,c}(h)$ and, later on, we will consider cases in which the behaviour of $E^P_{\lambda,y}$ and $N_{\lambda+rh,c}(h)$ are similar.

In the classical setting, i.e. when a differential operator is considered in place of $A(h)$ and the asymptotics are considered as $\lambda \to +\infty$, the asymptotics of the spectral function have already been extensively studied, see [SV]. There the authors consider the behaviour of the second asymptotic term of the spectral function, and conditions are found under which a two-term asymptotic formula for the spectral function may be written down, or under which so-called clustering of the spectral function occurs.

In [K] the asymptotics of the semi-classical spectral function were studied as $h \to 0$. There the author obtained the first asymptotic term of the spectral function but did not consider the second asymptotic term.

In this paper we use methods developed in [SV], adapted to the semi-classical setting, to study the second asymptotic term of $E^P_{\lambda,y}$ as $h \to 0$. In the classical (large λ) case, it is natural to impose the condition that the principal symbols of the operators involved are homogeneous with respect to the ξ variables. However, in the semi-classical case no such hypothesis can be made and this complicates our analysis considerably.

Remark. Due to the polarisation formula appearing in [SV] section 1.8, corresponding results for the kernel of the operator $P(h)\chi(A(h))Q(h)^*$, where $Q(h)$ denotes another, arbitrary h P.D.O. with compactly supported symbol, follow directly from results for $E^P_{\lambda,y}$ stated below.

1.2 h P.D.O., loop points and higher order loop points.

Let $A(h)$ denote an h P.D.O. as defined in [HR2] and suppose that $A(h)$ has Weyl symbol given by the formal series $a(h) = \sum_{j\geq 0} h^j a_j$, where the series is understood in the sense explained in [HR2]. From here on we assume that the symbol of $A(h)$ satisfies the following standard conditions.

- (H_1) $a(h)$ is real valued for all $h \in (0, h_0]$.
- (H_2) There exists a constant $\gamma_0 \in \mathbb{R}$ such that $a_0(x,\xi) \geq \gamma_0$ for all $(x,\xi) \in \mathbb{R}^{2n}$.
- (H_3) For some fixed $\gamma_1 \in \mathbb{R}$, there exist constants $C_0 \geq 0$ and $N_0 > 0$ such that,
$$a_0(x,\xi) - \gamma_1 \leq C_0(a_0(y,\eta) - \gamma_1)(1 + |x-y|^2 + |\xi - \eta|^2)^{\frac{N_0}{2}},$$
for all $(x,\xi), (y,\eta) \in \mathbb{R}^{2n}$.
- (H_4) With the same fixed γ_1 and for each $j \geq 0$ and each pair of multi-indices α, β, there exists a constant $C_{\alpha,\beta,j} > 0$ such that,
$$\left|\partial_x^\alpha \partial_\xi^\beta a_j\right| \leq C_{\alpha,\beta,j}(a_0 - \gamma_1),$$
for all $(x,\xi) \in \mathbb{R}^{2n}$.

Under these hypotheses, from [HR2], there exists an $h_1 \in (0, h_0)$ such that $A(h)$ is essentially self-adjoint for all $h \in (0, h_1]$, when considered as an operator in $L^2(\mathbb{R}^n)$. Re-defining h_0 to be equal to this h_1, the spectral theorem implies that there exist a family of (spectral) projection operators $E_\lambda(h)$, $\lambda \in \mathbb{R}$, $h \in (0, h_0]$ such that,
$$A(h) = \int_{\mathbb{R}} \lambda dE_\lambda(h),$$
where the integral is understood as a Steiltjes integral. Denoting the integral kernel of $E_\lambda(h)$ by $e(\lambda, x, y; h)$, [Hö1], for example, implies that the function e is C^∞ in the x and y variables for all $\lambda \in \mathbb{R}$, $h \in (0, h_0]$. Hence, as,
$$E^P_{\lambda,y}(r,c;h) = \Big(P(h)\tilde{P}(h)e(\lambda + rh + ch, \cdot, *; h)\Big)(x,y)\Big|_{x=y} - \Big(P(h)\tilde{P}(h)e(\lambda + rh - ch, \cdot, *; h)\Big)(x,y)\Big|_{x=y},$$

and $P(h)$ has compactly supported symbol, it follows that $E^P_{\lambda,y}$ is well defined for all $\lambda \in \mathbb{R}, h \in (0, h_0]$. (Here $P(h)$ acts with respect to $\cdot$ and $\widetilde{P}(h)$, the h P.D.O. with compactly supported symbol $\overline{p}(x, -\xi; h)$, $\overline{p}$ denoting complex conjugate and $p(x, \xi; h)$ denoting the symbol of $P(h)$, acts with respect to $*$.)

It turns out that the asymptotic behaviour of $E^P_{\lambda,y}$ is closely related to the set of so-called loop points of the Hamiltonian flow generated by the principal symbol a_0 of $A(h)$. We denote by $\Phi^t_{a_0} = \exp(tH_{a_0})$ the Hamiltonian flow generated by a_0 and, for a point $(y, \eta) \in \mathbb{R}^{2n}$, we define $(x^*(t, y, \eta), \xi^*(t, y, \eta)) = \Phi^t_{a_0}(y, \eta)$. Defining, for fixed $\lambda \in \mathbb{R}$ and fixed $y \in \mathbb{R}^n$,

$$\Sigma_{\lambda,y} = \{\xi \in \mathbb{R}^n \mid a_0(y, \xi) = \lambda\},$$

if $\eta \in \Sigma_{\lambda,y}$, then $a_0\left(\Phi^t_{a_0}(y, \eta)\right) = \lambda$ for all t. Furthermore, if $(\nabla_\xi a_0)(y, \eta) \neq 0$ for all $\eta \in \Sigma_{\lambda,y}$, then $\Sigma_{\lambda,y}$ is a smooth manifold and in this case we shall say that the pair (λ, y) is a regular value of a_0. For regular (λ, y) we define the surface measure on $\Sigma_{\lambda,y}$ defined, $d\tilde{\eta} = |\nabla_\xi a_0|^{-1} dS, dS$ being the induced Lebesgue measure on $\Sigma_{\lambda,y}$.

For $m \in \mathbb{Z}_+ := \{k \in \mathbb{Z} | k \geq 0\}$, a point $\eta_0 \in \Sigma_{\lambda,y}$ is said to be an mth order loop point if there exists a constant $T > 0$ such that,

$$\partial^\alpha_\eta \left(|y - x^*(T, y, \eta)|^2\right)\Big|_{\eta=\eta_0} = 0, \quad \text{for all } \alpha \in \mathbb{Z}^n_+ \text{ such that } \|\alpha\|_{l^1(\mathbb{Z}^n)} \leq 2m.$$

We denote by $\Pi^m_{\lambda,y} \subseteq \Sigma_{\lambda,y}$ the set of all such m th order loop points. We refer to the special cases $\Pi_{\lambda,y} := \Pi^0_{\lambda,y}$ and $\Pi^a_{\lambda,y} := \Pi^\infty_{\lambda,y}$ as the sets of loop and absolute loop points, respectively and, for $m_1 \leq m_2$, we have the inclusion $\Pi^a_{\lambda,y} \subseteq \Pi^{m_2}_{\lambda,y} \subseteq \Pi^{m_1}_{\lambda,y} \subseteq \Pi_{\lambda,y}$.

We denote by $T_y : \Pi_{\lambda,y} \longrightarrow \mathbb{R}_+$ the primitive (positive) loop 'time' function defined, for $\eta \in \Pi_{\lambda,y}$,

$$T_y(\eta) = \min\{t > 0 \mid x^*(t, y, \eta) = y\}.$$

Also, for $\eta \in \Pi_{\lambda,y}$ we define, $\Phi_y(\eta) = \xi^*(T_y(\eta), y, \eta)$ and, as $\lambda = a_0(\Phi^t_{a_0}(y, \eta))$ for all t, we have,

$$\lambda = a_0(\Phi^{T_y(\eta)}_{a_0}(y, \eta)) = a_0(y, \Phi_y(\eta)). \tag{1.3}$$

Hence, Φ_y is a function $\Phi_y : \Pi_{\lambda,y} \longrightarrow \Sigma_{\lambda,y}$.

For integer $m \in \mathbb{Z}_+$ we define the set,

$$\begin{aligned}\mathcal{O}^m_{\lambda,y} &= \left(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0\right) \\ &\cup \left(\bigcup_{k\geq 1} \{\Phi^k_y(\eta) | \eta \in \Pi^m_{\lambda,y} \cap \mathrm{supp}_y p_0 \text{ and } \Phi^j_y(\eta) \in \Pi^m_{\lambda,y} \text{ for all } 0 \leq j \leq (k-1)\}\right),\end{aligned}$$

and again we denote the special cases, $\mathcal{O}_{\lambda,y} := \mathcal{O}^0_{\lambda,y}$ and $\mathcal{O}^a_{\lambda,y} := \mathcal{O}^\infty_{\lambda,y}$. Here $\Phi^j_y(\eta)$ denotes Φ_y applied j times to η and $\mathrm{supp}_y p_0 := \{\eta \in \mathbb{R}^n | (y, \eta) \in \mathrm{supp} p_0\}$. We note that $\mathcal{O}^m_{\lambda,y}$ is the union of the images of the set $\Pi^m_{\lambda,y} \cap \mathrm{supp}_y p_0$ under the maps $\{\Phi^j_y\}^\infty_{j=0}$.

Later on we will make the following assumption,

(H_5) the principal symbols of $A(h)$ and $P(h)$ and the points $\lambda \in \mathbb{R}$ and $y \in \mathbb{R}^n$ are such that $(\nabla_\xi a_0)(y, \eta) \neq 0$ for all $\eta \in \mathcal{O}_{\lambda,y}$.

The structure (and size) of the set of m th order loop points becomes simpler (and smaller) as m increases. For this reason we will also make the following additional assumption,

(H_6) the principal symbols of $A(h)$ and $P(h)$ and the points λ and y are such that $\operatorname{meas}\left(\left(\mathcal{O}^1_{\lambda,y} \cap \Pi_{\lambda,y}\right) \setminus \Pi^1_{\lambda,y}\right) = 0,$

where meas refers to the surface measure on $\mathcal{O}_{\lambda,y}$ defined, $d\tilde{\eta} = |\nabla_\xi a_0|^{-1} dS$, dS being the induced Lebesgue measure on $\mathcal{O}_{\lambda,y}$.

The conditions in (H_5) and (H_6) appear unfamiliar and restrictive at first sight. However, as $\Phi_y(\eta) \in \Sigma_{\lambda,y}$ for all $\eta \in \Pi_{\lambda,y}$, it follows that,

$$\mathcal{O}^m_{\lambda,y} \subseteq \Sigma_{\lambda,y} \text{ for all } m \geq 0 \text{ and } p_0 \in C_c^\infty(\mathbb{R}^{2n}). \tag{1.4}$$

Hence hypothesis (H_5) is satisfied, for all $p_0 \in C_c^\infty(\mathbb{R}^{2n})$, in the case when,

(H_5') (λ, y) is a regular value of a_0.

Also, in view of (1.4), hypothesis (H_6) is satisfied, for all $p_0 \in C_c^\infty(\mathbb{R}^{2n})$, in the case when,

(H_6') the set $\Pi_{\lambda,y} \setminus \Pi^1_{\lambda,y}$ has zero measure.

The condition (H_5') is stronger than (H_5). However, we note that condition (H_5') is analogous to the corresponding condition imposed in [PP2] where the semi-classical asymptotics of the counting function was studied. There, the authors made the assumption that the set $\Sigma_\lambda := \{(y,\eta) \in \mathbb{R}^{2n} \,|\, a_0(y,\eta) = \lambda\}$ is compact and that the (full) gradient ∇a_0 is non-vanishing on Σ_λ.

Furthermore, the condition in (H_6') is analogous to the corresponding condition used when the classical case is considered, i.e. when the asymptotics of the spectral function of differential operators is considered at large energy values $\lambda \to +\infty$, see [SV]. In the classical case, the sets of loop points and absolute loop points always have equal measure (the same being true of the sets of periodic and absolutely periodic points). The proof of this fact relies heavily on the natural assumption that the principal symbol is homogeneous of degree 1 in the ξ variables.

No such assumption of homogeneity can be made in the semi-classical situation, and we must impose (H_6) or (H_6') as an additional assumption. However, applying a similar method of proof to that appearing in the proof of Lemma 1.8.3 in [SV], we can write down a sufficient condition, in terms of an auxiliary function, under which (H_6) is satisfied. First we require a lemma and a definition. (All proofs of the results in this section are given below in sections 2 and 3.)

Lemma 1.1. *Suppose that the principal symbol a_0 of $A(h)$, the points λ and y, and a compact set $X \subseteq \Sigma_{\lambda,y}$ are such that $(\nabla_\xi a_0)(y,\eta) \neq 0$ for all $\eta \in X$. Then the loop time function T_y is uniformly bounded away from zero on X. That is, there exists a constant $T > 0$ such that,*

$$T_y(\eta) > T > 0 \quad \text{for all} \quad \eta \in X.$$

In view of Lemma 1.1, it makes sense to refer to $t_\star(\eta)$ as the unique $t-$solution (if one exists) of the equation,

$$x^*(t,y,\eta)=y \quad \text{with} \quad (t,\eta)\in\left[t_0-\frac{T}{2},t_0+\frac{T}{2}\right]\times X. \tag{1.5}$$

In the special case when $\Sigma_{\lambda,y}$ is compact and (λ,y) is a regular value of a_0, this being similar to the hypothesis used in [PP2], if the following conditions are satisfied for all solutions $t_\star$ satisfying (1.5) (with $X=\Sigma_{\lambda,y}$), then hypothesis (H_6') (and hence (H_6)) is satisfied.

Theorem 1.2. *Suppose that a_0,λ and y are such that (λ,y) is a regular value of a_0. If, for all $t_0\in\mathbb{R}$, the $t-$solutions $t_\star(\eta)$ of (1.5) ((1.5) with $X=\Sigma_{\lambda,y}$) are such that, either $\nabla_\eta t_\star(\eta)=0$ or $\nabla_\eta t_\star(\eta)$ is not parallel to $(\nabla_\xi a_0)(y,\eta)$ for all $\eta\in\Pi_{\lambda,y}$, then (H_6') is satisfied.*

Later on—see sub-section 1.9 below—we show that hypotheses (H_5') and (H_6') are satisfied in the case when $a_0=|\xi|^2+Kx\cdot x$ where K is a real, symmetric, positive definite matrix (for example if $A(h)$ is a Schrödinger operator with quadratic potential), and λ and y have $\lambda>Ky\cdot y$.

In the case when a_0 is quasi-homogeneous (i.e. there exist $l,m,s\in\mathbb{N}$ such that $a_0(\mu^l x,\mu^m\xi)=\mu^s a_0(x,\xi)$ for all $(x,\xi)\in\mathbb{R}^{2n}$ and all $\mu>0$), we have the following corollary to Theorem 1.2.

Corollary 1.3. *Suppose there exist $l,m,s\in\mathbb{N}$ such that $a_0(\mu^l x,\mu^m\xi)=\mu^s a_0(x,\xi)$ for all $(x,\xi)\in\mathbb{R}^{2n}$ and all $\mu>0$. Let $y=0$. Then hypothesis (H_5') is satisfied for all $\lambda>0$. Furthermore, if $\lambda>0$ and $s=(m+l)$ then, for all $t_0\in\mathbb{R}$, the $t-$solutions $t_\star$ of (1.5) (with $X=\Sigma_{\lambda,0}$) are such that either, $\nabla_\eta t_\star(\eta)=0$ or $\nabla_\eta t_\star(\eta)$ is not parallel to $(\nabla_\xi a_0)(0,\eta)=0$, for all $\eta\in\Pi_{\lambda,0}$. Hence, the hypotheses of Theorem 1.2 (and therefore (H_6')) are satisifed.*

Other cases in which (H_6) (or (H_6')) is satisfied are not yet known. This is in sharp contrast with the corresponding condition on the sets of periodic and absolutely periodic points in the non-homogeneous case where many examples are known—see [PP1], [He] and the proof of Lemma 5 in [D]. All these examples rely on the contact structure of the energy hypersurface and it is difficult to see what could replace the rôle of the contact structure in the loop case.

1.3 Gutzwiller-type pre-trace formula.

For $\eta\in\Pi^1_{\lambda,y}$ we denote by γ_y the closed loop,

$$\gamma_y(\eta)=\{\Phi^t_{a_0}(y,\eta)|0\le t\le T_y(\eta)\}.$$

For such a closed loop we define the classical action (as the integral of the canonical 1 form ξdx along γ_y) and phase shift functions,

$$\begin{aligned} s_y(\eta)&=\int_{\gamma_y}\xi dx=\int_0^{T_y}\xi^*(t)\cdot\dot{x}^*(t)dt,\\ q_y(\eta)&=\int_0^{T_y}a_{\text{sub}}(x^*(t),\xi^*(t))dt-\frac{\pi}{2}m_y(\eta), \end{aligned} \tag{1.6}$$

where a_{sub} denotes the sub-principal symbol of $A(h)$, defined as the second term in the Weyl symbol of $A(h)$, see [R] for example, and $m_y : \Pi^1_{\lambda,y} \longrightarrow \mathbb{Z}_4$ denotes a Maslov index of $\gamma_y(\eta)$ which is defined in Definition 2.8 below.

When $\eta \in \Pi^1_{\lambda,y}$ is such that $\Phi^j_y(\eta) \in \Pi^1_{\lambda,y}$ for all $0 \leq j \leq (k-1)$, some $\mathbb{Z}_+ \ni k \geq 1$, we denote $T^{(k)}_y(\eta) = \sum_{0\leq j\leq k-1} T_y(\Phi^j_y(\eta))$. Analagously to [SV] we define the operator $U_{\lambda,y}$ acting on a suitable function f as,

$$(1.7) \qquad (U_{\lambda,y} f)(\eta) = \begin{cases} e^{-iq_y(\eta)} \left|\det \xi^*_\eta(T_y(\eta))\right|^{\frac{1}{2}} f(\Phi_y(\eta)) & \text{for all } \eta \in \Pi^1_{\lambda,y} \\ 0 & \text{for all } \eta \notin \Pi^1_{\lambda,y}, \end{cases}$$

where ξ^*_η denotes the matrix of derivatives with j,k th entry $(\xi^*_\eta)_{jk} = \frac{\partial \xi^*_k}{\partial \eta_j}$. We also define the operator $U_{\lambda,y}(h) = e^{ih^{-1}s_y(\eta)} U_{\lambda,y}$.

Similarly to [SV] we have the following lemma.

Lemma 1.4. *Let $a_0 \in C^\infty(\mathbb{R}^{2n}), p_0 \in C^\infty_c(\mathbb{R}^{2n}), \lambda \in \mathbb{R}$ and $y \in \mathbb{R}^n$ be such that $(\nabla_\xi a_0)(y,\eta) \neq 0$ for all $\eta \in \Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0$. Then the operators $U_{\lambda,y}$ and $U_{\lambda,y}(h)$ (for all $h > 0$) are partially isometric when considered as operators in the Hilbert space $L^2(\Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0) = L^2(\Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0, d\tilde{\eta})$. Furthermore, $U_{\lambda,y}$ and $U_{\lambda,y}(h)$ (for all $h > 0$) have equal kernel and image spaces and,*

$$\begin{aligned} \operatorname{Ker}\,(U_{\lambda,y}) &= \{f \in L^2(\Sigma_{\lambda,y}) | \operatorname{supp} f \cap \Phi_y(\Pi^1_{\lambda,y}) = \emptyset\}, \\ \operatorname{Im}\,(U_{\lambda,y}) &= \{f \in L^2(\Sigma_{\lambda,y}) | \operatorname{supp} f \subseteq \Pi^1_{\lambda,y}\}. \end{aligned}$$

With this notation, we have the following preliminary result which we refer to as a Gutzwiller-type pre-trace formula.

Theorem 1.5. *Let $A(h)$ denote an h P.D.O. which satisfies hypotheses (H_1) to (H_4) and let $P(h)$ denote an h P.D.O. with compactly supported symbol. Suppose that a_0, p_0, λ and y are such that hypotheses (H_5) and (H_6) are satisfied. Let $\rho \in \mathcal{S}(\mathbb{R})$ denote an arbitrary, real-valued Schwartz function with Fourier transform $\hat{\rho} \in C^\infty_c(\mathbb{R})$, $\rho > 0$ and $\hat{\rho}$ even. Then, denoting the integral kernel of the operator $P(h)\rho\left(h^{-1}(\lambda I - A(h))\right)P(h)^*$ by $\left[P(h)\rho\left(h^{-1}(\lambda I - A(h))\right)P(h)^*\right](x,y;h)$, we have the following. For all $r_0 > 0$ there exists an $h_0 > 0$ such that,*

$$(1.8) \quad \begin{aligned} h^{n-1}(2\pi)^n \left[P(h)\rho\left((\lambda h^{-1} + r)I - h^{-1}A(h)\right)P(h)^*\right](x,y;h)\big|_{x=y} &= \hat{\rho}(0)\|\tilde{p}_0\|^2 \\ + \int_{\Sigma_{\lambda,y}} \sum_{k\geq 1} \hat{\rho}(T^{(k)}_y) \left[e^{irT^{(k)}_y} \left(U_{\lambda,y}(h)\right)^k \tilde{p}_0 + \left(\left(U_{\lambda,y}(h)\right)^k\right)^* e^{-irT^{(k)}_y} \tilde{p}_0\right] \overline{\tilde{p}_0} d\tilde{\eta}& \\ + o(1),& \end{aligned}$$

for all $h \in (0, h_0]$ and $|r| \leq r_0$. Here $\|\cdot\|$ denotes norm in $L^2(\Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0)$, $o(1)$ denotes a function such that $o(1) \to 0$ as $h \to 0$ which may depend on the choice of ρ, but is independent of $r \in [-r_0, r_0]$, $\tilde{p}_0 := p_0|_{\Sigma_{\lambda,y}}$ and the sum in (1.8) is necessarily finite.

1.4 Main theorem.

We define the formal operator series,

(1.9)
$$Q_{\lambda,y}(r;h)=\sum_{k\geq 1}\left[(iT_y^{(k)})^{-1}e^{irT_y^{(k)}}\,(U_{\lambda,y}(h))^k-\left((U_{\lambda,y}(h))^k\right)^*(iT_y^{(k)})^{-1}e^{-irT_y^{(k)}}\right],$$

with $r\in\mathbb{R}$ a real parameter.

To make proper sense of the object $Q_{\lambda,y}$ we understand its rigerous definition in an analagous way to that of its classical counter part, see [SV]. We require the following lemma.

Lemma 1.6. *For all $h>0$, the series,*

$$\sum_{k\geq 1}\left[e^{irT_y^{(k)}}U_{\lambda,y}(h)^k+\left(U_{\lambda,y}(h)^k\right)^*e^{-irT_y^{(k)}}\right],\tag{1.10}$$

converges in the weak operator topology and defines a Borel measure (in r) which assumes values in the space of bounded operators on $L^2(\Sigma_{\lambda,y}\cap\operatorname{supp}_y p_0)$. This measure has Fourier transform, $\mathcal{F}_{r\to t}\sum_{k\geq 1}\left[e^{irT_y^{(k)}}U_{\lambda,y}(h)^k+\left(U_{\lambda,y}(h)^k\right)^*e^{-irT_y^{(k)}}\right]$, equal to zero for t in a small neighbourhood of the origin.

