# First KdV integrals and absolutely continuous spectrum for 1-D Schrödinger operator

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**Abstract:** We consider 1-D Schrödinger operators on $L^2(R_+)$ with slowly decaying potentials. Under some conditions on the potential, related to the first integrals of the KdV equation, we prove that the a.c. spectrum of the operator coincides with the positive semiaxis and the singular spectrum is unstable. Examples show that for special classes of sparse potentials these results can not be improved.

**Contents**



## Introduction

Let $H = H^\alpha$ be the Hamiltonian on $L^2(R_+)$ given by

$$H\psi = -\psi'' + V(x)\psi, \quad x \geq 0, \tag{1}$$

$$\psi(0)\cos\alpha - \psi'(0)\sin\alpha = 0, \quad \alpha \in S^1 = [0, \pi). \tag{2}$$

It is assumed that the potential $V$ vanishes at infinity in some sense (see (4)), and from this decay it follows that the Birman condition:

$$\lim_{x\to\infty}\int_x^{x+1}|V(x)|d\,x=0 \tag{3}$$

holds. Then (see the original paper [18] or the monograph [9]) the operator $H^\alpha$ is essentially self-adjoint, and:

a) the negative part of the spectrum of $H^\alpha$ is bounded from below and discrete;

b) the essential spectrum of $H^\alpha$ coincides with the positive semiaxis:

$$\Sigma_{ess}=[0,\infty).$$

Let $\mu^\alpha(d\,\lambda)$ be the spectral measure of $H^\alpha$, i.e. $H^\alpha$ is unitarily equivalent to the operator of multiplication by $\lambda$ in the space $L^2(R,\mu^\alpha(d\,\lambda))$. A direct method of construction of the measure will be recalled in the next section.

The paper concerns the positive part of the spectrum of the operator $H^\alpha$. Let $H^p$, $\quad p=0,1,2,..$, be the space of functions on $R_+$ whose derivative of order $p$ belongs to $L^2$. The simplest form of our result is given by

**Theorem 1** . *Let $V\in H^{p-1}\cap L^{p+1}$ for some integer $p\geq 1$, i.e*

$$J_p(V)=\int_0^\infty(|V^{(p-1)}(x)|^2+|V(x)|^{p+1})d\,x<\infty. \tag{4}$$

*Then:*

*1) for all $\alpha\in[0,\pi)$, the essential support of the absolutely continuous component of the spectral measure $\mu^\alpha(d\lambda)$ of the operator $H^\alpha$ coincides with $[0,\infty)$, i.e. $\mu^\alpha_{ac}(\Gamma)>0$ for any Borel set $\Gamma\subset[0,\infty),|\Gamma|>0$;*

*2) for a.e. $\alpha\in[0,\pi)$, the spectral measure $\mu^\alpha(d\lambda)$ of the operator $H^\alpha$ is absolutely continuous on $[0,\infty)$.*

For $p=1$, the first part of this theorem is identical to the main result of the paper [1] by P. Deift and R. Killip. The paper [1] by itself is a culmination of a long sequence of publications (see for instance [2], [3], [4]) on spectral $L^2-$ conjecture. The central point of the proof in [1] was a relationship between the transmission coefficients of the operators with fast decaying potentials and $L^2$- norm of their potentials. The $L^2$ - norm of a solution is a first integral of the KdV equation $V_t = 6VV_x - V_{xxx}$. The idea to use the high order first integrals:

$$I_k(V) = \int_0^\infty P_k(V, V', V'', ...,) d\,x \tag{5}$$

in a similar setting is sitting on the surface. Here $P_k$ are certain polynomials (see [4]). In particular, $P_1 = V^2$, $P_2 = (V')^2/2 + V^3$, $P_3 = \frac{1}{2}V_{xx}^2 - \frac{5}{2}V^2V_{xx} + \frac{5}{2}V^4$.

Our main goal here is an implementation of this idea and its development in several directions:

a) clarification of the proof by separation of the measure theory and KdV ideas;

b) simplification of the result (instead of very complicated expressions for $I_k, k = 1, 2, ..,$ we use very simple functionals $J_k, k = 1, 2, ..$);

c) construction of several classes of examples where the spectral bifurcation $\Sigma_{ac} \Rightarrow \Sigma_{pp}$ is related to the transition

$$J_k < \infty \quad \Rightarrow \quad J_k = \infty, \quad k = 1, 2, .. \quad .$$

The outline of the paper is the following. First we obtain a new test for the existence of the a.c. spectrum of the Schrödinger operator based on the boundedness of the transfer operator in average (with respect to the spectral parameter). Then a formula is given for the norm of the transfer operator in terms of the transmission coefficient. This provides the existence of the a.c. spectrum if there is a sequence $L_n \to \infty$ such that the transmission coefficients of the operators with potentials $V_n$ are bounded in average. Here $V_n$ is the restriction of $V$ to the segment $[0, L_n]$. In section 2, the transmission coefficients are estimated in terms of the first integrals of the KdV equation. Inequalities of the Kolmogorov type allow us to estimate the first integrals by the functionals $J_p$. A short section 3 contains the end of the proof of Theorem1. The last section provides examples showing the "exactness" of Theorem1.

In the spirit of general functional analysis one could expect that the first term of the integrand in condition (4) dominates and the second term can be omitted. However, the statement that $\int_0^\infty |V^{(p-1)}(x)|^2 d\,x < \infty$ for some $p \geq 2$ implies the existence of the a.c. spectrum on $[0, \infty)$, is obviously wrong. If, for instance , $p = 3$, $V_1$ is an infinitely smooth, compactly supported (or fast decaying) potential and

$$V(x) = V_0(x) + V_1(x), \quad V_0(x) = \alpha x, \quad \alpha > 0 \ ,$$

then $V'' \in L^2(R_+)$, but the spectrum of $H = -d^2/dx^2 + V(x)$ is discrete for any $\alpha \in [0, \pi)$. Using piecewise smooth (and mainly linear) functions $V_0(x)$ instead of $\alpha x$, one can construct examples when $V'' \in L^2(R_+)$, but the spectrum is a.c. above an arbitrary fixed level $h_0 = \sup V_0(x) > 0$, and is singular below this level.

We need some additional conditions on the potential $V(x)$ to guarantee that $\Sigma_{ac} = [0, \infty)$. We expect that the condition $|V(x)| \to 0$ together with $\int_0^\infty |V^{(p-1)}(x)|^2 d\,x < \infty$ for some $p \geq 1$ is sufficient for $\Sigma_{ac}(H) = [0, \infty)$ in many interesting cases . Even the more general statement is probably true:

If $||V||_\infty < \infty$ , $h_0 = \limsup_{x\to\infty} V(x)$ and $V^{(p)} \in L^2(R_+)$ for some $p \geq 1$, then the spectrum of $H = -d^2/dx^2 + V(x)$ is singular for $\lambda \leq h_0$ and contains the a.c. component which is essentially supported on $[h_0, \infty)$.

We shall return to the analysis of this conjecture in a different publication.

# 1 Tests for the a.c. spectrum

## 1.1 Spectral measure

Let $\psi = \psi_\lambda(x)$ be the solution of the following initial value problem on $R_+$ :

$$H\psi = \lambda\psi, \quad \lambda > 0, \quad \psi(0) = \sin\alpha, \quad \psi'(0) = \cos\alpha, \quad \alpha \in [0, \pi). \tag{6}$$

The spectral measure $\mu^\alpha(d\,\lambda)$ of the operator $H = H^\alpha$ can be defined in terms of the generalized Fourier transform and the Parseval equality [8]:

$$L^2(R_+) \ni \varphi(x) \to \hat{\varphi}(\lambda) = \int_0^\infty \hat{\varphi}(x)\psi_\lambda(x)d\,x, \tag{7}$$

$$\varphi(x) = \int_{\Sigma(H^\alpha)} \hat{\varphi}(\lambda)\psi_\lambda(x)\mu^\alpha(d\,\lambda), \tag{8}$$

$$(\varphi(x), \varphi(x))_{L^2(R_+)} = (\hat{\varphi}(\lambda), \hat{\varphi}(\lambda))_{L^2(\Sigma(H^\alpha),\mu^\alpha)}. \tag{9}$$

Here $\Sigma(H^\alpha)$ is the spectrum of $H^\alpha$. We denote by $T_\lambda(x_1, x_2)$ the transfer operator on the interval $[x_1, x_2]$:

$$T_\lambda(x_1, x_2) : (\psi_\lambda(x_1), \psi_\lambda'(x_1)) \to (\psi_\lambda(x_2), \psi_\lambda'(x_2)). \tag{10}$$

The following formula (attributed to M.Krein's school, see, for example, [10]) gives an expression for the spectral measure, and is very important for our analysis:

$$\mu^{\alpha}(d\,\lambda) = w - \lim_{L\to\infty} \mu^{\alpha}_{L}(d\,\lambda), \tag{11}$$

where

$$\mu^{\alpha}_{L}(d\,\lambda) = \frac{d\,\lambda}{\rho^2_{\lambda}(L)} \tag{12}$$

and

$$\rho^2_{\lambda}(x) = ||T_{\lambda}(0,x)e_{\alpha}||^2, \quad e_{\alpha} = (\sin\alpha, \cos\alpha). \tag{13}$$

## 1.2 The a.c. spectrum and the transfer operator

Let $\Delta \subset (0,\infty)$ be a closed interval. It is well-known that the spectrum of $H^{\alpha}$ is pure a.c. on $\Delta$ if the transfer operator $T_{\lambda}(0,L)$ is bounded uniformly in $L \in (0,\infty)$ and $\lambda \in \Delta$ (see [14], [15] and references there). A part of this statement, namely, the existence of the a.c. spectrum can be justified easily using (11) and (12). In fact, assume that

$$||T_{\lambda}(0,L_n)|| \le c_0, \quad \lambda \in \Delta \tag{14}$$

for some sequence $\{L_n\}, L_n \to \infty$, and a fixed interval $\Delta \subset (0,\infty)$, $|\Delta| > 0$. Then

$$\frac{\mu^{\alpha}_{L}(d\,\lambda)}{d\,\lambda} \ge \frac{1}{c_0^2}, \quad \lambda \in \Delta, \tag{15}$$

and therefore, the same estimate is valid for the limit measure $\mu^{\alpha}(d\,\lambda)$.

One of our central observations is that the uniform (14) or even a pointwise estimate can be replaced by boundedness in average.

**Theorem 2** . *Let $\Delta \subset (0,\infty)$ be a closed interval. If there is a sequence $\{L_n\}, \quad L_n \to \infty$, such that*

$$\int_{\Delta} \ln\left(||T_{\lambda}(0,L_n)||\right) d\,\lambda \le C(\Delta), \tag{16}$$

*then*

*i) for all $\alpha \in [0,\pi)$, the essential support of the measure $\mu^\alpha_{ac}(d\,\lambda)$ contains $\Delta$;*

*ii) for a.e. $\alpha \in [0,\pi)$, the spectral measure $\mu^\alpha(d\lambda)$ of the operator $H^\alpha$ is absolutely continuous on $\Delta$.*

**Remark.** Theorem 2 remains valid if logarithm function in (16) is replaced by any monotonically increasing on $[1,\infty)$ function $\Phi(\lambda)$.

In order to prove the theorem we need

**Lemma 1** . *Let $\nu_n(d\,\lambda), n = 1,2,...,$ be a sequence of measures on an interval $\Delta$, absolutely continuous with respect to $d\,\lambda$ with the densities*

$$\frac{\nu_n(d\,\lambda)}{d\,\lambda} = hI_{\Delta\setminus\Gamma_n}(\lambda)d\,\lambda, \quad n = 1,2,...$$

*Here $h$ is a constant, $I_B(\lambda)$ is the characteristic function of a Borel set $B$, and*

$$|\Gamma_n| \le \varepsilon.$$

*Then there is a subsequence $\nu_{n_l}(d\,\lambda)$ which converges weakly to a measure $\nu(d\,\lambda)$, and any such weak limit $\nu(d\,\lambda)$ has the following properties:*

*1) the measure $\nu(d\,\lambda)$ is absolutely continuous with respect to $d\,\lambda$, and $\nu(d\,\lambda)/d\,\lambda \le h$;*

*2) there exists a set $\Gamma$ such that $|\Gamma| \le 2\varepsilon$ and*

$$\frac{\nu(d\,\lambda)}{d\,\lambda} \ge \frac{h}{2} \quad on \quad \Delta\setminus\Gamma;$$

*Proof.* Let $F_n(\lambda) = \int_{-\infty}^{\lambda} \nu_n(d\,s)$ be the distribution function of the measure $\nu_n(d\,\lambda)$. By the second Helly theorem there exists a subsequence $\nu_{n_l}(d\,\lambda)$ which converges weakly to a measure $\nu(d\,\lambda)$, and for any such subsequence the distribution functions $F_{n_l}(\lambda)$ converge to the distribution function $F(\lambda)$ of the limit measure at all points $\lambda$ where $F(\lambda)$ is continuous. The set where $F$ is discontinuous is at most countable. Consider an interval $[\alpha,\beta) \subset \Delta$ with $\alpha,\beta$ outside of this set. Then

$$\nu_{n_l}(\alpha,\beta) = F_{n_l}(\beta) - F_{n_l}(\alpha) = \int_\alpha^\beta \nu_{n_l}(d\,\lambda) \le \int_\alpha^\beta hd\,\lambda = h(\beta-\alpha).$$

Passing to the limit $l \to \infty$, we get

$$\nu(\alpha, \beta) \leq h(\beta - \alpha).$$

This estimate implies that the measure $\nu(d\,\lambda)$ is absolutely continuous on $\Delta$ and $\nu(d\,\lambda)/d\,\lambda \leq h$;

To prove the second statement of the lemma we note that

$$\nu_{n_l}(\Delta) \geq h(|\Delta| - \varepsilon)$$

and therefore,

$$\nu(\Delta) \geq h(|\Delta| - \varepsilon). \tag{17}$$

On the other hand, if

$$\Gamma = \{\lambda : \frac{\nu(d\,\lambda)}{d\,\lambda} \leq \frac{h}{2}\}$$

then

$$\nu(\Delta) \leq \frac{h}{2}|\Gamma| + h(|\Delta| - |\Gamma|). \tag{18}$$

Inequalities (17) and (18) imply the second statement of Lemma.

This completes the proof of the Lemma.