In view of Lemma 1.6 we define the function $Q^P_{\lambda,y}(r;h)$ to be the distribution function of the Borel measure,

$$(2\pi)^{-n}\left(\sum_{k\geq 1}\left[e^{irT_y^{(k)}}U_{\lambda,y}(h)^k+\left(U_{\lambda,y}(h)^k\right)^*e^{-irT_y^{(k)}}\right]\tilde{p}_0,\tilde{p}_0\right),\tag{1.11}$$

with Fourier transform, $\mathcal{F}_{r\to t}$, equal to zero for t in a small neighbourhood of the origin. (Here $(\cdot,\cdot)$ denotes inner product in $L^2(\Sigma_{\lambda,y}\cap\operatorname{supp}_y p_0)$.)

Then, formally, we have,

$$Q^P_{\lambda,y}(r;h)=(2\pi)^{-n}\left(Q_{\lambda,y}(r;h)\tilde{p}_0,\tilde{p}_0\right),\tag{1.12}$$

and, as the left-hand side of (1.12) is well defined, we can make sense of the object in (1.9) according to (1.12) (i.e. we can make proper sense of the operator series in (1.9) by way of its quadratic form).

The next theorem is the main result of this paper.

Theorem 1.7. *Let $A(h)$ denote an h P.D.O. which satisfies hypotheses (H_1) to (H_4) and let $P(h)$ denote an h P.D.O. with compactly supported symbol. Suppose that a_0,p_0,λ and y are such that hypotheses (H_5) and (H_6) are satisfied. Then for all $r_0,c_0>0$ there exist $\epsilon_0,h_0>0$ such that,*

$$h^{1-n}\left[Q^P_{\lambda,y}(r+c-\epsilon;h)-Q^P_{\lambda,y}(r-c+\epsilon;h)\right]-C\epsilon h^{1-n}-o_\epsilon(h^{1-n})$$
$$\leq E^P_{\lambda,y}(r,c;h)-\frac{2c}{(2\pi)^n}h^{1-n}\|\tilde{p}_0\|^2\leq$$
$$h^{1-n}\left[Q^P_{\lambda,y}(r+c+\epsilon;h)-Q^P_{\lambda,y}(r-c-\epsilon;h)\right]+C\epsilon h^{1-n}+o_\epsilon(h^{1-n}),$$

for all $h\in(0,h_0]$, $|r|\leq r_0$, $c\in(0,c_0]$ and $\epsilon\in(0,\epsilon_0]$ and where $C>0$ denotes a constant, independent of h,c,r and ϵ. Here, $\|\cdot\|$ denotes norm in $L^2(\Sigma_{\lambda,y}\cap\operatorname{supp}_y p_0)$ and, in general, $o_\epsilon(h^{1-n})$ depends upon $\epsilon>0$ but is uniform for $|r|\leq r_0$ and $c\in(0,c_0]$.

1.5 Clustering or two-term asymptotics.

Theorem 1.7 is the pre-trace analogue of Theorem 1.1 appearing in [PP2]. Motivated by a definition in that paper, we say that *clustering* of the spectral function of $A(h)$ occurs at the energy level λ and point y on the diagonal if there exist a bounded energy shift function $r(h)$, an h P.D.O. $P(h)$ with compactly supported symbol and positive constants $C, c_0 > 0$ such that,

$$\liminf_{h\to 0}\left(h^{n-1}E^P_{\lambda,y}(r(h),c;h)\right) \geq C, \quad \text{for all } c\in(0,c_0]. \tag{1.13}$$

Hence, clustering of the spectral function occurs if we can find an interval of length $O(h)$ (whose centre may depend on h) such that the difference of the spectral function, calculated between the end points of the interval, is $O(h^{n-1})$ and not $o(h^{n-1})$ as might be expected.

Clustering must clearly be caused by discontinuities of the function $Q^P_{\lambda,y}$ with respect to the parameter r. The following corollary provides a useful sufficient condition under which clustering occurs.

Corollary 1.8. *Suppose there exist constants $\epsilon_0, h_0 > 0$, a bounded function $r(h)$ and an h P.D.O. $P(h)$ with compactly supported symbol such that,*

$$Q^P_{\lambda,y}(r(h)+\epsilon;h) - Q^P_{\lambda,y}(r(h)-\epsilon;h) \geq C(\epsilon),$$

for all $h\in(0,h_0]$, $\epsilon\in(0,\epsilon_0]$ and where $C(\epsilon)$ denotes a function which is independent of h, but may depend continuously on ϵ. If $0<\sup_{\epsilon\in(0,\epsilon_0]}C(\epsilon)<\infty$, then we have clustering of the spectral function. To be precise, with this $r(h)$ and for any $\delta>0$, there exists a $c_0>0$ such that,

$$\liminf_{h\to 0}\left(h^{n-1}E^P_{\lambda,y}(r(h),c;h)\right) \geq \sup_{\epsilon\in(0,\epsilon_0]} C(\epsilon)-\delta, \quad \text{for all } c\in(0,c_0]. \tag{1.14}$$

On the other hand, if the function $Q^P_{\lambda,y}$ is uniformly continuous with respect to r in a bounded interval, we obtain so-called *two-term* asymptotics of the spectral function. The following corollary is analagous to Corollary 1.2 in [PP2], the proof of which applies in our case.

Corollary 1.9. *Suppose that the function $Q^P_{\lambda,y}(r;h)$ is uniformly continuous for $r\in[r_1,r_2]$ and all $h\in(0,h_0]$. Let R_1,R_2 be such that $r_1<R_1<R_2<r_2$. Then there exists a constant $c_0>0$ such that,*

$$\begin{aligned} E^P_{\lambda,y}(r,c;h) &= \frac{2c}{(2\pi)^n}h^{1-n}\|\tilde{p}_0\|^2 \\ &\quad + h^{1-n}\left[Q^P_{\lambda,y}(r+c;h)-Q^P_{\lambda,y}(r-c;h)\right]+o(h^{1-n}), \end{aligned} \tag{1.15}$$

for all $h\in(0,h_0]$, $r\in[R_1,R_2]$ and $c\in(0,c_0]$.

We have the following trivial case of Corollary 1.9.

Corollary 1.10. *Suppose that the set* $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$ *has zero measure. Then* $Q^P_{\lambda,y}(r;h) = 0$ *and the hypotheses of Corollary 1.9 are trivially satisfied. In this case we obtain that, for all* $r_0, c_0 > 0$ *there exists an* h_0 *such that,*

$$E^P_{\lambda,y}(r,c;h) = \frac{2c}{(2\pi)^n} h^{1-n} \|\tilde{p}_0\|^2 + o(h^{1-n}),$$

for all $h \in (0,h_0], r \in [-r_0, r_0]$ *and* $c \in (0, c_0]$.

From here on we consider the case when $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$ has positive measure. Ideally we would like to write down simple necessary and sufficient conditions for when clustering occurs and for when we obtain two-term asymptotics. In general, however, this is very difficult as the definition of $Q^P_{\lambda,y}$ involves compositions of the operators $e^{irT_y(\eta)}$ and $U_{\lambda,y}(h)$. These two operators do not necessarily commute and thus it is difficult to write down general, necessary and sufficient conditions for clustering. We consider some special cases.

1.6 Constant loop time and connected $\Pi^1_{\lambda,y}$.

In [PP2] it was obsereved that, from Stokes formula, the classical action is constant on connected components of the set

$$\Pi_\lambda := \{\nu \in \mathbb{R}^{2n} | a_0(\nu) = \lambda \text{ and there exists a } T > 0 \text{ such that } \Phi^T_{a_0}(\nu) = \nu\}.$$

In our case we have the following.

Lemma 1.11. *The action* $s_y(\eta)$ *is constant on connected components of* $\Pi_{\lambda,y}$.

The following theorem is the semi-classical analogue of Theorem 1.8.17 in [SV]. In the semi-classical setting we require the additional assumption that the action s_y is constant almost everywhere on the set $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$. Lemma 1.11 provides a natural, sufficient condition under which this assumption is satisfied.

Theorem 1.12. *Suppose that the loop time and action functions* T_y *and* s_y *are constant almost everywhere on* $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$. *Then the function* $Q^P_{\lambda,y}(r;h)$ *is continuous with respect to* $r \in \mathbb{R}$ *(and therefore uniformly continuous as, in this case,* $Q^P_{\lambda,y}$ *is* $(2\pi)T_y^{-1}$ *periodic in* r*) for all* $h > 0$ *if and only if* $\tilde{p}_0$ *is orthogonal (in* $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$*) to all the* $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$ *eigenfunctions of* $U_{\lambda,y}$ *with corresponding eigenvalues of unit length.*

Thus, in the case when T_y and s_y are constant almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$, if $\tilde{p}_0$ is orthogonal to all the $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$ eigenfunctions of $U_{\lambda,y}$ with corresponding eigenvalues of unit length, then $Q^P_{\lambda,y}$ is uniformly continuous with respect to $r \in \mathbb{R}$ and for all $h > 0$. According to Corollary 1.9, in this case we have two-term asymptotics of the spectral function.

On the other hand, if $\tilde{p}_0$ is an $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$ eigenfunction of $U_{\lambda,y}$ with eigenvalue e^{-iq_y}, $q_y \in (-\pi, \pi]$, then in view of (1.9) and (1.12) and summing the trigonometric series we have,

$$\begin{aligned} Q^P_{\lambda,y}(r;h) &= \frac{2}{(2\pi)^n} \sum_{k \geq 1} (kT_y)^{-1} \sin\left(k(rT_y + h^{-1}s_y - q_y)\right) \|\tilde{p}_0\|^2 \\ &= (2\pi)^{-n} T_y^{-1} \left\{\pi - rT_y - h^{-1}s_y + q_y\right\}_{2\pi} \|\tilde{p}_0\|^2, \end{aligned} \tag{1.16}$$

where $\{\cdot\}_{2\pi}$ denotes the residuum modulo 2π defined, $\{\mu\}_{2\pi} = \mu - 2\pi k$, $k \in \mathbb{Z}$ being the integer such that $-\pi < (\mu - 2\pi k) \leq \pi$.

In this case the discontinuities of the function $Q^P_{\lambda,y}$ with respect to the parameter r give rise to clustering of the spectral function. Indeed, defining, for any integer $p \in \mathbb{Z}$ and all $h > 0$, the bounded function,

$$r(h) = \left(\{q_y - h^{-1}s_y\}_{2\pi} + 2\pi p\right) T_y^{-1}, \tag{1.17}$$

then for all $0 < \epsilon < \frac{2\pi}{T_y}$ and all $h \in (0, h_0]$ we have,

$$Q^P_{\lambda,y}(r(h)+\epsilon; h) - Q^P_{\lambda,y}(r(h)-\epsilon; h) = \frac{2(\pi - \epsilon T_y)}{(2\pi)^n T_y}\|\tilde{p}_0\|^2. \tag{1.18}$$

Corollary 1.8 then implies that, for any $\delta > 0$, there exists a $c_0 > 0$ such that we obtain (1.13) with $C = (2\pi)^{1-n}\|\tilde{p}_0\|^2 T_y^{-1} - \delta$, and hence, we obtain clustering if $\|\tilde{p}_0\| > 0$.

We conclude that, in the case when T_y and s_y are constant almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$, the $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$ eigenfunctions of $U_{\lambda,y}$ with unit length eigenvalues generate clusters of the spectral function of $A(h)$.

1.7 Case when $\Phi_y(\eta) = \eta$ for almost all $\eta \in \Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$.

Lemma 1.13. *Suppose that $\Phi_y(\eta) = \eta$ almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$. Then $\mathcal{O}^m_{\lambda,y} = \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0$ for all m and $p_0 \in C_c^\infty(\mathbb{R}^{2n})$ and hence, (H_5) is satisfied in the case when a_0, p_0, λ and y are such that $(\nabla_\xi a_0)(y,\eta) \neq 0$ for all $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0$. Furthermore, hypothesis (H_6) is satisified in the case when the set $(\Pi^1_{\lambda,y} \setminus \Pi_{\lambda,y}) \cap \mathrm{supp}_y p_0$ has zero measure and, for all $f \in L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$,*

$$\left(U_{\lambda,y} f\right)(\eta) = e^{-iq_y(\eta)} f(\eta), \tag{1.19}$$

for almost all $\eta \in \Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$. Similarly to (1.16) we obtain, in this case,

$$Q^P_{\lambda,y}(r;h) = (2\pi)^{-n}\int_{\Pi^1_{\lambda,y}} \left\{\pi - rT_y(\tilde{\eta}) - h^{-1}s_y(\tilde{\eta}) + q_y(\tilde{\eta})\right\}_{2\pi} \frac{|\tilde{p}_0|^2}{T_y(\tilde{\eta})} d\tilde{\eta}. \tag{1.20}$$

The case when $\Phi_y(\eta) = \eta$ almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$ is interesting for the following reason. If all the points $\{(y,\eta) | \eta \in \Pi^1_{\lambda,y}\}$ are *periodic* points of the Hamiltonian flow $\Phi^t_{a_0}$, then $\Phi_y(\eta) = \eta$ for all $\eta \in \Pi^1_{\lambda,y}$. Thus, the case when all the points in $\{(y,\eta) | \eta \in \Pi^1_{\lambda,y}\}$ are periodic is a particular example in which the hypothesis $\Phi_y(\eta) = \eta$ is satisfied.

When studying the semi-classical asymptotics of the counting function of eigenvalues of h P.D.O., see [PP2], in place of the set of loop points one must consider the set of periodic points of the Hamiltonian flow $\Phi^t_{a_0}$. We would therefore expect that, in the case when $\Phi_y(\eta) = \eta$ almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$, the asymptotics of the spectral function are similar to those of the counting function. The following corollary illustrates the similarities and, upon comparison with Corollary 1.3 in [PP2], says that, as might be expected, the asymptotics of the spectral function in this case are essentially the same as those of the counting function.

Corollary 1.14. *Suppose that $\Phi_y(\eta) = \eta$ almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$. Suppose also that there exist a subset $\Pi_1 \subset \Sigma_{\lambda,y}$, an integer $p \in \mathbb{Z}$, an $h_0 > 0$ and a bounded function $r(h)$ (bounded for all $h \in (0, h_0]$) such that the set $(\Pi_1 \cap \Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0)$ has positive ($d\tilde{\eta}$) measure and,*

$$r(h)T_y(\eta) + h^{-1}s_y(\eta) - q_y(\eta) = 2\pi p, \tag{1.21}$$

for all $h \in (0, h_0]$ and $\eta \in (\Pi_1 \cap \Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0)$. Then for each $\delta > 0$ there exist constants $\epsilon_0, h_0 > 0$ such that,

$$Q^P_{\lambda,y}(r(h) + \epsilon; h) - Q^P_{\lambda,y}(r(h) - \epsilon; h) \geq (2\pi)^{1-n} \int_{\Pi^1_{\lambda,y} \cap \Pi_1} \frac{|\tilde{p}_0|^2}{T_y(\tilde{\eta})} d\tilde{\eta} - \delta, \tag{1.22}$$

for all $\epsilon \in (0, \epsilon_0], h \in (0, h_0]$. According to Corollary 1.8, if $\int_{\Pi^1_{\lambda,y} \cap \Pi_1} \frac{|\tilde{p}_0|^2}{T_y(\tilde{\eta})} d\tilde{\eta} > 0$, then we obtain clustering of the spectral function. Indeed, with the $r(h)$ in (1.21), for any $\delta > 0$ there exists a constant $c_0 > 0$ such that we obtain (1.13) with $C = (2\pi)^{1-n} \int_{\Pi^1_{\lambda,y} \cap \Pi_1} \frac{|\tilde{p}_0|^2}{T_y(\tilde{\eta})} d\tilde{\eta} - \delta$.

1.8 Disconnected $\Pi^1_{\lambda,y}$ and non-constant Φ_y.

In this sub-section we consider a particular case when, by imposing some regularity conditions on the flow $\Phi^t_{a_0}$, it turns out that we can write down an explicit formula for $Q^P_{\lambda,y}$.

Suppose that, except for null sets, $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$ decomposes into a finite union of disjoint sets,

$$\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0 = \bigcup_{1 \leq j \leq m} \Pi_j, \tag{1.23}$$

where $m \geq 2$ is an integer. Suppose that for all integer $1 \leq j \leq m$ and $k \geq 1$ we have,

$$\Phi^l_y(\Pi_j) \subseteq \begin{cases} \Pi_{j+l} & \text{for all } 0 \leq l \leq m-j \\ \Pi_{j+l-km} & \text{for all } km - j + 1 \leq l \leq (k+1)m - j, \end{cases} \tag{1.24}$$

and assume that the functions T_y and s_y are constant almost everywhere on each Π_j. That is, suppose there exist constants $\{T_j\}^m_{j=1}, \{s_j\}^m_{j=1}$ such that, for each j, $T_y(\eta) = T_j$ and $s_y(\eta) = s_j$ almost everywhere on Π_j. Suppose that $P(h)$ has principal symbol p_0 such that there exists another sequence of constants $\{q_j\}^m_{j=1}$ ($q_j \in (-\pi, \pi]$ for all j) with, for each j,

$$(U_{\lambda,y}\tilde{p}_0)(\eta) = e^{-iq_j}(\tilde{p}_0)(\Phi_y(\eta)) \quad \text{almost everywhere on } \Pi_j. \tag{1.25}$$

Finally, suppose that p_0 is such that,

(1.26)

$$\int_{\Pi_j} \tilde{p}_0\left(\Phi^{km+l}_y(\tilde{\eta})\right) \overline{\tilde{p}_0}(\tilde{\eta}) d\tilde{\eta} = \int_{\Pi_j} \tilde{p}_0\left(\Phi^l_y(\tilde{\eta})\right) \overline{\tilde{p}_0}(\tilde{\eta}) d\tilde{\eta} \in \mathbb{R}$$
$$\text{for all } k \geq 0, 1 \leq j \leq m \text{ and } 0 \leq l \leq (m-1),$$

$$\text{and} \quad p_{j,k} := \int_{\Pi_j} \tilde{p}_0\left(\Phi^{k+1-j}_y(\tilde{\eta})\right) \overline{\tilde{p}_0}(\tilde{\eta}) d\tilde{\eta} = \int_{\Pi_{k+1}} \tilde{p}_0\left(\Phi^{j+m-k-1}_y(\tilde{\eta})\right) \overline{\tilde{p}_0}(\tilde{\eta}) d\tilde{\eta}$$
$$\text{for all } 1 \leq j \leq (m-1) \text{ and } j \leq k \leq (m-1).$$

(We note that (1.26) is satisfied in the case when p_0 is constant on $\Pi^1_{\lambda,y}$ or $\Sigma_{\lambda,y}$.)

For integers j and k with $1 \le j \le k \le m$ we define the constants,

$$T_{j,k} = \sum_{j\le l\le k} T_l\ ,\ s_{j,k} = \sum_{j\le l\le k} s_l\ ,\ q_{j,k} = \sum_{j\le l\le k} q_l.$$

Then considering (1.9) and (1.12) and summing the series we have:

Lemma 1.15. *Let the hypotheses of the second paragraph of this sub-section be satisfied. Then,*

(1.27)

$$Q^P_{\lambda,y}(r;h) = \frac{2}{(2\pi)^n}\sum_{k\ge 1}(kT_{1,m})^{-1}\sin\left(k(rT_{1,m}+h^{-1}s_{1,m}-q_{1,m})\right)\int_{\Pi^1_{\lambda,y}}|\tilde p_0|^2 d\tilde\eta$$

$$+\frac{2}{(2\pi)^n}\sum_{\substack{1\le j\le m-1\\ j\le k\le m-1}}\left\{\left(\sum_{l\in\mathbb{Z}}(lT_{1,m}+T_{j,k})^{-1}\sin(r(lT_{1,m}+T_{j,k})+h^{-1}(ls_{1,m}+s_{j,k}) -(lq_{1,m}+q_{j,k}))\right)p_{j,k}\right\}.$$

We can sum the first series on the right-hand side of (1.27) as in (1.16). To sum the second series we require the following lemma.

Lemma 1.16. *Let $\mu\in\mathbb{R}\setminus\mathbb{Z}$. Then,*

$$2\sum_{k\in\mathbb{Z}}(k+\mu)^{-1}\sin\left((k+\mu)x\right) = (2\pi)\frac{\sin\left(\mu(x-\{x-\pi\}_{2\pi})\right)}{\sin(\pi\mu)}, \tag{1.28}$$

and we note that the function in (1.28) is periodic in x if and only if $\mu\in\mathbb{Q}\setminus\mathbb{Z}$.

In the case when there exist constant $s,q\in\mathbb{R}$ such that,

$$s_j = sT_j \quad\text{and}\quad q_j = qT_j \quad\text{for all } 1\le j\le m, \tag{1.29}$$

we can use Lemma 1.16 to sum the second series on the right-hand side of (1.27). Combining Lemmas 1.15 and 1.16 we have:

Theorem 1.17. *Let the hypotheses of the second paragraph of this sub-section hold and suppose that there exists constants $s,q\in\mathbb{R}$ such that (1.29) holds. Then,*

(1.30)

$$Q^P_{\lambda,y}(r;h) = \frac{1}{(2\pi)^n T_{1,m}}\{\pi-(r+h^{-1}s-q)T_{1,m}\}_{2\pi}\int_{\Pi^1_{\lambda,y}}|\tilde p_0|^2\ d\tilde\eta$$

$$+\sum_{\substack{1\le j\le m-1\\ j\le k\le m-1}}\frac{p_{j,k}\sin\left(\frac{T_{j,k}}{T_{1,m}}\left((r+h^{-1}s-q)T_{1,m}-\{(r+h^{-1}s-q)T_{1,m}-\pi\}_{2\pi}\right)\right)}{(2\pi)^{n-1}T_{1,m}\sin\left(\frac{\pi T_{j,k}}{T_{1,m}}\right)}.$$

In view of Theorem 1.17, let $r(h)$ denote a bounded function such that,

$$(r(h)+h^{-1}s-q)T_{1,m}\equiv 0 \mod 2\pi, \tag{1.31}$$

for all $h \in (0, h_0]$. Then for all $0 < \epsilon < \frac{2\pi}{T_{1,m}}$ and $h \in (0, h_0]$ we have,
(1.32)

$$Q^P_{\lambda,y}(r(h)+\epsilon;h) - Q^P_{\lambda,y}(r(h)-\epsilon;h) = \frac{2(\pi - \epsilon T_{1,m})}{(2\pi)^n T_{1,m}} \int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 \, d\tilde{\eta} + \frac{(4\pi)}{(2\pi)^n T_{1,m}} \sum_{\substack{1\le j\le m-1\\ j\le k\le m-1}} \left(p_{j,k}\cos\left((r(h)+h^{-1}s-q)T_{j,k}\right)\right).$$

According to Corollary 1.8, we obtain clustering of the spectral function, in this case, if there exists a bounded function $r(h)$ which satisfies (1.31) and is such that, for all $h \in (0, h_0]$,

$$2\sum_{\substack{1\le j\le m-1\\ j\le k\le m-1}} \left(p_{j,k}\cos\left((r(h)+h^{-1}s-q)T_{j,k}\right)\right) + \int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta} > 0. \tag{1.33}$$

On the other hand, if,

$$2\sum_{\substack{1\le j\le m-1\\ j\le k\le m-1}} \left(p_{j,k}\cos\left(\frac{2\pi l T_{j,k}}{T_{1,m}}\right)\right) + \int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta} = 0, \quad \text{for all } l \in \mathbb{Z}, \tag{1.34}$$

then $Q^P_{\lambda,y}$ is continuous in r, for all $h \in (0, h_0]$. Furthermore, as $Q^P_{\lambda,y}$ is periodic in r if and only if $T_jT_1^{-1} \in \mathbb{Q}$ for all $2 \le j \le m$, if (1.34) holds and $T_jT_1^{-1} \in \mathbb{Q}$ for all $2 \le j \le m$, then $Q^P_{\lambda,y}$ is uniformly continuous in r for all $h \in (0, h_0]$. In this case, according to Corollary 1.9, we obtain two-term asymptotics of the spectral function.

Returning to the possibility of clustering, if there exists an integer *clustering shift constant*, $\omega \in \mathbb{Z}$, such that,

$$2\sum_{\substack{1\le j\le m-1\\ j\le k\le m-1}} \left(p_{j,k}\cos\left(\frac{2\pi \omega T_{j,k}}{T_{1,m}}\right)\right) + \int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta} > 0, \tag{1.35}$$

then we can always achieve (1.33). Indeed, in the case when $T_jT_1^{-1} \in \mathbb{Q}_+$ for all $2 \le j \le m$, we choose sequences of positive integers $\{a_{j,k}\}$, $\{b_{j,k}\}$ such that,

$$\frac{T_{j,k}}{T_{1,m}} = \frac{a_{j,k}}{b_{j,k}} \quad \text{and} \quad \text{g.c.d.}\,(a_{j,k}, b_{j,k}) = 1, \quad \text{for all } 1 \le j \le k \le (m-1),$$

(here, necessarily, $a_{j,k} < b_{j,k}$ and g.c.d. denotes greatest common divisor). We then define $\tilde{T} = \text{l.c.m.}_{1\le j\le k\le m-1}\{b_{j,k}\}$, where l.c.m. denotes least common multiple, and take $r(h)$ to be the bounded function,

$$r(h) = \frac{\tilde{T}}{T_{1,m}} \left\{\frac{T_{1,m}}{\tilde{T}}(q - h^{-1}s)\right\}_{2\pi} + \frac{2\pi\omega}{T_{1,m}}. \tag{1.36}$$

Then $r(h)$ satisfies (1.31) and $(r(h)+h^{-1}s-q)T_{j,k} \equiv \frac{2\pi\omega T_{j,k}}{T_{1,m}} \mod 2\pi$ for all j and k. According to (1.32), and Corollary 1.8, we obtain that, for any $\delta > 0$, there exists a $c_0 > 0$ such that,

(1.37)
$$(2\pi)^{n-1}T_{1,m} \liminf_{h\to 0} \left(h^{n-1}E^P_{\lambda,y}(r(h),c;h)\right) \geq 2 \sum_{\substack{1\leq j\leq m-1 \\ j\leq k\leq m-1}} \left(p_{j,k}\cos\left(\frac{2\pi\omega T_{j,k}}{T_{1,m}}\right)\right) + \int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta} - \delta,$$

for all $c \in (0, c_0]$, and we obtain clustering if (1.35) holds.

In the general case when we know only that $T_jT_1^{-1} \in \mathbb{R}_+$ we must work harder to acheive (1.33). However, using the $r(h)$ given by the following lemma, we again obtain (1.37).

Lemma 1.18. *For any $\delta' > 0$, there exists a bounded function $r(h)$ which satisfies (1.31) and is such that,*

$$\left|\left\{\left(r(h)+h^{-1}s-q\right)T_{j,k} - \frac{2\pi\omega T_{j,k}}{T_{1,m}}\right\}_{2\pi}\right| < \delta', \tag{1.38}$$

for all $1 \leq j \leq k \leq (m-1), h > 0$. With this $r(h)$, for any $\delta > 0$, there exists a $c_0 > 0$ such that we have (1.37) and clustering occurs if (1.35) is true.

Remark. We note that, in the case when $p_{j,k} = \frac{1}{m}\int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta}$ for all $1 \leq j \leq k \leq (m-1)$ (eg. in the case when p_0 is constant on $\Pi^1_{\lambda,y}$ and the Π_j have equal measure) and $\omega = 0$, the constant on the right-hand side of (1.37) becomes $m\int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta} - \delta$. In this case, on comparison with the result obtained in the case when T_y and s_y are constant almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$, the factor $\frac{T_{1,m}}{m}$ clearly plays the rôle of an average loop time. In the general case, when (1.35) is satisfied, it therefore makes sense to think of,

$$T_{1,m}\left(\int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta}\right)\left(2\sum_{\substack{1\leq j\leq m-1 \\ j\leq k\leq m-1}} \left(p_{j,k}\cos\left(\frac{2\pi\omega T_{j,k}}{T_{1,m}}\right)\right) + \int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta}\right)^{-1},$$

as a *weighted average loop time.*

1.9 Schrödinger operator with quadratic potential.

We consider the example of a Schrödinger operator $A(h) = -h^2\Delta + Kx\cdot x$ with $K > 0$ a real, symmetric, positive-definite matrix. Without loss of generality we may assume that K is diagonal (otherwise an orthogonal matrix may be used to diagonalise K and must be carried through all subsequent calculations), and we denote by $\{\lambda_j\}_{j=1}^n$ the n, positive eigenvalues of K. Defining the diagonal matrix $X(t)$ with entries $X_{jk}(t) = \frac{\delta_{jk}}{\sqrt{\lambda_j}}\sin(t\sqrt{\lambda_j})$ we have,

$$\begin{pmatrix} x^* \\ \xi^* \end{pmatrix} = \begin{pmatrix} \dot{X}(t) & X(t) \\ \ddot{X}(t) & \dot{X}(t) \end{pmatrix}\begin{pmatrix} y \\ \eta \end{pmatrix}. \tag{1.39}$$

If $\lambda \in \mathbb{R}$ and $y \in \mathbb{R}^n$ are such that $\lambda > Ky \cdot y$ then $\Sigma_{\lambda,y} = (\lambda - Ky \cdot y)^{\frac{1}{2}} \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ denotes the usual n-sphere, and $\nabla_\xi a_0 \neq 0$ on $\Sigma_{\lambda,y}$. Hence hypothesis (H_5') is satisfied in the case when $\lambda > Ky \cdot y$.

In what follows, in the case when $\sqrt{\frac{\lambda_j}{\lambda_1}} \in \mathbb{Q}$ for all $2 \leq j \leq n$, choosing any integer $p_j, q_j \in \mathbb{N}$ such that g.c.d.$(p_j, q_j) = 1$ and $\sqrt{\frac{\lambda_j}{\lambda_1}} = \frac{p_j}{q_j}$ for all $1 \leq j \leq n$, we denote by T the positive constant

$$T = \frac{\pi}{2\sqrt{\lambda_1}} \ \text{l.c.m.}\,(q_1, \ldots, q_n)\,, \tag{1.40}$$

and J the idempotent, diagonal matrix with entries,

$$J_{jk} = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } 2\sqrt{\lambda_j}T \equiv 0 \mod 2\pi, \\ -1 & \text{if } 2\sqrt{\lambda_j}T \equiv \pi \mod 2\pi. \end{cases} \tag{1.41}$$

Theorem 1.19. *Let $\lambda \in \mathbb{R}$ and $y \in \mathbb{R}^n$ be such that $\lambda > Ky \cdot y$. Then $\Pi_{\lambda,y}$ has positive $(d\tilde{\eta})$ measure if and only if $\sqrt{\frac{\lambda_j}{\lambda_1}} \in \mathbb{Q}$ for all $2 \leq j \leq n$. In the case when $\sqrt{\frac{\lambda_j}{\lambda_1}} \in \mathbb{Q}$ for all $2 \leq j \leq n$, we have,*

i) $\Pi_{\lambda,y} = \Pi^1_{\lambda,y} = \Sigma_{\lambda,y}$,

ii) *if $y \neq 0$ then, for almost all $\eta \in \Pi_{\lambda,y}$, we have: $T_y(\eta) = 2T$, $\Phi_y(\eta) = \eta$, $s_y(\eta) = 2\lambda T$, $q_y(\eta) = \{\pi + \mathrm{Tr}\,(\sqrt{K})T\}_{2\pi} - \pi$ and $(U_{\lambda,y} f)\,(\eta) = e^{-iq_y} f(\eta)$,*

iii) *if $y = 0$ then, for almost all $\eta \in \Pi_{\lambda,y}$, we have: $T_y(\eta) = T$, $\Phi_y(\eta) = J\eta$, $s_y(\eta) = \lambda T$, $q_y(\eta) = \{\pi + \frac{1}{2}\mathrm{Tr}\,(\sqrt{K})T\}_{2\pi} - \pi$ and $(U_{\lambda,y} f)\,(\eta) = e^{-iq_y} f(J\eta)$,*

where f denotes any function.

In view of Theorem 1.19 i), in the case when $A(h) = -h^2\Delta + Kx \cdot x$, hypothesis (H_6') is satisfied when K is any real, symmetric, positive definite matrix, and λ and y are any points such that $\lambda > Ky \cdot y$.

In the case when there exists a $2 \leq j \leq n$ such that $\sqrt{\frac{\lambda_j}{\lambda_1}} \notin \mathbb{Q}$, $\Pi_{\lambda,y}$ has zero measure and hence, Corollary 1.10 implies that, for all h P.D.O. $P(h)$ with compactly supported symbol and all $c_0 > 0$, there exists an $h_0 > 0$ such that,

$$E^P_{\lambda,y}(r, c; h) = \frac{2c}{(2\pi)^n} h^{1-n} \|\tilde{p}_0\|^2 + o(h^{1-n}), \tag{1.42}$$

for all $h \in (0, h_0]$ and $c \in (0, c_0]$.

Remark. In the case when, for example, $p_0 = f(a_0)$ for η in a neighbourhood of $\Sigma_{\lambda,y}$, with f some function, we note that $\|\tilde{p}_0\|^2 = |f(\lambda)|^2 (\lambda - Ky \cdot y)^{\frac{n-1}{2}} \omega_{n-1}$, where ω_{n-1} denotes the surface area of the unit n-sphere $\mathbb{S}^{n-1}$.

On the other hand, if $\sqrt{\frac{\lambda_j}{\lambda_1}} \in \mathbb{Q}$ for all $2 \leq j \leq n$, Theorem 1.19 implies that $\Pi^1_{\lambda,y}$ has positive measure and that the loop time and action functions, T_y and s_y, are constant almost everywhere on $\Pi^1_{\lambda,y}$. Furthermore, in the case when $y \neq 0$, any function in $L^2(\Sigma_{\lambda,y})$ is an eigenfunction of $U_{\lambda,y}$ with eigenvalue e^{-iq_y}, and we can apply Theorem 1.12. Taking $r(h)$ as in (1.17), we obtain that, for any $\delta > 0$, there exists a $c_0 > 0$ such that,

$$\liminf_{h\to 0} \left(h^{n-1} E^P_{\lambda,y}(r(h), c; h)\right) \geq \frac{(2\pi)^{1-n}}{2T} \|\tilde{p}_0\|^2 - \delta, \tag{1.43}$$

for all $c \in (0, c_0]$, and clustering of the spectral function occurs if $\|\tilde{p}_0\| > 0$.

Similarly, in the case when $y = 0$ and $2\sqrt{\lambda_j}T \equiv 0 \mod 2\pi$, for all $1 \leq j \leq n$, the matrix $J = I$ and hence, any function in $L^2(\Sigma_{\lambda,y})$ is an eigenfunction of $U_{\lambda,y}$ with eigenvalue e^{-iq_y}, and again we can apply Theorem 1.12. Choosing $r(h)$ as in (1.17), in this case we obtain (1.43) with the $2T$ replaced by T.

In the case when $y = 0$ and there exists a $1 \leq k \leq n$ such that $2\sqrt{\lambda_k}T \equiv \pi \mod 2\pi$, the matrix $J \neq I$ and the situation is more complicated. We must apply the results in sub-section 1.8.

Defining $\Pi_1 = \{\eta \in \Sigma_{\lambda,y} | \eta_k > 0\} \cap \operatorname{supp}_y p_0$ and $\Pi_2 = \{\eta \in \Sigma_{\lambda,y} | \eta_k < 0\} \cap \operatorname{supp}_y p_0$, and assuming that p_0 is such that $J\Pi_1 = \Pi_2$, then $\Pi^1_{\lambda,y}$ decomposes as in (1.23), with $m = 2$. Clearly (1.24) is satisfied with this choice of Π_j and, furthermore, $T_1 = T_2 = T$, $s_1 = s_2 = \lambda T$ and (1.25) is satisfied with $q_1 = q_2 = \{\pi + \frac{1}{2}\mathrm{Tr}\,(\sqrt{K})T\}_{2\pi} - \pi$. We note that (1.29) is also satisfied.

Furthermore if in addition, $\tilde{p}_0 \in \mathbb{R}$, then as J is idempotent, the first line in (1.26) is satisfied. Using the change of variables $\tilde{\eta} \mapsto J\tilde{\eta}$, it is easy to check that the second line in (1.26) is satisfied and,

$$p_{1,1} = \int_{\Pi_1} \tilde{p}_0(J\tilde{\eta})\overline{\tilde{p}_0}(\tilde{\eta})d\tilde{\eta} = \frac{1}{2}\int_{\Sigma_{\lambda,y}} \tilde{p}_0(J\tilde{\eta})\overline{\tilde{p}_0}(\tilde{\eta})d\tilde{\eta}. \tag{1.44}$$

In the non-trivial case when $\|\tilde{p}_0\| > 0$, either,

$$\|\tilde{p}_0\|^2 + 2p_{1,1} > 0 \quad \text{or} \quad \|\tilde{p}_0\|^2 - 2p_{1,1} > 0, \tag{1.45}$$

(if $\|\tilde{p}_0\|^2 + 2p_{1,1} \leq 0$ and $\|\tilde{p}_0\|^2 - 2p_{1,1} \leq 0$, then $\|\tilde{p}_0\|^2 \leq 0$), and thus, (1.34) cannot be true. We cannot, therefore, have two-term asymptotics of the spectral function in this case.

On the other hand, as one of the inequalities in (1.45) is always satisfied, we always have clustering of the spectral function. Indeed if $\|\tilde{p}_0\|^2 \pm 2p_{1,1} > 0$, then choosing $\omega = 0$ or $\omega = 1$, respectively, as the clustering shift constant and taking $r(h) = \frac{1}{T}\{T(q - h^{-1}s)\}_{2\pi} + \frac{\pi\omega}{T}$, we obtain that, for any $\delta > 0$, there exists a $c_0 > 0$ such that,

$$\liminf_{h\to 0} \left(h^{n-1} E^P_{\lambda,y}(r(h), c; h)\right) \geq \frac{(2\pi)^{1-n}}{2T} \left(\|\tilde{p}_0\|^2 \pm 2p_{1,1}\right) - \delta,$$

respectively, for all $c \in (0, c_0]$ and we obtain clustering.

2. Proof of Theorems 1.5 and 1.7

2.1 Tauberian result and Theorem 1.7.

Throughout what follows we let $\rho, \gamma \in \mathcal{S}(\mathbb{R})$ denote Schwartz functions satisfying the following conditions:

i) $\hat{\rho}, \hat{\gamma} \in C_c^\infty(\mathbb{R})$,
ii) $\rho, \gamma > 0$,
iii) $\hat{\rho}$ and $\hat{\gamma}$ are both even,
iv) $\hat{\rho}(0) = 1, \hat{\rho}'(0) = 0$,
v) $\hat{\gamma}(0) = 0$.

where $\hat{\rho}$ denotes Fourier transform and $\hat{\rho}'$ denotes differentiation.

For any $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 < \lambda_2$ we denote by $\mathcal{F}_+(\lambda_1, \lambda_2)$ the space of functions $\sigma_h(\lambda)$ such that:

i) for each $h \in (0, h_0]$, $\sigma_h(\lambda)$ is a real-valued, non-negative, non-decreasing function of λ,
ii) $\sigma_h(\lambda) = 0$ for all $\lambda \leq \lambda_1$,
iii) $\sigma_h(\lambda) = \sigma_h(\lambda_2)$ for all $\lambda \geq \lambda_2$,
iv) for each function $\sigma_h \in \mathcal{F}_+(\lambda_1, \lambda_2)$, there exists a $\mathbb{R} \ni p > 0$ such that, for all $\lambda \in \mathbb{R}$, $\sigma_h(\lambda) = o(h^{-p})$ as $h \to 0$.

For any $\delta \neq 0$ we define the dilation of a function f as $f_\delta(\lambda) = \frac{1}{\delta} f(\frac{\lambda}{\delta})$ and, rescaling the result in [SV], Theorem B.4.1, we have the following Tauberian result, see [B3] for proof.

Theorem 2.1. *Let $\sigma_{jh} \in \mathcal{F}_+(\lambda_1, \lambda_2)$, $j = 1, 2$. Suppose there exists a function ρ satisfying the conditions above such that,*

$$\frac{d}{d\lambda}(\sigma_{jh} * \rho_h)(\lambda) = O_j(h^{-n}), \qquad j = 1, 2,$$

where the O_j are uniform with respect to $\lambda \in \mathbb{R}$. With this ρ, suppose that for all $r_0 > 0$ and all functions γ satisfying the conditions above,

$$\begin{aligned}(\sigma_{1h} * \rho_h)(\lambda + rh) &= (\sigma_{2h} * \rho_h)(\lambda + rh) + o(h^{1-n}), \\ \frac{d}{d\lambda}(\sigma_{1h} * \gamma_h)(\lambda + rh) &= \frac{d}{d\lambda}(\sigma_{2h} * \gamma_h)(\lambda + rh) + o(h^{-n}),\end{aligned}$$

where the o are independent of $|r| \leq r_0$. (Of course the O_j and o may depend on the choice of ρ and γ.) Then for all $\epsilon > 0, |r| \leq r_0$ and $h \in (0, h_0]$,

$$\sigma_{2h}(\lambda + (r - \epsilon)h) - o_\epsilon(h^{1-n}) \leq \sigma_{1h}(\lambda + rh) \leq \sigma_{2h}(\lambda + (r + \epsilon)h) + o_\epsilon(h^{1-n}),$$

where the o_ϵ is independent of $|r| \leq r_0$ but may depend on ϵ.

With a view to applying the above theorem we define the functions,

$$\begin{aligned}\sigma_{1h}(\lambda) &= \left(P(h)\chi(A(h)P(h)^*)\,(x,y)\right|_{x=y}, \\ \sigma_{2h}(\lambda + rh) &= (\sigma_{1h} * \rho_h)(\lambda) + \frac{rh^{1-n}}{(2\pi)^n}\|\tilde{p}_0\|^2 + h^{1-n} Q^P_{\lambda,y}(r; h),\end{aligned} \tag{2.1}$$

where χ denotes the characteristic function of the interval $(-\infty, \lambda]$ and, as usual, $\|\cdot\|$ denotes norm in $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$.

Let $f \in C_c^\infty(\mathbb{R})$ denote a real-valued function which is equal to 1 in a neighbourhood of λ and has support contained in the set $[\lambda - \delta, \lambda + \delta], \delta > 0$. Then we define $\tilde{\sigma}_{1h}$ and $\tilde{\sigma}_{2h}$ to be the same functions in (2.1) but with the $P(h)$ replaced by $P(h)f(A(h))$.

For any ρ satisfying the assumptions above and having $\operatorname{supp}\hat{\rho} \subseteq [-T, T]$ where $T > 0$ is given in Lemma 1.1 (with $X = \Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0$), an explicit calculation gives,

$$(\tilde{\sigma}_{2h} * \rho_h)(\lambda + rh) = (\tilde{\sigma}_{1h} * \rho_h)(\lambda) + \frac{rh^{1-n}}{(2\pi)^n}\|\tilde{p}_0\|^2. \tag{2.2}$$

(Here we have used Lemma 1.1 and the fact that $\operatorname{supp}\hat{\rho} \subseteq [-T, T]$, to show that $Q^P_{\lambda,y}(\cdot; h) * \rho_h = 0$.) Then, assuming for a moment, that Theorem 1.5 is true and observing that,

(2.3)
$$\begin{aligned}&\frac{d}{d\lambda}\left(\tilde{\sigma}_{1h} * \rho_h\right)(\lambda + rh)\\ &\quad = h^{-1}\left[P(h)f(A(h))\rho\left((\lambda h^{-1} + r)I - h^{-1}A(h)\right)f(A(h))P(h)^*\right](x, y; h)\Big|_{x=y},\end{aligned}$$

from (2.2) we have,

$$(\tilde{\sigma}_{2h} * \rho_h)(\lambda + rh) = (\tilde{\sigma}_{1h} * \rho_h)(\lambda) + (rh)\frac{d}{d\lambda}\left(\tilde{\sigma}_{1h} * \rho_h\right)(\lambda) + o(h^{1-n}), \tag{2.4}$$

where the $o(h^{1-n})$ is independent of r in any bounded interval, $r \in [-r_0, r_0]$.

Using integration by parts and again assuming the statement of Theorem 1.5,

(2.5)
$$\begin{aligned}&(rh)\int_0^1 (1-t)\frac{d^2}{d\lambda^2}\left(\tilde{\sigma}_{1h} * \rho_h\right)(\lambda + rth)dt\\ &\qquad\qquad = -\frac{d}{d\lambda}\left(\tilde{\sigma}_{1h} * \rho_h\right)(\lambda) + \int_0^1 \frac{d}{d\lambda}(\tilde{\sigma}_{1h} * \rho_h)(\lambda + rth)dt = o(h^{-n}),\end{aligned}$$

where the $o(h^{-n})$ is independent of r in any bounded interval. Hence, applying Taylor's Theorem and (2.5), from (2.4) we obtain,

$$(\tilde{\sigma}_{2h} * \rho_h)(\lambda + rh) = (\tilde{\sigma}_{1h} * \rho_h)(\lambda + rh) + o(h^{1-n}), \tag{2.6}$$

again with the $o(h^{1-n})$ independent of r in any bounded interval.

Another explicit calculation, noting that (2.3) remains true with ρ replaced by any γ satisfying the assumptions above, and assuming the statement of Theorem 1.5 gives,

$$\frac{d}{d\lambda}(\tilde{\sigma}_{2h} * \gamma_h)(\lambda + rh) = \frac{d}{d\lambda}(\tilde{\sigma}_{1h} * \gamma_h)(\lambda + rh) + o(h^{-n}), \tag{2.7}$$

for any γ satisfying the assumptions above and with the $o(h^{-n})$ independent of r in any bounded interval.

Again assuming the statement of Theorem 1.5, $\frac{d}{d\lambda}(\tilde{\sigma}_{1h} * \rho_h)(\lambda) = O(h^{-n})$ and, as f has compact support, $\tilde{\sigma}_{1h} \in \mathcal{F}_+(\lambda_1, \lambda_2)$, for some λ_1, λ_2. Furthermore, $\frac{d}{d\lambda}(\tilde{\sigma}_{2h} * \rho_h)(\lambda) = O(h^{-n})$ and, in view of (2.6) and (2.7), $\tilde{\sigma}_{2h} \in \mathcal{F}_+(\lambda_1, \lambda_2)$, with the same λ_1, λ_2.

We have thus established that, assuming the statement of Theorem 1.5 to be true, $\tilde{\sigma}_{1h}$ and $\tilde{\sigma}_{2h}$ satisfy the hypotheses of Theorem 2.1. Applying Theorem 2.1 to $\tilde{\sigma}_{1h}(\lambda + rh + ch)$, $\tilde{\sigma}_{2h}(\lambda + rh + ch)$, $\tilde{\sigma}_{1h}(\lambda + rh - ch)$ and $\tilde{\sigma}_{2h}(\lambda + rh - ch)$, and observing that $E^P_{\lambda,y}(r, c; h) = \tilde{\sigma}_{1h}(\lambda+rh+ch) - \tilde{\sigma}_{1h}(\lambda+rh-ch)$ for small enough h, we obtain the statement in Theorem 1.7. Hence, Theorem 1.7 follows in corollary from Theorems 1.5 and 2.1., and we are reduced to proving only Theorem 1.5. The remainder of this section is devoted to a proof of Theorem 1.5.