*Proof of Theorem 2.* We shall prove Theorem 2 and the Remark simulteneously, i.e. we shall assume that the logarithm function in (16) is replaced by a positive monotonically increasing function $\Phi$. Fix $\varepsilon > 0$. Let $R_\varepsilon$ be the solution of the equation

$$\Phi(R_\varepsilon) = \frac{C(\Delta)}{\varepsilon}$$

where $C(\Delta)$ is defined in (16). Let

$$\Gamma_n(\varepsilon) = \{\lambda : \lambda \in \Delta, \quad ||T_\lambda(0, L_n)|| \geq R_\varepsilon\}.$$

Since $\Phi(t) > \frac{C(\Delta)}{\varepsilon}$ when $t > R_\varepsilon$, we have

$$\int_\Delta \Phi(||T_\lambda(0, L_n)||)d\,\lambda \geq \Phi(R_\varepsilon)|\Gamma_n| = \frac{C(\Delta)}{\varepsilon}|\Gamma_n|.$$

From here and (16), it immediately follows that $|\Gamma_n(\varepsilon)| \leq \varepsilon$.

Consider a sequence of measures $\nu_{n,\varepsilon}(d\lambda)$ on $\Delta$ defined by the formula

$$\nu_{n,\varepsilon}(d\lambda) = \frac{I_{\Delta\setminus\Gamma_n(\varepsilon)}(\lambda)}{R_\varepsilon} d\lambda.$$

Let

$$\mu_{L_n}(d\lambda) = \frac{d\lambda}{||T_\lambda(0, L_n)e_\alpha||^2}$$

be the measure defined in (12) with $L = L_n$. Then

$$\mu_{L_n}(d\lambda) \geq \nu_{n,\varepsilon}(d\lambda), \quad n = 1, 2, .., \quad \lambda \in \Delta. \tag{19}$$

Let $\nu_\varepsilon(d\lambda)$ be a weak limit of some subsequence $\nu_{n_l,\varepsilon}(d\lambda), \quad l \to \infty$. Its existence follows from Lemma 1. From (11) and (19) it follows that

$$\mu^\alpha(d\lambda) \geq \nu_\varepsilon(d\lambda), \quad \lambda \in \Delta.$$

Lemma 1 applied to the sequence $\nu_{n,\varepsilon}(d\lambda)$ leads to the estimate

$$\frac{\nu_\varepsilon(d\lambda)}{d\lambda} \geq \frac{I_{\Delta\setminus\Gamma_\varepsilon}}{R_\varepsilon}, \quad \lambda \in \Delta,$$

with some $\Gamma_\varepsilon$ such that $|\Gamma_\varepsilon| \leq 2\varepsilon$. Thus, the same estimate holds for $\mu^\alpha(d\lambda)$. Since $\varepsilon > 0$ is an arbitrary small number, the essential support of $\mu^\alpha_{ac}(d\lambda)$ contains the interval $\Delta$.

The part i) of Theorem 2 is proved.

Now we shall prove *part ii) of Theorem 2.* Let us consider first a specific Hamiltonian $H^{\pi/2}$ (Neumann boundary conditions) and let $\mu^{\pi/2}(d\lambda)$ be its spectral measure. Under Birman's condition (3) on the potential,

$$\int_{\lambda_0}^{\infty} \frac{\mu^{\pi/2}(d\lambda)}{1 + |\lambda|} < \infty \tag{20}$$

where $\lambda_0$ is the bottom of the spectrum of $H^{\pi/2}$. This estimates can be found in [8], (Chapter 4) for much more general potentials and for boundary condition (2) with $\alpha \neq 0$. Estimate (20) fails if $\alpha = 0$. One could also prove a stronger estimate for $\mu^\alpha(d\lambda)$:

$$\int_{\lambda_0}^{\lambda} \mu^\alpha(d\lambda) \leq C_\alpha(1 + \sqrt{|\lambda|}), \quad \lambda \geq \lambda_0(\alpha), \alpha \neq 0. \tag{21}$$

We shall not prove (21) here, since (20) is enough for our purpose.

Estimate (20) allows us to define the Borel transform $m^{\pi/2}(z)$ of the spectral measure $\mu^{\pi/2}(d\,\lambda)$:

$$m^{\pi/2}(z) = \int_{\lambda_0}^{\infty} \frac{\mu^{\pi/2}(d\,\lambda)}{z-\lambda}, \quad \Im z > 0.$$

Function $m^{\pi/2}(z)$ is analytic in the half- plane $\Im z > 0$. Due to the general theory of Borel transforms (see [13], Chapter I), $m^{\pi/2}(z)$ has the following properties:

a) for a.e. $\quad \lambda \in R$,

$$0 \leq \Im m^{\pi/2}(\lambda + i0) < \infty; \quad |\Re m^{\pi/2}(\lambda + i0)| < \infty. \tag{22}$$

b) the absolutely continuous component of $\mu^{\pi/2}(d\,\lambda)$ is equal to

$$\pi^{-1}\Im m^{\pi/2}(\lambda + i0)d\,\lambda;$$

c) the singular component of the measure $\mu^{\pi/2}(d\,\lambda)$ is supported on the set $\{\lambda : \Im m^{\pi/2}(\lambda + i0) = \infty\}$.

Consider the set

$$\Gamma = \{\lambda : 0 < \Im m^{\pi/2}(\lambda + i0) < \infty\}, \tag{23}$$

and let

$$\Omega = [0, \infty) \setminus \Gamma.$$

Part i) of Theorem 2 and the properties a) - c) above imply that $|\Omega| = 0$ and the singular component of $\mu^{\pi/2}(d\,\lambda)$ is supported on $\Omega$. We shall show that the singular components of all measures $\mu^{\alpha}(d\,\lambda)$, $\alpha \in (0,\pi)$, are supported on the same set $\Omega$. Recall (see [8]) that Weyl function $m^{\alpha}(z)$ of the operator $H^{\alpha}$ is defined by the relation

$$\psi_1(x,z) + m^{\alpha}(z)\psi_2(x,z) \in L^2[0,\infty), \quad \Im z > 0, \tag{24}$$

where $\psi_1(x,z)$ and $\psi_2(x,z)$ are solutions of the equation $-\psi'' + V(x)\psi = z\psi$ with initial conditions $\psi_1(0) = \sin\alpha$, $\psi_1'(0) = \cos\alpha$, and $\psi_2(0) = -\cos\alpha$, $\psi_2'(0) = \sin\alpha$. The following representation is valid (see [8]):

$$m^{\alpha}(z) = \cot\alpha + \int_{\Sigma(H^{\alpha})} \frac{\mu^{\alpha}(d\,\lambda)}{z-\lambda}, \quad \Im z > 0, \quad \alpha \in (0,\pi).$$

The properties a) - c) of the Borel transform of a measure are valid for $m^\alpha(z)$. Direct calculations based on (24) give

$$m^\alpha(z) = \frac{\cos\alpha + m^{\pi/2}(z)\sin\alpha}{\sin\alpha - m^{\pi/2}(z)\cos\alpha}. \tag{25}$$

Using (25), it is easy to check that

$$0 < \Im m^\alpha(\lambda + i0) < \infty$$

at points $\lambda$ where $0 < \Im m^{\pi/2}(\lambda + i0) < \infty$. Hence, $\mu^\alpha_{sing}$ is supported on $\alpha-$ independent set $\Omega$ of Lebesgue measure zero.