2.2 Approximation by h F.I.O.

Let $\mu > 0$ and let $P(h)$ and $Q(h)$ denote h P.D.O. with compactly supported symbols. As in [B2] we define Z to be the matrix of derivatives $Z(t, y, \eta, \mu) = \xi^*_\eta - i\mu x^*_\eta$, where $x^* = x^*(t, y, \eta)$, with the same for ξ^*, $(x^*, \xi^*) = \Phi^t_{a_0}(y, \eta)$ and x^*_η denotes the matrix of derivatives with j, k th entry $(x^*_\eta)_{jk} = \frac{\partial x^*_k}{\partial \eta_j}$. We re-call, from [B2], that $\det Z \neq 0$ for all t, y, η and $\mu > 0$, and we define the function,

$$u_0(t, y, \eta, \mu) = (\det Z)^{\frac{1}{2}} e^{-i\int_0^t a_{\mathrm{sub}}(x^*,\xi^*)ds} p_0(x^*, \xi^*)\overline{q_0}(y, \eta), \tag{2.8}$$

where p_0 and q_0 denote the principal symbols of $P(h)$ and $Q(h)$. Now $Z(0) = I$ and, as $\det Z(0) = 1$ and $\det Z \neq 0$ for all t, the square root $(\det Z)^{\frac{1}{2}}$ in (2.8) is understood as the branch of $(\det Z)^{\frac{1}{2}}$ which is continuous for all t and equal to 1 when $t = 0$. We also define ϕ to be the function,

$$\phi(t, x, y, \eta, \mu) = \int_0^t \xi^*(s) \cdot \dot{x}^*(s)ds - ta_0(x^*, \xi^*) + \xi^* \cdot (x - x^*) + \frac{i\mu}{2}|x - x^*|^2. \tag{2.9}$$

According to [B2], for any h P.D.O. $A(h)$ which is essentially self-adjoint in $L^2(\mathbb{R}^n)$ for all $h \in (0, h_0]$, $h_0 > 0$, the operator $U_{P,Q}(t; h) = P(h)e^{-ih^{-1}tA(h)}Q(h)^*$ is a so-called h Fourier integral operator, or h F.I.O. for short. Furthermore, denoting by $u_{P,Q}(t, x, y; h)$ the integral kernel of $U_{P,Q}$, [B2] gives, for all $T' > 0$ and $\mu > 0$,

$$\sup_{x,y\in\mathbb{R}^n} \max_{t\in[-T',T']} \left| u_{P,Q}(t, x, y; h) - (2\pi h)^{-n} \int u_0 e^{ih^{-1}\phi} d\eta \right| = O_\mu(h), \quad \text{as } h \to 0, \tag{2.10}$$

where the $O_\mu(h)$ may depend on μ. We observe that the left-hand side of (1.8) is equal to,

$$(2\pi h)^{n-1} \int \hat{\rho}(t) e^{irt} e^{ih^{-1}\lambda t} u_{P,P}(t, y, y; h)dt. \tag{2.11}$$

Hence, defining $\psi(\lambda, t, y, \eta, \mu) = \lambda t + \phi(t, y, y, \eta, \mu)$, in view of (2.10) and (2.11), in order to prove Theorem 1.5 we are required to consider the asymptotics, for all functions ρ satisfying the conditions in the statement of Theorem 1.5, of the integral,

$$(2\pi h)^{-1} \int \hat{\rho}(t) e^{irt} e^{ih^{-1}\psi} u_0(t, y, \eta, \mu) d\eta dt. \tag{2.12}$$

We shall use a stationary phase argument to study the asymptotics of the integral in (2.12).

2.3 Stationary points and Hessian matrix.

We are going to apply a stationary phase argument to estimate the integration in (2.12) with respect to two variables. We choose the variables t and $\tau(\eta) = a_0(y,\eta) - \lambda$ as, with this choice, we can show that the necessary non-degeneracy condition on the Hessian matrix,

$$\psi''_{t,\tau} = \begin{pmatrix} \psi_{tt} & \psi_{t\tau} \\ \psi_{\tau t} & \psi_{\tau\tau} \end{pmatrix}, \tag{2.13}$$

of second derivatives of ψ with respect to t and τ is met at all stationary points. This fact is not easily established—the in-homogeneity of the symbol a_0 creating problems not encountered in classical (large λ) case, see [SV].

In the large λ case, using the homogeneity of the principal symbol a_0, one can calculate the value of the relevant Hessian matrix, and it straightforward to see that this matrix has non-zero determinant. In our case we must employ a different technique. In particular we must use the extra parameter μ to establish that $\psi''_{t,\tau}$ has non-zero determinant. We also note that, as the function ψ can be complex-valued, we must apply a more complicated complex stationary phase argument.

Using the facts that $ta_0(x^*,\xi^*) = \int_0^t a(x^*,\xi^*)ds$ and (as $x^*(0) = y$) $x^*_\tau(0) = 0$ and differentiating we have,

$$\begin{aligned} \psi_t &= -\tau + (\dot{\xi}^* - i\mu\dot{x}^*)\cdot(y - x^*) \\ \psi_\tau &= (\xi^*_\tau - i\mu x^*_\tau)\cdot(y - x^*). \end{aligned} \tag{2.14}$$

Hence, ψ is real-valued and stationary (with respect to t and τ) if and only if $(t,\tau) = (t_\star(\lambda,y,\eta), \tau_\star(\lambda,y,\eta))$ where $(t_\star,\tau_\star)$ is a solution of the equations,

$$x^*(t_\star,y,\eta) = y \quad \text{and} \quad \tau_\star = 0. \tag{2.15}$$

We note that the point $(t_\star,\tau_\star) = (0,0)$ is always a solution of (2.15) but that, in the case when $\Pi_{\lambda,y} \neq \emptyset$, there may also exist non-zero solutions of (2.15). We also note that any solution of (2.15) is independent of μ.

Theorem 2.2. *Suppose that the principal symbols $a_0 \in C^\infty(\mathbb{R}^{2n})$ and $p_0, q_0 \in C_c^\infty(\mathbb{R}^{2n})$, and the points $\lambda \in \mathbb{R}$ and $y \in \mathbb{R}^n$ are such that $(\nabla_\xi a_0)(y,\eta) \neq 0$ for all $\eta \in \Sigma_{\lambda,y} \cap (\mathrm{supp}_y p_0 \cup \mathrm{supp}_y q_0)$. For $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$ such that there exists a solution $(t_\star,\tau_\star)$ of (2.15) with $\xi^*(t_\star) \in \mathrm{supp}_y p_0$, denote $\psi''_{t,\tau} := \psi''_{t,\tau}\big|_{(t,\tau)=(t_\star,\tau_\star)}$, and define the set,*

$$V_{\lambda,y} := \{\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 | \text{there exists a } t_\star(\eta) \text{ such that } x^*(t_\star) = y \\ \text{and } \xi^*(t_\star) \in \mathrm{supp}_y p_0\}.$$

Then:
i) There exists a $\mu_0 > 0$ such that $\psi''_{t,\tau}$ has non-zero determinant for all $\mu \geq \mu_0, \eta \in V_{\lambda,y}$.
ii) The matrix $-i\psi''_{t,\tau}$ has symmetric, positive semi-definite real part and hence, for all $\mu \geq \mu_0, \eta \in V_{\lambda,y}$, we can define the square root $\left(\det(-i\psi''_{t,\tau})\right)^{-\frac{1}{2}}$ as in [Hö1] *section 3.4.*

iii) In the case when $x^*_\tau(t_\star) = 0$, there exists a function $\theta : V_{\lambda,y} \times [\mu_0, +\infty) \to (-\pi, \pi]$ such that $\sup_{\eta \in V_{\lambda,y}} \theta(\eta,\mu) \to 0$ as $\mu \to +\infty$ and, according to the definition of the square root in [Hö2] *section 3.4,*

$$\left(\det(-i\psi''_{t,\tau})\right)^{-\frac{1}{2}} = e^{i\theta(\eta,\mu)},$$

for all $\mu \geq \mu_0, \eta \in V_{\lambda,y}$.

Proof. i) We define the real, 2×2 matrices,

$$A = \begin{pmatrix} |\dot{x}^*|^2 & x^*_\tau \cdot \dot{x}^* \\ x^*_\tau \cdot \dot{x}^* & |x^*_\tau|^2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \dot{\xi}^* \cdot \dot{x}^* & \xi^*_\tau \cdot \dot{x}^* \\ \xi^*_\tau \cdot \dot{x}^* & \xi^*_\tau \cdot x^*_\tau \end{pmatrix}, \tag{2.16}$$

where all entries are evaluated at $(t,\tau) = (t_\star, \tau_\star)$. We have $-i\psi''_{t,\tau} = \mu A + iB$. Using the chain rule for differentiation,

$$1 = \frac{\partial}{\partial \tau}\left(a_0(x^*(t), \xi^*(t))\right) = -x^*_\tau(t) \cdot \dot{\xi}^*(t) + \xi^*_\tau(t) \cdot \dot{x}^*(t). \tag{2.17}$$

Using this identity and the formula,

$$\det(\mu A + iB) = \left(\mu^2 \det A - \det B\right) + i\mu\left((\text{Tr } A)(\text{Tr } B) - \text{Tr } (AB)\right), \tag{2.18}$$

(which is valid for any real, 2×2 matrices A and B), in the case when $x^*_\tau(t_\star) = 0$, we have $\det(-i\psi''_{t,\tau}) = 1$.

Let $\eta \in V_{\lambda,y}$. Then, as $\lambda = a_0(x^*(t_\star), \xi^*(t_\star)) = a_0(y, \xi^*(t_\star))$, we have $\xi^*(t_\star) \in \Sigma_{\lambda,y}$ and hence, $\xi^*(t_\star) \in \Sigma_{\lambda,y} \cap \text{supp}_y p_0$. As $(\nabla_\xi a_0)(y, \eta') \neq 0$ for all $\eta' \in \Sigma_{\lambda,y} \cap \text{supp}_y p_0$,

$$\dot{x}^*(t_\star) = (\nabla_\xi a_0)(x^*(t_\star), \xi^*(t_\star)) = (\nabla_\xi a_0)(y, \xi^*(t_\star)) \neq 0. \tag{2.19}$$

Now $\det A = |\dot{x}^*|^2 |x^*_\tau|^2 - (x^*_\tau \cdot \dot{x}^*)^2$ and by the Cauchy-Schwartz inequality, there exists an $\omega \in \left[0, \frac{\pi}{2}\right]$ such that,

$$\det A = |\dot{x}^*|^2 |x^*_\tau|^2 (1 - \cos^2 \omega) \geq 0, \tag{2.20}$$

with $\omega = 0$ if and only if $\dot{x}^*(t_\star)$ is parallel to $x^*_\tau(t_\star)$. Suppose then that $x^*_\tau(t_\star) \neq 0$ and $\omega \neq 0$. Then, in view of (2.19) and (2.20), $\det A > 0$ and, taking account of (2.18), we can choose $\mu > 0$ large enough to ensure $\det(-i\psi''_{t,\tau}) \neq 0$ for all $\eta \in V_{\lambda,y}$ such that $x^*_\tau(t_\star) \neq 0$ and $\omega \neq 0$.

On the other hand, if $x^*_\tau(t_\star) \neq 0$ and $\omega = 0$, then there exists $\alpha \in \mathbb{R}$ such that $\dot{x}^*(t_\star) = \alpha x^*_\tau(t_\star)$ and, in view of (2.19), $\alpha \neq 0$. Then using the fact that $\dot{x}^*(t_\star) = \alpha x^*_\tau(t_\star)$ and the identity in (2.17),

$$\text{Tr } (A)\text{Tr } (B) - \text{Tr } (AB) = -\alpha |x^*_\tau|^2 \neq 0. \tag{2.21}$$

According to (2.18), in this case $\det(-i\psi''_{t,\tau}) \neq 0$.

ii) Taking account of (2.20), $\det A$ is non-negative and, as the diagonal entries of A are also non-negative, A is a positive semi-definite matrix.

iii) In the case when $x^*_\tau(t_\star) = 0$, $A = \begin{pmatrix} |\dot{x}^*|^2 & 0 \\ 0 & 0 \end{pmatrix}$. Furthermore, using (2.17), the matrix B has eigenvalues $\frac{c \pm \sqrt{c^2+4}}{2}$, $(c = \dot{x}^* \cdot \dot{\xi}^*)$ and hence has zero signature. From the proof of i) above, $\left|\det(-i\psi''_{t,\tau})\right| = 1$. Furthermore, from (2.19) and as $\Sigma_{\lambda,y} \cap \text{supp}_y p_0$ is compact, $\inf_{\eta \in V_{\lambda,y}} |\dot{x}^*(t_\star)|^2 > 0$. Hence the statement in part iii) follows according to [Hö2] section 3.4. $\square$

2.4 Malgrange Preparation Theorem and complex stationary phase formula.

It is necessary to briefly introduce some of the notation used in [Hö2], sections 7.5 and 7.7. The ideal of C^∞ smooth functions generated by the functions $f_j(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, where $1 \leq j \leq r$, will be denoted $I(f_1, \dots f_r)$. The ideal $I(f_1, \dots f_r)$ is the set of all functions g which are C^∞ in a neighbourhood of zero and such that,

$$g(x,y) = \sum_{1 \leq j \leq r} q_j(x,y) f_j(x,y),$$

where the $q_j \in C^\infty$ denote some other functions. We shall say that two functions $g_1(x,y)$ and $g_2(x,y)$ belong to the same residue class modulo the ideal generated by $f_1, \dots, f_n$ if and only if $(g_1(x,y) - g_2(x,y)) \in I(f_1, \dots, f_n)$.

Let $f(x,y)$ denote a C^∞ function and suppose that,

$$\mathrm{Im} f \geq 0, \ \mathrm{Im} f(x_0,y_0) = 0, \ f_x(x_0,y_0) = 0 \text{ and } \det f_{xx}(x_0,y_0) \neq 0, \tag{2.22}$$

for some (x_0, y_0), and where f_{xx} denotes the Hessian matrix of second derivatives of f with respect to x. Then by the Malgrange Preparation Theorem—see [Hö2], Theorems 7.5.6–7.5.9—there exist n, C^∞ functions $X_j(y)$ with $X_j(y_0) = (x_0)_j$ for all j, and such that,

$$I(f_{x_1}, \dots, f_{x_n}) = I(x_1 - X_1(y), \dots, x_n - X_n(y)), \tag{2.23}$$

for (x,y) in a neighbourhood of (x_0,y_0), and where f_{x_j} denotes differentiation with respect to x_j.

Moreover, according to the Malgrange Preparation Theorem and the remarks preceding Lemma 7.7.8 in [Hö2], for f satisfying (2.22) and any other C^∞ function $g(x,y)$, there exists another function of y only, which we denote $g^0(y)$, and which belongs to the same residue class as g modulo the ideal $I(f_{x_1}, \dots, f_{x_n})$. When $X(y)$ is real-valued, $X(y)$ corresponds to a stationary point of f (ie. $f_x(X(y),y) = 0$), and thus, one may think of $X(y)$ as a stationary point of f in the general case. (Of course, strictly speaking, this is not correct as $X(y)$ may be complex-valued, in which case the preceding statement makes no sense.) We require the following lemma.

Lemma 2.3. *Let f satisfy the conditions in (2.22).*
i) Let $X(y)$ denote a function such that $I(f_x) = I(x - X(y))$, and suppose that $X(y)$ is real-valued for y in some set, $y \in \Omega$ say. Let $g(x,y)$ denote any other C^∞ function. Then, for all $y \in \Omega$, the function of y only, $g(X(y),y)$ belongs to the same residue class as g modulo the ideal $I(f_x)$.
ii) Let $f^0(y)$ denote a function which belongs to the same residue class as f modulo the ideal $I(f_x)$. If $\mathrm{Im} f^0(y) = 0$ for y in some set $y \in \Omega$, then there exists an $x_\star(y)$ which is real-valued for $y \in \Omega$ and is such that $\mathrm{Im} f(x_\star(y),y) = 0$ and $f_x(x_\star(y),y) = 0$ for all $y \in \Omega$.
iii) Suppose that there exists a function $x_\star(y)$ which is real-valued for y in some set $y \in \tilde{\Omega}$, and is such that $\mathrm{Im} f(x_\star(y),y) = 0$, $f_x(x_\star(y),y) = 0$ and $\det f_{xx}(x_\star(y),y) \neq 0$ for all y in some subset of $\tilde{\Omega}$, $y \in \Omega \subseteq \tilde{\Omega}$ say. Then $I(f_x) = I(x - x_\star(y))$ for

all $y \in \Omega$ and, for any function $f^0(y)$ which belongs to the same residue class as f modulo the ideal $I(f_x)$, we have $Imf^0(y) = 0$ for all $y \in \Omega$.

Proof. i) By repeated use of the Malgrange Preparation Theorem, [Hö2], Theorem 7.5.7 and [Hö2], equation (7.7.16), there exist C^∞ functions $g^\alpha(y)$ such that for any $N \geq 1$,

$$g(x,y) = \sum_{|\alpha|<N} g^\alpha(y)(x - X(y))^\alpha \mod I^N.$$

Here the g^α are not necessarily derivatives of g and I^N denotes the ideal of functions $h(x,y)$ which are such that,

$$h(x,y) = \sum_{|\alpha|=N} q_\alpha f_x^\alpha,$$

where the $q_\alpha(x,y)$ are some other C^∞ functions. If $X(y)$ is real-valued then,

$$g(x,y) - g(X(y),y) = \sum_{1\leq|\alpha|<N} g^\alpha(y)(x - X(y))^\alpha \mod I^N,$$

and hence, $(g(x,y) - g(X(y),y)) \in I(x - X(y)) = I\left(f_x\right)$.

ii) An application of [Hö2], Lemma 7.7.8 implies that, for any $X(y)$ such that $I(f_x) = I(x - X(y))$, we have $\mathrm{Im}X(y) = 0$ for all $y \in \Omega$. For any $f^0(y)$ in the same residue class as f modulo $I(f_x)$, according to [Hö2], equation (7.7.16), there exist C^∞ functions $f^\alpha(y)$ such that,

$$f(x,y) = f^0(y) + \sum_{2\leq|\alpha|<N} f^\alpha(y)(x - X(y))^\alpha \mod I^N. \tag{2.24}$$

Differentiating with respect to x, and putting $x = X(y)$ in (2.24) for $y \in \Omega$ gives $f_x(X(y),y) = 0$, and putting $x = X(y)$ in (2.24) for $y \in \Omega$ gives $\mathrm{Im}f(X(y),y) = \mathrm{Im}f^0(y) = 0$. Choosing $x_\star = X$ completes the proof.

iii) Taylor's Theorem implies that, as $f_x(x_\star) = 0$, each $f_{x_j}(x_\star(y),y) \in I(x - x_\star(y))$ for all $y \in \Omega$. Furthermore, by the hypothesis on the determinant of f_{xx}, all the $(f_{x_j})_x(x_\star(y),y)$ are linearly independent for all $y \in \Omega$. An application of [Hö2], Lemma 7.5.8 implies that $I(f_x) = I(x - x_\star(y))$ for all $y \in \Omega$. Thus, for any $f^0(y)$ in the same residue class as f modulo $I(f_x)$, (2.24) with X replaced by $x_\star$ is valid for all $y \in \Omega$. Putting $x = x_\star = X$ in (2.24) gives, for all $y \in \Omega$,

$$\mathrm{Im}f(x_\star(y),y) = \mathrm{Im}f^0(y),$$

this being zero by hypothesis. □

Applying Theorem 2.2, and Lemma 2.3 in the particular case when $f = \psi$, we have the following result. In what follows we denote by $B(r,x) = \{y \in \mathbb{R}^n \,|\, |x-y| \leq r\}$ the usual Euclidean ball of radius $r \geq 0$ and centre $x \in \mathbb{R}^n$.

Corollary 2.4. *Suppose that the principal symbols a_0, p_0 and q_0, and the points λ and y are such that $(\nabla_\xi a_0)(y,\eta) \neq 0$ for all $\eta \in \Sigma_{\lambda,y} \cap (\mathrm{supp}_y p_0 \cup \mathrm{supp}_y q_0)$. Suppose that $t_0 \in \mathbb{R}$ and $\eta_0 \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$ are such that $x^*(t_0,y,\eta_0) = y$ and $\xi^*(t_0,y,\eta_0) \in \mathrm{supp}_y p_0$. Denoting by $\tilde\eta$ coordinates on $\Sigma_{\lambda,y}$ in a neighbourhood of η_0, for any function $\psi^0(\lambda,y,\tilde\eta)$ which belongs to the same residue class as ψ modulo the ideal $I(\psi_t,\psi_\tau)$, we define the set $\Omega_{\lambda,y} = \{\tilde\eta \in \Sigma_{\lambda,y} |\ \mathrm{Im}\ \psi^0(\lambda,y,\tilde\eta) = 0\}$. Then there exist $\mu_0, \epsilon_0 > 0$ (the same μ_0 in Theorem 2.2) such that, for all $\mu \geq \mu_0$ and $\epsilon \in (0,\epsilon_0]$, the set $\Omega_{\lambda,y} \cap B(\epsilon,\eta_0)$ is independent of the choice of ψ^0 and furthermore,*
(2.25)

$$\begin{aligned}&\Omega_{\lambda,y} \cap B(\epsilon,\eta_0)\\ &\quad= \{\tilde\eta \in \Sigma_{\lambda,y} \cap B(\epsilon,\eta_0) |\ \text{there exists a } t_\star(\lambda,y,\tilde\eta) \in \mathbb{R} \text{ such that } x^*(t_\star,y,\tilde\eta) = y\}.\end{aligned}$$

In this case, for any other C^∞ function $g(t,\tau,\lambda,y,\tilde\eta)$, according to Lemma 2.3 i), for all $\tilde\eta \in \Omega_{\lambda,y} \cap B(\epsilon,\eta_0)$, $g(t_\star,0,\lambda,y,\tilde\eta)$ belongs to the same residue class as g modulo the ideal $I(\psi_t,\psi_\tau)$.

Proof. At the point (t_0,y_0,η_0) we have $\mathrm{Im}\psi = \psi_t = \psi_\tau = 0$ and, according to Theorem 2.2 i), $\det(\psi''_{t,\tau}) \neq 0$. We can thus apply Lemma 2.3 in the case when $f = \psi$. From Lemma 2.3 ii), for all $\tilde\eta \in \Omega_{\lambda,y}$, there exist real-valued functions $t_\star(\lambda,y,\tilde\eta)$ and $\tau_\star(\lambda,y,\tilde\eta)$ such that $\mathrm{Im}\psi(t_\star,\tau_\star) = \psi_t(t_\star,\tau_\star) = \psi_\tau(t_\star,\tau_\star) = 0$. From (2.15), this $(t_\star,\tau_\star)$ must have $x^*(t_\star) = y$ and $\tau_\star = 0$ and hence, $\Omega_{\lambda,y} \cap B(\epsilon,\eta_0)$ is contained in the set on the right-hand side of (2.25).

On the other hand, if $x^*(t_\star) = y$ and $\tau_\star = 0$ then, from (2.15), $\mathrm{Im}\psi(t_\star,\tau_\star) = \psi_t(t_\star,\tau_\star) = \psi_\tau(t_\star,\tau_\star) = 0$. Now at the point (t_0,y_0,η_0), $\det(\psi''_{t,\tau}) \neq 0$ and hence, by continuity, there exists an $\epsilon > 0$ such that $\det\left(\psi''_{t,\tau}\big|_{(t_\star,\tau_\star)}\right) \neq 0$ for all $\tilde\eta$ in the set on the right-hand side of (2.25). Thus, by Lemma 2.3 iii), $I(\psi_t,\psi_\tau) = I(t-t_\star,\tau-0)$ and $\mathrm{Im}\psi^0 = 0$ for all $\tilde\eta$ in the set on the right-hand side of (2.25). ☐

We can now apply the complex stationary phase formula appearing in [Hö2] Theorem 7.7.12.