The spectral averaging of $\mu^\alpha(d\,\lambda)$ over $\alpha \in (0, \pi)$ gives (see [10] or [13])

$$\frac{1}{\pi}\int_0^\pi \mu^\alpha(d\,\lambda) = d\,\lambda.$$

This formula applied to the set $\Omega$ leads to

$$\frac{1}{\pi}\int_0^\pi \mu^\alpha(\Omega) d\,\alpha = |\Omega| = 0.$$

Then $\mu^\alpha(\Omega) = 0$ for a.e. $\alpha$ , i.e. $\mu^\alpha_{sing}(d\lambda) = 0$ for a.e. $\alpha$.

Theorem 2 is proved.

### 1.3 The a.c. spectrum and the transmission coefficient

First of all we need a simple proposition which provides a connection between the transfer matrix and the scattering theory. In this section, it is convenient to replace spectral parameter $\lambda > 0$ by $k^2$. Let $\psi_k = \psi_{k^2} = \psi_\lambda$, and let a new transfer operator $P_k(x_1, x_2)$, $\quad k = \sqrt{\lambda}$, (Pruffer transformation) be given by

$$P_k(x_1, x_2) : \left(\psi_k(x_1), \frac{\psi_k'(x_1)}{k}\right) \to \left(\psi_k(x_2), \frac{\psi_k'(x_2)}{k}\right). \tag{26}$$

It follows from the definition of $P_k$ that, for $\lambda = k^2$ and $D = \mathrm{diag}(1, k)$,

$$T_\lambda = D^{-1}(k) P_k D(k). \tag{27}$$

**Lemma 2** . *Let us consider a compactly supported on* $(0, L)$ *continuous potential* $V(x)$*, and let* $\psi(x,k)$ *be the Jost solution of the equation* $\psi''(x,k) + V(x)\psi(x,k) = k^2\psi(x,k)$*,* $x \in R$*, such that*

$$\psi(x,k) = \begin{cases} \exp(-ikx), & x \le 0, \\ a(k)\exp(-ikx) + b(k)\exp(ikx), & x \ge L. \end{cases}$$

*Then the norm of the transfer operator* $P_k$ *is given by*

$$||P_k(0,L)|| = |a(k)| + |b(k)|. \tag{28}$$

**Remark.** Everywhere below, we shall refer to $a(k)$ as the transmission coefficient, although this term is more often used for $1/a(k)$, which is the corresponding coefficient in the asymptotic expansion of the solution $\psi/a$ of the scattering problem.

*Proof.* Denote $a(k) = a_1 + ia_2$; $\quad b(k) = b_1 + ib_2$. Vector $(\Re\psi(x,k), \Im\psi(x,k))$ is equal to $(\sin kx, \cos kx)$ if $x \le 0$, and it is equal to

$$((a_1+b_1)\cos kx + (a_2-b_2)\sin kx, (a_2+b_2)\cos kx + (b_1-a_1)\sin kx)$$

if $x \ge L$. Hence, the matrix of the transfer operator $P_k(0,x), x \ge L$ is equal to

$$\left[ \begin{array}{cc} (a_1+b_1)\cos kx + (a_2-b_2)\sin kx & (a_2+b_2)\cos kx + (b_1-a_1)\sin kx \\ (a_2-b_2)\cos kx - (a_1+b_1)\sin kx & (b_1-a_1)\cos kx + (a_2-b_2)\sin kx \end{array} \right]. \tag{29}$$

Then

$$\mathrm{Tr}(P_k(0,x)P_k^*(0,x)) = 2(|a|^2 + |b|^2). \tag{30}$$

The norm $||P_k||$ is equal to $\sqrt{\mu}$ where $\mu$ is the larger eigenvalue of the matrix $P_kP_k^*$. Since $\det P_k = 1$, the characteristic equation for $\mu$ is

$$\mu^2 - 2(|a|^2 + |b|^2)\mu + 1 = 0. \tag{31}$$

The identity $|a|^2 = 1 + |b|^2$ holds for coefficients $a$ and $b$; and therefore, the roots of (31) are $\mu_{1,2} = (|a| \pm |b|)^2$. Then

$$||P_k(0,x)|| = \sqrt{\mu_1} = |a| + |b|.$$

Lemma is proved.

The identity $|a|^2 = 1 + |b|^2$ and formulas (28), (27) allow us to reformulate Theorem 2 as follows. Define by $I_{[a,b]}(x)$ the characteristic function of the interval $[a,b]$.

**Theorem 3** . *Let $\Delta \subset (0,\infty)$ be a closed interval. Assume that there exists a sequence $L_n \to \infty$ such that the transmission coefficients $a_n(k)$ of the operators*

$$H = -d^2/dx^2 + V_n(x), \quad \textit{where} \quad V_n(x) = V(x)I_{[0,L_n]}(x),$$

*satisfy the condition*

$$\int_\Delta \ln|a_n(k)|d\,k \le C(\Delta), \quad n = 1, 2, \dots \tag{32}$$

*Then*

*i) for all $\alpha \in [0,\pi)$, the essential support of the measure $\mu^\alpha_{ac}(d\,\lambda)$ contains $\Delta$;*

*ii) for a.e. $\alpha \in [0,\pi)$, the spectral measure $\mu^\alpha(d\lambda)$ of the operator $H^\alpha$ is absolutely continuous on $\Delta$.*

# 2 The estimates for the first integrals of KdV equation.

Our next goal is to further simplify Theorem 2 by rewriting the estimate (32) in terms of the potential $V$ and its derivatives. The central point here is the estimation of $\int_\Delta \ln|a(k)|d\,k$ for compactly supported or fast decaying potentials.

Let $V(t,x)$ be a solution of the Cauchy problem for the KdV- equation

$$V_t = 6VV_x - V_{xxx}, \quad V(0,x) = V(x),$$

in the class of the fast decaying functions. Let $\lambda_j = -(\kappa_j)^2, j = 1, 2, .., q$, be negative eigenvalues and let $a(k)$ be the transmission coefficient for the Hamiltonian $H = -d^2/dx^2 + V(t,x)$. Then (as was discovered by Gardner, Green, Kruskal and Miura, see details in [5] or [7]) the scattering data $\{\lambda_j, j = 1, 2, ...q, \quad a(k), k \in R\}$ do not depend on $t$. They are invariants of the KdV - dynamics.

It is also well known that the function $a(k)$ admits analytic continuation in the half plane $\Im z > 0$, and $i \ln a(z)$ has the asymptotic expansion

$$i \ln a(z) \sim -\sum_{m=0}^{\infty} \frac{I_m(u)}{z^{2m+1}}, \quad \Im z > 0, \quad z \to \infty. \tag{33}$$

Coefficients $I_m$ are invariants of the KdV equation also. They have two different representations. First, they can be written in the form

$$I_m = (-1)^{m+1}\sum_{j=0}^{q}\frac{2\kappa_j^{2m+1}}{2m+1} + \frac{1}{\pi}\int_{-\infty}^{+\infty} k^{2m}\ln|a(k)|)d\,k, \quad m = 0, 1, 2, ... \quad (34)$$

This representation can be found in [7] and can be proved in the following simple way.

Let $B_q(z)$ be the Blashke product with the same zeros $\{i\kappa_k\}_{k=1}^q$ as the zeros of the function $a(z)$ in the upper half plane:

$$B_q(z) = \prod_{j=1}^{q}\frac{z - i\kappa_j}{z + i\kappa_j}. \quad (35)$$

For real $z$, we have $|B(z)| = 1$. Hence,

$$\Im[i\ln\frac{a(z)}{B_q(z)}] = \ln|a(z)|, \quad z \in R. \quad (36)$$

Using the Gerglotz formula, we get

$$i\ln a(z) = i\ln B_q(z) + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\ln|a(k)|}{k - z}d\,k. \quad (37)$$

It is clear that

$$i\ln\frac{z - i\kappa}{z + i\kappa} \sim \sum_{m=0}^{\infty}\frac{2\kappa^{2m+1}}{2m+1}\frac{(-1)^m}{z^{2m+1}}, \quad z \to \infty. \quad (38)$$

Using (37) and (38), we get (34).

Denote by $r = (r_0, r_1, ...., r_{2m})$ a vector with nonnegative integer coordinates, and let

$$d(r) = 2r_0 + 3r_1 + \cdots + (2m+2)r_{2m} \quad . \quad (39)$$

The second representation for $I_m$ is given by

**Lemma 3** . *The following formulas hold:*

$$I_m = \int_{-\infty}^{\infty} P_m(V, V', ...)d\,x, \quad n = 0, 1, 2, 3, .., \quad (40)$$

*where $P_m$ are the polynomials in the function $V$ and its derivatives:*

$$P_m(V, V', ...) = \sum_{\{r:\, d(r)=2m+2\}} c_m(r) \prod_{l=0}^{2m} [V(x)^{(l)}]^{r_l}. \tag{41}$$

**Remark 1.** The coefficients $c_m(r)$ can be calculated by the formula

$$c_m(r) = \mathbf{E} \prod_{l=0}^{m} \frac{\xi_l^{r_l}}{r_l!}.$$

Here

$$\xi_l(x) = \int_0^1 \omega^l(\tau) d\tau \tag{42}$$

and $w(\tau)$ is the "Wiener bridge", i.e. conditional Wiener process determined by the condition $w(0) = w(1) = 0$ with zero means and the correlation function $B(s,t) = (s \wedge t)(1 - s \vee t)$.

**Remark 2.** The structure (41) of the polynomials $P_m$ is the consequence of the invariance of KdV equation under scaling $V(t,x) \to \varepsilon^2 V(\varepsilon^3 t, \varepsilon x)$. This invariance implies that the polynomial $P_m(V, V', ...)$ must be homogeneous of order $2m+2$ under the transformation $V(x) \to \varepsilon^2 V(\varepsilon x)$. Specific values of the coefficients $c_m(r)$ in the "Minakshisundaram type" expansion (41) are not important for our analysis. However, the estimates for $c_m(r)$, $m \to \infty$, can be fundamental for some applications.

Proof of this lemma uses the formula ([7], p. 169):

$$\mathrm{tr}[(H - z^2 I)^{-1} - (H_0 - z^2 I)^{-1}] = -\frac{1}{2z} \frac{d}{dz} \ln a(z) \tag{43}$$

and the Kac-Feynman formula (for complete proof see [20]).

The following inequalities are crucial.

**Theorem 4** . *The functional $I_m$ has the estimates:*

*a)*

$$I_m \le c_m \int_0^\infty (|V^{(m-1)}(x)|^2 + |V(x)|^{m+1}) dx, \qquad m = 1, 2, .., \tag{44}$$

*b)*

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} k^{4m+2}\ln|a(k)|d\,k \le I_{2m+1}, \qquad m = 0,1,,2,.. \quad , \tag{45}$$

*c)*

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} k^{4m}\ln|a(k)|d\,k \le I_{2m} + d_m\int_0^{\infty}|V(x)|^{2m+1}d\,x, \quad m = 1,,2,\dots \tag{46}$$

*Here $c_m$ and $d_m$ are positive constants which do not depend on $V$.*

**Corollary 1** . *For any $m = 1,2,..,$ and any closed interval $\Delta \in (0,\infty)$, $|\Delta| < \infty$,*

$$\int_{\Delta}\ln|a(k)|d\,k \le c_{m,\Delta}\int_0^{\infty}(|V^{(m-1)}(x)|^2 + |V(x)|^{m+1})d\,x, \quad m = 1,2,\dots \quad . \tag{47}$$

*Proof.* Inequalities b) and c) follow from representation (34). If $m$ is odd, then both terms in (34) are nonnegative which implies b). If $m$ is even, we can use the one - dimensional version of the Lieb-Thirring estimate (see [17] or [19], p. 368): for any number $\gamma > \frac{1}{2}$

$$\sum_{j=1}^{q}|\lambda_j|^{\gamma} \le C_{\gamma}\int_{-\infty}^{+\infty}|V_{-}(x)|^{\frac{1}{2}+\gamma}d\,x. \tag{48}$$

Here $V_{-}(x)$ is the negative part of the potential $V(x)$. This estimate with $\gamma = 2m + \frac{1}{2}$ together with (34) implies c).

To prove a) we use the second representation for $I_m$. Let us denote by $||\cdot||_s$ the norm in the space $L^s(R)$ and by $A_m(r)$ the integral

$$A_m(r) = \int_{-\infty}^{\infty}\prod_{k=0}^{2m}[V^{(k)}(x)]^{r_k}d\,x, \quad r = (r_0, r_1, ...., r_{2m}). \tag{49}$$

¿From (40 ) and (41 ) it follows that the estimate

$$|A_m(r)| \le C(||V||_{m+1}^{m+1} + ||V^{(m-1)}||_2^2), \quad C = C(r,m), \tag{50}$$

implies a). It is sufficient to prove (50) for integrals $A_m(r)$ with an additional condition

$$r_m = r_{m+1} = ... = r_{2m} = 0, \tag{51}$$

since an arbitrary $A_m(r)$ can be reduced via integration by parts to a linear combination of similar integrals for which $d(r)$ is the same and condition (51) holds. Applying generalized Holder inequality

$$|\int_{-\infty}^{\infty} \prod_{k=0}^{m-1} f_k(x)dx| \leq \prod_{k=0}^{m-1} \left(\int_{-\infty}^{\infty} |f_k(x)|^{\lambda_k} dx\right)^{1/\lambda_k}, \quad \sum_{k=0}^{m-1} 1/\lambda_k = 1, \tag{52}$$

with $\lambda_k = (2m+2)/(k+2)r_k$ to functions $f_k = [V^{(k)}(x)]^{r_k}$ we get

$$|A_m(r)| \leq \prod_{k=0}^{m-1} ||V^{(k)}||_{\frac{2m+2}{k+2}}^{r_k}. \tag{53}$$

We also need a Kolmogorov type inequality ( see, for example, [16] ):

$$||f^{(k)}||_q \leq C(q,p,r,n,k)||f||_p{}^{1-\alpha}||f^{(n)}||_r{}^{\alpha} \tag{54}$$

which is valid if

$$\frac{1}{q} = \frac{n-k}{np} + \frac{k}{nr}, \qquad \alpha = \frac{k - 1/q + 1/p}{n - 1/r + 1/p}, \qquad 0 \leq k \leq n. \tag{55}$$

Parameters $q = (2m+2)/(k+2)$, $p = m+1$, $r = 2$, $n = m-1$ satisfy the first condition of (55). With these values of the parameters, $\alpha = k/(m-1)$, and (55) implies

$$||V^{(k)}||_{(2m+2)/(k+2)} \leq C(m,k)||V||_{m+1}^{\frac{m-k-1}{m-1}}||V^{(m-1)}||_2{}^{\frac{k}{m-1}}. \tag{56}$$

If

$$\kappa = r_0 + r_1 + .... + r_{m-1}, \tag{57}$$

then (53) and (56) imply

$$|A_m(r)| \leq C\left(||V||_{m+1}^{m+1}\right)^{\frac{\kappa-2}{m-1}}\left(||V^{(m-1)}||_2^2\right)^{\frac{m-\kappa+1}{m-1}}. \tag{58}$$

Holder inequality $ab \leq a^p/p + a^q/q$ with parameters $p = (m-1)/(\kappa-2)$ and $q = (m-1)/(m-\kappa+1)$ applied to (58) gives (50).