Theorem 2.5. *Let the principal symbols a_0, p_0 and q_0, and the points λ and y be such that $(\nabla_\xi a_0)(y,\eta) \neq 0$ for all $\eta \in \Sigma_{\lambda,y} \cap (\mathrm{supp}_y p_0 \cup \mathrm{supp}_y q_0)$. Suppose $t_0 \in \mathbb{R}$ and $\eta_0 \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$ are such that $x^*(t_0,y,\eta_0) = y$ and $\xi^*(t_0,y,\eta_0) \in \mathrm{supp}_y p_0$. Let $\rho \in C^\infty(\mathbb{R})$ have Fourier transform $\hat\rho \in C_c^\infty(\mathbb{R})$ and $\mathrm{supp}\hat\rho \subseteq \left[t_0 - \frac{T}{2}, t_0 + \frac{T}{2}\right]$ where $T > 0$ is the constant given in Lemma 1.1 with $X = \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$. Then the function $t_\star : \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 \longrightarrow \mathbb{R}$, defined,*
(2.26)

$$t_\star(\eta) = \begin{cases} t_1 \in \mathrm{supp}\hat\rho \text{ such that } x^*(t_1,y,\eta) = y & \text{if } \eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 \text{ is such} \\ & \text{that such a } t_1 \text{ exists,} \\ T_1 & \text{otherwise,} \end{cases}$$

where $T_1 > (t_0 + \frac{T}{2})$ can be any constant greater than $(t_0 + \frac{T}{2})$, is well-defined.

Let $\chi \in C_c^\infty(\mathbb{R}^n)$ have support contained in a sufficiently small neighbourhood of η_0. In the case when there exists an $\eta_1 \in \Pi_{\lambda,y} \cap \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi$ (not necessarily equal to η_0) such that $0 \neq t_\star(\eta_1) \in \mathrm{supp}\hat\rho$, we make the additional assumption *that the set* $(\Pi_{\lambda,y} \setminus \Pi^1_{\lambda,y}) \cap \mathrm{supp}_y q_0$ *has zero measure. Then for all* r_0*, there exist* $\mu_0, h_0 > 0$ *such that,*

(2.27)
$$(2\pi h)^{-1} \int \hat{\rho}(t) e^{irt} e^{ih^{-1}\psi} \chi(\eta) u_0(t,y,\eta,\mu) d\eta dt$$
$$= \int_{\Sigma_{\lambda,y}} e^{i\theta} \hat{\rho}(t_\star) e^{irt_\star} e^{i\varphi} \left(\det Z(t_\star)\right)^{\frac{1}{2}} \chi(\tilde{\eta}) p_0(y, \xi^*(t_\star)) \overline{q_0}(y,\tilde{\eta}) d\tilde{\eta} + o_\mu(1),$$

for all $h \in (0, h_0]$, $|r| \le r_0$ *and* $\mu \ge \mu_0$*. Here the* $o_\mu(1)$ *may depend on the choices of* μ, ρ *and* χ*, but is independent of* $r \in [-r_0, r_0]$, φ *denotes the function* $\varphi(\lambda, y, \tilde{\eta}; h) = \int_0^{t_\star} \left(h^{-1}\xi^* \cdot \dot{x}^* - i a_{\mathrm{sub}}(x^*, \xi^*)\right) ds$ *and* $\theta(\tilde{\eta}, \mu)$ *denotes the function in Theorem 2.2.*

Proof. Let $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$ and suppose that $t_1, t_2 \in \mathbb{R}$ are such that $t_1 < t_2$ and $x^*(t_1, y, \eta) = x^*(t_2, y, \eta) = y$. Then $x^*(t - t_1, y, \eta) = x^*(t, y, \xi^*(t_1))$ and thus, according to Lemma 1.1 (with $X = \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$), $(t_2 - t_1) > T$. Hence, for each $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$ such that $x^*(t, y, \eta) = y$ for some $t \in \mathrm{supp}\hat{\rho}$, there exists a unique $t_1 \in \mathrm{supp}\hat{\rho}$ such that $x^*(t_1, y, \eta) = y$ and the function $t_\star$ is well-defined.

Now $(\nabla_\xi a_0)(y, \eta) \ne 0$ for all $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$. Hence, provided $\mathrm{supp}\chi$ is contained in a sufficiently small neighbourhood of η_0, we can choose (according to [St] Lemma 5.3.11, for example) coordinate functions $\{u_j\}_{j=1}^{n-1}$, $u_j : \mathrm{supp}\chi \longrightarrow \mathbb{R}$, such that, for all $1 \le j, k \le (n-1)$, $j \ne k$ and $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}\chi$,

$$\begin{aligned} (\nabla_\eta \tau)(\eta) \cdot (\nabla_\eta u_j)(\eta) &= 0, \\ (\nabla_\eta u_j)(\eta) \cdot (\nabla_\eta u_k)(\eta) &= 0, \\ \text{and} \quad |(\nabla_\eta u_j)(\eta)|^2 &= 1. \end{aligned} \tag{2.28}$$

An explicit calculation and using (2.28) gives, $\left|\det\left(\frac{\partial \eta}{\partial(\tau,u)}\right)\Big|_{\tau=0}\right| = \left|\nabla_\xi a_0|_{\Sigma_{\lambda,y}}\right|^{-1}$, and hence,

$$\left|\det\left(\frac{\partial \eta}{\partial(\tau,u)}\right)\Big|_{\tau=0}\right| du = d\tilde{\eta}, \tag{2.29}$$

where du denotes Lebesgue measure in $\mathbb{R}^{n-1}$. (Here $\frac{\partial \eta}{\partial(\tau,u)}$ denotes the matrix with entries $\left(\frac{\partial \eta}{\partial(\tau,u)}\right)_{1k} = \frac{\partial \eta_k}{\partial \tau}$, $1 \le k \le n$, and $\left(\frac{\partial \eta}{\partial(\tau,u)}\right)_{jk} = \frac{\partial \eta_k}{\partial u_{j-1}}$, $2 \le j \le n$, $1 \le k \le n$.) For future reference we define the set $U_{\lambda,y}$,

$$\mathbb{R}^{n-1} \supseteq U_{\lambda,y} = \{(u_1(\eta), \ldots, u_{n-1}(\eta)) | \eta \in \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi\}. \tag{2.30}$$

We denote $u_0 = u(\eta_0)$ and, as $\eta_0 \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$, $\tau(\eta_0) = 0$. Hence, in the new coordinate system, η_0 has coordinates $(0, u_0)$.

According to (2.15), when evaluated at $(\lambda, t, \tau, y, u, \mu) = (\lambda, t_0, 0, y, u_0, \mu)$ we have,

$$\mathrm{Im}\psi = \psi_t = \psi_\tau = 0.$$

Choosing μ_0 large enough, by Theorem 2.2 i), for all $\mu \ge \mu_0$, $\det(\psi''_{t,\tau}) \ne 0$ when evaluated at the same point $(\lambda, t_0, 0, y, u_0, \mu)$. Thus, provided $\mathrm{supp}\chi$ is sufficiently small, changing variables $\eta \mapsto (\tau, u)$ in the integral in the right-hand side of (2.27), we can apply the stationary phase formula, [Hö2] Theorem 7.7.12., for the integration with respect to τ and u. We obtain that the integral on the left-hand side of (2.27) is equal to,

$$(2.31)\quad \int_{U_{\lambda,y}} \left(\hat{\rho} e^{irt} u_0 \chi \left|\det\left(\frac{\partial \eta}{\partial(\tau,u)}\right)\right|\right)^0 \left(\left(\det(-i\psi''_{t,\tau})\right)^0\right)^{-\frac{1}{2}} e^{ih^{-1}\psi^0} du + O_\mu(h),$$

where the $O_\mu(h)$ may depend on $\mu \geq \mu_0$, but is independent of $|r| \leq r_0$. Here, each function marked with a superscript 0 denotes another function which belongs to the same residue class as the original function, modulo the ideal $I(\psi_t, \psi_\tau)$.

For $\epsilon \geq 0$ we define the set $\Omega_\epsilon = \{u \in U_{\lambda,y} | 0 \leq \mathrm{Im}\psi^0(\lambda, y, u) \leq \epsilon\}$, and for all $\epsilon > 0$, we have $U_{\lambda,y} = \Omega_0 \cup (\Omega_\epsilon \setminus \Omega_0) \cup \Omega_\epsilon^c$, where ^{c} denotes set complement. We consider the integration in (2.31) over each of the sets Ω_0, $(\Omega_\epsilon \setminus \Omega_0)$ and Ω_ϵ^c separately.

According to Corollary 2.4, provided $\mathrm{supp}\chi$ is sufficiently small,

$$(2.32)\quad \Omega_0 = \{\tilde{\eta} \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi |\ \text{there exists a } t_\star(\tilde{\eta}) \in \mathbb{R} \text{ such that } x^*(t_\star, y, \tilde{\eta}) = y\}.$$

Furthermore, from Corollary 2.4, for any smooth function $g(t, \tau, \lambda, y, \tilde{\eta})$, the function $g(t_\star, 0, \lambda, y, \tilde{\eta})$ belongs to the same residue class as g modulo the ideal $I(\psi_t, \psi_\tau)$, for all $\tilde{\eta} \in \Omega_0$. Hence, taking account of (2.29) and the fact that $e^{ih^{-1}\psi}\, u_0|_{t_\star,0} = e^{i\varphi} \left(\det Z(t_\star)\right)^{\frac{1}{2}} p_0(y, \xi^*(t_\star))\overline{q_0}(y, \tilde{\eta})$, the integral in (2.31), with $U_{\lambda,y}$ replaced by Ω_0, is equal to,

$$(2.33)\quad \int_{\Sigma_{\lambda,y}} \hat{\rho}(t_\star) e^{irt_\star} \left(\det\left(-i\psi''_{t,\tau}\right)\right)^{-\frac{1}{2}} e^{i\varphi} \left(\det Z(t_\star)\right)^{\frac{1}{2}} \chi(\tilde{\eta}) p_0(y, \xi^*(t_\star))\overline{q_0}(y, \tilde{\eta}) d\tilde{\eta},$$

where we have used the fact that $\hat{\rho}(t_\star) = 0$ if $\tilde{\eta} \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$ is such that $x^*(t, y, \tilde{\eta}) \neq y$ for all $t \in \mathrm{supp}\hat{\rho}$.

If $\eta_1 \in \Pi^1_{\lambda,y}$ then $x^*_\eta(T_y(\eta), y, \eta)\big|_{\eta=\eta_1} = 0$ and hence, by differentiation, if $\eta_1 \in \Pi^1_{\lambda,y}$ then $x^*_\tau(t_\star(\eta_1), y, \eta_1) = 0$. But, according to the hypotheses of the Theorem, if there exists an $\eta_1 \in \Pi_{\lambda,y} \cap \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi$ such that $t_\star(\eta_1) \neq 0$, then the set $(\Pi_{\lambda,y} \setminus \Pi^1_{\lambda,y}) \cap \mathrm{supp}_y q_0$ has zero measure. Hence, according to Theorem 2.2 iii), for all $\mu \geq \mu_0$, the term $\left(\det(-i\psi''_{t,\tau})\right)^{-\frac{1}{2}}$ in (2.33) is equal to $e^{i\theta(\tilde{\eta},\mu)}$ for almost all $\tilde{\eta} \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0$ such that $\hat{\rho}(t_\star) \neq 0$. The term $\left(\det(-i\psi''_{t,\tau})\right)^{-\frac{1}{2}}$ in (2.33) can therefore be replaced by $e^{i\theta(\tilde{\eta},\mu)}$.

In the case when $t_\star = 0$ for all $\tilde{\eta} \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi$, $x^*_\tau(t_\star)$ is always zero. We may therefore replace the term $\left(\det(-i\psi''_{t,\tau})\right)^{-\frac{1}{2}}$ in (2.33) by $e^{i\theta(\tilde{\eta},\mu)}$ without appealing to the additional hypothesis.

Now $|e^{ih^{-1}\psi^0}| \leq 1$ for all $h > 0$ and $u \in U_{\lambda,y}$. Hence, as $\mathrm{meas}(\Omega_\epsilon \setminus \Omega_0) \to 0$ as $\epsilon \to 0$, for each $\delta > 0$ and $\mu \geq \mu_0$, there exists an $\epsilon_0 > 0$ such that

$$(2.34)\quad \left|\int_{\Omega_\epsilon \setminus \Omega_0} \left(\hat{\rho} e^{irt} u_0 \chi \left|\det\left(\frac{\partial \eta}{\partial(\tau,u)}\right)\right|\right)^0 \left(\left(\det(-i\psi''_{t,\tau})\right)^0\right)^{-\frac{1}{2}} e^{ih^{-1}\psi^0} du\right| < \frac{\delta}{2},$$

for all $h \in (0, h_0], \epsilon \in (0, \epsilon_0]$ and $|r| \leq r_0$.

Finally, as $\mathrm{Im}\psi^0 > \epsilon$ for all $u \in \Omega_\epsilon^c$, according to [Hö2] Theorem 7.7.1, for each $\mu \geq \mu_0, \epsilon > 0$ and $N \in \mathbb{Z}_+$,

(2.35)

$$\int_{\Omega_\epsilon^c} \left(\hat{\rho} e^{irt} u_0 \chi \left| \det \left(\frac{\partial \eta}{\partial(\tau, u)} \right) \right| \right)^0 \left(\left(\det(-i\psi''_{t,\tau}) \right)^0 \right)^{-\frac{1}{2}} e^{ih^{-1}\psi^0} du = O_{\mu,\epsilon}(h^N),$$

where the $O_{\mu,\epsilon}(h^N)$ may depend on ϵ and μ, but is independent of $|r| \leq r_0$.

Combining all the above (with $N = 1$), for each $\mu \geq \mu_0$ and $\delta > 0$, there exists an $\epsilon > 0$ such that

(2.36)

$$\begin{aligned} \Bigg| (2\pi h)^{-1} \int \hat{\rho}(t) e^{irt} e^{ih^{-1}\psi} \chi(\eta) u_0(t, y, \eta, \mu) d\eta dt \\ - \int_{\Sigma_{\lambda,y}} e^{i\theta} \hat{\rho}(t_\star) e^{irt_\star} e^{i\varphi} \left(\det Z(t_\star) \right)^{\frac{1}{2}} \chi(\tilde{\eta}) p_0(y, \xi^*(t_\star)) \overline{q_0}(y, \tilde{\eta}) d\tilde{\eta} \Bigg| \\ \leq \frac{\delta}{2} + |O_{\mu,\epsilon}(h)| + |O_\mu(h)|, \end{aligned}$$

for all $h \in (0, h_0]$, the right-hand side being independent of $|r| \leq r_0$. With this μ and ϵ fixed, there exists an $h_1 > 0$ such that the right-hand side of (2.36) is less than δ for all $h \in (0, h_1]$. The right-hand side of (2.36) can hence be replaced by $o_\mu(1)$ which completes the proof. □

In the case when $x^*(t, y, \eta) \neq y$ or $\xi^*(t, y, \eta) \notin \mathrm{supp}_y p_0$, for all $t \in \mathrm{supp}\hat{\rho}$ and $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi$, the phase ψ has positive imaginary part or is non-stationary everywhere on $\mathrm{supp}(\hat{\rho} u_0 \chi)$. In this case we have the following result.

Theorem 2.6. *Suppose that either:*

i) $x^*(t, y, \eta) \neq y$ *or* $\xi^*(t, y, \eta) \notin \mathrm{supp}_y p_0$, *for all* $t \in \mathrm{supp}\hat{\rho}$ *and* $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi$ *or,*

ii) $\Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi = \emptyset$.

Then for all $\mu > 0$ *and* $N \in \mathbb{Z}_+$,

$$(2\pi h)^{-1} \int \hat{\rho}(t) e^{irt} e^{ih^{-1}\psi} \chi(\eta) u_0(t, y, \eta, \mu) d\eta dt = O_\mu(h^N), \tag{2.37}$$

as $h \to 0$, *where the* $O_\mu(h^N)$ *may depend on* μ, *but is independent of* $r \in [-r_0, r_0]$.

Proof. If $x^*(t, y, \eta) \neq y$ then, according to (2.15), $\mathrm{Im}\psi > 0$. If $x^*(t, y, \eta) = y$ and $\eta \notin \Sigma_{\lambda,y}$ then, according to (2.14), $\psi_t \neq 0$. On the other hand, if $x^*(t, y, \eta) = y$ for $t \in \mathrm{supp}\hat{\rho}$ and $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y q_0 \cap \mathrm{supp}\chi$ then, by hypothesis, $\xi^*(t, y, \eta) \notin \mathrm{supp}_y p_0$ and thus $(t, \eta) \notin \mathrm{supp}(\hat{\rho} u_0 \chi)$. Hence, $\mathrm{Im}\psi > 0$ or $(\psi_t, \psi_\eta) \neq 0$ for all $(t, \eta) \in \mathrm{supp}(\hat{\rho} u_0 \chi)$ and (2.37) follows from a straightforward application of [Hö2] Theorem 7.7.1. □

2.5 Different cases of $\mathrm{supp}\hat{\rho}$**.**

In the case when $\mathrm{supp}\hat{\rho}$ is contained in a sufficiently small neighbourhood of zero we obtain simpler asymptotics, due to the fact that $x^*(0, y, \eta) = y$ for all η.

Theorem 2.7. *Let $A(h)$ denote an h P.D.O. which satisfies hypotheses (H_1) to (H_4) and let $P(h)$ denote an h P.D.O. with compactly supported symbol. Suppose that a_0, p_0, λ and y are such that $(\nabla_\xi a_0)(y,\eta) \neq 0$ for all $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0$. Let $\rho \in C^\infty(\mathbb{R})$ have Fourier transform $\hat{\rho} \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}\hat{\rho} \subseteq \left[-\frac{T}{2}, \frac{T}{2}\right]$ where $T > 0$ is the constant in Lemma 1.1 with $X = \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0$. Then for each $r_0 > 0$, there exists an $h_0 > 0$ such that,*

(2.38)
$$h^{n-1}(2\pi)^n \left[P(h)\rho\left((\lambda h^{-1} + r)I - h^{-1}A(h)\right) P(h)^*\right](x,y;h)\big|_{x=y} = \hat{\rho}(0)\|\tilde{p}_0\|^2 + o(1)$$

for all $h \in (0, h_0]$ and $|r| \leq r_0$. The $o(1)$ may depend on the choice of ρ, but is independent of $r \in [-r_0, r_0]$.

Proof. Choose $\delta > 0$ arbitrary and let $f \in C_c^\infty(\mathbb{R}^n)$ have $f(\lambda) = 1$ and $\mathrm{supp} f \subseteq [\lambda - \delta, \lambda + \delta]$. We denote by $P_j(h)$ the h P.D.O. with compactly supported symbols, $P_1(h) = P(h)f(A(h))$ and $P_2(h) = P(h) - P_1(h)$, and by $p_0^{(1)}$ and $p_0^{(2)}$ their respective principal symbols. Taking δ small enough (ensuring that $\mathrm{supp}_y p_0^{(1)}$ is contained in a sufficiently small neighbourhood of $\Sigma_{\lambda,y}$), we choose finite sequences of points $\{\eta_j\}_{j=1}^M$ and functions $\{\chi_j\}_{j=1}^M, \chi_j \in C_c^\infty(\mathbb{R}^n)$, such that: i) $\sum_j \chi_j = 1$ for all $\eta \in \mathrm{supp}_y p_0^{(1)}$ and, ii) each χ_j has support contained in a sufficiently small neighbourhood of η_j such that, for each j, we can apply Theorem 2.5 with $P(h) = Q(h) = P_1(h), t_0 = 0, \eta_0 = \eta_j$ and $\chi = \chi_j$.

As $x^*(0) = y$ and $\xi^*(0) = \eta$ for all $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0^{(1)}$, the function $t_\star$ in (2.26), in this case, is identically zero and also, $(\det Z(0))^{\frac{1}{2}} = 1$ and $\varphi(0) = 0$ (and $V_{\lambda,y} = \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0^{(1)}$). Applying Theorem 2.5 with $t_0 = 0, \eta_0 = \eta_j$ and $\chi = \chi_j$, summing the result (for $1 \leq j \leq M$), taking account of (2.10), (2.11) and the fact that $p_0^{(1)}\big|_{\Sigma_{\lambda,y}} = \tilde{p}_0$ then, for each $r_0 > 0$, there exist $\mu_0, h_0 > 0$ such that,

(2.39)
$$h^{n-1}(2\pi)^n \left[P_1(h)\rho\left((\lambda h^{-1} + r)I - h^{-1}A(h)\right) P_1(h)^*\right](x,y;h)\big|_{x=y}$$
$$= \hat{\rho}(0)\sum_{j=1}^M \int_{\Sigma_{\lambda,y}} e^{i\theta_j(\tilde{\eta},\mu)}|\tilde{p}_0(\tilde{\eta})|^2\chi_j(\tilde{\eta})d\tilde{\eta} + o_\mu(1),$$

for all $h \in (0, h_0], |r| \leq r_0$ and $\mu \geq \mu_0$. Here the $o_\mu(1)$ may depend on μ (and δ), but is independent of $r \in [-r_0, r_0]$ and each θ_j is a function such that $\sup_{\tilde{\eta}\in\Sigma_{\lambda,y}\cap\mathrm{supp}_y p_0} \theta_j(\tilde{\eta},\mu) \to 0$ as $\mu \to \infty$. Also $\mu_0 \geq \max(\mu_0(1), \ldots, \mu_0(M))$ where the $\mu_0(j)$ denotes the μ_0 given by Theorem 2.5 in the case when $\chi = \chi_j$.

Now $\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0^{(2)} = \emptyset$. Hence, applying Theorem 2.6 (with $N = 1$) in the cases when $P(h)$ and $Q(h)$ are replaced by $P_1(h)$ and $P_2(h)$, $P_2(h)$ and $P_1(h)$, and $P_2(h)$ and $P_2(h)$, and summing the results and (2.39) we obtain that, for all $r_0 > 0$, there exist $\mu_0, h_0 > 0$ such that,

(2.40)
$$h^{n-1}(2\pi)^n \left[P(h)\rho\left((\lambda h^{-1} + r)I - h^{-1}A(h)\right) P(h)^*\right](x,y;h)\big|_{x=y}$$
$$= \hat{\rho}(0)\sum_{j=1}^M \int_{\Sigma_{\lambda,y}} e^{i\theta_j(\tilde{\eta},\mu)}|\tilde{p}_0(\tilde{\eta})|^2\chi_j(\tilde{\eta})d\tilde{\eta} + O_\mu(h) + o_\mu(1),$$

for all $h \in (0, h_0], |r| \le r_0$ and $\mu \ge \mu_0$. Again the $O_\mu(h)$ may depend on μ (and δ), but is independent of $r \in [-r_0, r_0]$. Finally, using the fact that $\sum_j \chi_j = 1$ on $\mathrm{supp}_y p_0$, there exists a constant $C > 0$, independent of μ such that,

(2.41)

$$\left| \sum_{j=1}^{M} \int_{\Sigma_{\lambda,y}} e^{i\theta_j(\tilde{\eta},\mu)} |\tilde{p}_0(\tilde{\eta})|^2 \chi_j(\tilde{\eta}) d\tilde{\eta} - \|\tilde{p}_0\|^2 \right| \le C \sum_{j=1}^{M} \int_{\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0} \left| e^{i\theta_j(\tilde{\eta},\mu)} - 1 \right| d\tilde{\eta}.$$

But $\sup_{\tilde{\eta} \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0} \theta_j(\tilde{\eta}, \mu) \to 0$ as $\mu \to +\infty$ and hence, for all $\epsilon > 0$, there exists a $\mu \ge \mu_0$ such that the right-hand side of (2.41) is less than $\frac{\epsilon}{2|\hat{\rho}(0)|}$. With this ϵ and μ fixed, there exists an $0 < h_1 \le h_0$ such that $|O_\mu(h)| + |o_\mu(1)| < \frac{\epsilon}{2}$ for all $h \in (0, h_1]$ and hence, the right-hand side of (2.40) can be replaced by $\hat{\rho}(0)\|\tilde{p}_0\|^2 + o(1)$. $\square$

The case when $\mathrm{supp}\hat{\rho}$ is separated from zero is more complicated. First we define the Maslov index $m_y : \Pi^1_{\lambda,y} \longrightarrow \mathbb{Z}_4$ which was mentioned in section 1.