Theorem 4 is proved.

# 3 Theorem 1 and its generalization

In order to prove theorem 1 we need

**Lemma 4** . *If $V(x) \in H^{p-1}(R_+) \cap L^{p+1}(R_+)$ then it can be represented in the form*

$$V = V_B + V_R \tag{59}$$

*where :*

*a) a "bump" potential $V_B$ has the form*

$$V_B = \sum_{k=0}^{\infty} v_k. \tag{60}$$

*Here potentials $v_k, k \geq 1$, have compact supports without intersection , and the support of $v_{k+1}$ is located to the right of the support of $v_k$;*

*b) $V_B$ continued by zero for $x \leq 0$ belongs to $H^{p-1}(R) \cap L^{p+1}(R)$;*

*c) the remainder potential $V_R$ is "small" in the following sense:*

$$\int_0^\infty |V_R(x)| d\,x < \infty. \tag{61}$$

*Proof.* The potential $V \in L^{p+1}$ and, as a result, satisfies the Birman condition (3). Hence, there exists a sequence $\{L_m\}_0^\infty$ such that $L_0 = 0$, $L_{m+1} - L_m \geq 2$ and

$$\int_{L_m-1}^{L_m+1} |V(x)| d\,x < \frac{1}{2^m}; \quad \int_{L_m-1}^{L_m+1} (|V(x)|^{p+1} + |V^{(p-1)}|^2) d\,x < \frac{1}{2^m}, \quad m \geq 1. \tag{62}$$

Consider the following partition of unity $1 = \sum_{k=0}^\infty \varphi_k$ on $R$. Let $\varphi \in C_0^\infty(R)$, $|\varphi| \leq 1$, $\varphi = 1$ in a neighborhood of $x = 0$, and $\varphi = 0$ when $|x| > 1$. Then $\varphi_{2k}(x) = \varphi(x - L_k)$, $\varphi_{2k+1}(x) = 1 - \varphi_{2k}(x) - \varphi_{2k+2}(x)$ if $x \in [L_k, L_{k+1}]$, and $\varphi_{2k+1}(x) = 0$ if $x \notin [L_k, L_{k+1}]$. Obviously $||\varphi_k^{(n)}(x)||_\infty < C$, uniformly in $k = 1, 2, ..$ and $n = 0, 1, ..., p$. Define

$$v_k = V\varphi_{2k+1}, \quad V_R(x) = V(x) \sum_{k=0}^\infty \varphi_{2k}(x).$$

Then the statement a) is obvious. According to (62), the potential $V_R(x)$ belongs to $L^1(0,\infty)$. It remains to prove that

$$||V_B||_{p+1}^{p+1} + ||V_B^{(p-1)}||_2^2 < \infty.$$

Since $V \in H^{p-1}(R_+) \cap L^{p+1}(R_+)$, it is sufficient to show that

$$\int_0^1 |V^{(k)}| d\,x \le C, \quad \int_{L_m-1}^{L_m+1} |V^{(k)}|^2 d\,x < \frac{C}{2^m}, \quad m \ge 1 \tag{63}$$

when $0 \le k \le p-1$. Let $S = S_m, m \ge 0$, be the interval of integration in (63), and let $||\cdot||_{l,S}$ be the norm in $L^l(S)$. Then (63) follows from (62) and the estimate (see, for example [16])

$$||V^{(k)}||_{2,S} \le C_1||V||_{2,S} + C_2||V^{(p-1)}||_{2,S}, \quad 0 \le k \le p-1. \tag{64}$$

Lemma 4 is proved.

*Proof of Theorem 1.* We need to prove only the first statement of the theorem, since it implies the second one (see the proof of Theorem 2).

Since the essential support of the spectral measure is invariant under $L^1$- perturbations of the potentials (see [6]), it is enough to prove the first statement of the theorem only for $V = V_B$.

Let $L_k$ be an arbitrary point in the gap between supports of the potentials $v_k$ and $v_{k+1}$ and let

$$V_m = \sum_{k=1}^m v_k, \quad x > 0; \quad V_m = 0, \quad x < 0.$$

Then

$$J_p(V_m) \le J_p(V) = C < \infty. \tag{65}$$

According to Theorem 3, it is sufficient to prove that for any closed interval $\Delta \subset (0,\infty)$ there is a constant $C(\Delta)$ such that

$$\int_\Delta \ln(|a_m(k)|) d\,k \le C(\Delta) \tag{66}$$

for any $m = 1, 2, \ldots$ Here $a_m(k)$ is the transmission coefficient associated with operator $H = -d^2/dx^2 + V_m(x)$. Inequality (66) follows immediately from (65), (47), and Theorem 3.

Theorem 1 is proved.

The following statement can be proved similarly to Theorem 1.

**Theorem 5** . *Let*

$$V = V_B + V_R, \tag{67}$$

*where*

$$V_B = \sum_{k=1}^{\infty} v_k, \quad v_k \in H^{p_k - 1}(R_+) \cap L^{p_k+1}(R_+), \qquad V_R \in L^1(R_+), \tag{68}$$

*and the supports of* $v_k$ *do not intersect each other. Let* $p_k \leq p$, $k = 1, 2, ..$, *and*

$$\sum_{k=1}^{\infty} J_{p_k}(v_k) < \infty. \tag{69}$$

*Then:*

*1) for all* $\alpha \in [0, \pi)$, *the essential support of the absolutely continuous component of the spectral measure* $\mu^\alpha(d\lambda)$ *of the operator* $H^\alpha$ *coincides with* $[0, \infty)$, *i.e.* $\mu^\alpha_{ac}(\Gamma) > 0$ *for any Borel set* $\Gamma \subset [0, \infty)$, $|\Gamma| > 0$.

*2) for a.e.* $\alpha \in [0, \pi)$, *the spectral measure* $\mu^\alpha(d\lambda)$ *of the operator* $H^\alpha$ *is absolutely continuous on* $[0, \infty)$.

# 4 The spectral bifurcation. Examples.

The examples below show the "exactness" of the statement of Theorem 1. We shall construct a family of potentials $V = V(x, \varepsilon; m)$, $m$ is integer, $m \geq 2$, with the following properties:

a) for each $m$, the potential $V$ depends continuously on parameter $\varepsilon$ (in the uniform norm);

b) $J_p(V) = \infty$ for all $\varepsilon$ if $p \neq m$, and therefore neither Theorem 1 with $p \neq m$, nor the results of [1] are applicable to such potentials;

c) if $\varepsilon < 0$ then $J_m(V) < \infty$, and Theorem 1 with $p = m$ can be applied. Hence, the spectrum of $H^\alpha$ on semiaxis $\lambda \geq 0$ is a.c. for a.e. $\alpha$. If $\varepsilon \geq 0$ then $J_m(V) = \infty$, and the spectrum of $H^\alpha$ is singular for a.e. $\alpha$.

In order to make the idea of the examples transparent we shall discuss in detail the case $m = 2$. Then, we shall breifly discribe the changes which are needed to construct $V$ for arbitrary $m$.

Let us consider a sparse potential $V$ of the form

$$V = \sum_{n=1}^{\infty} h_n(\varepsilon) v_n(x - x_n), \quad x \in R, \tag{70}$$

where "bumps" $v_n(x)$ have compact supports and $x_n \to \infty$ as $n \to \infty$. We assume that the functions $v_n = v_n(x)$ have the following properties: $v_n \in C^\infty(R)$, $v_n$ are even (for simplicity), $|v_n(x)| \leq 1$, $v_n(x) = 1$ when $|x| \leq L_n$, and $v_n(x) = 0$ when $|x| \geq L_n + 1$. Let the behavior of all the functions $v_n(x)$ be the same when $L_n \geq |x| \geq L_n+1$, i.e. $\omega(x) = v_n(L_n+\frac{1}{2}+x)$ does not depend on $n$ when $|x| \leq \frac{1}{2}$ . Moreover, we assume that $\omega'(x)$ is even when $|x| \leq \frac{1}{2}$.

We choose

$$L_n = n^\beta, \quad h_n = n^{-1/2+\varepsilon}, \quad |\varepsilon| < \varepsilon_0, \quad x_n = \exp(cn \ln n), \quad c > 1 \tag{71}$$

with arbitrary $\beta \in (0, 1/4)$ and $\varepsilon_0 = \min(\frac{1-4\beta}{6}, \frac{\beta}{2})$.

**Theorem 6** . *1) Potential $V$ with $v_n$ described above has the following properties:*

$$\int_0^\infty |V(x)|^2 dx = \infty,$$

$$\int_0^\infty (|V^{(p-1)}(x)|^2 + |V(x)|^{p+1}) dx < \infty \quad \textit{if} \quad \varepsilon < 0, \quad p > 1, \tag{72}$$

$$\int_0^\infty (|V^{(p-1)}(x)|^2 + |V(x)|^{p+1}) dx = \infty \quad \textit{if} \quad 0 \leq \varepsilon < \varepsilon_0, \quad p > 1. \tag{73}$$

*2) For a.e. $\alpha$, the spectrum of the operator $H^\alpha$ with the potential $V$ is pure singular if $\varepsilon \geq 0$ and is a.c. on the semiaxis $\lambda > 0$ if $\varepsilon < 0$.*

**Remark.** Theorem 1 with any $p > 1$ can be applied to potential (70) if $\varepsilon < 0$. Hence, this potential provides an example when Theorem 1 garantees the existence of the a.c. spectrum in spite of the fact that $V \notin L^2$. Then, $V(x, \varepsilon, m)$ with $m = 2$ can be realized as

$$\tilde{V} = V + V_1$$

where $V$ is given by (70), $V_1$ is a compactly supported function such that $|V_1'| < C$, $V_1'' \notin L^2$. Obviously, $J_p(\tilde{V}) < \infty$ only if $p = 2$, $\varepsilon < 0$. On the other hand, the a.c. spectrum of operators $H^\alpha$ with potentials $V$ and $\tilde{V}$ coincide, since the perturbation by $V_1$ is of the trace class.

*Proof.* First of all note that

$$\int_{-\infty}^{\infty} |V(x)|^2 dx \geq \sum_{n=1}^{\infty} h_n^2 L_n = \infty.$$

Further,

$$\int_{-\infty}^{\infty} |V(x)|^{p+1} dx = 2 \sum_{n=1}^{\infty} h_n^{p+1}(L_n + a), \quad \text{where} \quad a = \int_0^1 |V(L_n + x)|^{p+1} dx. \tag{74}$$

Thus , if $p > 1$, then

$$\int_{-\infty}^{\infty} |V(x)|^{p+1} dx < \infty \tag{75}$$

Similarly, if $p > 1$, then

$$\int_{-\infty}^{\infty} |V^{(p-1)}(x)|^2 dx = b \sum_{n=1}^{\infty} h_n^2, \quad \text{where} \quad b = 2 \int_0^1 |V^{(p-1)}(L_n + x)|^2 dx. \tag{76}$$

Together with (71) and (75) this implies (72) and (73).

From (72) and Theorem 1 it follows that the spectrum of $H^\alpha$ on the semiaxis $\lambda > 0$ is a.c. for a.e. $\alpha$ if $\varepsilon < 0$. Since $V$ vanishes at infinity, the spectrum of $H^\alpha$ on the semiaxis $\lambda > 0$ is discrete. It only remains to show that $\Sigma(H^\alpha)$ on the semiaxis $\lambda > 0$ is pure singular for a.e. $\alpha$ if $0 \leq \varepsilon < \varepsilon_0$.

Let $\psi$ be the solution of the problem

$$-\psi''(x) + h_n v_n \psi(x) = k^2 \psi(x), \quad x \in R, \quad \psi(x) = e^{-ikx}, \quad x < -L_n - 1. \tag{77}$$

Then

$$\psi(x) = a_n e^{-ikx} + b_n e^{+ikx}, \quad x > L_n + 1. \tag{78}$$

The function $\psi(x)/a_n$ describes the scattering of the plane wave, coming from $x=\infty$, on the "one bump" potential $v_n$ . The spectrum of one-dimensional Schrödinger operators with sparse potentials was studied in [11], [12], where it was proved that under an assumption on $x_n$

$$\sum_{n=1}^{\infty}\frac{x_n}{x_{n+1}}<\infty \tag{79}$$

(which is called super lacunarity ), the spectrum of $H^\alpha$ with a sparse potential is singular on any interval $\Delta\in(0,\infty)$ for a.e. $\alpha$ if the series

$$\sum_{n=1}^{\infty}|b_n|^2 \tag{80}$$

diverges for almost all $k^2\in\Delta$ ( and the spectrum is a.c. if the series converges for $k^2\in\Delta$). Thus, in order to complete the proof of Theorem 6, it is only left to show the divergence of the series (80) when $\varepsilon>0$.

We fix a closed interval $\Delta\in(0,\infty)$. We will show that

$$b_n(k)=h_n\frac{\alpha(k)}{k}\sin(2k(L_n+1/2))+O(h_n^2L_n^2),\quad n\to\infty, \tag{81}$$

where $\alpha(k)$ is an entire function and the remainder decays uniformly in $k^2\in\Delta$, i.e.

$$|O(h_n^2L_n^2)|\leq C(\Delta)h_n^2L_n^2,\quad n=1,2,3,\ldots \tag{82}$$

Here $C(\Delta)$ does not depend on $n$ and $k^2\in\Delta$. Since $\sum_{n=1}^{\infty}h_n^3L_n^2<\infty$, $\sum_{n=1}^{\infty}h_n^4L_n^4<\infty$, and $\alpha(k)=0$ only at a discrete set of points, the divergence of (80) when $0\leq\varepsilon<\varepsilon_0$ follows from the divergence of the series

$$\sum_{n=1}^{\infty}h_n^2|\sin(2k(L_n+1/2))|^2=\sum_{n=1}^{\infty}h_n^2[1-\cos(4k(L_n+1/2))],\quad 0\leq\varepsilon<\varepsilon_0. \tag{83}$$

The divergence of the last series is obvious, since $\sum_{n=1}^{\infty}h_n^2=\infty$ if $\varepsilon>0$ and

$$\sum_{n=1}^{\infty}n^{-1+2\varepsilon}\cos(4k(n^\beta+1/2))<\infty,\quad 2\varepsilon<\beta,\quad k\neq0. \tag{84}$$

Finally, it is only left to justify (81).