Definition 2.8. Let $\eta \in \Pi^1_{\lambda,y}$. Then $x^*_\eta(T_y(\eta), y, \eta) = 0$ and hence, for each $\mu > 0$ and as t evolves from zero to $T_y(\eta)$, $(\det Z(t, y, \eta, \mu))^2$ makes an integer number of revolutions about the origin in the complex plane. (Here we use the standard convention that one complete anti-clockwise revolution about the origin counts $+1$, and one complete clockwise revolution about the origin counts -1.) We denote this integer number $r(\eta)$ and note that $r(\eta)$ is independent of the choice of μ. The Maslov index $m_y(\eta)$ (at least the Maslov index which is used here) is then defined to be the integer $m_y \in [0, 3]$ such that,

$$m_y(\eta) \equiv r(\eta) \mod 4.$$

Remark. We refer to [SV] Appendix D.6. for proof, in the classical case (i.e. when $a_0(x, \xi)$ is homogeneous in the ξ variables), of the fact that this Maslov index coincides with other Maslov indices, defined in more traditional ways.

We require the following lemma.

Lemma 2.9. *a) Let $\eta \in \Pi_{\lambda,y}$ be such that $\Phi^j_y(\eta) \in \Pi_{\lambda,y}$ for all $0 \le j \le (k-1)$, some $\mathbb{Z}_+ \ni k \ge 1$. Then, for all $1 \le l \le k$,*

i) $\xi^*(T^{(l)}_y(\eta), y, \eta) = \Phi^l_y(\eta)$,

ii) $\int_0^{T^{(l)}_y(\eta)} \xi^*(t, y, \eta) \cdot \dot{x}^*(t, y, \eta) dt = \sum_{j=0}^{l-1} s_y(\Phi^j_y(\eta))$,

iii) $\int_0^{T^{(l)}_y(\eta)} a_{\mathrm{sub}}(x^*(t, y, \eta), \xi^*(t, y, \eta)) dt$
$= \sum_{j=0}^{l-1} \int_0^{T_y(\Phi^j_y(\eta))} a_{\mathrm{sub}}\left(x^*(t, y, \Phi^j_y(\eta)), \xi^*(t, y, \Phi^j_y(\eta))\right) dt.$

b) Let a_0, p_0, λ and y satisfy hypotheses (H_5) and (H_6). Then for all $\mathbb{Z}_+ \ni k \ge 1$ and $1 \le l \le k$,

iv)

$$\left(\det Z(T^{(l)}_y(\eta), y, \eta, \mu)\right)^{\frac{1}{2}} = \prod_{j=0}^{l-1} \left|\det \xi^*_\eta\left(T_y(\Phi^j_y(\eta)), y, \Phi^j_y(\eta)\right)\right|^{\frac{1}{2}} e^{\frac{\pi i}{2} \sum_{j=0}^{l-1} m_y(\Phi^j_y(\eta))},$$

for all $\mu > 0$ and almost all $\eta \in \Pi_{\lambda,y} \cap \operatorname{supp}_y p_0$ such that $\Phi_y^j(\eta) \in \Pi_{\lambda,y}$ for all $0 \leq j \leq (k-1)$,

v)

$$T_y^{(l)}(\eta) > \sharp\{j \in \mathbb{Z} | 0 \leq j \leq (l-1) \textit{ and } \Phi_y^j(\eta) \in \Pi_{\lambda,y} \cap \operatorname{supp}_y p_0\} T,$$

for all $\eta \in \Pi_{\lambda,y} \cap \operatorname{supp}_y p_0$ such that $\Phi_y^j(\eta) \in \Pi_{\lambda,y}$ for all $0 \leq j \leq (k-1)$.

The branch of the square root in iv) is that discussed in sub-section 2.2 and the T in v) is that given in Lemma 1.1 with $X = \Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0$.

Proof. By the definition of Φ_y and the fact that $x^*(T_y(\eta), y, \eta) = y$,

$$x^*(t+T_y(\eta), y, \eta) = x^*(t, y, \Phi_y(\eta)) \quad \text{and} \quad \xi^*(t+T_y(\eta), y, \eta) = \xi^*(t, y, \Phi_y(\eta)), \tag{2.42}$$

for all t. Using these relations inductively,

$$x^*(t + T_y^{(l)}(\eta), y, \eta) = x^*(t, y, \Phi_y^l(\eta)) \quad \text{and} \quad \xi^*(t + T_y^{(l)}(\eta), y, \eta) = \xi^*(t, y, \Phi_y^l(\eta)), \tag{2.43}$$

and putting $t = 0$ in (2.43) immediately gives i). Using (2.43) and a change of variables, for all $2 \leq l \leq k$,

$$\int_{T_y^{(l-1)}(\eta)}^{T_y^{(l)}(\eta)} \xi^*(t, y, \eta) \cdot \dot{x}^*(t, y, \eta) dt = s_y(\Phi_y^{l-1}(\eta)).$$

An induction and this identity then give ii). Similary, (2.43) yields, for all $2 \leq l \leq k$,

$$\int_{T_y^{(l-1)}(\eta)}^{T_y^{(l)}(\eta)} a_{\text{sub}}\left(x^*(t, y, \eta), \xi^*(t, y, \eta)\right) dt$$
$$= \int_0^{T_y(\Phi_y^{l-1}(\eta))} a_{\text{sub}}\left(x^*(t, y, \Phi_y^{l-1}(\eta)), \xi^*(t, y, \Phi_y^{l-1}(\eta))\right) dt,$$

and a similar induction and this identity give iii).

Now suppose that $\eta \in \Pi_{\lambda,y}^1$ and that $(\nabla_\xi a_0)(y, \Phi_y(\eta)) \neq 0$. Then,

$$\dot{x}^*(T_y(\eta), y, \eta) = (\nabla_\xi a_0)(y, \Phi_y(\eta)) \neq 0. \tag{2.44}$$

Differentiating the identity $x^*(T_y(\eta), y, \eta) = y$ with respect to η and using the fact that $x_\eta^*(T_y(\eta), y, \eta) = 0$ (as $\eta \in \Pi_{\lambda,y}^1$), $\dot{x}_j^*(T_y(\eta), y, \eta)\frac{\partial T_y(\eta)}{\partial \eta_k} = 0$, for all j, k. According to (2.44), we therefore have,

$$\nabla_\eta T_y(\eta) = 0 \quad \text{for all } \eta \in \Pi_{\lambda,y}^1 \text{ such that } (\nabla_\xi a_0)(y, \Phi_y(\eta)) \neq 0. \tag{2.45}$$

The function Φ_y was defined, $\Phi_y(\eta) = \xi^*(T_y(\eta), y, \eta)$. Hence differentiating and using (2.45),

$$\partial_\eta \Phi_y(\eta) = \xi_\eta^*(T_y(\eta), y, \eta) \quad \text{for all } \eta \in \Pi_{\lambda,y}^1 \text{ such that } (\nabla_\xi a_0)(y, \Phi_y(\eta)) \neq 0. \tag{2.46}$$

Then using (2.45) and (2.46) and differentiating the identities in (2.42) we obtain,

$$Z(t+T_y(\eta),y,\eta,\mu)=\xi^*_\eta(T_y(\eta),y,\eta)Z(t,y,\Phi_y(\eta),\mu), \tag{2.47}$$

for all $t\in\mathbb{R},\mu>0$ and $\eta\in\Pi^1_{\lambda,y}$ such that $(\nabla_\xi a_0)(y,\Phi_y(\eta))\neq 0$.

Using (2.47) inductively, if $\eta\in\Pi^1_{\lambda,y}$ is such that $(\nabla_\xi a_0)(y,\Phi^{j+1}_y(\eta))\neq 0$ and $\Phi^j_y(\eta)\in\Pi^1_{\lambda,y}$ for all $0\le j\le(k-1)$ then, for all $1\le l\le k$,

$$Z(t+T^{(l)}_y(\eta),y,\eta,\mu)=\left(\prod_{j=0}^{l-1}\xi^*_\eta(T_y(\Phi^j_y(\eta)),y,\Phi^j_y(\eta))\right)Z(t,y,\Phi^l_y(\eta),\mu). \tag{2.48}$$

In view of hypothesis (H_5), for all $\eta\in\Pi_{\lambda,y}\cap\mathrm{supp}_y p_0$ such that $\Phi^j_y(\eta)\in\Pi_{\lambda,y}$ for all $0\le j\le(k-1)$ we have, $(\nabla_\xi a_0)(y,\Phi^{m+1}_y(\eta))\neq 0$ for all $0\le m\le(k-1)$. Furthermore, in view of hypothesis (H_6), almost all points $\eta\in\mathcal{O}^1_{\lambda,y}\cap\Pi_{\lambda,y}$ are 1st order loop points. Hence, for almost all $\eta\in\Pi_{\lambda,y}\cap\mathrm{supp}_y p_0$ such that $\Phi^j_y(\eta)\in\Pi_{\lambda,y}$ for all $0\le j\le(k-1)$ we have, $\Phi^m_y(\eta)\in\Pi^1_{\lambda,y}$ for all $0\le m\le(k-1)$. Thus, if (H_5) and (H_6) are satisifed, we have (2.48) for almost all $\eta\in\Pi_{\lambda,y}\cap\mathrm{supp}_y p_0$ such that $\Phi^j_y(\eta)\in\Pi_{\lambda,y}$ for all $0\le j\le(k-1)$.

Setting $t=0$ in (2.48) and observing that $Z(0)=I$ for all η gives,

$$\left|\det Z(T^{(l)}_y(\eta),y,\eta,\mu)\right|^{\frac12}=\prod_{j=0}^{l-1}\left|\det\xi^*_\eta\left(T_y(\Phi^j_y(\eta)),y,\Phi^j_y(\eta)\right)\right|^{\frac12}.$$

Also from (2.48) (and again noting that $Z(0)=I$ for all η), for each $2\le l\le k$, as t evolves from $T^{(l-1)}_y(\eta)$ to $T^{(l)}_y(\eta)$, counted modulo 4, $(\det Z)^2$ makes $m_y(\Phi^{l-1}_y(\eta))$ full rotations about zero in the complex plane. This fact and an induction imply that the appropriate branch of the square root in iv) has argument equal to, $\frac{\pi}{2}\sum_{j=0}^{l-1}m_y(\Phi^j_y(\eta))$.

According to hypothesis (H_5), $(\nabla_\xi a_0)(y,\eta)\neq 0$ for all $\eta\in\Sigma_{\lambda,y}\cap\mathrm{supp}_y p_0$. Hence, by Lemma 1.1, $T_y(\Phi^j_y(\eta))>T$ for all j such that $\Phi^j_y(\eta)\in\Pi_{\lambda,y}\cap\mathrm{supp}_y p_0$. The statement in v) follows from these observations. □

Theorem 2.10. *Let $A(h)$ denote an h P.D.O. which satisfies hypotheses (H_1) to (H_4) and let $P(h)$ denote an h P.D.O. with compactly supported symbol. Suppose that a_0,p_0,λ and y are such that hypotheses (H_5) and (H_6) are satisfied. Let $\rho\in C^\infty(\mathbb{R})$ have Fourier transform $\hat\rho\in C^\infty_c((0,T_+])$, any $T_+>0$. Then, for each r_0, there exists an $h_0>0$ such that,*

(2.49)

$$h^{n-1}(2\pi)^n\left[P(h)\rho\left((\lambda h^{-1}+r)I-h^{-1}A(h)\right)P(h)^*\right](x,y;h)\big|_{x=y}$$
$$=\int_{\Sigma_{\lambda,y}}\sum_{k\ge1}\hat\rho(T^{(k)}_y(\tilde\eta))\left(e^{irT^{(k)}_y(\tilde\eta)}\left(U_{\lambda,y}(h)\right)^k p_0\right)(\tilde\eta)\overline{\tilde p_0}(\tilde\eta)d\tilde\eta+o(1),$$

for all $h\in(0,h_0]$ and $|r|\le r_0$. The $o(1)$ may depend on the choice of ρ, but is independent of $r\in[-r_0,r_0]$ and the sum in (2.49) is finite. (Due to the compactness *of* supp$\hat{\rho}$ *and the lower bound given in Lemma 2.9 v), we note that the sum in (2.49) is obviously finite.)*

Proof. As supp$\hat{\rho} \subset (0, T_+]$, there exist sequences of points $\{t_j\}_{j=1}^N$, $\{t'_j\}_{j=1}^N$, $t_j, t'_j \in \mathbb{R}$, such that, denoting by T the constant from Lemma 1.1. (with $X = \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0$),

i) $t_j \le t'_j$ for all $1 \le j \le N$,
ii) $(t'_j - t_j) \le \frac{T}{2}$ for all $1 \le j \le N$,
iii) $t_{j-1} < t_j$ and $t'_{j-1} < t'_j$ for all $2 \le j \le N$,
iv) $[t_{j-1}, t'_{j-1}] \cap [t_{j+1}, t'_{j+1}] = \emptyset$ for all $2 \le j \le N-1$,
v) supp$\hat{\rho} \subseteq \bigcup_{j=1}^N [t_j, t'_j]$.

With such a choice of $\{t_j\}$ and $\{t'_j\}$, there exists a sequence of functions $\{\hat{\rho}_j\}_{j=1}^N$, such that $\hat{\rho}_j \in C_c^\infty(\mathbb{R})$ and supp$\hat{\rho}_j \subseteq [t_j, t'_j]$ for all j, and $\sum_j \hat{\rho}_j(t) = \hat{\rho}(t)$ for all t.

For arbitrary $\delta > 0$, the h P.D.O. $P_j(h), j = 1, 2$, are defined as in the proof of Theorem 2.7. Then taking $\delta > 0$ small enough (again ensuring $\mathrm{supp}_y p_0^{(1)}$ is contained in a sufficiently small neighbourhood of $\Sigma_{\lambda,y}$), there exist sequences $\{\eta_j\}_{j=1}^M$, $\eta_j \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0^{(1)}$, and $\{\chi_j\}_{j=1}^M$, $\chi_j \in C_c^\infty(\mathbb{R}^n)$, $\sum_j \chi_j = 1$ for all $\eta \in \mathrm{supp}_y p_0^{(1)}$, each χ_j having support contained in a sufficiently small neighbourhood of η_j such that, for each $1 \le j \le M$ and $1 \le l \le N$, either: a) there exists a $t''_l \in \mathrm{supp}\hat{\rho}_l$ and $\eta'_j \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0^{(1)} \cap \mathrm{supp}\chi_j$ such that we can apply Theorem 2.5 with $P(h) = Q(h) = P_1(h)$, $\hat{\rho} = \hat{\rho}_l$, $t_0 = t''_l$, $\eta_0 = \eta'_j$ and $\chi = \chi_j$, or b) we can apply Theorem 2.6 with $P(h) = Q(h) = P_1(h)$, $\hat{\rho} = \hat{\rho}_l$ and $\chi = \chi_j$. (We note that, due to ii), supp$\hat{\rho}_l \subseteq \left[t''_l - \frac{T}{2}, t''_l + \frac{T}{2}\right]$ for all $t''_l \in \mathrm{supp}\hat{\rho}_l$.)

In case a), applying Theorem 2.5, there exists a function $(t_\star)_l(\tilde{\eta})$ (defined as in (2.26) with T_1 chosen such that $T_1 > T_+$) and constants $\mu_{j,l}, h_{j,l} > 0$ such that,

$$
\begin{aligned}
&(2\pi h)^{-1} \int \hat{\rho}_l(t) e^{irt} e^{ih^{-1}\psi} \chi_j(\eta) u_0(t, y, \eta, \mu) d\eta dt \\
&\quad = \int_{\Sigma_{\lambda,y}} e^{i\theta_{jl}} \hat{\rho}_l((t_\star)_l) e^{ir(t_\star)_l} e^{i\varphi_l} \left(\det Z((t_\star)_l)\right)^{\frac{1}{2}} \chi_j(\tilde{\eta}) \tilde{p}_0(\xi^*((t_\star)_l)) \overline{\tilde{p}_0}(\tilde{\eta}) d\tilde{\eta} \\
&\qquad\qquad + o_{j,l,\mu}(1),
\end{aligned} \tag{2.50}
$$

for all $h \in (0, h_{j,l}]$, $|r| \le r_0$ and $\mu \ge \mu_{j,l}$. Here the $o_{j,l,\mu}(1)$ may depend on j, l and μ (and ρ), but is independent of $r \in [-r_0, r_0]$. The θ_{jl} is a function such that $\sup_{\tilde{\eta} \in V_{\lambda,y}} \theta_{jl}(\tilde{\eta}, \mu) \to 0$ as $\mu \to \infty$, and $\varphi_l = \int_0^{(t_\star)_l} \left(h^{-1}\xi^* \cdot \dot{x}^* - a_{\mathrm{sub}}(x^*, \xi^*)\right) dt$.

In case b) either $x^*(t, y, \eta) \ne y$ or $\xi^*(t, y, \eta) \notin \mathrm{supp}_y p_0$ for all $t \in \mathrm{supp}\hat{\rho}_l$ and $\eta \in \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0 \cap \mathrm{supp}\chi_j$, or $\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0 \cap \mathrm{supp}\chi_j = \emptyset$. Hence,

$$
\int_{\Sigma_{\lambda,y}} \hat{\rho}_l((t_\star)_l) e^{ir(t_\star)_l} e^{i\varphi_l} \left(\det Z((t_\star)_l)\right)^{\frac{1}{2}} \chi_j(\tilde{\eta}) \tilde{p}_0(\xi^*((t_\star)_l)) \overline{\tilde{p}_0}(\tilde{\eta}) d\tilde{\eta} = 0, \tag{2.51}
$$

where $(t_\star)_l(\tilde{\eta})$ denotes a function defined as in (2.26) with T_1 chosen, $T_1 > T_+$.

Applying Theorem 2.6, in case b) we have,

$$(2.52)\quad (2\pi h)^{-1}\int \hat{\rho}_l(t)e^{irt}e^{ih^{-1}\psi}\chi_j(\eta)u_0(t,y,\eta,\mu)d\eta dt \\ = \int_{\Sigma_{\lambda,y}} e^{i\theta_{jl}}\hat{\rho}_l((t_\star)_l)e^{ir(t_\star)_l}e^{i\varphi_l}\left(\det Z((t_\star)_l)\right)^{\frac{1}{2}}\chi_j(\tilde{\eta})\tilde{p}_0(\xi^*((t_\star)_l))\overline{\tilde{p}_0}(\tilde{\eta})d\tilde{\eta} \\ + O_{j,l,\mu}(h),$$

where the $O_{j,l,\mu}(h)$ is independent of r and the θ_{jl} denotes a function such that $\theta_{jl}(\tilde{\eta},\mu)=0$ for all $\tilde{\eta}$ and μ.

As $\Sigma_{\lambda,y}\cap \mathrm{supp}_y p_0^{(2)}=\emptyset$, we can apply Theorem 2.6 in the case when $P(h)$ and $Q(h)$ are replaced by $P_1(h)$ and $P_2(h)$, $P_2(h)$ and $P_1(h)$, and $P_2(h)$ and $P_2(h)$, and summing the results and (2.50) and (2.52) (for all j,l) we obtain that, for each $r_0>0$, there exist $\mu_0,h_0>0$ such that the left-hand side of (2.49) is equal to,

$$(2.53)\quad \int_{\Sigma_{\lambda,y}}\sum_{j,l} e^{i\theta_{jl}}\hat{\rho}_l((t_\star)_l)e^{ir(t_\star)_l}e^{i\varphi_l}\left(\det Z((t_\star)_l)\right)^{\frac{1}{2}}\chi_j(\tilde{\eta})\tilde{p}_0(\xi^*((t_\star)_l))\overline{\tilde{p}_0}(\tilde{\eta})d\tilde{\eta} \\ + o_\mu(1)+O_\mu(h),$$

for all $h\in(0,h_0],|r|\le r_0$ and $\mu\ge\mu_0$. Here $h_0:=\min_{jl}\{h_{j,l}\}$ and $\mu_0:=\max_{jl}\{\mu_{j,l}\}$.

By definition, for each $1\le l\le N$, there exists a unique $\mathbb{Z}_+\ni k\ge 1$ such that $(t_\star)_l(\tilde{\eta})=T_y^{(k)}(\tilde{\eta})$, and we denote by $f:[1,N]\longrightarrow\mathbb{Z}_+$ the map which takes each l to its corresponding k. (We note that f may not be injective. If, for example, $(t_\star)_l\in[t_l,t'_{l-1}]$ for some $2\le l\le(N-1)$ then, as $\sum_l\hat{\rho}_l(t)=\hat{\rho}(t)$, we must have $\hat{\rho}((t_\star)_l)=\hat{\rho}_{l-1}((t_\star)_l)+\hat{\rho}_l((t_\star)_l)$. Hence, $(t_\star)_{l-1}=(t_\star)_l$ or $f(l-1)=f(l)$.)