The solution $\psi$ of (77) satisfies the integral equation

$$\psi(x)=e^{-ikx}+\int_{-L_n-1}^{x}\frac{\sin k(x-\xi)}{k}h_n v_n(\xi)\psi(\xi)d\xi. \tag{85}$$

The norm of the integral operator in the right-hand side of (85) in the space $C(R)$ does not exceed

$$h_n(L_n+2)/|k|\le C_1(\Delta)h_nL_n.$$

It is less than 1 if $n$ is large enough. Thus,

$$\psi(x)=e^{-ikx}+\int_{-L_n-1}^{x}\frac{\sin k(x-\xi)}{k}h_n v_n(\xi)e^{-ik\xi}d\xi+O(h_n^2L_n^2), \tag{86}$$

and therefore,

$$b_n(k)=\frac{h_n}{2k}\int_{-\infty}^{\infty}v_n(\xi)e^{-2ik\xi}d\xi+O(h_n^2L_n^2),\quad k^2\in\Delta,\quad n\to\infty. \tag{87}$$

Since the function $v_n'$ is supported on the set $\{\xi : L_n\le|\xi|\le L_n+1\}$ and is an odd function, after integration by parts, (87) takes the form

$$b_n(k)=\frac{-ih_n}{(2k)^2}\int_{-\infty}^{\infty}v_n'(\xi)e^{-2ik\xi}d\xi+O(h_n^2L_n^2)=$$

$$-\frac{h_n}{2k^2}\int_{L_n}^{L_n+1}v_n'(\xi)\sin 2ik\xi d\xi+O(h_n^2L_n^2)=$$

$$-\frac{h_n}{2k^2}\int_{-1/2}^{1/2}\omega'(\xi)\sin[2k(\xi+L_n+\frac{1}{2})]d\xi+O(h_n^2L_n^2)=$$

$$-\frac{h_n}{2k^2}\int_{-1/2}^{1/2}\omega'(\xi)\cos 2k\xi d\xi\sin 2k(L_n+\frac{1}{2})+O(h_n^2L_n^2). \tag{88}$$

The last equality is the consequence of the fact that $\omega'(\xi)$ is even on the interval $|\xi|<1/2$. Equality (88) implies (81).

The proof of Theorem 6 is complete.

In order to construct potentials $V(x, \varepsilon; m)$ for $m > 2$ one can consider first the family of potentials:

$$W = \sum_{n=1}^{\infty} h_n(\varepsilon) v_n(x - x_n) \varphi(x - x_n, n), \quad x \in R,$$

where $h_n$, $v_n$ are the same as in (70), and $\varphi$ is a $C^\infty$ - function such that $|\varphi^{(j)}| < C_j$ if $j \geq m$, and

$$\varphi(x - x_n, n) = (L_n - |x - x_n|)^{m-2}, \quad |x - x_n| \geq 1.$$

We assume that (71) holds with small enough $\beta > 0$ and $\varepsilon_0 = \varepsilon_0(\beta) > 0$. Then one can easily check that

$$J_p(W) = \infty, \quad \text{if} \quad p < m \quad \text{or} \quad p \geq m, \quad \varepsilon \geq 0;$$

$$J_p(W) < \infty, \quad \text{if} \quad p \geq m, \quad \varepsilon < 0.$$

Thus, Theorem 1 with any $p \geq m$ can be applied to the operator $H^\alpha$ with potential $W$ when $\varepsilon < 0$. Arguments similar to those used in the proof of Theorem 6 can be applied to prove that operator $H^\alpha$ with potential $W$, has singular spectrum for a.e $\alpha$ if $\varepsilon \geq 0$. Then, $V(x, \varepsilon; m)$, $m > 2$, can be realized as $V = W + V_1$ where $V_1$ is a compactly supported function such that $|V_1^{(m-1)}| < C$, $V_1^{(m)} \notin L^2$ (compare with the Remark following Theorem6).

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# References

[1] Deift,P., Killip,R.: On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials, Comm. Math. Phys. **203**, 341- 347 (1999).

[2] Christ,M., Kisilev,A., Remling,C.: The absolutely continuous spectrum for one-dimensional Schrödinger operators with decaying potentials, Math.Res.Lett. **4**, no.5, 719-723(1997).

[3] Christ,M., Kisilev,A.: The absolutely continuous spectrum for one-dimensional Schrödinger operators with slowly decaying potentials : some optimal results, J.Am.Math.Soc.**193**, no.1, 151-170(1998).

[4] Remling,C.: The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials, Comm. Math. Phys. **203**, 341- 347 (1999).

[5] Gardner, C., Greene, J., Kruskal, R., Miura, R.: Korteweg - de Vries equation and generalizations, VI, Methods for exact solution, Comm. Pure Appl. Math. **27**, 97- 133(1974).

[6] Kato,T. :*Perturbation theory for linear operators.* Reprint of the 1980 edition, Berlin, Springer, 1995.

[7] Dodd, R. K., Eilenbeck, J. C., Gibbon, J. D., Morris, H. C.: *Solitons and nonlinear wave equations*, Academic Press, 1982.

[8] Levitan, B., Sargsyan, I.:*Introduction to spectral theory,* Translation Math. Monographs,**39**, AMS, Providence, RI, 1976.

[9] Glazman, I.: *Direct methods of qualitative spectral analysis of singular differential operators,* Israeli Progr. Scient. Transl., Jerusalem, 1965.

[10] Atkinson, F. V.:*Discrete and continuous boundary problems*, Mathematics in Science and Engineering, **8**, Academic Press, New York-London, 1964.

[11] Molchanov, S.: Multiscattering on sparse bumps, Contemp. Math., **217**, 157-181(1998).

[12] Molchanov, S.: Multiscale averaging for ordinary differential equations. *Homogenization,* Series on advances in mathematics for applied sciences,**50**, (Kozlov,S., Berdichevskii,V., Papanicolaou,G., eds.), World Scientific, 316-397(1999).

[13] Simon, B.: Spectral analysis of rank one perturbations and applications, CRM Lecture Notes, **8**, (Feldman,J., Froese,R., Rosen,L., eds.), AMS, Providence,RI, 1995.

[14] Simon, B.: Bounded eigenvunctions and absolutely continuous spectra for one-dimensional Schrödinger operators, **124**, no.11, 3361–3369(1996).

[15] Last, Y., Simon, B.: Eigenfunctions, transfer matrices , and absolutely continuous spectra of one-dimensional Schrödinger operators, Invent. Math., **135**, no. 2, 329–367(1999).

[16] Gabushin, V.: Inequalities for norms of the functions and their derivatives in the $L_p$ metrics, Math. Notes, **1**, 194-198(1967).

[17] Lieb, E., Thirring, W.: Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. Studies in Mathematical Physics: Essay in Honor of Valentine Bargmann, Princeton Univ. Press, Princeton, 1976, 269-303.

[18] Birman, M.: Perturbations of quadratic forms and the spectrum of singular boundary value problems, Dokl. Akad. Nauk SSSR(Russian), **125**, 471–474(1959).

[19] Reed, M., Simon, B. :*Methods of modern mathematical physics, IV. Analysis of operators*, New York-London, Academic Press,1972.

[20] Novitskii, M.: *Spectral invariants of the Schrödinger operator families, inverse problems and related fuctionals,* Doctoral theses, Kharkov, 1997.