Due to the fact that $\sum_l\hat{\rho}_l(t)=\hat{\rho}(t)$, for each $k\in f([1,N])$ we must have, $\sum_{l\in f^{-1}(k)}\hat{\rho}_l((t_\star)_l(\tilde{\eta}))=\sum_{l\in f^{-1}(k)}\hat{\rho}_l(T_y^{(k)}(\tilde{\eta}))=\hat{\rho}(T_y^{(k)}(\tilde{\eta}))$ for all $\tilde{\eta}\in\Pi_{\lambda,y}\cap\mathrm{supp}_y p_0$. Furthermore, $\theta_{jl}=\theta_{jl'}$ for all $l,l'\in f^{-1}(k)$. According to Lemma 2.9 and the definition of $U_{\lambda,y}(h)$,

$$(2.54)\qquad e^{i\varphi_l}\left(\det Z((t_\star)_l)\right)^{\frac{1}{2}}p_0(y,\xi^*((t_\star)_l))=\left((U_{\lambda,y}(h))^k\tilde{p}_0\right)(\tilde{\eta}),$$

for all $1\le j\le N$, $l\in f^{-1}(k)$ and almost all $\tilde{\eta}\in\Pi_{\lambda,y}\cap\mathrm{supp}_y p_0$ such that $\Phi_y^m(\tilde{\eta})\in\Pi_{\lambda,y}$ for all $0\le m\le(k-1)$. Hence, we can replace the integrand in (2.53) by,

$$(2.55)\qquad \sum_{j=1,k\in f([1,N])}^{M} e^{i\theta_{jk}(\tilde{\eta},\mu)}\hat{\rho}(T_y^{(k)}(\tilde{\eta}))e^{irT_y^{(k)}(\tilde{\eta})}\left((U_{\lambda,y}(h))^k\tilde{p}_0\right)(\tilde{\eta})\overline{\tilde{p}_0}(\tilde{\eta})\chi_j(\tilde{\eta}).$$

The $(t_\star)_l$ where all chosen to have $T_1>T_+$. Hence, for each $1\le j\le M$ and $k\in f([1,N])$, $\sup\theta_{jk}(\tilde{\eta},\mu)\to 0$ as $\mu\to+\infty$, where the sup can be taken over all $\tilde{\eta}$ such that,

$$\hat{\rho}(T_y^{(k)}(\tilde{\eta}))\left((U_{\lambda,y}(h))^k\tilde{p}_0\right)(\tilde{\eta})\overline{\tilde{p}_0}(\tilde{\eta})\chi_j(\tilde{\eta})\ne 0.$$

But $|e^{irT_y^{(k)}}| = |e^{ih^{-1}s_y^{(k)}}| = 1$ for all r and h. Hence, there exists a constant $C > 0$ such that, for all $\epsilon' > 0$ there exists a $\mu > 0$ such that,
(2.56)

$$\left| \int_{\Sigma_{\lambda,y}} \sum_{j=1, k\in f([1,N])}^{M} \hat{\rho}(T_y^{(k)}) e^{irT_y^{(k)}} \left((U_{\lambda,y}(h))^k \tilde{p}_0\right) \overline{\tilde{p}_0} \left(e^{i\theta_{jk}(\mu)} - 1\right) \chi_j d\tilde{\eta} \right| \leq C\epsilon',$$

for all r and h. (Here C is independent of μ, r, h and ϵ'.) Taking account of (2.53), (2.55), (2.56) and the fact that $\sum_j \chi_j = 1$ on $\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0$, for each $\epsilon > 0$, there exists a $\mu > 0$ such that,
(2.57)

$$\begin{aligned} \Big| h^{n-1}(2\pi)^n \left[P(h)\rho\left((\lambda h^{-1} + r)I - h^{-1}A(h)\right) P(h)^*\right](x, y; h)\big|_{x=y} \\ - \int_{\Sigma_{\lambda,y}} \sum_{k\in f([1,N])} \hat{\rho}(T_y^{(k)}(\tilde{\eta})) \left(e^{irT_y^{(k)}(\tilde{\eta})} \left(U_{\lambda,y}(h)\right)^k p_0\right)(\tilde{\eta}) \overline{\tilde{p}_0}(\tilde{\eta}) d\tilde{\eta} \Big| \\ \leq \frac{\epsilon}{2} + |o_\mu(1)| + |O_\mu(h)|, \end{aligned}$$

for all $h \in (0, h_0]$ and $|r| \leq r_0$. With this μ and ϵ fixed, there exists an $0 < h_1 \leq h_0$ such that $|o_\mu(1)| + |O_\mu(h)| \leq \frac{\epsilon}{2}$ for all $h \in (0, h_1]$ and $|r| \leq r_0$. Hence, the right-hand side of (2.57) can be replaced by $o(1)$. Finally, observing that $\hat{\rho}(T_y^{(k)})((U_{\lambda,y}(h))^k \tilde{p}_0) = 0$ for all $1 \leq k \in \mathbb{Z}_+ \setminus f([1, N])$, without changing the value of (2.57) we can replace the sum by a sum over all $k \geq 1$. We have therefore established (2.49). $\square$

The proof of Theorem 1.5 is completed by combining Theorems 2.7 and 2.10.

Proof (of Theorem 1.5). Suppose that $\rho \in \mathcal{S}(\mathbb{R})$ is real-valued, postive and has Fourier transform $\hat{\rho} \in C_c^\infty(\mathbb{R})$ such that $\hat{\rho}$ is even. Then for arbitrary $\delta > 0$, there exist real-valued functions $\gamma_1, \gamma_2 \in \mathcal{S}(\mathbb{R})$ such that $\hat{\gamma}_1, \hat{\gamma}_2 \in C_c^\infty(\mathbb{R})$, $\mathrm{supp}\hat{\gamma}_1 \subseteq [-\delta, \delta]$, $\mathrm{supp}\hat{\gamma}_2 \subset (0, +\infty)$ and $\rho(\lambda) = \gamma_2(-\lambda) + \gamma_1(\lambda) + \gamma_2(\lambda)$. (It follows also that $\hat{\rho}(t) = \hat{\gamma}_2(-t) + \hat{\gamma}_1(t) + \hat{\gamma}_2(t)$ for all t.)

In view of the fact that $A(h)$ is self-adjoint, we have,

$$\overline{u_P}(-t, y, y; h) = u_P(t, y, y; h) \text{ for all } t,$$

where $u_P := u_{P,P}$ denotes the kernel mentioned in sub-section 2.2. Hence, replacing $\hat{\rho}(t)$ by $\hat{\gamma}_2(-t)$ in (2.11), employing the change of variables $t \mapsto -t$ and using the fact that $\hat{\gamma}_2$ is real-valued we have,
(2.58)

$$\begin{aligned} \left[P(h)\gamma_2\left(-(\lambda h^{-1} + r)I + h^{-1}A(h)\right) P(h)^*\right](x, y; h)\big|_{x=y} \\ = \overline{\left[P(h)\gamma_2\left((\lambda h^{-1} + r)I - h^{-1}A(h)\right) P(h)^*\right](x, y; h)}\Big|_{x=y}. \end{aligned}$$

Choosing $\delta \leq \frac{T}{2}$, where T is the constant given by Lemma 1.1 in the case when $X = \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0$, applying Theorem 2.7 with ρ replaced by γ_1, applying Theorem 2.10 with ρ replaced by γ_2, taking account of (2.58) and using the fact that $\rho(\lambda) = \gamma_2(-\lambda) + \gamma_1(\lambda) + \gamma_2(\lambda)$ yields (1.8). $\square$

3. Proof of auxillary results

In this section we give proof of the auxillary results presented in the first section.

Proof (of Lemma 1.1). As $\dot{x}^*(0) = (\nabla_\xi a_0)(y, \eta)$ then, by hypothesis, $\dot{x}^*(0) \neq 0$ for all $\eta \in X$. Hence, for each $\eta_0 \in X$, there exists a neighbourhood U_0 of η_0 and a j such that $\inf_{\eta \in U_0} \dot{x}^*_j(0) > 0$. An application of the implicit function theorem implies that there exists a $t_0 > 0$ such that $x^*_j(t) \neq y_j$ for all $0 < t \leq t_0$ and $\eta \in U_0$. By compactness, we can cover X by a finite collection of such neighbourhoods and, taking the minimum of the t_0 obtained, provides the lower bound for $T_y(\eta)$. □

Proof (of Theorem 1.2). Let $\eta \in \Pi_{\lambda,y} \setminus \Pi^1_{\lambda,y}$. Then, from the definition of $\Pi^1_{\lambda,y}$, there exists a function $t_\star(\eta)$ such that $x^*(t_\star(\eta), y, \eta) = y$ and $x^*_\eta(t_\star(\eta), y, \eta) \neq 0$. Differentiating the identity $x^*_j(t_\star(\eta), y, \eta) = y_j$ with respect to η_k we have,

$$\dot{x}^*_j(t_\star) \frac{\partial t_\star}{\partial \eta_k} = -(x^*_\eta)_{kj}(t_\star). \tag{3.1}$$

Now (λ, y) is a regular value of a_0 and hence, there exists a j such that $\dot{x}^*_j(t_\star) = (\nabla_{\xi_j} a_0)(y, \xi^*(t_\star)) \neq 0$. (3.1) then implies that $\nabla_\eta t_\star(\eta) \neq 0$ for all $\eta \in \Pi_{\lambda,y} \setminus \Pi^1_{\lambda,y}$.

As in the proof of Theorem 2.5, due to that fact that $\nabla_\xi a_0 \neq 0$ on $\Sigma_{\lambda,y}$, there exists a neighbourhood $U_0 \subseteq \mathbb{R}^n$ of η and coordinate functions $\{u_j\}_{j=1}^{n-1}$, $u_j : U_0 \longrightarrow \mathbb{R}$ such that (2.28) holds. According to the chain rule,

$$\frac{\partial t_\star}{\partial \tau} = \frac{\partial t_\star}{\partial \tau} \frac{\partial \tau}{\partial \eta_j} + \sum_k \frac{\partial t_\star}{\partial u_k} \frac{\partial u_k}{\partial \eta_j}. \tag{3.2}$$

But $\nabla_\eta \tau = (\nabla_\xi a_0)(y, \eta)$ and as, by hypothesis, $\nabla_\eta t_\star$ is not parallel to $\nabla_\xi a_0$, there exist a k and a neighbourhood $U_1 \subseteq U_0$ of η such that, $\frac{\partial t_\star}{\partial u_k} \neq 0$ for all $\eta \in (\Pi_{\lambda,y} \setminus \Pi^1_{\lambda,y}) \cap U_1$. It thus follows that the set $(\Pi_{\lambda,y} \setminus \Pi^1_{\lambda,y}) \cap U_1$ forms a sub-manifold of codimension 1 in $\Sigma_{\lambda,y}$, and hence has zero measure. Repeating the argument, we can cover $\Pi_{\lambda,y} \setminus \Pi^1_{\lambda,y}$ by a union of such neighbourhoods which completes the proof. □

Proof (of Corollary 1.3). Firstly, $a_0(0, \mu^m \xi) = \mu^s a_0(0, \xi)$ for all $\xi \in \mathbb{R}^n$ and $\mu > 0$. Differentiation with respect to μ and setting $\mu = 1$ gives,

$$m(\nabla_\xi a_0)(0, \xi) \cdot \xi = s a_0(0, \xi), \tag{3.3}$$

for all ξ. If $\lambda > 0$, then (3.3) implies that $(\nabla_\xi a_0)(0, \xi) \cdot \xi \neq 0$ for all $\xi \in \Sigma_{\lambda,0}$ and hence, hypothesis (H'_5) is satisfied.

As a_0 is quasi-homogeneous we have,

$$\Phi^t_{a_0} = I(\mu^{-1}) \circ \Phi^{\mu^{m+l-s} t}_{a_0} \circ I(\mu), \tag{3.4}$$

for all y, η, t and $\mu > 0$ and where $I(\mu)$ denotes the linear map given by the matrix $\begin{pmatrix} \mu^l I & 0 \\ 0 & \mu^m I \end{pmatrix}$. Hence, in the case when $y = 0$ and $s = (m+l)$, any $t-$solution $t_\star$ of

(1.5) has, $t_\star(\eta) = t_\star(\mu^m \eta)$, for all $\eta \in \Pi_{\lambda,0}$ and $\mu > 0$. Differentiating with respect to μ and setting $\mu = 1$,

$$(\nabla_\eta t_\star)(\eta) \cdot \eta = 0, \tag{3.5}$$

for all $\eta \in \Pi_{\lambda,0}$. Now suppose that $\nabla_\eta t_\star \neq 0$ and $\nabla_\eta t_\star$ is parallel to $\nabla_\xi a_0$ at $\eta \in \Pi_{\lambda,0}$. Then there exists $\alpha(\eta) \neq 0$ such that $(\nabla_\eta t_\star)(\eta) = \alpha(\eta)(\nabla_\xi a_0)(y,\eta)$. In view of (3.5), $(\nabla_\xi a_0)(y,\eta)\cdot\eta \neq 0$ and (3.3) implies that $a_0(0,\eta) = 0$. This provides a condtradiction as $\eta \in \Sigma_{\lambda,0}$ and $\lambda > 0$. $\square$

Proof (of Lemma 1.4). Firstly, the statements about the kernel and image space follow by definition. For any functions $f, g \in L^2(\Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0)$ we have,

$$(U_{\lambda,y} f, U_{\lambda,y} g) = \int_{\Pi^1_{\lambda,y} \cap \operatorname{supp}_y p_0} |\det \xi^*_\eta(T_y(\tilde\eta))|^{\frac{1}{2}} f(\Phi_y(\tilde\eta)) \overline{g}(\Phi_y(\tilde\eta)) d\tilde\eta. \tag{3.6}$$

By hypothesis, $(\nabla_\xi a_0)(y,\tilde\eta) \neq 0$ for all $\tilde\eta \in \Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0$. Hence, for all $\tilde\eta \in \Pi^1_{\lambda,y} \cap \operatorname{supp}_y p_0$ such that $\Phi_y(\tilde\eta) \in \Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0$ we have, $(\nabla_\xi a_0)(y, \Phi_y(\tilde\eta)) \neq 0$. According to (2.46) then, $\partial_\eta \Phi_y(\tilde\eta) = \xi^*_\eta(T_y(\tilde\eta))$ for all $\tilde\eta \in \Pi^1_{\lambda,y} \cap \operatorname{supp}_y p_0$ such that $\Phi_y(\tilde\eta) \in \Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0$. Using this fact and changing variables $\Phi_y(\tilde\eta) \mapsto \tilde\xi$ in (3.6), if $f, g \in L^2(\Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0) \setminus \operatorname{Ker}\,(U_{\lambda,y})$ we have, $(U_{\lambda,y} f, U_{\lambda,y} g) = (f,g)$ and $U_{\lambda,y}$ is a partial isometry. The same argument applies for $U_{\lambda,y}(h)$. $\square$

Proof (of Lemma 1.6). We follow essentially the same arguments appearing in the proofs of [SV] Lemmas 1.8.11 and 1.8.12, which we reproduce for completeness. In what follows we denote by $\tilde U_r$ the operator $\tilde U_r = e^{irT_y(\eta)} U_{\lambda,y}(h)$, and we note that $\tilde U_r^k = e^{irT_y^{(k)}(\tilde\eta)} U_{\lambda,y}(h)^k$. For any $f \in L^2(\Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0)$ and any function $\gamma \in \mathcal{S}(\mathbb{R})$ with Fourier transform $\hat\gamma \in C_c^\infty(\mathbb{R})$ we have,

$$\int_{\mathbb{R}} \sum_{k\geq 1} \left(\tilde U_r^k f, f\right) \gamma(r) dr = \int_{\Sigma_{\lambda,y}} \sum_{k \geq 1} \hat\gamma(T_y^{(k)}(\tilde\eta)) \left(\tilde U_0^k f\right)(\tilde\eta) \overline{f}(\tilde\eta) d\tilde\eta, \tag{3.7}$$

$(\cdot,\cdot)$ denoting the inner product in $L^2(\Sigma_{\lambda,y} \cap \operatorname{supp}_y p_0)$, and the sum in (3.7) being finite due to the lower bound in Lemma 2.9 v) and the fact that $\hat\gamma$ has compact support. (3.7) implies that $\sum_{k\geq 1} (\tilde U_r^k f, f)$ belongs to the dual of the space $\mathcal{S}(\mathbb{R})$, denoted $\mathcal{S}'(\mathbb{R})$, or in other words, that $\sum_{k\geq 1}(\tilde U_r^k f, f)$ is a tempered distribution in r. Furthermore, $\|f\|^2 + \sum_{k\geq 1} 2\mathrm{Re}(\tilde U_r^k f, f)$ coincides with the $\mathcal{S}'(\mathbb{R})$ limit of,

$$\left(f + \sum_{k\geq 1} \left[\tilde U^k_{r+i\epsilon} + \left(\tilde U^k_{r+i\epsilon}\right)^* \right] f, f \right), \tag{3.8}$$

as $\epsilon \searrow 0$.

Since the norm of the operator $\tilde U_{r+i\epsilon}$ is strictly less than 1, for all $\epsilon > 0$, we may expand $(I - \tilde U_{r+i\epsilon})^{-1}$ in an operator power series. Doing this we obtain, for all $\epsilon > 0$ and analogously to [SV],

$$
\begin{aligned}
I + \sum_{k\geq 1} \left[\tilde{U}^k_{r+i\epsilon} + \left(\tilde{U}^k_{r+i\epsilon}\right)^*\right] \\
&= (I-\tilde{U}_{r+i\epsilon})^{-1}(I-\tilde{U}_{r+i\epsilon})(I-\tilde{U}^*_{r+i\epsilon})(I-\tilde{U}^*_{r+i\epsilon})^{-1} \\
&\quad + (I-\tilde{U}_{r+i\epsilon})^{-1}\tilde{U}_{r+i\epsilon} + \tilde{U}^*_{r+i\epsilon}(I-\tilde{U}^*_{r+i\epsilon})^{-1} \\
&= (I-\tilde{U}_{r+i\epsilon})^{-1}(I-\tilde{U}^*_{r+i\epsilon}\tilde{U}_{r+i\epsilon})(I-\tilde{U}^*_{r+i\epsilon})^{-1} \geq 0.
\end{aligned}
$$

Hence, the functions in (3.8) (for all $\epsilon > 0$) are non-negative. In turn, as the quantity $\|f\|^2 + \sum_{k\geq 1} 2\mathrm{Re}(\tilde{U}^k_r f, f)$ coincides with the $\mathcal{S}'(\mathbb{R})$ limit as $\epsilon \searrow 0$ of (3.8), we conclude that $\|f\|^2 + \sum_{k\geq 1} 2\mathrm{Re}(\tilde{U}^k_r f, f)$ is a non-negative distribution, and is therefore a positive Borel measure. Hence, the series in (1.10) converges in the weak operator topology and defines a Borel measure.

That the Fourier transform of the object in (1.10) vanishes for t in a small neighbourhood of zero follows from (3.7) and taking γ with $\mathrm{supp}\hat{\gamma} \subset [-T, T]$ where $T > 0$ is the constant from Lemma 1.1 (with $X = \Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0$). $\square$

Proof (of Corollary 1.8). Let $C > 0$ denote the constant in Theorem 1.7. By the continuity of $C(\epsilon)$, for any $\delta > 0$, there exists a $c_1 > 0$ such that,

$$
C\left(\frac{c}{2}\right) - \sup_{\epsilon\in(0,\epsilon_0]} C(\epsilon) - \frac{Cc}{2} > -\frac{\delta}{2}, \tag{3.9}
$$

for all $c \in (0, c_1]$. Let then $c \in (0, c_1]$. Setting the ϵ in Theorem 1.7 equal to $\frac{c}{2}$, the left-hand inequality there and (3.9) give,

$$
h^{n-1}E^P_{\lambda,y}(r(h), c; h) - \sup_{\epsilon\in(0,\epsilon_0]} C(\epsilon) \geq \frac{2c}{(2\pi)^n}\|\tilde{p}_0\|^2 - \frac{\delta}{2} - o_{\frac{c}{2}}(1) \geq -\frac{\delta}{2} - o_{\frac{c}{2}}(1). \tag{3.10}
$$

There exists an $0 < h_1(c) \leq h_0$ such that $|o_{\frac{c}{2}}(1)| < \frac{\delta}{2}$ for all $h \in (0, h_1(c)]$ and, applying this bound in (3.10), and letting $h \to 0$ yields (1.14). $\square$

Proof (of Corollary 1.9). The proof is the same as that appearing in the proof of Corollary 1.2 in [PP2]. It suffices to observe that, for $c_0 > 0$ small enough and $c \in (0, c_0]$ fixed, for any $\delta > 0$ there exists an $\epsilon > 0$ such that $C\epsilon < \frac{\delta}{2}$ and,

$$
|Q_{\lambda,y}(r + c; h) - Q_{\lambda,y}(r + c \pm \epsilon; h)| < \frac{\delta}{4},
$$

for all $h \in (0, h_0]$ and $r \in [R_1, R_2]$, the same inequality being valid with c replaced by $-c$. (Here $C > 0$ denotes the constant from Theorem 1.7.) With this ϵ fixed, there exists $0 < h_1 \leq h_0$ such that $h^{n-1}|o_\epsilon(h^{1-n})| < \frac{\delta}{4}$ for all $h \in (0, h_1]$, and (1.15) follows from these inequalities and Theorem 1.7. $\square$

Proof (of Corollary 1.10). The statement follows from the observation that, by definition, for all $k \geq 1$, $\left((U_{\lambda,y}(h))^k\tilde{p}_0\right)(\tilde{\eta})\overline{\tilde{p}_0}(\tilde{\eta}) \neq 0$ only if $\tilde{\eta} \in \Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$. $\square$

Proof (of Lemma 1.11). Let $\eta_0, \eta_1 \in \Pi_{\lambda,y}$ denote two points in the same connected component of $\Pi_{\lambda,y}$. Choose a smooth path $\vartheta : [0,1] \mapsto \Pi_{\lambda,y}$ such that $\vartheta(0) = \eta_0$, $\vartheta(1) = \eta_1$ and $\vartheta(t)$ lies in the same connected component of $\Pi_{\lambda,y}$ for all $t \in [0,1]$. Now,

$$s_y(\eta) = \int_0^{T_y(\eta)} (\xi^*(s) \cdot \dot{x}^*(s) - a_0(x^*(s), \xi^*(s)))\, ds + \lambda T_y(\eta),$$

and hence, differentiating,

$$\begin{aligned}\frac{d}{dt} s_y(\vartheta(t)) = \int_0^{T_y(\vartheta(t))} & \left(\xi^* \cdot \dot{x}^*_\eta \vartheta_t(t) + \dot{\xi}^* \cdot x^*_\eta \vartheta_t(t)\right) ds \\ & + \left(\xi^* \cdot \dot{x}^*|_{s=T_y(\vartheta(t))}\right) (\nabla_\eta T_y)(\vartheta(t)) \cdot \vartheta_t(t),\end{aligned} \tag{3.11}$$

where ϑ_t denotes the derivative of ϑ with respect to t and all trajectory points, x^* and ξ^*, in (3.11) (and the rest of this proof) are evaluated at the point $(s, y, \vartheta(t))$. Using the fact that $x^*_\eta(0) = 0$ we have,

$$\frac{d}{dt} s_y(\vartheta(t)) = \xi^* \cdot x^*_\eta \vartheta_t(t)\big|_{s=T_y(\vartheta(t))} + \left(\xi^* \cdot \dot{x}^*|_{s=T_y(\vartheta(t))}\right) (\nabla_\eta T_y)(\vartheta(t)) \cdot \vartheta_t(t). \tag{3.12}$$

Differentiating the identity $\xi^* \cdot (y - x^*)|_{s=T_y(\vartheta(t))} = 0$ with respect to t we obtain that the quantity in (3.12) is zero, which completes the proof. $\square$

Proof (of Theorem 1.12). The result follows essentially from the same proof as that of Theorem 1.8.17 in [SV]. In our case we must take care of the extra parameters r and h. We sketch the proof, highlighting only the parts which are different in our case, and refer the reader to the proof of Theorem 1.8.17 in [SV] for more details.

Let $\tilde{U}_{\lambda,y}$ denote the minimal unitary dilation of $U_{\lambda,y}$ (which always exists) acting in the extended Hilbert space $\tilde{\mathcal{H}} \supseteq L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$. Then it follows from [SV] that $\tilde{p}_0 \in L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$ is orthogonal to all the $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$ eigenfunctions of $U_{\lambda,y}$ with unit length eigenvalues if and only if $\tilde{p}_0$ is orthogonal to all the $\tilde{\mathcal{H}}$ eigenfunctions of $\tilde{U}_{\lambda,y}$. Furthermore, introducing the measure $\mu_{\tilde{p}_0}$ (given by the spectral theorem) such that,

$$\left(\tilde{U}^k_{\lambda,y} \tilde{p}_0, \tilde{p}_0\right)_{\tilde{\mathcal{H}}} = \int_{\mathbb{S}} e^{-ikq} d\mu_{\tilde{p}_0}(q), \text{ for all } k \in \mathbb{Z},$$

$\tilde{p}_0$ is orthogonal to all the $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$ eigenfunctions of $U_{\lambda,y}$ with unit length eigenvalues if and only if the measure $\mu_{\tilde{p}_0}$ is continuous (i.e. $\mu_{\tilde{p}_0}(\{q\}) = 0$ for all points $q \in \mathbb{S}$).

Using the argument appearing in the proof of Proposition 1.8.16 in [SV] and summing the trigonometric series as in (1.16), understood in the sense of the Sz.-Nagy Foias calculus, see [SNF], we have,

$$Q^P_{\lambda,y}(r; h) = (2\pi)^{-n} T_y^{-1} \int_{\mathbb{S}} \{\pi - rT_y - h^{-1} s_y + q\}_{2\pi} d\mu_{\tilde{p}_0}(q). \tag{3.13}$$

For sufficiently small $\epsilon > 0$,

$$\begin{aligned}\{\pi - (r+\epsilon)T_y - h^{-1}s_y + q\}_{2\pi} - \{\pi - rT_y - h^{-1}s_y + q\}_{2\pi} &= (2\pi)\chi^+_\epsilon(q) - \epsilon T_y \\ \{\pi - rT_y - h^{-1}s_y + q\}_{2\pi} - \{\pi - (r-\epsilon)T_y - h^{-1}s_y + q\}_{2\pi} &= (2\pi)\chi^-_\epsilon(q) - \epsilon T_y,\end{aligned}$$

where χ_ϵ^+ and χ_ϵ^- denote the characteristic functions of the intervals,

$$(rT_y + h^{-1}s_y - k_{r,h}, rT_y + h^{-1}s_y + \epsilon T_y - k_{r,h}] \text{ and,}$$
$$(rT_y + h^{-1}s_y - \epsilon T_y - k_{r,h}, rT_y + h^{-1}s_y - k_{r,h}],$$

and $k_{r,h} := (rT_y + h^{-1}s_y) - \{rT_y + h^{-1}s_y\}_{2\pi}$. Then from (3.13), as $\epsilon \to 0$,

$$\begin{aligned}
&Q^P_{\lambda,y}(r+\epsilon;h) - Q^P_{\lambda,y}(r;h) \to 0,\\
&Q^P_{\lambda,y}(r;h) - Q^P_{\lambda,y}(r-\epsilon;h) \to (2\pi)^{1-n} T_y^{-1} \mu_{\tilde{p}_0}(\{rT_y + h^{-1}s_y\}_{2\pi}),
\end{aligned}$$

and hence, $Q^P_{\lambda,y}$ is continuous (with respect to r) if and only if $\tilde{p}_0$ is orthogonal to all the $L^2(\Sigma_{\lambda,y} \cap \mathrm{supp}_y p_0)$ eigenfunctions of $U_{\lambda,y}$ with unit length eigenvalues. □

Proof (of Lemma 1.13). From (2.46) and in view of hypothesis (H_5), for all $\eta \in \Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$, $\partial_\eta \Phi_y = \xi^*_\eta(T_y(\eta), y, \eta)$. Hence in the case when $\Phi_y(\eta) = \eta$ almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$, we have $\xi^*_\eta(T_y(\eta), y, \eta) = I$ almost everywhere on $\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0$ and (1.19) follows. We obtain (1.20) by taking account of (1.9) and (1.12) and summing the trigonometric series as in (1.16). □

Proof (of Corollary 1.14). The proof is similar to that of Corollary 1.3 in [PP2]. Due to the fact that,

$$\begin{aligned}
&\mathrm{meas}\left(\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0\right)\\
&\qquad = \sum_{m \geq 0} \mathrm{meas}\left(\{\tilde{\eta} \in \Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0 | m \leq T_y(\tilde{\eta}) < (m+1)\}\right),
\end{aligned}$$

defining $U = \{\tilde{\eta} \in \Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0 | T_y(\tilde{\eta}) < T'\}$, it follows that $\mathrm{meas}((\Pi^1_{\lambda,y} \cap \mathrm{supp}_y p_0) \setminus U) \to 0$ as $T' \to +\infty$. For any $h \in (0, h_0]$ and $\epsilon > 0$ we define U_1^h to be the set of all $\tilde{\eta} \in U$ for which there exists an integer k such that,

$$2\pi k \leq -h^{-1}s_y(\tilde{\eta}) + q_y(\tilde{\eta}) - (r(h) \pm \epsilon)T_y(\tilde{\eta}) < 2\pi(k+1),$$

and we define $U_2^h = U \setminus U_1^h$. By definition, $\Pi_1 \cap U \subseteq U_2^h$ for all $h \in (0, h_0]$ and $\epsilon > 0$. Now provided $\epsilon < \frac{\pi}{T'}$, we have,

$$\begin{aligned}
&\big(\{\pi - (r(h)+\epsilon)T_y(\tilde{\eta}) - h^{-1}s_y(\tilde{\eta}) + q_y(\tilde{\eta})\}_{2\pi}\\
&-\{\pi - (r(h)-\epsilon)T_y(\tilde{\eta}) - h^{-1}s_y(\tilde{\eta}) + q_y(\tilde{\eta})\}\big)\\
&\qquad = \begin{cases} -2\epsilon T_y(\tilde{\eta}) & \text{for all } \tilde{\eta} \in U_{1,h} \\ (2\pi) - 2\epsilon T_y(\tilde{\eta}) & \text{for all } \tilde{\eta} \in U_{2,h}. \end{cases}
\end{aligned}$$

Hence, using the fact that $-\pi < \{\cdot\}_{2\pi} \leq \pi$, for all $\epsilon < \frac{\pi}{T'}$,

(3.14)
$$\begin{aligned}
&Q^P_{\lambda,y}(r(h)+\epsilon;h) - Q^P_{\lambda,y}(r(h)-\epsilon;h)\\
&\quad \geq -(2\pi)^{1-n}\int_{\Pi^1_{\lambda,y}\setminus U} \frac{|\tilde{p}_0|^2}{T_y(\tilde{\eta})}d\tilde{\eta} - 2\epsilon(2\pi)^{1-n}\int_U |\tilde{p}_0|^2 d\tilde{\eta} + (2\pi)^{1-n}\int_{U_2^h} \frac{|\tilde{p}_0|^2}{T_y(\tilde{\eta})}d\tilde{\eta}.
\end{aligned}$$

We recall that $\mathrm{meas}((\Pi^1_{\lambda,y}\cap \mathrm{supp}_y p_0)\setminus U)\to 0$ as $T'\to+\infty$. Hence, as $T_y(\tilde\eta)>T$ (T the constant from Lemma 1.1), for each $\delta>0$ there exists a T' such that $(2\pi)^{1-n}\int_{\Pi^1_{\lambda,y}\setminus U}\frac{|\tilde p_0|^2}{T_y(\tilde\eta)}d\tilde\eta<\frac{\delta}{3}$ and $(2\pi)^{1-n}\int_{\Pi_1\setminus U}\frac{|\tilde p_0|^2}{T_y(\tilde\eta)}d\tilde\eta<\frac{\delta}{3}$. With this T' fixed, we can choose an $\epsilon_0<\frac{\pi}{T'}$ such that $2\epsilon(2\pi)^{1-n}\int_U|\tilde p_0|^2d\tilde\eta<\frac{\delta}{3}$ for all $\epsilon\in(0,\epsilon_0]$. Recalling that $\Pi_1\cap U\subseteq U^h_2$, and using these inequalities in (3.14) yields (1.22). $\square$

Proof (of Lemma 1.15). In view of (1.24), if $\tilde\eta\in\Pi_j$ then,

(3.15)
$$T_y^{(km+l)}(\tilde\eta)=\begin{cases} kT_{1,m} & \text{if } k\geq 1, l=0\\ kT_{1,m}+T_{j,j+l-1} & \text{if } k\geq 0, 1\leq l\leq (m-j)\\ (k+1)T_{1,m}-T_{l-m+j,j-1} & \text{if } k\geq 0, (m+1-j)\leq l\leq (m-1),\end{cases}$$

the same equalities being true with the T_y replaced by s_y. Further, from (1.24) and (1.25), if $\tilde\eta\in\Pi_j$ then,

(3.16)
$$\left(U^{km+l}_{\lambda,y}\tilde p_0\right)(\tilde\eta)=\begin{cases} e^{-ikq_{1,m}}\tilde p_0(\Phi^{km}_y(\tilde\eta)) & \text{if } k\geq 1, l=0\\ e^{-i(kq_{1,m}+q_{j,j+l-1})}\tilde p_0(\Phi^{km+l}_y(\tilde\eta)) & \text{if } k\geq 0, 1\leq l\\ & \quad\leq (m-j)\\ e^{-i((k+1)q_{1,m}-q_{l-m+j,j-1})}\tilde p_0(\Phi^{km+l}_y(\tilde\eta)) & \text{if } k\geq 0, (m+1-j)\\ & \quad\leq l\leq (m-1).\end{cases}$$

Taking account of (1.9) and the first line in (1.26) we have, for $2\leq j\leq (m-1)$,

(3.17)
$$\begin{aligned}\int_{\Pi_j}&\left(Q_{\lambda,y}(r;h)\tilde p_0\right)(\tilde\eta)\overline{\tilde p_0}(\tilde\eta)d\tilde\eta\\ &=\sum_{k\geq1}\frac{2}{(kT_{1,m})}\sin\left(k(rT_{1,m}+h^{-1}s_{1,m}-q_{1,m})\right)\int_{\Pi_j}|\tilde p_0|^2d\tilde\eta\\ &+\sum_{\substack{j\leq l\leq(m-1)\\ k\geq0}}\frac{2p_{j,l}}{(kT_{1,m}+T_{j,l})}\sin(r(kT_{1,m}+T_{j,l})+h^{-1}(ks_{1,m}+s_{j,l})\\ &\qquad\qquad -(kq_{1,m}+q_{j,l}))\\ &+\sum_{\substack{1\leq l\leq(j-1)\\ k\geq1}}\frac{2p_{j,l+m-1}}{(kT_{1,m}-T_{l,j-1})}\sin(r(kT_{1,m}-T_{l,j-1})+h^{-1}(ks_{1,m}-s_{l,j-1})\\ &\qquad\qquad -(kq_{1,m}-q_{l,j-1})).\end{aligned}$$

In the case when $j=1$ we obtain (3.17) without the third series, and in the case when $j=m$ we obtain (3.17) without the second series. Using the fact that the sine function is odd and summing (3.17) for $1 \leq j \leq m$,

(3.18)
$$\begin{aligned}\frac{(2\pi)^n}{2} Q^P_{\lambda,y}(r;h) &= \sum_{k\geq 1} (kT_{1,m})^{-1} \sin\left(k(rT_{1,m} + h^{-1}s_{1,m} - q_{1,m})\right) \int_{\Pi^1_{\lambda,y}} |\tilde{p}_0|^2 d\tilde{\eta} \\ &+ \sum_{\substack{1\leq j\leq (m-1)\\ j\leq l\leq (m-1)\\ k\geq 0}} \frac{p_{j,l}}{(kT_{1,m}+T_{j,l})} \sin(r(kT_{1,m}+T_{j,l}) + h^{-1}(ks_{1,m}+s_{j,l}) \\ &\qquad\qquad - (kq_{1,m}+q_{j,l})) \\ &+ \sum_{\substack{2\leq j\leq m\\ 1\leq l\leq (j-1)\\ k\leq -1}} \frac{p_{j,l+m-1}}{(kT_{1,m}+T_{l,j-1})} \sin(r(kT_{1,m}+T_{l,j-1}) + h^{-1}(ks_{1,m}+s_{l,j-1}) \\ &\qquad\qquad - (kq_{1,m}+q_{l,j-1})).\end{aligned}$$

Changing the order of the j and l summation in one of the sums in (3.18) and using the hypothesis (from the second line of (1.26)) that $p_{j,l} = p_{l+1,j+m-1}$ then yields (1.27). $\square$

Proof (of Lemma 1.16). We denote the series on the left-hand side of (1.28) by $f(x)$. Formal differentiation gives,

$$f'(x) = (4\pi)\cos(\mu x)\sum_{k\in\mathbb{Z}} \delta(x-2\pi k), \tag{3.19}$$

where $\delta(x-2\pi k)$ denotes a delta function located at the point $2\pi k$. From (3.19) we conclude that, formally, $f(x)$ is constant on each of the intervals $(2\pi l, 2\pi(l+1))$, $l\in\mathbb{Z}$, and that $f(x)$ makes a 'jump' of $(4\pi)\cos(2\pi k\mu)$ at each of the points $x = 2\pi k$. Then using the fact that sine is an odd function we have,

$$f(x) = \begin{cases} 2\pi & \text{for } 0 < x < 2\pi \\ -2\pi & \text{for } -2\pi < x < 0. \end{cases} \tag{3.20}$$

Formally integrating (3.19) and using (3.20) to evaluate the constant of integration we obtain,

$$f(x) = (2\pi)\left(1 + 2\sum_{1\leq j\leq l}\cos(2\pi j\mu)\right) \quad \text{for } 2\pi l < x < 2\pi(l+1), \text{ all } l\geq 1. \tag{3.21}$$

Writing $2\cos x = e^{ix}+e^{-ix}$, using a geometric summation and trigonometric identities,

$$1 + 2\sum_{1\leq j\leq l}\cos(2\pi j\mu) = \frac{\cos(2\pi\mu l) - \cos(2\pi\mu(l+1))}{1-\cos(2\pi\mu)} = \frac{\sin(\mu(2\pi l+\pi))}{\sin(\pi\mu)}. \tag{3.22}$$

As $x - \{x-\pi\}_{2\pi} = 2\pi l + \pi$ for all $2\pi l < x \leq 2\pi(l+1)$, (3.22) and (3.21) imply that (1.28) is true, formally, for all $x \in \bigcup_{l\geq 1}(2\pi l, 2\pi(l+1))$. The function f is odd and hence (1.28) is also true, formally, for all $x \in \bigcup_{l\geq 1}(-2\pi(l+1), -2\pi l)$. Taking account of (3.20), we have established (1.28), formally, for all $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

From the general theory of Fourier series, see [To] for example, we know that the series on the left-hand side of (1.28) converges and has sum equal to a function which is continuous for $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and may have dis-continuities at the points $2\pi\mathbb{Z}$. In view of this fact the calculations above, though formal, prove the equality in (1.28). $\square$

Proof (of Theorem 1.17). Due to the equalities in (1.29), $s_{j,k} = sT_{j,k}$ and $q_{j,k} = qT_{j,k}$ for all $1 \leq j \leq k \leq m$. Taking out a factor $T_{1,m}$ in (1.27) we can use Lemma 1.16 (with $\mu = \frac{T_{j,k}}{T_{1,m}}, 0 < \mu < 1$) to sum each of the series in curly brackets in (1.27). Summing the first series on the right-hand side of (1.27) as in (1.16) then yields (1.30). $\square$

Proof (of Lemma 1.18). We define $X \subset \mathbb{Z}_+^2$ to be the set of integer pairs,

$$X = \{(j,k) | 1 \leq j \leq k \leq (m-1) \text{ and } \frac{T_{j,k}}{T_{1,m}} \in \mathbb{Q}\},$$

and for $(j,k) \in X$ we define integer $a_{j,k}, b_{j,k}$ such that $\frac{a_{j,k}}{b_{j,k}} = \frac{T_{j,k}}{T_{1,m}}$ and g.c.d. $(a_{j,k}, b_{j,k}) = 1$. Set,

$$r(h) = \frac{\text{l.c.m.}(b_{j,k})}{T_{1,m}} \left\{ \frac{T_{1,m}}{\text{l.c.m.}(b_{j,k})}(q - h^{-1}s) \right\}_{2\pi} + \frac{2\pi l \text{ l.c.m.}(b_{j,k})}{T_{1,m}} + \frac{2\pi\omega}{T_{1,m}} \tag{3.23}$$

for all $h > 0$ and where $l \in \mathbb{Z}$. In (3.23) the l.c.m. is taken over all pairs $(j,k) \in X$ and, in the case when $X = \emptyset$, the l.c.m. term is replaced by 1. This $r(h)$ satisfies (1.31) and we have,

(3.24)

$$\begin{aligned}(r(h) + h^{-1}s - q)T_{j,k} &= \frac{2\pi\omega T_{j,k}}{T_{1,m}} \\ &+ \frac{\text{l.c.m.}(b_{j,k})T_{j,k}}{T_{1,m}} \left(2\pi l + \left\{ \frac{T_{1,m}}{\text{l.c.m.}(b_{j,k})}(q - h^{-1}s) \right\}_{2\pi} - \frac{T_{1,m}}{\text{l.c.m.}(b_{j,k})}(q - h^{-1}s) \right).\end{aligned}$$

In the case when $(j,k) \in X$, the quantity in (3.24), counted modulo 2π, is equal to $\frac{2\pi\omega T_{j,k}}{T_{1,m}}$ for all $l \in \mathbb{Z}$.

Letting $\{\rho_j\}_{j=1}^N$ denote a finite collection of arbitrary, irrational numbers, $\rho_j \notin \mathbb{Q}$, for all $\delta'' > 0$, there exists a set $Y = \{1, \ldots, M\} \subset \mathbb{Z}$ such that, for each $x \in \mathbb{R}$, there exists an $l \in Y$ such that,

$$|\{x - 2\pi l \rho_j\}_{2\pi}| < \delta'', \text{ for all } 1 \leq j \leq N.$$

Taking account of this fact and (3.24), for each $\delta' > 0$, there exists a finite set $Y \subset \mathbb{Z}$ such that, for each $h > 0$, there exists an $l \in Y$ such that the $r(h)$ in (3.23) satisfies (1.38), for all $(j,k) \notin X$. As Y is a finite set, the $r(h)$ in (3.23), with l chosen in this way, remains bounded for all $h > 0$.

For all $\delta > 0$, we can therefore find a bounded function $r(h)$ which satisfies (1.31) and is such that,

$$\left|\cos\left((r(h)+h^{-1}s-q)T_{j,k}\right)-\cos\left(\frac{2\pi\omega T_{j,k}}{T_{1,m}}\right)\right| < \frac{2\delta}{m(m-1)},$$

for all $1 \leq j \leq k \leq (m-1), h > 0$. Hence, in view of (1.33) and (1.35), we obtain (1.37) and clustering occurs. □

Proof (of Theorem 1.19). For each j, y and η we define the numbers,

$$\theta_j(\eta) = \arctan\left(\frac{\eta_j}{\sqrt{\lambda_j}y_j}\right), \tag{3.25}$$

where each θ_j is chosen such that $0 \leq \theta_j < \pi$, $\theta_j = \frac{\pi}{2}$ if $y_j = 0$ and $\theta_j = 0$ if $\eta_j = 0$. From (1.39), $x_j^*(t) = y_j$ if and only if $t\sqrt{\lambda_j} \equiv 0 \mod 2\pi$ or $t\sqrt{\lambda_j} \equiv 2\theta_j(\eta) \mod 2\pi$. Hence, considering the equations $x_j^* = y_j$ with $j = 1, 2$, if $\eta \in \Pi_{\lambda,y}$ we must have one of the following possibilities,

i) $\frac{l}{\sqrt{\lambda_1}} = \frac{m}{\sqrt{\lambda_2}}$ for some $l, m \in \mathbb{Z}$
ii) $\frac{\pi l}{\sqrt{\lambda_1}} = \frac{\theta_2(\eta)+\pi m}{\sqrt{\lambda_2}}$ for some $l, m \in \mathbb{Z}$
iii) $\frac{\theta_1(\eta)+\pi l}{\sqrt{\lambda_1}} = \frac{\pi m}{\sqrt{\lambda_2}}$ for some $l, m \in \mathbb{Z}$
iv) $\frac{\theta_1(\eta)+\pi l}{\sqrt{\lambda_1}} = \frac{\theta_2(\eta)+\pi m}{\sqrt{\lambda_2}}$ for some $l, m \in \mathbb{Z}$.

The intersection of the set $\{\eta \in \mathbb{R}^n | \eta_1 = 0 \text{ or } \eta_2 = 0\}$ with $\Sigma_{\lambda,y}$ forms a sub-manifold of co-dimension 1 and therefore has zero surface measure. Hence, we need only consider $\tilde{\Sigma}_{\lambda,y} = \{\eta \in \Sigma_{\lambda,y} | \eta_1 \neq 0 \text{ and } \eta_2 \neq 0\}$. We have the following cases for y,

a) $y_1 = y_2 = 0$
b) $y_1 = 0$ and $y_2 \neq 0$
c) $y_1 \neq 0$ and $y_2 = 0$
d) $y_1 \neq 0$ and $y_2 \neq 0$.

In case a), we can have i), ii), iii) or iv) only if $\sqrt{\frac{\lambda_1}{\lambda_2}} \in \mathbb{Q}$. In case b), the set $\{\eta \in \tilde{\Sigma}_{\lambda,y} |$ ii) or iv) is true$\}$ consists of a countable union of sub-manifolds of co-dimension 1 and therefore has zero measure. In case b) we can have i) or iii) only if $\sqrt{\frac{\lambda_1}{\lambda_2}} \in \mathbb{Q}$. Similarly in case c), the set $\{\eta \in \tilde{\Sigma}_{\lambda,y} |$ iii) or iv) is true$\}$ has zero measure, and we can have i) or ii) only if $\sqrt{\frac{\lambda_1}{\lambda_2}} \in \mathbb{Q}$. Finally in case d), the set $\{\eta \in \tilde{\Sigma}_{\lambda,y} |$ ii), iii) or iv) is true$\}$ has zero measure, and we can have i) only if $\sqrt{\frac{\lambda_1}{\lambda_2}} \in \mathbb{Q}$. By an inductive argument then, $\Pi_{\lambda,y}$ can have positive measure only if $\sqrt{\frac{\lambda_j}{\lambda_1}} \in \mathbb{Q}$ for all $2 \leq j \leq n$.

Let $\sqrt{\frac{\lambda_j}{\lambda_1}} \in \mathbb{Q}$ for all $2 \leq j \leq n$. By definition, $\eta \in \Pi^1_{\lambda,y}$ if and only if there exists a $T' > 0$ such that,

$$x^*(T', y, \eta) = y \quad \text{and} \quad x^*_\eta(T', y, \eta) = 0. \tag{3.26}$$

According to (1.39), in the case when $y \neq 0$, (3.26) occurs with $T' = 2T$, for all $\eta \in \Sigma_{\lambda,y}$ (the T being defined in (1.40) and $2T$ being the least such T'). In the case when $y = 0$, (3.26) occurs with $T' = T$, for all $\eta \in \Sigma_{\lambda,y}$ (again T being the least such T'). Hence, if $\sqrt{\frac{\lambda_j}{\lambda_1}} \in \mathbb{Q}$ for all $2 \leq j \leq n$, then $\Pi^1_{\lambda,y}$ has positive measure, and the statements about the function T_y follow. The statements about the function Φ_y then follow by taking account of (1.39) and (1.41), and the statements about the action s_y follow by straightforward integration according to (1.6).

The matrix Z was defined in sub-section 2.2 and, in this case,

$$Z(t, y, \eta, \mu) = (\dot{X}(t))^t - i\mu(X(t))^t,$$

for all t, y, η and μ. The matrix Z is thus diagonal and has entries $Z_{jj} = \cos(t\sqrt{\lambda_j}) - \frac{i\mu}{\sqrt{\lambda_j}} \sin(t\sqrt{\lambda_j})$. The Z_{jj} sweep out ellipses in the complex plane and hence, for all $\mu > 0$, as t evolves from 0 to lT, $l \in \mathbb{Z}$, $(\det Z)^2$ makes $-\frac{lT}{\pi}\mathrm{Tr}\,(\sqrt{K})$ full rotations about the origin in the complex plane. Counting this number modulo 4, taking account of Definition 2.8 and the fact that $a_{\mathrm{sub}} = 0$ yields the expressions for q_y. Finally, the statements about the operator $U_{\lambda,y}$ follow from the fact that $\xi^*_\eta(T_y(\eta)) = I$ or J, both matrices having determinant of absolute value 1. $\square$

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